Some problems involving fractional order operators

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Definition $(-\Delta)^\gamma$

- Infinitesimal generator of a Levy process
- Pseudo-differential operator, principal symbol $|\xi|^{2\gamma}$.
- $$(-\Delta)^\gamma f(x) = C_{n,\gamma} \int_{\mathbb{R}^n} \frac{f(x) - f(\xi)}{|x - \xi|^{n+2\gamma}}\, d\xi$$
- **Caffarelli-Silvestre's extension**: For $\gamma \in (0,1)$, $\gamma = \frac{1-a}{2}$,

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- Dirichlet-to-Neumann operator, non-local
- $\gamma = 1/2$ (half-Laplacian)

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- **Examples:**
  - Conformal Laplacian (order 2), $L_g := -\Delta_g + c_n R_g$
  - Paneitz operator (order 4), $P_g := (-\Delta_g)^2 + \ldots$
  - Order $2k$ (Graham-Jenne-Mason-Sparling)

Conformally compact Einstein manifolds

(X^{n+1}, g) smooth manifold of dimension $n+1$ with metric g
Smooth boundary $M^n = \partial X$
Defining function: $\rho > 0$ in X, $\rho = 0$ on M, $d\rho \neq 0$ on M.

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Def: (X, g) is conformally compact if there exists ρ such that $(X, \bar{g})$ is compact for $\bar{g} = \rho^2 g$.
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Model example: Hyperbolic space

$$(\mathbb{H}^{n+1}, g_{\mathbb{H}}) = \left(B^{n+1}, \left(\frac{2}{1-|y|^2}\right)^2 |dy|^2\right) = \left(\mathbb{R}^{n+1}_+, \frac{|dx|^2+dy^2}{y^2}\right)$$

Scattering theory

Theorem (Graham-Zworski, Mazzeo)

There exists a meromorphic family of pseudo-differential operators on M, called $\{S(s)\}$, $s \in \mathbb{C}$, $Re(s) > n/2$ such that

$$\sigma\left(S(s)\right) = c_{n,s}\sigma\left((-\Delta_M)^{s-n/2}\right)$$

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- **Construction:** Given f on M, solve the eigenvalue equation

$$-\Delta_g u_s - s(n-s)u_s = 0 \text{ in } X \quad (1)$$

$$u_s = F\rho^s + G\rho^{n-s}, \quad F|_{\rho=0} = f. \quad (2)$$

Then $S(s)f = G|_M$.

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- **Operators of fractional order:**

  Define $\boxed{P_\gamma := c_{n,\gamma} S\left(\frac{n}{2}+\gamma\right)}$

  Conformal: $\quad g_u = u^{\frac{4}{n-2\gamma}} g \quad \rightarrow \quad P_\gamma^{g_u} = u^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma^g(u\,\cdot)$

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- **Operators of fractional order:**

  Define $\boxed{P_\gamma := c_{n,\gamma} S\left(\frac{n}{2} + \gamma\right)}$

  Conformal: $\quad g_u = u^{\frac{4}{n-2\gamma}} g \quad \to \quad P_\gamma^{g_u} = u^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma^g(u\,\cdot)$
- **Model example:** in $\mathbb{R}^n$, $\gamma \notin \mathbb{Z}$: $\quad P_\gamma = (-\Delta)^\gamma$

Results (Chang-G.)

Theorem A

Caffarelli-Silvestre extension problem in $\mathbb{R}^{n+1}_+$, $\quad \gamma \in (0,1)$

$$\Updownarrow$$

scattering theory in $\mathbb{H}^{n+1}$

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Theorem B

For $\gamma \in (0,1)$, on general c.c. Einstein manifolds,

$$\begin{cases} \operatorname{div}(\rho^a \nabla U) + E(\rho) U = 0 & \text{in } (X, \bar{g}) \\ U = f, & \text{at } \rho = 0 \end{cases} \quad \Rightarrow$$

$$P_\gamma f = c_\gamma \lim_{\rho \to 0} \rho^a \partial_\rho U, \quad \gamma = \frac{1-a}{2}$$

Here $E(\rho) = -\Delta_{\bar{g}} \left(\rho^{\frac{a}{2}}\right) \rho^{\frac{a}{2}} + \left(\gamma^2 - \frac{1}{4}\right) \rho^{-2+a} + \frac{n-1}{4n} R_{\bar{g}} \rho^a$

Theorem C

For $\gamma = m + \alpha$, $m \in \mathbb{N}$, $\alpha \in (0,1)$

$$\begin{cases} \operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u = f, & \text{at } y = 0 \end{cases} \quad \Rightarrow$$

$$P_\gamma f = c_\gamma \lim_{y \to 0} y^{1-2\alpha} \underbrace{\partial_y \left(y^{-1} \left(\partial_y \left(\ldots y^{-1} \partial_y u \right) \right) \right)}_{m+1 \text{ derivatives}}$$

Q_γ curvature

Def: $Q^g_\gamma := P^g_\gamma(1)$.

Conformal invariance:

$$g_w = w^{\frac{4}{n-2\gamma}} g \quad \leftrightarrow \quad P^g_\gamma(w) = w^{\frac{n+2\gamma}{n-2\gamma}} Q^{g_w}_\gamma.$$

$\gamma-$**Yamabe problem:** find $g_w = w^{\frac{4}{n-2\gamma}} g$ such that $Q^{g_w}_\gamma = cst$

Proposition (Qing-G.)

If the γ-Yamabe constant of M is strictly less than S^n, then there exists a minimizer for $I_\gamma[U] = \dfrac{\int_X \rho^a |\nabla U|^2 + \int_X \rho^a E(\rho) U^2}{\left(\int_M U^{\frac{2n}{n-2\gamma}}\right)^{\frac{n-2\gamma}{n}}}$

- Second order degenerate elliptic equations
- Weighted Sobolev trace inequalities
- $\gamma = 1/2:$ constant mean curvature on the boundary (Escobar)

Theorem: (G.-Mazzeo-Sire)

The model case $S^n \backslash S^k$ with the metric given by $S^{n-k-1} \times \mathbb{H}^{k+1}$ has constant Q_γ-curvature. Moreover, if $k < n/2$, Q_γ is positive when $k < \frac{n-2\gamma}{2}$.

Renormalized volume

We have that

$$vol_{g^+}(\{\rho > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \ldots + c_{n-1}\epsilon^{-1} + V + o(1)$$

Computation (Fefferman-Graham)
Let u_s be the solution of

$$-\Delta_g u_s - s(n-s)u_s = 0 \text{ in } X \tag{3}$$

$$u_s = F\rho^s + G\rho^{n-s}, \quad F|_{\rho=0} = 1. \tag{4}$$

Define $v = -\left.\frac{d}{ds}\right|_{s=n} u_s$.
It is a solution of $-\Delta_{g^+} v = n \quad \text{in } X$.
Integrate on $\rho > \epsilon$. Then

$$V = \frac{1}{d_{\frac{n}{2}}} \int_M Q_{\frac{n}{2}}.$$

Weighted renormalized volume

How about if we use u_s instead??
Then we define $vol_{g^+,\gamma}(\{\rho > \epsilon\}) := \int_{\{\rho>\epsilon\}} (\rho^*)^{\frac{n}{2}-\gamma}\, dvol_{g^+}$

Chang-G

It has an asymptotic expansion

- If $\gamma < \frac{1}{2}$, then

$$vol_{g^+,\gamma}(\{\rho > \epsilon\}) = \epsilon^{-\frac{n}{2}-\gamma}\left[\left(\tfrac{n}{2}+\gamma\right)^{-1} vol(M) + \epsilon^{2\gamma} V_\gamma + l.o.t.\right]$$

  where $V_\gamma := \frac{1}{d_\gamma}\frac{1}{\frac{n}{2}-\gamma}\int_M Q_\gamma$.

- If $\gamma > \frac{1}{2}$, then

$$vol_{g^+,\gamma}(\{\rho > \epsilon\}) = \epsilon^{-\frac{n}{2}-\gamma}\left[\left(\tfrac{n}{2}+\gamma\right)^{-1} vol(M) + \epsilon W_0 + l.o.t.\right]$$

  for $W_0 := \left(\frac{n}{2}+\gamma\right)^{-1}(2n-1)\int_M J_0$.

Thank you!!