

Cooperative games with a priori unions: introduction of a wide set of solutions*

José Miguel Giménez[†]

December, 2006

[†] Department of Applied Mathematics III. Engineering School of Manresa.
Technical University of Catalonia.
E-mail: jose.miguel.gimenez@upc.edu

Abstract

The semivalues are solution concepts for cooperative games that assign to each player a weighted sum of his/her marginal contributions to the coalitions, where the weights only depend on the coalition size. The Shapley value and the Banzhaf value are semivalues. The solutions introduced here are modifications of the semivalues when we consider a priori coalition blocks in the player set. A first semivalue is used among the coalition blocks and a second semivalue acts within each block. For all these solutions, we offer a computation procedure based on suitable modifications of the multilinear extension of the game and a product of matrices.

Keywords: Cooperative Game; Solution; Semivalue; Coalition Structure; Multilinear Function.

Mathematics Subject Classification 2000: 91A12.

* Research partially supported by Grant MTM 2006-06064 of the Education and Science Spanish Ministry and the European Regional Development Fund and Grant SGR 2005-00651 of the Catalonia Government (*Generalitat de Catalunya*).

1 Introduction

The probabilistic values form a wide family of solutions for cooperative games. These values introduced by Weber [12] evolve from a probabilistic payoff process where each of the players possesses a probabilistic distribution over the coalitions he is member of. If according to this distribution a coalition is chosen then the payoff to the player is the marginal contribution of that player to the chosen coalition. Thus, the corresponding probabilistic value is obtained as the expected payoff of the players.

A particular type of probabilistic values are the semivalues, introduced and axiomatically characterized by Dubey et al. [5]. Now the probabilistic distributions only depend on the coalition size and coincide for all players. The semivalues represent a natural generalization of both important solution concepts, the Shapley value [11] and the Banzhaf value [2, 8].

The main objective of this work consists of constructing solutions based on marginal contributions for games with structure of coalition blocks. These blocks formed in the set of players can be economic, political or social groups of agents with affinities or common interests. The coalition blocks act as one unit in a first bargaining process. Later, a new bargaining process occurs among the members of each block.

The introduced solutions are modifications of the semivalues for situations with coalition structure. Here, two semivalues are used at two different levels: a first semivalue among the coalition blocks considered as players in a modified quotient game and a second semivalue among the players of each coalition block. According to their construction, these solutions are called *mixed modified semivalues*.

In addition, we offer a computation procedure for the mixed modified semivalues based on suitable modifications of the *Owen's multilinear extension* [7] and a product of matrices. For this computation we use reference systems of semivalues and we consider a particular type of semivalues whose allocations can be easily obtained from the multilinear extension of each game.

The organization of the paper is as follows. After a preliminary Section 2 devoted to cooperative games and semivalues, in Section 3 we introduce reference systems for semivalues and we compute allocations by semivalues to all players of a given game by means of a product of matrices. This section contains notations and procedures that are tools to reach the central objective of this work and it allows us to offer a self-contained paper. Section 4 provides the general method to construct the wide family of mixed modified semivalues for games with coalition structure, whereas Section 5 offers a computation procedure for this family of solutions. Finally, in Section 6, a generalization of these solutions is proposed.

2 Cooperative games and semivalues

A *cooperative game* with transferable utility is a pair (N, v) , where N is a finite set of *players* and $v : 2^N \rightarrow \mathbb{R}$ is the so-called *characteristic function*, which assigns to every *coalition* $S \subseteq N$ a real number $v(S)$, the *worth* of coalition S , and satisfies the natural condition $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$. A player $i \in N$ is a *dummy* in v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \setminus \{i\}$. Two players $i, j \in N$ are *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for

all $S \subseteq N \setminus \{i, j\}$.

From now on we suppose $N = \{1, 2, \dots, n\}$ and we denote with G_N the set of all cooperative games on N . For a given set of players N , we identify each game (N, v) with its characteristic function v .

A *solution* on the set of cooperative games G_N is an allocation rule that assigns a payoff to each game player, i.e., a function $\Psi : G_N \rightarrow \mathbb{R}^N$, where $\Psi[v] = (\Psi_1[v], \dots, \Psi_n[v])$. It represents a method to measure the negotiation strength of the players in the game. The payoff vector space \mathbb{R}^N is also called the allocation space.

In order to calibrate the importance of each player i in the different coalitions S , we can look at his/her *marginal contribution* $v(S) - v(S \setminus \{i\})$. If these marginal contributions are weighted by means of coefficients depending only on the coalition size, we arrive at the solution concept known as semivalue.

The semivalues were introduced and axiomatically characterized in Dubey et al. [5]. These solutions are characterized by means of four axioms:

- A1. *Additivity.* $\Psi[u + v] = \Psi[u] + \Psi[v] \forall u, v \in G_N$.
- A2. *Symmetry.* $\Psi_{\pi i}[\pi v] = \Psi_i[v] \forall v \in G_N, \forall i \in N, \forall \pi$ permutation of N , where game πv is defined by $(\pi v)(\pi S) = v(S) \forall S \subseteq N$.
- A3. *Positivity.* Game v monotonic implies $\Psi_i[v] \geq 0 \forall i \in N$.
- A4. *Projection.* $\Psi_i[v] = v(\{i\}) \forall v \in A_N$, where A_N denotes the set of additive games in G_N : games v such that $v(S \cup T) = v(S) + v(T)$ if $S \cap T = \emptyset$ and $S, T \subseteq N$.

Theorem 2.1 (Dubey et al., 1981 [5]) (a) *Every weighting vector (p_1, p_2, \dots, p_n) verifying conditions*

$$\sum_{s=1}^n \binom{n-1}{s-1} p_s = 1 \quad \text{and} \quad p_s \geq 0 \quad \text{for } 1 \leq s \leq n \quad (1)$$

defines a semivalue $\psi : G_N \rightarrow \mathbb{R}^N$ whose allocations are given by

$$\psi_i[v] = \sum_{S \subseteq N: i \in S} p_s [v(S) - v(S \setminus \{i\})] \quad \forall i \in N \quad (\text{where } s = |S|).$$

(b) *Conversely, every semivalue defined on G_N is of this form, so that, there exists a one-to-one map between the semivalues on G_N and the vectors $(p_s)_{s=1}^n$ that verify conditions (1).*

The number of coalitions of size s that contain a player $i \in N$ is $\binom{n-1}{s-1}$. Conditions (1) give a probability distribution on the set of these coalitions, assuming equal weight for equal size. With $Sem(G_N)$ we denote the set of all semivalues defined on G_N .

Well known solutions like the Shapley value [11] and the Banzhaf value [2, 8] are semivalues. In the case of the Banzhaf value each marginal contribution has an equal weight, namely $1/2^{n-1}$, whereas in the case of the Shapley value, all marginal

contributions to coalitions with a same size have an equal weight, i.e., $1/[n\binom{n-1}{s-1}]$ for $1 \leq s \leq n$.

Definition 2.2 *A semivalue on G_N is called binomial semivalue if its weighting coefficients are in geometric progression.*

We assume $p_{s+1} = kp_s$ for $1 \leq s \leq n-1$ and $k > 0$. Every positive number can be written as $k = \alpha/(1-\alpha)$ with $0 < \alpha < 1$, so that the first condition in (1) allows us to determine a parametric expression for the weighting coefficients:

$$1 = \sum_{s=1}^n \binom{n-1}{s-1} p_s = p_1 \sum_{s=1}^n \binom{n-1}{s-1} k^{s-1} = p_1 [1+k]^{n-1} \Rightarrow p_1 = (1-\alpha)^{n-1}$$

and $p_s = \alpha^{s-1}(1-\alpha)^{n-s}$ for $2 \leq s \leq n$.

The binomial semivalues are related with the numbers $\alpha \in (0, 1)$. Given a number $\alpha \in \mathbb{R}$, $0 < \alpha < 1$, we call α -binomial semivalue ψ_α to the semivalue on G_N whose weighting coefficients are

$$p_{\alpha,s} = \alpha^{s-1}(1-\alpha)^{n-s} \quad \text{for } 1 \leq s \leq n.$$

The extreme cases of binomial semivalues correspond to values $\alpha = 0$ and $\alpha = 1$. For $\alpha = 0$ we obtain the dictatorial index ψ_0 , with coefficients $(1, 0, \dots, 0)$, whereas for $\alpha = 1$ we obtain the marginal index ψ_1 , with coefficients $(0, \dots, 0, 1)$. The respective allocations are

$$(\psi_0)_i[v] = v(\{i\}) \quad \forall i \in N \quad \text{and} \quad (\psi_1)_i[v] = v(N) - v(N \setminus \{i\}) \quad \forall i \in N.$$

The Banzhaf value is the binomial semivalue for $\alpha = 1/2$.

3 Reference systems for semivalues

In this Section, since there exists a one-to-one map between semivalues and weighting vectors, we introduce a concept of reference system for semivalues related with reference systems for weighting vectors in a geometric sense. Then, the computation of allocations by any semivalue reduces to computations by semivalues belonging to a reference system. We will prove that n different binomial semivalues form a reference system for semivalues on cooperative games with n players and their allocations can be easily obtained.

Definition 3.1 *The family of semivalues on G_N $\{\psi_j\}_{j=1}^n$, with respective weighting coefficients $(p_{j,s})_{s=1}^n$, $1 \leq j \leq n$, forms a reference system of $Sem(G_N)$ if, and only if, the family of points*

$$\{P_j (p_{j,s})_{s=1}^n\}_{j=1}^n$$

forms a reference system of the hyperplane of \mathbb{R}^n with equation $\sum_{s=1}^n \binom{n-1}{s-1} p_s = 1$.

Proposition 3.2 ([6], [1]) *For $n > 1$, given n real numbers $\alpha_j \in [0, 1]$, such that $\alpha_j \neq \alpha_k$ if $j \neq k$, the family of binomial semivalues $\{\psi_{\alpha_j}\}_{j=1}^n$ forms a reference system of $Sem(G_N)$ ($n = |N|$).*

Proof We denote by P_j the point corresponding to the weighting coefficients of semivalue ψ_{α_j} , $1 \leq j \leq n$. According to Definition 3.1, we should prove that $\{P_j\}_{j=1}^n$ forms a reference system; taking P_1 as origin point and P_j , $2 \leq j \leq n$ as unit points, we should prove that the family $\{\overrightarrow{P_1 P_j}\}_{j=2}^n$ forms a basis of the hyperplane direction. It is enough to verify that this family is linearly independent.

In order to assure this last condition, it suffices to prove that the family $\{\overrightarrow{OP_j}\}_{j=1}^n$ is linearly independent, where O denotes the origin of coordinates. Indeed, we can write

$$\sum_{j=2}^n \beta_j \overrightarrow{P_1 P_j} = \sum_{j=2}^n \beta_j (\overrightarrow{OP_j} - \overrightarrow{OP_1}) = \left(- \sum_{j=2}^n \beta_j \right) \overrightarrow{OP_1} + \sum_{j=2}^n \beta_j \overrightarrow{OP_j} = \vec{0}.$$

The linear independence of $\{\overrightarrow{OP_j}\}_{j=1}^n$ implies $-\sum_{j=2}^n \beta_j = \beta_2 = \dots = \beta_n = 0$ and, in particular, $\beta_2 = \dots = \beta_n = 0$, which implies the linear independence of $\{\overrightarrow{P_1 P_j}\}_{j=2}^n$.

We denote by Δ the determinant of the matrix whose columns are the components of the vectors $\overrightarrow{OP_1}, \dots, \overrightarrow{OP_n}$,

$$\Delta = \begin{vmatrix} (1 - \alpha_1)^{n-1} & (1 - \alpha_2)^{n-1} & \dots & (1 - \alpha_n)^{n-1} \\ \alpha_1(1 - \alpha_1)^{n-2} & \alpha_2(1 - \alpha_2)^{n-2} & \dots & \alpha_n(1 - \alpha_n)^{n-2} \\ \dots & \dots & \dots & \dots \\ \alpha_1^{n-2}(1 - \alpha_1) & \alpha_2^{n-2}(1 - \alpha_2) & \dots & \alpha_n^{n-2}(1 - \alpha_n) \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{vmatrix}.$$

If $\alpha_j \neq 1$ for $1 \leq j \leq n$, we have

$$\begin{aligned} \Delta &= (1 - \alpha_1)^{n-1} (1 - \alpha_2)^{n-1} \dots (1 - \alpha_n)^{n-1} \prod_{1 \leq j < k \leq n} \left[\frac{\alpha_k}{1 - \alpha_k} - \frac{\alpha_j}{1 - \alpha_j} \right] \\ &= (1 - \alpha_1)^{n-1} (1 - \alpha_2)^{n-1} \dots (1 - \alpha_n)^{n-1} \prod_{1 \leq j < k \leq n} \frac{\alpha_k - \alpha_j}{(1 - \alpha_k)(1 - \alpha_j)} \\ &= \prod_{1 \leq j < k \leq n} (\alpha_k - \alpha_j) \end{aligned}$$

Since $\alpha_j \neq \alpha_k$ if $j \neq k$, $\Delta \neq 0$ from which the linear independence follows.

If, for instance, $\alpha_n = 1$, the last column is $(0, \dots, 0, 1)$. We can compute Δ developing by this column and we obtain a similar determinant as in the above cases. ■

Remark 3.3 According to Proposition 3.2, once we have fixed a reference system of binomial semivalues $\{\psi_{\alpha_j}\}_{j=1}^n$ in $Sem(G_N)$, for each semivalue ψ defined on G_N there exists a unique family of real numbers λ_j , $1 \leq j \leq n$, such that

$$\psi = \sum_{j=1}^n \lambda_j \psi_{\alpha_j} \quad \text{with} \quad \sum_{j=1}^n \lambda_j = 1.$$

The components of semivalue ψ in the reference system $\{\psi_{\alpha_j}\}_{j=1}^n$ are grouped according to the following notation:

$$\Lambda^t = (\lambda_1 \ \lambda_2 \ \dots \ \lambda_n). \quad (2)$$

Definition 3.4 (Owen, 1972 [7]) *The multilinear extension (MLE, in the sequel) of a game $v \in G_N$ is the function $f_v : [0, 1]^n \rightarrow \mathbb{R}$ defined by*

$$f_v(x_1, x_2, \dots, x_n) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \in N \setminus S} (1 - x_j) v(S), \quad (3)$$

It is known that the allocations according to the Shapley value Sh can be computed from the MLE of each game [7]:

$$Sh_i[v] = \int_0^1 \frac{\partial f_v}{\partial x_i}(t, t, \dots, t) dt \quad \forall i \in N, \quad \forall v \in G_N,$$

and, also, according to the Banzhaf value Bh [8]:

$$Bh_i[v] = \frac{\partial f_v}{\partial x_i}(1/2, 1/2, \dots, 1/2) \quad \forall i \in N, \quad \forall v \in G_N.$$

Now, we would generalize these results to all semivalues.

Theorem 3.5 ([6], [1]) *Let $f_v = f_v(x_1, x_2, \dots, x_n)$ be the MLE of game $v \in G_N$.*

(a) *The payoff vector that the binomial semivalue ψ_α assigns to the players of game $v \in G_N$ is*

$$\psi_\alpha[v] = \nabla f_v(\bar{\alpha}) \quad \forall \alpha \in [0, 1] \quad \text{where } \bar{\alpha} = (\alpha, \dots, \alpha).$$

(b) *The payoff vector that every semivalue ψ assigns to the players of game $v \in G_N$ is*

$$\psi[v] = B \Lambda$$

where the matrix $B = (b_{ij})$ depends on each reference system of semivalues $\{\psi_{\alpha_j}\}_{j=1}^n$,

$$b_{ij} = (\psi_{\alpha_j})_i[v] = \frac{\partial f_v}{\partial x_i}(\bar{\alpha}_j), \quad 1 \leq i, j \leq n,$$

and Λ is the matrix of the components of ψ in the reference system (as in (2)).

Proof (a) In the expression (3) of the MLE of game $v \in G_N$, we separate terms according to whether coalitions S contain or not player $i \in N$ and we compute the partial derivative with respect to variable x_i .

$$\begin{aligned} f_v(x_1, \dots, x_n) &= \sum_{S \subseteq N: i \in S} x_i \prod_{j \in S \setminus \{i\}} x_j \prod_{k \in N \setminus S} (1 - x_k) v(S) \\ &+ \sum_{S \subseteq N: i \notin S} (1 - x_i) \prod_{j \in S} x_j \prod_{k \in N \setminus (S \cup \{i\})} (1 - x_k) v(S); \end{aligned}$$

$$\begin{aligned} \frac{\partial f_v}{\partial x_i}(x_1, \dots, x_n) &= \sum_{S \subseteq N: i \in S} \prod_{j \in S \setminus \{i\}} x_j \prod_{k \in N \setminus S} (1 - x_k) v(S) \\ &- \sum_{S \subseteq N: i \notin S} \prod_{j \in S} x_j \prod_{k \in N \setminus (S \cup \{i\})} (1 - x_k) v(S); \end{aligned}$$

Replacing the variables by α for $0 \leq \alpha \leq 1$, if $\bar{\alpha} = (\alpha, \dots, \alpha)$, we obtain

$$\begin{aligned} \frac{\partial f_v}{\partial x_i}(\bar{\alpha}) &= \sum_{S \subseteq N: i \in S} \alpha^{s-1}(1-\alpha)^{n-s}v(S) - \sum_{S \subseteq N: i \notin S} \alpha^s(1-\alpha)^{n-s-1}v(S) \\ &= \sum_{S \subseteq N: i \in S} \alpha^{s-1}(1-\alpha)^{n-s}v(S) - \sum_{S \subseteq N: i \in S} \alpha^{s-1}(1-\alpha)^{n-s}v(S \setminus \{i\}) \\ &= \sum_{S \subseteq N: i \in S} \alpha^{s-1}(1-\alpha)^{n-s}[v(S) - v(S \setminus \{i\})] = (\psi_\alpha)_i[v]. \end{aligned}$$

(b) We consider a reference system of binomial semivalues $\{\psi_{\alpha_j}\}_{j=1}^n$ in $Sem(G_N)$. According to Remark 3.3, given a semivalue ψ on G_N , for every game $v \in G_N$ and every player $i \in N$ we have

$$\psi_i[v] = \left(\sum_{j=1}^n \lambda_j \psi_{\alpha_j} \right)_i[v] = \sum_{j=1}^n \lambda_j (\psi_{\alpha_j})_i[v] = \sum_{j=1}^n \lambda_j \frac{\partial f_v}{\partial x_i}(\bar{\alpha}_j),$$

where $f_v = f_v(x_1, \dots, x_n)$ is the MLE of game v . It suffices to identify each term with the elements of the matrices that have been defined, and hence we conclude that the expression in the statement follows. ■

Once a reference system of semivalues is chosen, a matrix summarizes the payoffs by any semivalue to all players of a given game.

Example 3.6 Four stores have a common supplier and obtain discounts according to its record of purchases. By performing joint orders, they can reach the discounts detailed in the following function v :

$$\begin{aligned} v(\{1\}) = 2, \quad v(\{2\}) = v(\{3\}) = v(\{4\}) = 1, \quad v(\{1, 2\}) = 5, \quad v(\{1, 3\}) = v(\{1, 4\}) = 4, \\ v(\{2, 3\}) = v(\{2, 4\}) = 3, \quad v(\{3, 4\}) = 2, \quad v\{1, 2, 3\} = 7, \quad v\{1, 2, 4\} = v\{1, 3, 4\} = 6, \\ v\{2, 3, 4\} = 5 \text{ and } v\{1, 2, 3, 4\} = 9. \end{aligned}$$

The four stores decide to act together –cooperation– and obtain the common discount: 9 %. Nevertheless, the stores are competitors in a same market. The problem consists in how to distribute these 9 units of utility among them.

The MLE of game v is

$$\begin{aligned} f_v(x_1, x_2, x_3, x_4) &= 2x_1 + x_2 + x_3 + x_4 + 2x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 \\ &\quad - x_1x_2x_3 - 2x_1x_2x_4 + x_1x_2x_3x_4 \end{aligned}$$

from which

$$\nabla f_v(\bar{\alpha}) = (2+4\alpha-3\alpha^2+\alpha^3, 1+4\alpha-3\alpha^2+\alpha^3, 1+2\alpha-\alpha^2+\alpha^3, 1+2\alpha-2\alpha^2+\alpha^3).$$

If we choose as reference system the α -binomial semivalues for values $\alpha = 0, 1/3, 2/3$ and 1, the matrix that summarizes the action of all the semivalues on 4-person games for game v is

$$B = \begin{pmatrix} 2 & 82/27 & 98/27 & 4 \\ 1 & 55/27 & 71/27 & 3 \\ 1 & 43/27 & 59/27 & 3 \\ 1 & 40/27 & 47/27 & 2 \end{pmatrix}$$

The Shapley value (w. coefficients $p_1 = 1/4, p_2 = 1/12, p_3 = 1/12, p_4 = 1/4$) in the reference system $\{\psi_0, \psi_{1/3}, \psi_{2/3}, \psi_1\}$ is

$$Sh = \frac{1}{8}\psi_0 + \frac{3}{8}\psi_{1/3} + \frac{3}{8}\psi_{2/3} + \frac{1}{8}\psi_1$$

so that $\Lambda^t = (1/8 \ 3/8 \ 3/8 \ 1/8)$ and the allocation to the players of game v according to the Shapley value is

$$Sh[v] = B\Lambda = \begin{pmatrix} 13/4 \\ 9/4 \\ 23/12 \\ 19/12 \end{pmatrix}.$$

A solution concept $\Psi : G_N \rightarrow \mathbb{R}^N$ verifies *efficiency* if the sum of allocations to the players agrees with the utility obtained by the grand coalition:

$$\sum_{i \in N} \Psi_i[v] = v(N) \quad \forall v \in G_N.$$

It is known that the Shapley value is the unique efficient semivalue. Thus, the utility distribution equals the allocation.

$$\text{Shapley value} \longrightarrow (3.25, 2.25, 1.92, 1.58)$$

On the other hand, if we consider, for instance, the binomial semivalue $\psi_{1/3}$ (w. coefficients $p_1 = 8/27, p_2 = 4/27, p_3 = 2/27, p_4 = 1/27$), the allocation to the players in game v corresponds with the second column of matrix B . Since $\psi_{1/3}$ is not efficient, the distribution of the total utility is obtained proportionally to the allocation, $82 : 55 : 43 : 40$.

$$1/3\text{-binomial semivalue} \longrightarrow (3.35, 2.25, 1.76, 1.64)$$

4 Mixed modified semivalues

From now on, given a semivalue, a superscript will denote the number of players in the cooperative games: $n = |N|$ for games in G_N .

Definition 4.1 Let ψ^n be a semivalue on G_N with weighting coefficients $(p_s^n)_{s=1}^n$. The family of induced semivalues by ψ^n on sets of games with less than n players is

$$\{\psi^m \in \text{Sem}(G_M) \text{ with } 1 \leq m \leq n \text{ and } m = |M| \},$$

where the respective weighting coefficients are recursively obtained according to the Pascal triangle (inverse) formula

$$p_s^m = p_s^{m+1} + p_{s+1}^{m+1} \quad 1 \leq s \leq m < n. \quad (4)$$

We can find the above definition in Dragan (1999) [4]. By convenience, we have included the initial semivalue in its induced family. It is not difficult to see that the induced semivalues of the Shapley value, the Banzhaf value or, in general, the α -binomial semivalues are of the same initial types.

$$\text{Shapley value: } p_s^{m+1} + p_{s+1}^{m+1} = \frac{1}{(m+1)\binom{m}{s-1}} + \frac{1}{(m+1)\binom{m}{s}} = \frac{1}{m\binom{m-1}{s-1}} = p_s^m;$$

α -binomial semivalues:

$$p_s^{m+1} + p_{s+1}^{m+1} = \alpha^{s-1}(1-\alpha)^{m+1-s} + \alpha^s(1-\alpha)^{m-s} = \alpha^{s-1}(1-\alpha)^{m-s} = p_s^m.$$

By applying expression (4) successively, the weighting coefficients of any induced semivalue ψ^m become

$$p_s^m = \sum_{j=0}^{n-m} \binom{n-m}{j} p_{s+j}^n \quad \text{for } 1 \leq s \leq m < n. \quad (5)$$

Proposition 4.2 *Let $\{\psi_{\alpha_j}^n\}_{j=1}^n$ be a reference system of binomial semivalues in $Sem(G_N)$.*

$$\text{If } \psi^n \in Sem(G_N) \text{ is } \psi^n = \sum_{j=1}^n \lambda_j \psi_{\alpha_j}^n, \text{ then } \psi^t = \sum_{j=1}^n \lambda_j \psi_{\alpha_j}^t \text{ for } 1 \leq t < n.$$

Proof Let us assume that $(p_s^n)_{s=1}^n$ are the weighting coefficients of semivalue $\psi^n \in Sem(G_N)$. We write these coefficients based on the coefficients of the binomial semivalues $\{\psi_{\alpha_j}^n\}$, $1 \leq j \leq n$:

$$p_s^n = \sum_{j=1}^n \lambda_j p_{\alpha_j, s}^n = \sum_{j=1}^n \lambda_j \alpha_j^{s-1} (1-\alpha_j)^{n-s} \quad \text{for } 1 \leq s \leq n.$$

In the equalities from $s = 1$ to $s = n - 1$, we separate a factor of type $(1 - \alpha_j)$,

$$p_s^n = \sum_{j=1}^n \lambda_j (1 - \alpha_j) \alpha_j^{s-1} (1 - \alpha_j)^{n-s-1}, \quad 1 \leq s \leq n - 1,$$

$$p_s^n + \sum_{j=1}^n \lambda_j \alpha_j^s (1 - \alpha_j)^{n-(s+1)} = \sum_{j=1}^n \lambda_j \alpha_j^{s-1} (1 - \alpha_j)^{n-1-s}, \quad 1 \leq s \leq n - 1,$$

from which

$$p_s^n + p_{s+1}^n = p_s^{n-1} = \sum_{j=1}^n \lambda_j \alpha_j^{s-1} (1 - \alpha_j)^{n-1-s} \quad \text{for } 1 \leq s \leq n - 1.$$

We have checked the property for level $t = n - 1$. By induction, it is easy to prove the statement for the remaining levels, $t = n - 2, \dots, 1$. ■

Remark 4.3 According to Proposition 4.2, the relative position of each semivalue with respect to a reference system of binomial semivalues remains the same for all its induced semivalues.

Thus, there are two possibilities to obtain the weighting coefficients of induced semivalues: (i) successively from the weighting coefficients of the initial semivalue by using the Pascal triangle formula (4) and (ii) by means of the coordinates in a reference system of binomial semivalues by using the weighting coefficients of induced binomial semivalues.

Let us consider cooperative games v defined on a given finite set of players N . We suppose that several groups of players with affinities or common interests act together and form a priori coalition blocks. We want to obtain allocation rules based on marginal contributions for games with structure of a priori coalition blocks.

Definition 4.4 *A structure of coalition blocks in the player set is a partition of N , $B = \{B_1, \dots, B_m\}$. With \mathcal{B}_N we denote the set of all coalition structures defined in N . A solution for cooperative games with coalition structure is a function $\Psi : G_N \times \mathcal{B}_N \rightarrow \mathbb{R}^N$ that assigns a payoff to each player, $\Psi[v; B] = (\Psi_1[v; B], \dots, \Psi_n[v; B])$.*

Let us suppose that two semivalues ψ^n and φ^n are defined on games with n players (eventually $\varphi^n = \psi^n$). The consideration of induced semivalues allows us to define a concept of *mixed modified semivalue for games with coalition structure* following a similar process to the one used by Owen to derive the coalition value [9] from the Shapley value [11] or the modified Banzhaf value for games with coalition structure [10] from the Banzhaf value [2, 8].

Let v be a cooperative game on N . Given a coalition structure $B = \{B_1, \dots, B_m\}$, we denote by M the set of classes in N given by the coalition structure B . For each subset $K \subseteq B_j$, we define a modified quotient game

$$u_{B_j|K}(L) = v\left(\bigcup_{l \in L} B_l \setminus K'\right) \quad \forall L \subseteq M,$$

where $K' = B_j \setminus K$. This is the game played by the partition classes with the exception of B_j , that is replaced by the subset K . Given a semivalue $\psi^n \in \text{Sem}(G_N)$, since the game $u_{B_j|K}$ is defined on a set M with m players ($1 \leq m \leq n$), we can apply the induced semivalue ψ^m :

$$w_j(K) = (\psi^m)_j[u_{B_j|K}] \quad \forall K \subseteq B_j. \quad (6)$$

The value $w_j(K)$ shows the strategic position of the subset $K \subseteq B_j$ if this subset directly negotiate with the other classes as players in the quotient game –according to the semivalue ψ^n – in absence of $K' = B_j \setminus K$.

Next, since the game w_j is defined on B_j , a set with $b_j = |B_j|$ players ($1 \leq b_j \leq n$), we can apply the induced semivalue φ^{b_j} and we define the mixed semivalue ψ^n/φ^n modified by the coalition structure B as

$$(\psi^n/\varphi^n)_i[v; B] = (\varphi^{b_j})_i[w_j] \quad \forall i \in B_j.$$

The semivalue ψ^n has acted among the coalition blocks whereas the distribution among the players of B_j is obtained according to the semivalue induced by φ^n . The next result offers an explicit expression for the allocations according to mixed modified semivalues.

Theorem 4.5 Let v be a game on N and let ψ^n, φ^n be two semivalues defined on G_N with respective weighting coefficients $(p_s^n)_{s=1}^n$ and $(q_s^n)_{s=1}^n$. Given a coalition structure $B = \{B_1, B_2, \dots, B_m\}$, the payoff to every player i in a coalition block $B_j \in B$ according to the mixed semivalue ψ^n/φ^n modified by B is

$$(\psi^n/\varphi^n)_i[v; B] = \sum_{S \subseteq B_j \setminus \{i\}} \sum_{T \subseteq M \setminus \{j\}} q_{s+1}^{b_j} p_{t+1}^m \left[v \left(\bigcup_{t' \in T} B_{t'} \cup S \cup \{i\} \right) - v \left(\bigcup_{t' \in T} B_{t'} \cup S \right) \right], \quad (7)$$

where $p_{t+1}^m = \sum_{h=0}^{n-m} \binom{n-m}{h} p_{t+1+h}^n$ ($t = |T|$) and $q_{s+1}^{b_j} = \sum_{h=0}^{n-b_j} \binom{n-b_j}{h} q_{s+1+h}^n$ ($s = |S|$).

Proof We first determine the game w_j to obtain the explicit formula for the modified semivalue. According to expression (6), for each $K \subseteq B_j$ we have

$$\begin{aligned} w_j(K) &= \sum_{L \subseteq M: j \in L} p_l^m [u_{B_j|K}(L) - u_{B_j|K}(L \setminus \{j\})] \\ &= \sum_{L \subseteq M: j \in L} p_l^m \left[v \left(\bigcup_{l^* \in L} B_{l^*} \setminus K' \right) - v \left(\bigcup_{l^* \in L \setminus \{j\}} B_{l^*} \setminus K' \right) \right], \end{aligned}$$

where $K' = B_j \setminus K$. If we write $L = T \cup \{j\}$ with $j \notin T$, then $l = t+1$ and therefore

$$w_j(K) = \sum_{T \subseteq M: j \notin T} p_{t+1}^m \left[v \left(\bigcup_{t' \in T} B_{t'} \cup K \right) - v \left(\bigcup_{t' \in T} B_{t'} \right) \right].$$

The mixed modified semivalue is obtained from game w_j . For every player i in block B_j ,

$$(\psi^n/\varphi^n)_i[v; B] = (\varphi^{b_j})_i[w_j] = \sum_{S' \subseteq B_j: i \in S'} q_{s'}^{b_j} [w_j(S') - w_j(S' \setminus \{i\})].$$

Writing $S' = S \cup \{i\}$ with $i \notin S$, the last equality becomes

$$(\psi^n/\varphi^n)_i[v; B] = \sum_{S \subseteq B_j: i \notin S} q_{s+1}^{b_j} [w_j(S \cup \{i\}) - w_j(S)].$$

If in this formula we replace $w_j(S \cup \{i\})$ and $w_j(S)$ with the expressions obtained from $w_j(K)$, then, for $i \in B_j$,

$$\begin{aligned} (\psi^n/\varphi^n)_i[v; B] &= \sum_{S \subseteq B_j: i \notin S} q_{s+1}^{b_j} \left[\sum_{T \subseteq M: j \notin T} p_{t+1}^m \left[v \left(\bigcup_{t' \in T} B_{t'} \cup S \cup \{i\} \right) - v \left(\bigcup_{t' \in T} B_{t'} \cup S \right) \right] \right] \\ &= \sum_{S \subseteq B_j \setminus \{i\}} \sum_{T \subseteq M \setminus \{j\}} q_{s+1}^{b_j} p_{t+1}^m \left[v \left(\bigcup_{t' \in T} B_{t'} \cup S \cup \{i\} \right) - v \left(\bigcup_{t' \in T} B_{t'} \cup S \right) \right]. \end{aligned}$$

The coefficients p_{t+1}^m are weighting coefficients of the induced semivalue ψ^m whereas the coefficients $q_{s+1}^{b_j}$ correspond to the induced semivalue ψ^{b_j} ; their respective expressions directly follow from formula (5). ■

Corollary 4.6 *If the semivalues are, respectively, α and β binomial semivalues, we denote the mixed modified semivalue $\psi_\alpha^n/\psi_\beta^n$ with the more compact notation ψ_β^α . Then, $\forall i \in B_j$*

$$(\psi_\beta^\alpha)_i[v; B] = \sum_{S \subseteq B_j \setminus \{i\}} \sum_{T \subseteq M \setminus \{j\}} \beta^s (1 - \beta)^{b_j - s - 1} \alpha^t (1 - \alpha)^{m - t - 1} \left[v \left(\bigcup_{t' \in T} B_{t'} \cup S \cup \{i\} \right) - v \left(\bigcup_{t' \in T} B_{t'} \cup S \right) \right]. \quad \blacksquare$$

Several properties of the solution concept for games with coalition structure here introduced are now detailed.

Proposition 4.7 *Let ψ^n and φ^n be two semivalues defined on cooperative games with set of players N . Let \mathcal{B}_N be the set of all coalition structures defined in N . The mixed modified semivalue ψ^n/φ^n verifies:*

- (a) $\psi^n/\varphi^n[u + v; B] = \psi^n/\varphi^n[u; B] + \psi^n/\varphi^n[v; B] \forall u, v \in G_N, \forall B \in \mathcal{B}_N$.
- (b) *If game $v \in G_N$ is monotonic, then $(\psi^n/\varphi^n)_i[v; B] \geq 0 \forall i \in N, \forall B \in \mathcal{B}_N$.*
- (c) *If player $i \in N$ is a dummy in game $v \in G_N$, then $(\psi^n/\varphi^n)_i[v; B] = v(\{i\}) \forall B \in \mathcal{B}_N$.*
- (d) *If $v(S \cup \{i_1\}) \geq v(S \cup \{i_2\}) \forall S \subseteq N \setminus \{i_1, i_2\}$, with $i_1, i_2 \in B_j$ and B_j coalition block of B , then $(\psi^n/\varphi^n)_{i_1}[v; B] \geq (\psi^n/\varphi^n)_{i_2}[v; B]$.*
- (e) *In case of coalition structure $N_I = \{\{1\}, \{2\}, \dots, \{n\}\}$ (individual blocks), $\psi^n/\varphi^n[v; N_I] = \psi^n[v] \forall v \in G_N$.*
- (f) *In case of coalition structure $\{N\} = \{\{1, 2, \dots, n\}\}$ (grand coalition), $\psi^n/\varphi^n[v; \{N\}] = \varphi^n[v] \forall v \in G_N$.*

Proof All sections of the statement can be easily checked. We only prove section (c). If player $i \in N$ is a dummy in game $v \in G_N$, $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \setminus \{i\}$, and therefore

$$(\psi^n/\varphi^n)_i[v; B] = \sum_{S \subseteq B_j \setminus \{i\}} \sum_{T \subseteq M \setminus \{j\}} q_{s+1}^{b_j} p_{t+1}^m v(\{i\}).$$

Since p_{t+1}^m and $q_{s+1}^{b_j}$ are weighting coefficients of induced semivalues,

$$\sum_{T \subseteq M \setminus \{j\}} p_{t+1}^m = \sum_{t=0}^{m-1} \binom{m-1}{t} p_{t+1}^m = 1 \quad \text{and} \quad \sum_{S \subseteq B_j \setminus \{i\}} q_{s+1}^{b_j} = \sum_{s=0}^{b_j-1} \binom{b_j-1}{s} q_{s+1}^{b_j} = 1,$$

from which $(\psi^n/\varphi^n)_i[v; B] = v(\{i\})$. \blacksquare

5 Computation procedure of mixed modified semivalues

According to Section 3, the multilinear extension (MLE) of a game has been an adequate tool to compute allocations according to all semivalues by means of reference systems of binomial semivalues. Now, we want to offer a similar procedure for the mixed modified semivalues. For it, we need a modified MLE for each coalition block obtained from the MLE of the initial game.

Definition 5.1 *Let $B = \{B_1, \dots, B_m\}$ be a coalition structure in N and let $M = \{1, \dots, m\}$ be the set of classes in N according to the coalition structure B . From the MLE f_v of game v , a modified multilinear extension for each coalition block $B_j \in B$ can be obtained by means of the following rules:*

- (1) *For each $t \in M$, $t \neq j$, and each $u \in B_t$ replace in f_v the variable x_u with y_t .*
- (2) *In the above function, reduce all exponents that appear in y_t to 1, i.e., replace y_t^r ($r > 1$) with y_t , obtaining another MLE*

$$f_{v,j}(x_k, y_t) \quad k \in B_j \text{ and } t \in M \setminus \{j\}$$

Remark 5.2 The rule (2) provides a linear function in the variables y_t , $t \in M \setminus \{j\}$, when y_t^r ($r > 1$) is replaced with y_t . We find this procedure for obtaining a modified multilinear extension in [3], where a computation for the Banzhaf solution for games with coalition structure is offered. Now, we will use each modified multilinear extension $f_{v,j}$ for obtaining allocations by mixed modified semivalues to the players of block B_j .

Theorem 5.3 *Let us assume that $\{\psi_{\alpha_k}^n\}_{k=1}^n$ is a reference system of binomial semivalues in $\text{Sem}(G_N)$. If v is a cooperative game on N , ψ^n and φ^n are two semivalues on G_N with respective expressions*

$$\psi^n = \sum_{k=1}^n \lambda_k \psi_{\alpha_k}^n \quad \varphi^n = \sum_{l=1}^n \tilde{\lambda}_l \psi_{\alpha_l}^n$$

and $B = \{B_1, \dots, B_m\}$ is a structure of coalition blocks defined in N , then the allocation for each player i in block B_j according to the modified solution ψ^n/φ^n can be computed by means of the following expression

$$(\psi^n/\varphi^n)_i[v; B] = \Lambda^t A(i) \tilde{\Lambda}, \quad (8)$$

where the matrix $A(i) = (a_{kl}(i))$ depends on the reference system of $\text{Sem}(G_N)$ and can be obtained from the MLE of block B_j

$$a_{kl}(i) = (\psi_{\alpha_l}^n/\varphi_{\alpha_k}^n)_i[v; B] = \frac{\partial f_{v,j}}{\partial x_i}(\bar{\alpha}_l, \bar{\alpha}_k), \quad 1 \leq k, l \leq n.$$

Λ and $\tilde{\Lambda}$ are, respectively, the matrices of the components of semivalues ψ^n and φ^n in the reference system $\{\psi_{\alpha_k}^n\}_{k=1}^n$: $\Lambda^t = (\lambda_1 \dots \lambda_n)$, $\tilde{\Lambda}^t = (\tilde{\lambda}_1 \dots \tilde{\lambda}_n)$.

Proof We first prove that the allocations to the players according to a mixed modified semivalue $\psi_{\alpha_l}^{\alpha_k}$, $1 \leq k, l \leq n$, can be obtained from the modified MLE:

$$(\psi_{\alpha_l}^{\alpha_k})_i[v; B] = \frac{\partial f_{v,j}}{\partial x_i}(\bar{\alpha}_l, \bar{\alpha}_k), \quad \forall i \in B_j. \quad (9)$$

Let us suppose that $f_v = f_v(x_1, \dots, x_n)$ is the MLE of game $v \in G_N$,

$$f_v(x_1, \dots, x_n) = \sum_{Q \subseteq N} \prod_{u \in Q} x_u \prod_{u \in N \setminus Q} (1 - x_u) v(Q).$$

Let B_t be a coalition block of $B = \{B_1, \dots, B_m\}$ with $t \in M \setminus \{j\}$, where $M = \{1, \dots, m\}$ denotes the set of classes according to the coalition structure B .

If $|B_t| = b$, after applying rule (1) in Definition 5.1 to obtain the modified MLE of coalition block B_j , $f_{v,j}$, for each $t \in M \setminus \{j\}$, three different types of products with the variable y_t can appear: (i) y_t^b , (ii) $(1 - y_t)^b$ and (iii) $y_t^c(1 - y_t)^d$ with $c + d = b$ and $0 < c, d < b$.

According to rule (2), products of type (i) are replaced with y_t . For type (ii):

$$(1 - y_t)^b = \sum_{h=0}^b \binom{b}{h} (-1)^h y_t^h = 1 + \sum_{h=1}^b \binom{b}{h} (-1)^h y_t^h;$$

replacing y_t^h with y_t for $1 \leq h \leq b$, all these products reduce to

$$1 + y_t \sum_{h=1}^b \binom{b}{h} (-1)^h = 1 + y_t \left[-1 + \sum_{h=0}^b \binom{b}{h} (-1)^h \right] = 1 - y_t.$$

Finally, the products of type (iii) vanish:

$$y_t^c(1 - y_t)^d = \sum_{h=0}^d \binom{d}{h} (-1)^h y_t^{c+h} \quad \rightarrow \quad y_t \sum_{h=0}^d \binom{d}{h} (-1)^h = 0.$$

These considerations imply that in the modified MLE $f_{v,j}$ only appear coalitions Q containing all elements of B_t or none of them. Thus, the coalitions Q with nonnull coefficient in $f_{v,j}$ adopt the form

$$Q = \bigcup_{t' \in T} B_{t'} \cup S \quad \text{with } T \subseteq M \setminus \{j\} \text{ and } S \subseteq B_j.$$

Since the variables for players in block B_j does not change, the modified MLE of coalition block B_j becomes

$$f_{v,j}((x_u)_{u \in B_j}, (y_{t'})_{t' \in M \setminus \{j\}}) = \sum_{S \subseteq B_j} \sum_{T \subseteq M \setminus \{j\}} \left[\prod_{u \in S} x_u \prod_{u \in B_j \setminus S} (1 - x_u) \prod_{t' \in T} y_{t'} \prod_{t' \in M \setminus (T \cup \{j\})} (1 - y_{t'}) \right] v \left(\bigcup_{t' \in T} B_{t'} \cup S \right).$$

Differentiating with respect to variable x_i , the marginal contributions of player i appear:

$$\frac{\partial f_{v,j}}{\partial x_i}((x_u)_{u \in B_j}, (y_{t'})_{t' \in M \setminus \{j\}}) = \sum_{S \subseteq B_j \setminus \{i\}} \sum_{T \subseteq M \setminus \{j\}} \left[\prod_{u \in S} x_u \prod_{u \in B_j \setminus (S \cup \{i\})} (1 - x_u) \prod_{t' \in T} y_{t'} \prod_{t' \in M \setminus (T \cup \{j\})} (1 - y_{t'}) \right] \left[v \left(\bigcup_{t' \in T} B_{t'} \cup S \cup \{i\} \right) - v \left(\bigcup_{t' \in T} B_{t'} \cup S \right) \right].$$

Replacing each x_u , $u \in B_j$, with α_l and each $y_{t'}$, $t' \in M \setminus \{j\}$, with α_k , we obtain

$$\frac{\partial f_{v,j}}{\partial x_i}(\bar{\alpha}_l, \bar{\alpha}_k) = \sum_{S \subseteq B_j \setminus \{i\}} \sum_{T \subseteq M \setminus \{j\}} \alpha_l^s (1 - \alpha_l)^{b_j - s - 1} \alpha_k^t (1 - \alpha_k)^{m - t - 1} \left[v \left(\bigcup_{t' \in T} B_{t'} \cup S \cup \{i\} \right) - v \left(\bigcup_{t' \in T} B_{t'} \cup S \right) \right].$$

According to expression in Corollary 4.6, we conclude that the allocation to players i in a coalition block B_j can be computed by means of formula (9), i.e.,

$$(\psi_{\alpha_l}^{\alpha_k})_i[v; B] = \frac{\partial f_{v,j}}{\partial x_i}(\bar{\alpha}_l, \bar{\alpha}_k), \quad \forall i \in B_j.$$

It only remains to prove the general expression (8) for any mixed modified semi-value. The n^2 mixed allocations $(\psi_{\alpha_l}^{\alpha_k})_i[v; B]$ with $1 \leq k, l \leq n$ form the elements of matrix $A(i)$. Since both semivalues ψ^n and φ^n are linear combinations of the binomial semivalues in the reference system $\{\psi_{\alpha_k}^n\}_{k=1}^n$, we can write for each player i in coalition block B_j :

$$(\psi^n / \varphi^n)_i[v; B] = \left(\sum_{k=1}^n \lambda_k \psi_{\alpha_k}^n / \sum_{l=1}^n \tilde{\lambda}_l \psi_{\alpha_l}^n \right)_i[v; B] = \sum_{k=1}^n \sum_{l=1}^n \lambda_k \tilde{\lambda}_l (\psi_{\alpha_k}^{\alpha_l})_i[v; B]$$

from which

$$(\psi^n / \varphi^n)_i[v; B] = \sum_{l=1}^n \left(\sum_{k=1}^n \lambda_k a_{kl}(i) \right) \tilde{\lambda}_l = \Lambda^t A(i) \tilde{\Lambda} \quad \forall i \in B_j. \quad \blacksquare$$

Once a reference system is chosen, a matrix summarizes the payoffs by any mixed modified semi-value to each player, given both a game and a coalition structure.

Example 5.4 We return to game v in Example 3.6. Now, we consider the structure of coalition blocks $B = \{B_1, B_2\}$ with $B_1 = \{1\}$ and $B_2 = \{2, 3, 4\}$.

From the MLE f_v of game v ,

$$f_v(x_1, x_2, x_3, x_4) = 2x_1 + x_2 + x_3 + x_4 + 2x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 - x_1x_2x_3 - 2x_1x_2x_4 + x_1x_2x_3x_4,$$

we obtain the MLE related with the coalition block B_2 applying rules (1) and (2) according to Definition 5.1:

$$f_{v,2}(x_2, x_3, x_4, y_1) = 2y_1 + x_2 + x_3 + x_4 + 2y_1x_2 + y_1x_3 + y_1x_4 + x_2x_3 + x_2x_4 - y_1x_2x_3 - 2y_1x_2x_4 + y_1x_2x_3x_4.$$

Then $\frac{\partial f_{v,2}}{\partial x_2}(x_2, x_3, x_4, y_1) = 1 + 2y_1 + x_3 + x_4 - y_1x_3 - 2y_1x_4 + y_1x_3x_4$ and

$$\frac{\partial f_{v,2}}{\partial x_2}(\bar{\alpha}_l, \bar{\alpha}_k) = 1 + 2\alpha_k + 2\alpha_l - 3\alpha_k\alpha_l + \alpha_k\alpha_l^2.$$

If we choose as reference system of semivalues on 4-person games $\{\psi_0^4, \psi_{1/3}^4, \psi_{2/3}^4, \psi_1^4\}$, the matrix that summarizes all allocations by mixed modified semivalues to player 2 is

$$A(2) = \begin{pmatrix} 1 & 5/3 & 7/3 & 3 \\ 5/3 & 55/27 & 67/27 & 3 \\ 7/3 & 65/27 & 71/27 & 3 \\ 3 & 25/9 & 25/9 & 3 \end{pmatrix}$$

For instance, if we want to use the 1/3-binomial semivalue on the modified quotient and the Banzhaf value (coefficients of Bh^4 : $p_s^4 = 1/2^3$, $1 \leq s \leq 4$) within each coalition block, the matrices of components are, respectively,

$$\Lambda^t = (0 \ 1 \ 0 \ 0) \quad \tilde{\Lambda}^t = \frac{1}{16}(-1 \ 9 \ 9 \ -1)$$

and

$$(\psi_{1/3}^4/Bh^4)_2[v; B] = \Lambda^t A(2) \tilde{\Lambda} = \frac{9}{4}.$$

Similar computations allow us to obtain the allocations to the remaining players. For instance, for player 3, we consider again the MLE $f_{v,2}$, but now we differentiate with respect to variable x_3 .

$$\frac{\partial f_{v,2}}{\partial x_3}(x_2, x_3, x_4, y_1) = 1 + y_1 + x_2 - y_1x_2 + y_1x_2x_4;$$

$$\frac{\partial f_{v,2}}{\partial x_3}(\bar{\alpha}_l, \bar{\alpha}_k) = 1 + \alpha_k + \alpha_l - \alpha_k\alpha_l + \alpha_k\alpha_l^2.$$

In the same reference system $\{\psi_0^4, \psi_{1/3}^4, \psi_{2/3}^4, \psi_1^4\}$, the matrix that summarizes all allocations by mixed modified semivalues to player 3 is

$$A(3) = \begin{pmatrix} 1 & 4/3 & 5/3 & 2 \\ 4/3 & 43/27 & 52/27 & 7/3 \\ 5/3 & 50/27 & 59/27 & 8/3 \\ 2 & 19/9 & 22/9 & 3 \end{pmatrix}$$

from which

$$(\psi_{1/3}^4/Bh^4)_3[v; B] = \Lambda^t A(3) \tilde{\Lambda} = \frac{7}{4}.$$

For player 4, the matrix $A(4)$ is obtained in a similar way from the MLE $f_{v,2}$:

$$A(4) = \begin{pmatrix} 1 & 4/3 & 5/3 & 2 \\ 4/3 & 40/27 & 46/27 & 2 \\ 5/3 & 44/27 & 47/27 & 2 \\ 2 & 16/9 & 16/9 & 2 \end{pmatrix}.$$

Then,

$$(\psi_{1/3}^4/Bh^4)_4[v; B] = \Lambda^t A(4) \tilde{\Lambda} = \frac{19}{12}.$$

On the other hand, to obtain allocations by mixed modified semivalues to player 1, we need the modified MLE $f_{v,1}$ since player 1 belongs to coalition block B_1 . Applying rule (1) in Definition 5.1 on MLE f_v we have

$$2x_1 + y_2 + y_2 + y_2 + 2x_1y_2 + x_1y_2 + x_1y_2 + y_2^2 + y_2^2 - x_1y_2^2 - 2x_1y_2^2 + x_1y_2^3$$

so that, after applying rule (2), we obtain

$$f_{v,1}(x_1, y_2) = 2x_1 + 5y_2 + 2x_1y_2.$$

Differentiating with respect to variable x_1 and using the reference system $\{\psi_0^4, \psi_{1/3}^4, \psi_{2/3}^4, \psi_1^4\}$ we construct the matrix that summarizes all allocations by mixed modified semivalues to player 1:

$$A(1) = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 8/3 & 8/3 & 8/3 & 8/3 \\ 10/3 & 10/3 & 10/3 & 10/3 \\ 4 & 4 & 4 & 4 \end{pmatrix}.$$

Now, the allocation to player 1 can be obtained:

$$(\psi_{1/3}^4/Bh^4)_1[v; B] = \Lambda^t A(1) \tilde{\Lambda} = \frac{8}{3}.$$

The mixed solution “1/3-binomial semivalue/Banzhaf value” for game v with coalition structure B becomes

$$\psi_{1/3}^4/Bh^4[v; B] = \left(\frac{8}{3}, \frac{9}{4}, \frac{7}{4}, \frac{19}{12} \right).$$

Again, the mixed modified semivalue “Shapley/Shapley” is efficient, i.e., the coalition value derived by Owen in 1977 (see [9]). For solution concepts without efficiency, the utility distribution among the players is obtained according to the proportion given by the allocation. In our example, the distribution of the total utility (9 units) follows the proportion 32 : 27 : 21 : 19.

$$1/3\text{-binomial semivalue/Banzhaf value} \longrightarrow (2.91, 2.45, 1.91, 1.73)$$

6 Concluding remark and extension

Although the semivalues form a wide family of solutions for cooperative games, we have shown that, once a reference system of binomial semivalues is chosen, one matrix summarizes the payoff by any semivalue to all players of a given game. The harmonious relation between binomial semivalues and multilinear extension allows us to obtain every allocation by semivalues from the multilinear extension, according to Theorem 3.5.

A similar behavior is found when we consider solution concepts based on semivalues for cooperative games with structure of coalition blocks. If we denote with G_N the set of cooperative games with finite set of players N and with \mathcal{B}_N the set of all coalition structures in N , the *mixed modified semivalues* are solutions $\Psi : G_N \times \mathcal{B}_N \rightarrow \mathbb{R}^n$ obtained in two steps: (i) a modified quotient game where a first (induced) semivalue acts, and (ii) a game played within the block containing a given player where a second (induced) semivalue acts.

It has been proved in Section 5 that, once a reference system of binomial semivalues is chosen, one $(n \times n)$ -matrix summarizes the payoff by any mixed modified semivalue to each player of a given game with coalition structure, where $n = |N|$.

Now, the elements of these $(n \times n)$ -matrices are computed from a modified multilinear extension related with each coalition block that it is derived from the multilinear extension of each game according to the steps in Definition 5.1. Then, Theorem 5.3 shows that every allocation to a given player by mixed modified semivalues is obtained by means of a product of three matrices:

$$(\psi^n / \varphi^n)_i[v; B] = \Lambda^t A(i) \tilde{\Lambda} \quad \forall i \in N, \quad (10)$$

where ψ^n is the semivalue that acts in the modified quotient game, φ^n the semivalue that acts within the coalition blocks, $A(i)$ the $(n \times n)$ -matrix related with the selected reference system of binomial semivalues and Λ and $\tilde{\Lambda}$, respectively, the matrices of the components of semivalues ψ^n and φ^n in the selected reference system.

According to formula 10, we can see that the computation of allocations by mixed modified semivalues has two stages: (i) a fixed computation for obtaining matrices $A(i)$ related with each game and each coalition structure, and (ii) a variable computation according to the selected semivalues ψ^n and φ^n , given by means of their respective matrices of components Λ and $\tilde{\Lambda}$.

In our work we have always supposed that a unique semivalue acts within each coalition block. It is not difficult to assume that different (induced) semivalues act in this second level giving rise to a concept of *generalized mixed modified semivalue*. More exactly, let us assume that $B \in \mathcal{B}_N$ is a coalition structure with m coalition blocks. If $\psi^n, \varphi_1^n, \dots, \varphi_m^n$ are $m+1$ semivalues in $Sem(G_N)$, we define the *generalized mixed modified semivalue* $\psi^n / \varphi_1^n \cdots \varphi_m^n$ by

$$(\psi^n / \varphi_1^n \cdots \varphi_m^n)_i[v; B] = (\psi^n / \varphi_j^n)_i[v; B] \quad \forall i \in B_j \quad (j = 1, \dots, m).$$

This last expression can be computed by means of a product of three matrices as in formula 10, where now the third matrix depends on each coalition block by means of its respective semivalue.

References

- [1] Amer, R., Giménez, J.M., 2003. Modification of semivalues for games with coalition structures. *Theory and Decision* 54, 185–205.
- [2] Banzhaf, J.F., 1965. Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review* 19, 317–343.
- [3] Carreras, F., Magaña, A. 1994. The multilinear extension and the modified Banzhaf-Coleman index. *Mathematical Social Sciences* 28, 215–222.
- [4] Dragan, I., 1999. Potential and consistency for semivalues of finite cooperative TU games. *International Journal of Mathematics, Game Theory and Algebra* 9, 85–97.
- [5] Dubey, P., Neyman, A., Weber, R.J., 1981. Value theory without efficiency. *Mathematics of Operations Research* 6, 122–128.
- [6] Giménez, J.M., 2001. Contributions to the study of solutions for cooperative games (in Spanish). Ph.D. Thesis. Technical University of Catalonia, Spain.
- [7] Owen, G., 1972. Multilinear extensions of games. *Management Science* 18, 64–79.
- [8] Owen, G., 1975. Multilinear extensions and the Banzhaf value. *Naval Research Logistics Quarterly* 22, 741–750.
- [9] Owen, G., 1977. Values of games with a priori unions. In: Henn, R., Moeschlin, O., (Eds.), *Essays in Mathematical Economics and Game Theory*. Springer-Verlag, pp. 76–88.
- [10] Owen, G., 1981. Modification of the Banzhaf-Coleman index for games with a priori unions. In: Holler, M.J., (Ed.), *Power, Voting and Voting Power*. Physica-Verlag, pp. 232–238.
- [11] Shapley, L.S., 1953. A value for n -person games. In: Kuhn, H.W., Tucker, A.W., (Eds.), *Contributions to the Theory of Games II*. Princeton University Press, pp. 307–317.
- [12] Weber, R.J., 1988. Probabilistic values for games. In: Roth, A.E., (Ed.), *The Shapley Value: Essays in Honor of Lloyd S. Shapley*. Cambridge University Press, pp. 101–119.