SKINNER-RUSK FORMALISM FOR OPTIMAL CONTROL

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Abstract

In 1983, the dynamics of a mechanical system was represented by a first-order system on a suitable phase space by R. Skinner and R. Rusk [8]. The corresponding unified formalism developed for optimal control systems allows us to formulate geometrically the necessary conditions given by Pontryagin's Maximum Principle, as long as the differentiability with respect to controls is assumed (see [1]).

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1 Previous geometric concepts

Let $\pi\colon E \longrightarrow \mathbf{R}$ be the configuration bundle of a non-autonomous system ($\dim E = n+1$), and $J^1\pi$ the first-order jet bundle [7]. We introduce the *extended momentum phase space* T^*E, with natural coordinates (t, q^i, p_0, p_i); the *restricted momentum phase space* $J^1\pi^* := \mathrm{T}^*E/\pi^*\mathrm{T}^*\mathbf{R}$, with natural coordinates (t, q^i, p_i); the *extended jet-momentum bundle* $\mathcal{W} := J^1\pi \times_E \mathrm{T}^*E$, with natural coordinates (t, q^i, v^i, p_0, p_i), and the *restricted jet-momentum bundle* $\mathcal{W}_r := J^1\pi \times_E J^1\pi^*$, with natural coordinates (t, q^i, v^i, p_i). There is also the natural projection $\mu_{\mathcal{W}}\colon \mathcal{W} \to \mathcal{W}_r$.

Definition 1 *Let $\rho_E\colon \mathcal{W} \to E$, $\rho_{\mathbf{R}}\colon \mathcal{W} \to \mathbf{R}$, $\rho_2\colon \mathcal{W} \to T^*E$ the canonical projections. Let Θ and Ω be the canonical forms of T^*E.*

1. *The* coupling 1-form *in $\mathcal{W}$, is the $\rho_{\mathbf{R}}$-semibasic 1-form $\hat{\mathcal{C}} \in \mathit{\Omega}^1(\mathcal{W})$, which is defined as follows: for every $w = (j^1_t\phi, \alpha) \in \mathcal{W}$ (that is, $\alpha \in T^*_{\rho_E(w)}E$ and $V \in \mathrm{T}_w\mathcal{W}$), then $\hat{\mathcal{C}}(V) := \alpha(\mathrm{T}_w(\phi \circ \pi \circ \rho_E)V)$.*

2. *The* canonical 1-form *is $\Theta_{\mathcal{W}} := \rho_2^*\Theta \in \mathit{\Omega}^1(\mathcal{W})$, and it is a ρ_E-semibasic form. The* canonical 2-form *is $\Omega_{\mathcal{W}} := -\mathrm{d}\Theta_{\mathcal{W}} = \rho_2^*\Omega \in \mathit{\Omega}^2(\mathcal{W})$.*

The couple $(\mathcal{W}, \Omega_{\mathcal{W}})$ is a presymplectic manifold. We have the local expressions:

$$\hat{\mathcal{C}} = (p_0 + p_i v^i)\mathrm{d}t,\ \Theta_{\mathcal{W}} = p_i\mathrm{d}q^i + p_0\mathrm{d}t,\ \Omega_{\mathcal{W}} = -\mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}p_0 \wedge \mathrm{d}t.$$

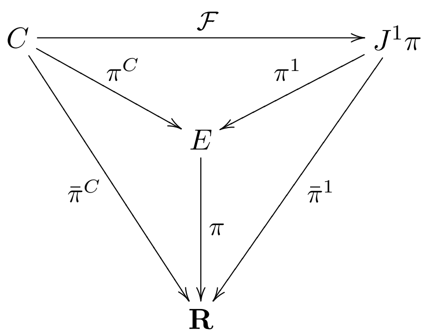
2 Optimal control theory

Let C be the bundle of controls. Denote by $\pi^C\colon C \longrightarrow E$, $\pi : E \longrightarrow \mathbf{R}$ the fiber bundle structures. We have natural coordinates adapted to the bundle structure: (t, q^i) for E and (t, q^i, u^a) for C, where t is time, q^j denote the *state variables* and u^a, $1 \leq a \leq m$, the *control inputs*.

Statement 1 *The* non-autonomous optimal control problem *consists in finding a C^2-piecewise smooth curve $\gamma(t) = (t, q^j(t), u^a(t))$ and $T \in \mathbf{R}^+$ satisfying:*

1. *the conditions for the state variables at time 0 and T;*
2. *the* state equations*: $\dot{q}^i = \mathcal{F}^i(t, q^j(t), u^a(t))\,,\ 1 \leq i \leq n$;*
3. *it minimizes the functional $\mathcal{J}(\gamma) = \int_0^T L(t, q^j(t), u^a(t))\,\mathrm{d}t\,.$*

Geometrically, $\mathcal{F} : C \longrightarrow J^1\pi$ makes the following diagram commutative:



Locally, $\mathcal{F}(t,q^i,u^a)=(t,q^i,\mathcal{F}^i(t,q^i,u^a))$.

Theorem 1 *(Pontryagin's Maximum Principle): If* $\gamma:[0,T]\to C$, $\gamma(t)=(t,q^i(t),u^a(t))$*, with* $\gamma(0)$ *and* $\gamma(T)$ *fixed, is an optimal trajectory, then there exist functions* $p_i(t)$, $1\leq i\leq n$*, verifying:*

$$\frac{dq^i}{dt}=\frac{\partial\mathcal{H}}{\partial p_i}(t,q^i(t),u^a(t),p_i(t))\ ,\ \frac{dp_i}{dt}=-\frac{\partial\mathcal{H}}{\partial q^i}(t,q^i(t),u^a(t),p_i(t))$$
$$\mathcal{H}(t,q^i(t),u^a(t),p_i(t))=\max_{u^a}\mathcal{H}(t,q^i(t),u^a,p_i(t)),\quad t\in[0,T]\,, \tag{1}$$

where $\mathcal{H}(t,q^i,u^a,p_i)=p_j\mathcal{F}^j(t,q^i,u^a)+pL(t,q^i,u^a)$*, and* $p\in\{-1,0\}$.

Condition (1) is usually replaced by the weaker condition $\frac{\partial\mathcal{H}}{\partial u^a}=0$, $1\leq a\leq m$. In this weaker form, the Maximum Principle only applies to optimal trajectories with optimal controls interior to the control fibres.

We only consider the case $p=-1$ (*normal extremal trajectories*), and *regular optimal control problems* (the matrix $\left(\frac{\partial^2\mathcal{H}}{\partial u^a\partial u^b}\right)$ has maximal rank).

3 Unified geometric framework for optimal control

This formulation of optimal control systems is based on the unified formalism developed for time-dependent systems in [2, 4], and for field theories in [3, 5].

Geometrically, an *optimal control system* is determined by the pair $(\mathcal{L},\mathcal{F})$ where $\mathcal{L}\in\Omega^1(C)$ is a $\bar\pi^C$-semibasic 1-form. Then $\mathcal{L}=L\bar\pi^{C*}\mathrm{d}t$, with $L\in\mathrm{C}^\infty(C)$ (the *cost function*); and $\mathcal{F}$ is the jet field introduced above.

As $\mathrm{Graph}\,\mathcal{F}$ is a subset of $C\times_E J^1\pi$, we can define the *extended control-jet-momentum bundle* $\mathcal{W}^\mathcal{F}=\mathrm{Graph}\,\mathcal{F}\times_E T^*E$, which is a submanifold of $C\times_E\mathcal{W}$, and the *restricted control-jet-momentum bundle* $\mathcal{W}_r^\mathcal{F}=\mathrm{Graph}\,\mathcal{F}\times_E J^1\pi^*$, which is a submanifold of $C\times_E\mathcal{W}_r$. Natural coordinates are (t,q^i,u^a,p_0,p_i) and (t,q^i,u^a,p_i), respectively. We have the immersions:

$$\begin{array}{lll} i^\mathcal{F} & : & \mathcal{W}^\mathcal{F}\hookrightarrow C\times_E\mathcal{W}\,,\quad i^\mathcal{F}(t,q^i,u^a,p_0,p_i)=(t,q^i,u^a,\mathcal{F}^i(t,q^j,u^b,),p_0,p_i)\ ,\\ i_r^\mathcal{F} & : & \mathcal{W}_r^\mathcal{F}\hookrightarrow C\times_E\mathcal{W}_r\,,\quad i_r^\mathcal{F}(t,q^i,u^a,p_i)=(t,q^i,u^a,\mathcal{F}^i(t,q^j,u^b),p_i)\ ,\end{array}$$

and the natural projections (submersions):

$$\begin{array}{lllll} \rho_1^\mathcal{F}:\mathcal{W}^\mathcal{F}\longrightarrow C & , & \rho_2^\mathcal{F}:\mathcal{W}^\mathcal{F}\longrightarrow \mathrm{T}^*E & , & \rho_E^\mathcal{F}:\mathcal{W}^\mathcal{F}\longrightarrow E\\ \rho_\mathbf{R}^\mathcal{F}:\mathcal{W}^\mathcal{F}\longrightarrow \mathbf{R} & , & \rho_1^{r\mathcal{F}}:\mathcal{W}_r^\mathcal{F}\longrightarrow C & , & \mu_{\mathcal{W}^\mathcal{F}}:\mathcal{W}^\mathcal{F}\longrightarrow \mathcal{W}_r^\mathcal{F}\\ \rho_\mathbf{R}^{r\mathcal{F}}:\mathcal{W}_r^\mathcal{F}\longrightarrow \mathbf{R} & , & \rho_E^{r\mathcal{F}}:\mathcal{W}_r^\mathcal{F}\longrightarrow E & . & \rho_2^{r\mathcal{F}}:\mathcal{W}_r^\mathcal{F}\longrightarrow J^1\pi^* \end{array} \tag{2}$$

We also have the submersion $\sigma_{\mathcal{W}}\colon C \times_E \mathcal{W} \longrightarrow \mathcal{W}$. Now, let

$$\mathcal{C}_{\mathcal{W}^{\mathcal{F}}} = (\sigma_{\mathcal{W}} \circ i^{\mathcal{F}})^* \hat{\mathcal{C}} \ , \ \Theta_{\mathcal{W}^{\mathcal{F}}} = (\sigma_{\mathcal{W}} \circ i^{\mathcal{F}})^* \Theta_{\mathcal{W}} \ , \ \Omega_{\mathcal{W}^{\mathcal{F}}} = (\sigma_{\mathcal{W}} \circ i^{\mathcal{F}})^* \Omega_{\mathcal{W}} \ .$$

Denote by $\mathcal{H}_{\mathcal{W}^{\mathcal{F}}} : \mathcal{W}^{\mathcal{F}} \longrightarrow \mathbf{R}$ the unique function such that $\mathcal{C}_{\mathcal{W}^{\mathcal{F}}} - (\rho_1^{\mathcal{F}})^* \mathcal{L} = \mathcal{H}_{\mathcal{W}^{\mathcal{F}}} (\rho_{\mathbf{R}}^{\mathcal{F}})^* \mathrm{d}t$. Locally

$$\begin{array}{rcl} \mathcal{C}_{\mathcal{W}^{\mathcal{F}}} & = & (p_0 + p_i \mathcal{F}^i(t, q^j, u^a))\mathrm{d}t \\ \Theta_{\mathcal{W}^{\mathcal{F}}} = p_i \mathrm{d}q^i + p_0 \mathrm{d}t & , & \Omega_{\mathcal{W}^{\mathcal{F}}} = -\mathrm{d}p_i \wedge \mathrm{d}q^i - \mathrm{d}p_0 \wedge \mathrm{d}t \\ \mathcal{H}_{\mathcal{W}^{\mathcal{F}}}(t, q^i, u^a, p_0, p_i) & = & p_0 + p_i \mathcal{F}^i(t, q^j, u^a) - L(t, q^j, u^a) \, . \end{array}$$

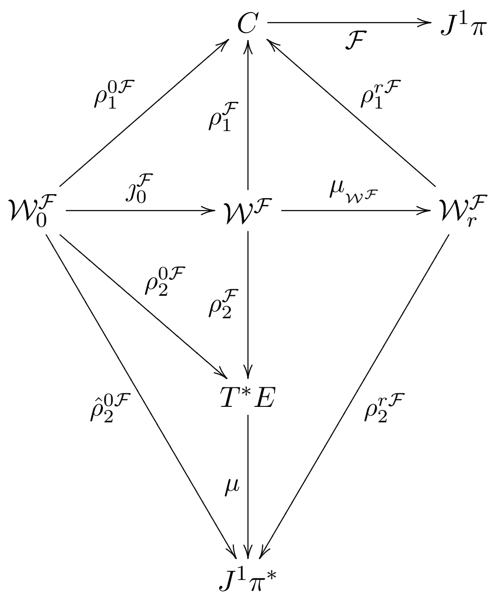
The triad $(\mathcal{W}^{\mathcal{F}}, \Omega_{\mathcal{W}^{\mathcal{F}}}, \mathcal{H}_{\mathcal{W}^{\mathcal{F}}})$ is a presymplectic Hamiltonian system, and the initial submanifold for the constraint algorithm (see [3, 5]) is $\mathcal{W}_0^{\mathcal{F}} = \{w \in \mathcal{W}^{\mathcal{F}} \mid \mathcal{H}_{\mathcal{W}^{\mathcal{F}}}(w) = 0\}$. Taking (t, q^i, u^a, p_i) as natural coordinates in $\mathcal{W}_0^{\mathcal{F}}$, the natural embedding $\jmath_0^{\mathcal{F}}\colon \mathcal{W}_0^{\mathcal{F}} \longrightarrow \mathcal{W}^{\mathcal{F}}$ is locally given by

$$\jmath_0^{\mathcal{F}}(t, q^i, u^a, p_i) = (t, q^i, u^a, L(t, q^j, u^b) - p_i \mathcal{F}^i(t, q^j, u^b), p_j) \, .$$

We have the following restrictions to $\mathcal{W}_0^{\mathcal{F}}$ of some of the projections (2):

$$\rho_1^{0\mathcal{F}}\colon \mathcal{W}_0^{\mathcal{F}} \longrightarrow C \ , \ \rho_2^{0\mathcal{F}}\colon \mathcal{W}_0^{\mathcal{F}} \longrightarrow \mathrm{T}^*E \ , \ \rho_E^{0\mathcal{F}}\colon \mathcal{W}_0^{\mathcal{F}} \longrightarrow E \ , \ \rho_{\mathbf{R}}^{0\mathcal{F}}\colon \mathcal{W}_0^{\mathcal{F}} \longrightarrow \mathbf{R} \, .$$

and so, we have the diagram



Proposition 1 *$\mathcal{W}_0^{\mathcal{F}}$ is a 1-codimensional $\mu_{\mathcal{W}^{\mathcal{F}}}$-transverse submanifold of $\mathcal{W}^{\mathcal{F}}$ diffeomorphic to $\mathcal{W}_r^{\mathcal{F}}$.*

Finally, we define the forms $\Theta_{\mathcal{W}_0^{\mathcal{F}}} = (\jmath_0^{\mathcal{F}})^*\Theta_{\mathcal{W}^{\mathcal{F}}}$ and $\Omega_{\mathcal{W}_0^{\mathcal{F}}} = (\jmath_0^{\mathcal{F}})^*\Omega_{\mathcal{W}^{\mathcal{F}}}$, with local expressions

$$\Theta_{\mathcal{W}_0^{\mathcal{F}}} = (L - p_i\mathcal{F}^i)\mathrm{d}t + p_i\mathrm{d}q^i \ , \ \Omega_{\mathcal{W}_0^{\mathcal{F}}} = -\mathrm{d}(L - p_i\mathcal{F}^i)\wedge \mathrm{d}t - \mathrm{d}p_i \wedge \mathrm{d}q^i \ ,$$

and we can set the Optimal Control equations geometrically as follows:

Theorem 2 *If $\gamma(t) = (t, q^i(t), u^a(t))$ is a solution to the regular optimal control problem given by $(\mathcal{L}, \mathcal{F})$, then there exists an integral curve of a vector field $Z \in \mathfrak{X}(\mathcal{W}_0^{\mathcal{F}})$, whose projection to C is $\gamma(t)$; such that Z is a solution to*

$$i(Z)\Omega_{\mathcal{W}_0^{\mathcal{F}}} = 0 \quad , \quad i(Z)(\rho_{\mathbf{R}}^{\mathcal{F}} \circ \jmath_0^{\mathcal{F}})^*\mathrm{d}t \neq 0$$

in a submanifold of $\mathcal{W}_0^{\mathcal{F}}$ which is given by the constraint algorithm.

This is a geometrical version of the weak form of the Maximum Principle.

4 Implicit optimal control problems

The above formalism is valid for optimal control problems described by *implicit state equations* $\Phi^\alpha(t, q, \dot{q}, u) = 0$, $1 \leq \alpha \leq s$, which are constraints determining a submanifold M_C of $C \times_E J^1\pi$.

An *implicit optimal control system* is given by $(\mathcal{L}, M_C)$, where now $\mathcal{L} \in \mathit{\Omega}^1(M_C)$. The so-called *extended control-jet-momentum bundle* $\mathcal{W}^{M_C} = M_C \times_E T^*E$ and *restricted control-jet-momentum bundle* $\mathcal{W}_r^{M_C} = M_C \times_E J^1\pi^*$ are submanifolds of $C \times_E \mathcal{W}$ and $C \times_E \mathcal{W}_r$ respectively, with canonical embeddings

$$i^{M_C}\colon \mathcal{W}^{M_C} \hookrightarrow C \times_E \mathcal{W} \, , \quad i_r^{M_C}\colon \mathcal{W}_r^{M_C} \hookrightarrow C \times_E \mathcal{W}_r \, .$$

We have the submersions

$$\mu_{\mathcal{W}^{M_C}}\colon \mathcal{W}^{M_C} \longrightarrow \mathcal{W}_r^{M_C} \ , \ \rho_1^{M_C}\colon \mathcal{W}^{M_C} \longrightarrow M_C \ , \ \rho_{\mathbf{R}}^{M_C}\colon \mathcal{W}^{M_C} \longrightarrow \mathbf{R} \ .$$

Now, let

$$\mathcal{C}_{\mathcal{W}^{M_C}} = (\sigma_{\mathcal{W}}\circ i^{M_C})^*\hat{\mathcal{C}} \ , \ \Theta_{\mathcal{W}^{M_C}} = (\sigma_{\mathcal{W}}\circ i^{M_C})^*\Theta_{\mathcal{W}} \ , \ \Omega_{\mathcal{W}^{M_C}} = (\sigma_{\mathcal{W}}\circ i^{M_C})^*\Omega_{\mathcal{W}} \, .$$

and the function $\mathcal{H}_{\mathcal{W}^{M_C}} \in \mathrm{C}^\infty(\mathcal{W}^{M_C})$ defined by $\mathcal{C}_{\mathcal{W}^{M_C}} - (\rho_1^{M_C})^*\mathcal{L} = \mathcal{H}_{\mathcal{W}^{M_C}}(\rho_{\mathbf{R}}^{M_C})^*\mathrm{d}t$. The initial submanifold for the constraint algorithm is $\mathcal{W}_0^{M_C} = \{w \in \mathcal{W}^{M_C} \mid \mathcal{H}_{\mathcal{W}^{M_C}}(w) = 0\}$, with natural embedding $\jmath_0^{M_C}\colon \mathcal{W}_0^{M_C} \hookrightarrow \mathcal{W}^{M_C}$.

Proposition 2 *$\mathcal{W}_0^{M_C}$ is a 1-codimensional $\mu_{\mathcal{W}^{M_C}}$-transversal submanifold of $\mathcal{W}^{M_C}$, diffeomorphic to $\mathcal{W}_r^{M_C}$.*

Finally, we can define the forms $\Theta_{\mathcal{W}_0^{M_C}} = (\jmath_0^{M_C})^*\Theta_{\mathcal{W}^{M_C}}$ and $\Omega_{\mathcal{W}_0^{M_C}} = (\jmath_0^{M_C})^*\Omega_{\mathcal{W}^{M_C}}$, and we can set the Optimal Control equations as the solutions (where there exist) of the equation

$$i(Z)\Omega_{\mathcal{W}_0^{M_C}} = 0, \qquad i(Z)(\rho_{\mathbf{R}}^{M_C} \circ \jmath_0^{M_C})^*\mathrm{d}t \neq 0$$

We characterize these solutions as follows:

Proposition 3 *Fixed $w \in \mathcal{W}_0^{M_C}$, the following conditions are equivalent:*

1. *There exists a vector $Z_w \in T_w\mathcal{W}_0^{M_C}$ verifying $\Omega_{\mathcal{W}_0^{M_C}}(Z_w, Y_w) = 0$, $\forall Y_w \in T_w\mathcal{W}_0^{M_C}$.*
2. *There exists a vector $Z_w \in T_w(C\times_E\mathcal{W})$ verifying (i) $Z_w \in T_w\mathcal{W}_0^{M_C}$, and (ii) $i(Z_w)(\sigma_{\mathcal{W}}^*\Omega_{\mathcal{W}})_w \in (T_w\mathcal{W}_0^{M_C})^0$.*

Condition *(ii)* means that $(i(Z)\sigma_{\mathcal{W}}^*\Omega_{\mathcal{W}})|_w = (\lambda_\alpha d\Phi^\alpha + \lambda d(\hat{C} - \hat{L}))|_w$, for some Lagrange multipliers λ_α's and λ to be determined.

An application of these results are the *Descriptor systems* (see [1, 6]).

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