

Bounds on the first non-null eigenvalue for self-adjoint boundary value problems on networks

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Abstract

We aim here at obtaining bounds on the first non-null eigenvalue for self-adjoint boundary value problems on a weighted network by means of equilibrium measures, that includes the study of Dirichlet, Neumann and Mixed problems. We also show the sharpness of these bounds throughout the analysis of some known examples. In particular, we emphasize the case of distance-regular graphs, and we show that the bounds obtained are better than the known until now.

Keywords: Networks, self-adjoint eigenvalue problems, discrete laplacian, equilibrium measures, distance-regular graphs.

1 Introduction

In this paper we consider the different self-adjoint eigenvalue problem on a subset of a weighted network for the Laplace-Beltrami operator. Specifically, we study lower and upper bound on the first non-null eigenvalue associated with each problem.

Eigenvalues have many applications in combinatorics and in other fields of mathematics. In the literature the problems that are mainly considered are those that concern with the Dirichlet eigenvalue problem and with the Poisson equation, see [3, 4, 6, 7, 9]. Some works involve the study of Neumann eigenvalues, see [3, 8] but no-one consider the case of Dirichlet-Neumann boundary value problem. Here we firstly show that the study of eigenvalue problems can be reduced to the study of either a Dirichlet eigenvalue problem or a Poisson eigenvalue problem in a suitable network associated with the initial problem. Hence, we obtain a new variational characterization of the first non-null eigenvalue associated with each problem.

The techniques used here are the usual in this context; that is, to apply to a particular function a discrete version of Green's Identity and the variational characterization of eigenvalues. The novelty lies in the functions we consider, namely the equilibrium measure for suitable subsets of the network. The use of equilibrium measures in the context of finite networks was introduced by the authors in [1], where it was proved that these measures contain valuable information about the connection between vertices of a subset as well as the connection between the set and its complementary. These properties were also made clear in [2] where we showed that the Green's Function of any self-adjoint boundary value problem on a network can be expressed, in a simple form, in terms of equilibrium measures. In addition it must be noted that the equilibrium measure can be obtained as either the solution of a linear programming problem in which the Laplacian acts as the coefficient matrix of the general linear constraints or the solution of a quadratic convex programming problem in which the Laplacian defines the objective function.

If one thinks about what functions are naturally associated with an arbitrary set in a general network, the only possible candidates seem to be the Dirac's measures and the characteristic function of the set. But they only express if a vertex is in or out of the set, and they say nothing about the connectivity between vertices of the set. Hereby, if we try to consider functions that should take into account both aspects, the natural choice is not other one than the equilibrium measure of the set. We will make the efficacy of this choice clear throughout some examples. Moreover we will pay special attention on distance-regular graphs, since in this type of graphs the equilibrium measures can be computed by hand.

2 Preliminaries

Along the paper, $\Gamma = (V, E)$ denote a simple and finite connected graph without loops, with vertex set V , edge set E and *order* n and *size* m . Two different vertices, $x, y \in V$, are called *adjacent*, which is represented by $x \sim y$, if $\{x, y\} \in E$. The cardinality of $F \subset V$ is denoted by $|F|$.

For each $x \in V$ and for each $j \in \mathbb{N}$ we denote by $S_j(x)$ and by $B_j(x)$ the *sphere and the ball of center x and radius j* ; that is, the sets $S_j(x) = \{y \in V : d(x, y) = j\}$ and $B_j(x) = \{y \in V : d(x, y) \leq j\}$, where $d(x, y)$ is the length of the shortest path joining x and y .

Fixed a vertex subset $F \subset V$, we denote by F^c its complementary in V and we also consider the following vertex subsets associated with F :

- (i) *Interior of F* : $\overset{\circ}{F} = \{x \in V : B_1(x) \subset F\}$.
- (ii) *Boundary of F* : $\delta(F) = \{x \in V : d(x, F) = 1\}$.
- (iii) *Closure of F* : $\bar{F} = \{x \in V : d(x, F) \leq 1\} = F \cup \delta(F)$.

A *dominating set* in Γ is a subset $F \subset V$ such that each element of F^c is adjacent to a vertex of F . Clearly, a set is dominating iff $\bar{F} = V$ or equivalently $\overset{\circ}{F} = \emptyset$.

We denote by $\mathcal{C}(V)$ or $\mathcal{C}(V \times V)$ the set of real functions defined on the sets V and $V \times V$, respectively. In addition, for each non empty set $F \subset V$ we denote by χ_F its characteristic function and by $\mathcal{C}(F)$ the set of real functions on V that vanish in F^c . If $u \in \mathcal{C}(V)$, the value $\sum_{x \in F} u(x)$ is denoted by $\int_F u dx$.

We call *weighted network* a triple (Γ, c, ν) where $c \in \mathcal{C}(V \times V)$ is a symmetric function such that $c(x, y) > 0$ when $x \sim y$ and $c(x, y) = 0$ otherwise and $\nu \in \mathcal{C}(V)$ verifies that $\nu(x) > 0$ for each $x \in V$. If $x \in V$, the number $k(x) = \int_V c(x, y) dy$ is called (*generalized*) *degree of x* . In addition, if $F \subset V$ is a proper subset, for any $x \in F$ the value $k_F^+(x) = \int_{\delta(F)} c(x, y) dy$ is called *out-degree of x* , whereas when $x \in F^c$ the value $k_F^-(x) = \int_{\delta(F^c)} c(x, y) dy$ is called *int-degree of x* . Observe that, F is a dominating set iff $k_F^- > 0$ or equivalently iff $k_{F^c}^+ > 0$.

In what follows if $F \subset V$ is non empty, we consider for any $u \in \mathcal{C}(F)$ the values

$$\|u\|_{1,\nu} = \int_F |u| \nu dx \quad \text{and} \quad \|u\|_{2,\nu} = \left(\int_F u^2 \nu dx \right)^{\frac{1}{2}}$$

and we define the *volume of F* as $\text{vol}_\nu(F) = \|\chi_F\|_{1,\nu}$. We omit the subscript ν in all above expressions when $\nu(x) = 1$ for all $x \in V$. In this case, $\text{vol}(F) = |F|$.

The *Laplace-Beltrami* operator of a weighted network (Γ, c, ν) is the linear operator $\mathcal{L}: \mathcal{C}(V) \rightarrow \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \frac{1}{\nu(x)} \int_V c(x, y) (u(x) - u(y)) dy, \quad x \in V. \quad (1)$$

If $F \subset V$ is a non empty subset, for each $u \in \mathcal{C}(\bar{F})$ we define the *conormal derivative of u* as the function belonging to $\mathcal{C}(\delta(F))$ given by

$$\frac{\partial u}{\partial \mathbf{n}_F}(x) = \frac{1}{\nu(x)} \int_{\delta(F^c)} c(x, y) (u(x) - u(y)) dy, \quad x \in \delta(F). \quad (2)$$

In [2] it was proved the so-called *Green's Identity*:

$$\int_F v \mathcal{L}(u) \nu dx - \int_F u \mathcal{L}(v) \nu dx = \int_{\delta(F)} u \frac{\partial v}{\partial \mathbf{n}_F} \nu dx - \int_{\delta(F)} v \frac{\partial u}{\partial \mathbf{n}_F} \nu dx, \quad u, v \in \mathcal{C}(\bar{F}). \quad (3)$$

It was also proved, for the case $\nu = 1$, the existence of the so-called equilibrium measures for any proper set. The techniques used there can easily be extended to the general case.

Specifically, for any proper subset $F \subset V$ there exists a unique function $\gamma^F \in \mathcal{C}(F)$, called *the equilibrium measure of F* , such that $\gamma^F(x) > 0$ for any $x \in F$ and $\mathcal{L}(\gamma^F) = 1$ on F . Moreover, $\gamma^F = I(F)^{-1}\sigma^F$ where $(I(F), \sigma^F)$ is the solution of the following quadratic convex programming problem:

$$I(F) = \min_{u \in \mathcal{C}(F)} \left\{ \int_F u \mathcal{L}(u) \nu \, dx : u \geq 0, \|u\|_{1,\nu} = 1 \right\}.$$

Alternatively, the pair $(I(F), \sigma^F)$ is also the solution of the linear programming problem

$$I(F) = \min_{u \in \mathcal{C}(F)} \{a : u \geq 0, \|u\|_{1,\nu} = 1, \mathcal{L}(u) \leq a\chi_F\}.$$

The following result shows the relevance of the equilibrium measures in studying topological properties of a subset.

Lemma 2.1 *If $F \subset V$ is a proper subset, then γ^F is constant on F iff k_F^+ is a positive multiple of ν .*

Proof. If $\gamma^F = a\chi_F$, then $1 = \mathcal{L}(\gamma^F) = \frac{a}{\nu}k_F^+$ on F . Conversely, if $k_F^+ = a\nu$, $a > 0$, then $\gamma^F = \frac{1}{a}\chi_F$ is the equilibrium measure of F . ■

Note that the above Lemma says that a necessary condition so that the equilibrium measure of F is constant is that F^c is a dominating set.

Throughout the paper the so-called *distance-regular graphs* will play an important role. Therefore, we introduce here its definition and the value of some equilibrium measures associated with them, see [2].

A connected k -regular graph $\Gamma = (V, E)$ with diameter D is called distance-regular if there exists integers b_i, c_i , $i = 0, \dots, D$ such that for any two vertices $x, y \in V$ at distance $d(x, y) = i$ there are exactly c_i neighbors of x in $S_{i-1}(y)$ and b_i neighbors of x in $S_{i+1}(y)$. Then, for any vertex $y \in V$, the values $|S_i(y)|$ and $|B_i(y)|$ do not depend on y and they will be denoted by $k_i = |S_i|$ and $|B_i|$, respectively. Moreover, $|B_i| = \sum_{j=0}^i k_j$, $i = 0, \dots, D$.

On the other hand, the equilibrium measure of any ball in a distance-regular graph has the following expression

$$\gamma^{B_r}(x) = \sum_{s=|x|}^r \frac{|B_s|}{k_s b_s}, \quad \text{for any } x \in B_r, \quad (4)$$

where $|x|$ denotes the distance between x and the center of the ball. In addition, if γ_x denotes the equilibrium measure for the subset $V \setminus \{x\}$ we have for any $x, y \in V$ that

$$\gamma_x(y) = \sum_{j=0}^{d(x,y)-1} \frac{n - |B_j|}{k_j b_j}. \quad (5)$$

3 Eigenvalues for self-adjoint Boundary Value Problems

In [2], general self-adjoint boundary value problems were introduced in the context of finite networks and an exhaustive study of its associated Green functions was also carried out. In this paper we are concerned with another aspect of this type of problems, namely the study of eigenvalue problems.

Let (Γ, c, ν) a weighted network and $F \subset V$ a non empty connected subset with vertex boundary $\delta(F) = H_1 \cup H_2$ where $H_1 \cap H_2 = \emptyset$. A *self-adjoint eigenvalue problem on F for the Laplace-Beltrami operator*, consists in finding $\lambda \in \mathbb{R}$ such that there exists $u \in \mathcal{C}(\bar{F})$ non-null verifying

$$\mathcal{L}(u) = \lambda u \text{ on } F, \quad \frac{\partial u}{\partial \mathbf{n}_F} = 0 \text{ on } H_1, \quad u = 0 \text{ on } H_2. \quad (6)$$

Problem (6) summarizes the different self-adjoint eigenvalue problems that appears in the literature with proper name; that is,

- (i) *Dirichlet-Neumann eigenvalue problem* when $H_1, H_2 \neq \emptyset$.
- (ii) *Dirichlet eigenvalue problem* when $H_2 = \delta(F) \neq \emptyset$.
- (iii) *Neumann eigenvalue problem* when $H_1 = \delta(F) \neq \emptyset$.
- (iv) *Poisson eigenvalue problem* when $\delta(F) = \emptyset$; that is when $F = V$.

It is well-known that the lower eigenvalue of (6) is simple, non negative and the corresponding eigenfunction can be chosen to be a positive function on $\mathcal{C}(F \cup H_1)$. Moreover, the lower eigenvalue is null for Neumann and Poisson problems and its corresponding eigenfunctions are constant on \bar{F} .

In the sequel, we will denote by $\lambda(F, H_1, H_2)$ the first non-null eigenvalue for problem (6). It is also well-known that $\lambda(F, H_1, H_2)$ can be characterized from a variational point of view as

$$\lambda(F, H_1, H_2) = \min_{\substack{u \in \mathcal{C}(F \cup H_1) \\ u \neq 0}} \left\{ \frac{\int_F u \mathcal{L}(u) \nu \, dx}{\int_F u^2 \nu \, dx} : \frac{\partial u}{\partial \mathbf{n}_F} = 0 \text{ on } H_1 \text{ and } a \int_F u \nu \, dx = 0 \right\}, \quad (7)$$

where $a = 1$ if $H_2 = \emptyset$, which corresponds to either Neumann or Poisson problems, and $a = 0$ otherwise.

The question of bounding the first non-null eigenvalue for both the Dirichlet and Poisson problems on a network has been widely treated. However, this is not the case for the other eigenvalue problems, specially in the case of Dirichlet-Neumann problem whose consideration is omitted in the literature. Some authors have deal with the Neumann eigenvalue problem, see for instance [3, 5, 8], but the lower bounds for the eigenvalue are obtained only under strong constrains on the type of considered subsets.

Our objective is to determine bounds for $\lambda(F, H_1, H_2)$ in terms of the equilibrium measures. For this, we will proceed analogously to [2] and hence we first reduce problem (6) to either a Dirichlet eigenvalue problem or a Poisson eigenvalue problem in a suitable network associated with the initial problem. The key idea is to notice that if the conormal derivative of a function is null at $x \in \delta(F)$, then the value of the function at this vertex is univocally determined by the values of the function on F . Therefore, in all cases, the space of functions verifying the boundary conditions is isomorphic to $\mathcal{C}(F)$. Of course, this result is only relevant when $H_1 \neq \emptyset$.

Lemma 3.1 *The function $\boldsymbol{\eta}_{F, H_1} : \mathcal{C}(F) \longrightarrow \mathcal{C}(F \cup H_1)$ given by $\boldsymbol{\eta}_{F, H_1}(u) = u - \frac{\nu}{k_F^-} \frac{\partial u}{\partial \mathbf{n}_F} \chi_{H_1}$, establishes an isomorphism between $\mathcal{C}(F)$ and $\left\{ u \in \mathcal{C}(F \cup H_1) : \frac{\partial u}{\partial \mathbf{n}_F} = 0 \text{ on } H_1 \right\}$. ■*

From the above lemma and using the variational characterization of $\lambda(F, H_1, H_2)$ we obtain that

$$\lambda(F, H_1, H_2) = \min_{\substack{v \in \mathcal{C}(F) \\ v \neq 0}} \left\{ \frac{\int_F v \mathcal{L}(\boldsymbol{\eta}_{F, H_1}(v)) \nu dx}{\|v\|_{2, \nu}^2}, \quad a \int_F v \nu dx = 0 \right\}, \quad (8)$$

since $\boldsymbol{\eta}_{F, H_1}(v) = v$ on F . Next, we show that the quadratic functional $\int_F v \mathcal{L}(\boldsymbol{\eta}_{F, H_1}(v)) \nu dx$ is in fact the quadratic functional associated with the Laplace-Beltrami operator of a suitable network with vertex set $F \cup H_2$. Therefore $\lambda(F, H_1, H_2)$ appears as the first Dirichlet eigenvalue for the new network when $H_2 \neq \emptyset$ or as the first non-null Poisson eigenvalue for the new network when $H_2 = \emptyset$. Specifically, given (Γ, c, ν) a weighted network and $F \subset V$ a proper connected subset, we define the function $b: (F \cup H_2) \times (F \cup H_2)$ as

$$b(x, y) = c(x, y) + \begin{cases} \int_{H_1} \frac{c(x, z)c(y, z)}{k^-(z)} dz, & x, y \in F \text{ and } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we consider the weighted network $(\bar{\Gamma}_F, b, \nu)$ whose vertex and edge sets are $F \cup H_2$ and $\bar{E} = \{\{x, y\} \in (F \cup H_2) \times (F \cup H_2) : b(x, y) > 0\}$, respectively. Note that in the new network the adjacencies between vertices in $F \cup H_2$ are maintained but new adjacency can appear between vertices of $\delta(F^c)$ that have a common neighbor in H_1 . Therefore, F is a connected subset whose boundary is now H_2 . We also remark that if (6) is a Poisson or a Dirichlet eigenvalue problem, then the network $(\bar{\Gamma}_F, b, \nu)$ coincides with the network (Γ, c, ν) .

Proposition 3.2 *Let (Γ, c, ν) be a weighted network, \mathcal{L} its Laplace-Beltrami operator and $F \subset V$ a proper connected subset. Then for any $u, v \in \mathcal{C}(F)$,*

$$\int_F v \mathcal{L}(\boldsymbol{\eta}_{F, H_1}(u)) \nu dx = \int_F v \bar{\mathcal{L}}(u) \nu dx,$$

where $\bar{\mathcal{L}}$ is the Laplace-Beltrami operator of the weighted network $(\bar{\Gamma}_F, b, \nu)$.

Proof. Clearly it suffices to prove the equality

$$\int_F \varepsilon_x \mathcal{L}(\boldsymbol{\eta}_{F, H_1}(\varepsilon_y)) \nu dz = \int_F \varepsilon_x \bar{\mathcal{L}}(\varepsilon_y) \nu dz,$$

for any $x, y \in F$, where ε_x stands for the Dirac's measure on x . If we consider $x, y \in F$, then

$$\begin{aligned} \int_F \varepsilon_x \mathcal{L}(\boldsymbol{\eta}_{F, H_1}(\varepsilon_y)) \nu dz &= \int_V c(x, z) (\varepsilon_y(x) - \boldsymbol{\eta}_{F, H_1}(\varepsilon_y)(z)) dz \\ &= \left[k(x) \varepsilon_y(x) - \int_F c(x, z) \varepsilon_y(z) dz - \int_{\delta(F)} c(x, z) \boldsymbol{\eta}_{F, H_1}(\varepsilon_y)(z) dz \right] \\ &= \left[k(x) \varepsilon_y(x) - c(x, y) - \int_{H_1} \frac{c(x, z) c(y, z)}{k_F^-(z)} dz \right]. \end{aligned}$$

Therefore, if $x \neq y$, we obtain that

$$\int_F \varepsilon_x \mathcal{L}(\boldsymbol{\eta}_{F, H_1}(\varepsilon_y)) \nu dz = -b(x, y) = \int_F \varepsilon_x \bar{\mathcal{L}}(\varepsilon_y) \nu dz,$$

whereas when $x = y$,

$$\int_F \varepsilon_x \mathcal{L}(\boldsymbol{\eta}_{F, H_1}(\varepsilon_x)) \nu dz = \left[k(x) - \int_{H_1} \frac{c(x, z)^2}{k_F^-(z)} dz \right].$$

On the other hand,

$$\begin{aligned}
\int_F \varepsilon_x \bar{\mathcal{L}}(\varepsilon_x) \nu dz &= \int_{F \cup H_2} b(x, y) dy \\
&= \left[\int_{F \cup H_2} c(x, y) dy + \int_{(F \setminus \{x\}) \times H_1} \frac{c(x, z)c(y, z)}{k^-(z)} dz dy \right] \\
&= \left[\int_{F \cup H_2} c(x, y) dy + \int_{H_1} \frac{c(x, z)}{k^-(z)} \left(\int_F c(y, z) dy \right) dz - \int_{H_1} \frac{c(x, z)^2}{k_F^-(z)} dz \right] \\
&= \left[k(x) - \int_{H_1} \frac{c(x, z)^2}{k_F^-(z)} dz \right]. \quad \blacksquare
\end{aligned}$$

Applying now the above proposition to identity (8) we obtain that

$$\lambda(F, H_1, H_2) = \min_{\substack{v \in \mathcal{C}(F) \\ v \neq 0}} \left\{ \frac{\int_F v \bar{\mathcal{L}}(v) \nu dx}{\|v\|_{2, \nu}^2}, \quad a \int_F v \nu dx = 0 \right\}; \quad (9)$$

that is, $\lambda(F, H_1, H_2)$ is the first Dirichlet eigenvalue for F on the network $(\bar{\Gamma}_F, b, \nu)$ when $H_2 \neq \emptyset$ or it is the first non-null Poisson eigenvalue of the network $(\bar{\Gamma}_F, b, \nu)$ when $H_2 = \emptyset$. Indeed, the equality between the bilinear forms considered in the above proposition implies that the self-adjoint boundary problems raised in any of the networks are equivalent. Specifically, we get the following result

Corollary 3.3 *Let (Γ, c, ν) be a weighted network, $F \subset V$ a proper connected subset and suppose that $\delta(F) = H_1 \cup H_2$ where $H_1 \cap H_2 = \emptyset$. Then, $u \in \mathcal{C}(F)$ satisfies $\bar{\mathcal{L}}(u) = f$ on F iff $v = \boldsymbol{\eta}_{F, H_1}(u)$ satisfies $\mathcal{L}(v) = f$ on F , $\frac{\partial v}{\partial n_F} = 0$ on H_1 . In addition λ is an eigenvalue and $u \in \mathcal{C}(F)$ is the associated eigenfunction on $(\bar{\Gamma}_F, b, \nu)$ iff λ is an eigenvalue and $\boldsymbol{\eta}_{F, H_1}(u)$ is the associated eigenfunction on (Γ, c, ν) for eigenvalue problem (6).*

4 Poisson eigenvalues

In this section we study bounds for the first non-null eigenvalue of the Laplace-Beltrami operator of a weighted network, in terms of the equilibrium measures. Recall that the Poisson eigenvalue problem can be formulated as finding $\lambda \in \mathbb{R}$ and $u \in \mathcal{C}(V)$ non-null such that $\mathcal{L}(u) = \lambda u$ on V . Therefore, Poisson eigenvalues are nothing else than the so-called *network's eigenvalues* and for this reason $\lambda(V, \emptyset, \emptyset)$ is usually denoted as $\lambda(\Gamma)$. This value contains valuable information about the connectivity of Γ and is often called *algebraic connectivity* of Γ , [4].

The following result displays a generalization of the most popular lower and upper bounds for $\lambda(\Gamma)$. When $\nu = 1$ the upper bound becomes $\lambda(\Gamma) \leq \frac{n}{n-1} \min_{x \in V} \{k(x)\}$ obtained by M. Friedler in [4], whereas when $\nu = k$ and $c(x, y) = 1$ for $x \sim y$, the lower bound gives $\lambda(\Gamma) \geq \frac{1}{2Dm}$ obtained by F. Chung in [3].

Lemma 4.1 *If D is the diameter of Γ , then*

$$\frac{1}{D \text{vol}_\nu(V)} \min_{x \sim y} \{c(x, y)\} \leq \lambda(\Gamma) \leq \min_{x \in V} \left\{ \frac{k(x)}{\nu(x)} \frac{\text{vol}_\nu(V)}{\text{vol}_\nu(V) - \nu(x)} \right\}.$$

Proof. The upper bound follows by considering the function $u = \varepsilon_x - \frac{\nu(x)}{\text{vol}_\nu(V)}$ in the variational characterization of $\lambda(\Gamma)$, for any $x \in V$. The lower bound follows the guidelines of [3, Lemma 1.9]. ■

The equilibrium measures that will play an essential role in this section are the equilibrium measures for sets of the form $V \setminus \{x\}$ with $x \in V$. As for any $x \in V$ it is verified that $\mathcal{L}(\gamma_x) = 1$ on $V \setminus \{x\}$, applying the Green's Identity we obtain that

$$\text{vol}_\nu(V) = \int_V \mathcal{L}(\gamma_x) \nu dy + \nu(x) (1 - \mathcal{L}(\gamma_x)(x)) = \nu(x) (1 - \mathcal{L}(\gamma_x)(x))$$

and therefore

$$\mathcal{L}(\gamma_x) = 1 - \frac{\text{vol}_\nu(V)}{\nu(x)} \varepsilon_x.$$

Theorem 4.2 *If (Γ, c, ν) is a weighted network, then*

$$\min_{x \in V} \left\{ \frac{\text{vol}_\nu(V)}{\|\gamma_x\|_{1,\nu}} \right\} < \lambda(\Gamma) \leq \min_{x \in V} \left\{ \frac{\text{vol}_\nu(V) \|\gamma_x\|_{1,\nu}}{\text{vol}_\nu(V) \|\gamma_x\|_{2,\nu}^2 - \|\gamma_x\|_{1,\nu}^2} \right\}.$$

Proof. If we fix $x \in V$ and we take $v = \|\gamma_x\|_{1,\nu} - \text{vol}_\nu(V) \gamma_x$, then $\int_V v \nu dy = 0$ and v is a non-null function, since $v(x) = \|\gamma_x\|_{1,\nu} > 0$. Moreover,

$$\int_V v \mathcal{L}(v) \nu dy = \text{vol}_\nu(V)^2 \|\gamma_x\|_{1,\nu} \quad \text{and} \quad \|v\|_{2,\nu}^2 = \text{vol}_\nu(V)^2 \|\gamma_x\|_{2,\nu}^2 - \text{vol}_\nu(V) \|\gamma_x\|_{1,\nu}^2$$

and hence, the upper bound follows from (7).

On the other hand, if $u \in \mathcal{C}(V)$ is a non-null eigenfunction, then applying the Green's Identity we obtain that for each $x \in V$ it is verified

$$\lambda(\Gamma) \int_V u \gamma_x \nu dy = \int_V \mathcal{L}(u) \gamma_x \nu dy = \int_V u \left(1 - \frac{\text{vol}_\nu(V)}{\nu(x)} \varepsilon_x \right) \nu dy = -\text{vol}_\nu(V) u(x).$$

Moreover, as $\lambda(\Gamma) > 0$ it is also true that

$$\text{vol}_\nu(V)|u(x)| = \lambda(\Gamma) \left| \int_V u \gamma_x \nu dy \right| \leq \lambda(\Gamma) \int_V |u| \gamma_x \nu dy, \quad x \in V.$$

Hence, taking $x_0 \in V$ such that $|u(x_0)| = \max_{x \in V} |u(x)|$, the result follows keeping in mind that $\int_V |u| \gamma_{x_0} \nu dy \leq |u(x_0)| \|\gamma_{x_0}\|_{1,\nu}$. ■

If the upper bound is attained then, there exists $x \in V$ such that $v = \|\gamma_x\|_{1,\nu} - \text{vol}_\nu(V) \gamma_x$ is an eigenfunction and hence γ_x is constant. Moreover, we have the following result

Lemma 4.3 *Given $x \in V$, γ_x is constant iff $v = \|\gamma_x\|_{1,\nu} - \text{vol}_\nu(V) \gamma_x$ is an eigenfunction. Moreover, the value $\frac{k(x)}{\nu(x)} \frac{\text{vol}_\nu(V)}{\text{vol}_\nu(V) - \nu(x)}$ is the corresponding eigenvalue.*

Proof. Suppose that $\gamma_x = a\chi_{V \setminus \{x\}}$, then from Lemma 4.3 a $c(x, y) = \nu(y)$ for any $y \neq x$. Moreover,

$$1 - \frac{\text{vol}_\nu(V)}{\nu(x)} = \mathcal{L}(\gamma_x)(x) = -\frac{k(x)a}{\nu(x)}.$$

Therefore, $a = \frac{\text{vol}_\nu(V) - \nu(x)}{k(x)}$. Let $v = \|\gamma_x\|_{1,\nu} - \text{vol}_\nu(V) \gamma_x = a(\text{vol}_\nu(V)\varepsilon_x - \nu(x))$, then for any $y \neq x$

$$\mathcal{L}(v)(y) = a\text{vol}_\nu(V)\mathcal{L}(\varepsilon_x)(y) = -a\text{vol}_\nu(V)\frac{c(x, y)}{\nu(y)} = \text{vol}_\nu(V)\frac{1}{a\nu(x)}v(y),$$

whereas

$$\mathcal{L}(v)(x) = a\text{vol}_\nu(V)\mathcal{L}(\varepsilon_x)(x) = a\text{vol}_\nu(V)\frac{k(x)}{\nu(x)} = \text{vol}_\nu(V)\frac{1}{a\nu(x)}v(x).$$

Conversely, if $v = \|\gamma_x\|_{1,\nu} - \text{vol}_\nu(V) \gamma_x$ is an eigenfunction associated with $\lambda \neq 0$, then

$$\mathcal{L}(v) = -\frac{\text{vol}_\nu(V)}{\nu(x)}(\nu(x) - \text{vol}_\nu(V)\varepsilon_x) = \lambda(\|\gamma_x\|_{1,\nu} - \text{vol}_\nu(V)\gamma_x)$$

and hence γ_x is constant. ■

In the case of distance-regular graphs from Theorem 4.2 we get bounds on $\lambda(\Gamma)$ in terms of the parameters associated with the graph.

Proposition 4.4 *If Γ is a distance-regular graph, then*

$$\frac{n}{\sum_{j=0}^{D-1} \frac{(n-|B_j|)^2}{k_j b_j}} < \lambda(\Gamma) \leq \frac{n \sum_{j=0}^{D-1} \frac{(n-|B_j|)^2}{k_j b_j}}{\sum_{j=0}^{D-1} \frac{|B_j| (n-|B_j|)^3}{k_j^2 b_j^2} + 2 \sum_{0 \leq i < j \leq D-1} \frac{|B_i| (n-|B_i|)(n-|B_j|)^2}{k_i k_j b_i b_j}}.$$

Proof. From (5) we get that $\|\gamma_x\|_1 = \sum_{j=0}^{D-1} \frac{(n-|B_j|)^2}{k_j b_j}$ and

$$\|\gamma_x\|_2^2 = \sum_{j=0}^{D-1} \frac{(n-|B_j|)^3}{k_j^2 b_j^2} + 2 \sum_{0 \leq i < j \leq D-1} \frac{(n-|B_i|)(n-|B_j|)^2}{k_i k_j b_i b_j}.$$

Therefore, the results follows by applying Theorem 4.2. \blacksquare

The above bounds are better than the well-known for general graphs

$$\frac{1}{nD} \leq \lambda(\Gamma) \leq \frac{n}{n-1} k,$$

since for a distance-regular graph $\frac{(n-1)^2}{k} \leq \|\gamma_x\|_1 \leq n^2 D$.

4.1 Dirichlet Eigenvalues

In this paragraph we obtain bounds for the first Dirichlet eigenvalue of a proper subset $F \subset V$ in terms of the equilibrium measure of F . Recall that if $F \subset V$ is a proper subset, from (6) the Dirichlet eigenvalue problem on F consist on finding $\lambda \in \mathbb{R}$ and $u \in \mathcal{C}(F)$ non-null such that $\mathcal{L}(u) = \lambda u$ on F . For this reason $\lambda(F, \emptyset, \delta(F))$ is usually denoted as $\lambda_d(F)$.

Next we obtain bounds on $\lambda_d(F)$ in terms of γ^F , the equilibrium measure of F . In spite of the simplicity of their proofs, compare for instance with the technique used by H. Urakawa in [9, Theorem 2.1], we will see throughout some examples that they are tight bounds which shows again the good properties of the equilibrium measures.

Theorem 4.5 *Let (Γ, c, ν) be a weighted network. Then for each proper subset $F \subset V$ the following inequalities hold:*

$$\min_{x \in F} \left\{ \frac{1}{\gamma^F(x)} \right\} \leq \lambda_d(F) \leq \frac{\|\gamma^F\|_{1,\nu}}{\|\gamma^F\|_{2,\nu}^2}.$$

Moreover, any of the above inequalities is an identity iff k_F^+ is a multiple of ν .

Proof. The upper bound follows directly by taking $u = \gamma^F$ in the variational characterization of $\lambda_d(F)$.

To obtain a lower bound for $\lambda_d(F)$, consider now u a positive eigenfunction corresponding to $\lambda_d(F)$. As $u \in \mathcal{C}(F)$ verifies that $\mathcal{L}(u) = \lambda_d(F)u$ on F , applying the Green's Identity we obtain

$$\lambda_d(F) \int_F u \gamma^F \nu dx = \int_F \gamma^F \mathcal{L}(u) \nu dx = \int_F u \mathcal{L} \gamma^F \nu dx = \|u\|_{1,\nu}$$

and hence

$$\frac{1}{\max_{x \in F} \{\gamma^F(x)\}} \leq \lambda_d(F) \leq \frac{1}{\min_{x \in F} \{\gamma^F(x)\}},$$

which in particular gives the claimed lower bound.

Finally, the lower bound is attained iff γ^F is constant and the upper bound is attained iff γ^F is an eigenfunction and hence iff γ^F is constant. Therefore, the result follows from Lemma 4.3. \blacksquare

Note that the upper bound in the above proposition is better than the elemental upper bound obtained in the end of its proof since

$$\frac{\|\gamma^F\|_{1,\nu}}{\|\gamma^F\|_{2,\nu}^2} \leq \frac{\|\gamma^F\|_{1,\nu}}{\min_{x \in F} \{\gamma^F(x)\} \|\gamma^F\|_{1,\nu}} = \max_{x \in F} \left\{ \frac{1}{\gamma^F(x)} \right\}.$$

In fact, both bounds coincide iff k_F^+ is a multiple of ν . In particular, this happens for any F proper subset of a complete graph, since $\lambda_d(F) = n - |F| = \frac{1}{\gamma^F}$.

The following simple example shows the sharpness of the above bounds. Let P_{n+2} be a path on $n+2$ vertices and $F = \{x_1, \dots, x_n\}$ where $k(x_i) = 2$ for any $i = 1, \dots, n$. Then $\gamma^F(x_i) = \frac{i(n+1) - i^2}{2}$, $i = 1, \dots, n$, and therefore,

$$\|\gamma^F\|_1 = \frac{n(n+1)(n+2)}{12} \quad \text{and} \quad \|\gamma^F\|_2^2 = \frac{n(n+1)(n+2)((n+1)^2 + 1)}{120}.$$

If we apply the above theorem we get

$$\frac{2}{\left\lceil \frac{(n+1)^2}{4} \right\rceil} \leq \lambda_d(F) \leq \frac{10}{(n+1)^2 + 1},$$

whereas it is well-known that $\lambda_d(F) = 2 - 2 \cos \left(\frac{\pi}{n+1} \right)$.

From (4) and the above Theorem we get bounds on the Dirichlet eigenvalue of either the complementary of a vertex or a ball of a distance-regular graph.

Proposition 4.6 *Let Γ be a distance-regular graph. Then, for each $1 \leq r \leq D - 1$,*

$$\frac{1}{\sum_{s=0}^r \frac{|B_s|}{k_s b_s}} < \lambda_d(B_r) < \frac{\sum_{s=0}^r \frac{|B_s|^2}{k_s b_s}}{\sum_{i=0}^r k_i \left(\sum_{s=i}^r \frac{|B_s|}{k_s b_s} \right)^2}.$$

Proposition 4.7 *Let Γ be a distance-regular graph, then for any $x \in V$,*

$$\frac{1}{\sum_{s=0}^{D-1} \frac{n - |B_s|}{k_s b_s}} < \lambda_d(V \setminus \{x\}) < \frac{\sum_{j=0}^{D-1} \frac{(n - |B_j|)^2}{k_j b_j}}{\sum_{j=0}^{D-1} \frac{(n - |B_j|)^3}{k_j^2 b_j^2} + 2 \sum_{0 \leq i < j \leq D-1} \frac{(n - |B_i|)(n - |B_j|)^2}{k_i k_j b_i b_j}}.$$

4.2 Neumann eigenvalues

In this section we study the Neumann eigenvalue problem on a proper subset $F \subset V$, that is to find $\lambda \in \mathbb{R}$ and $u \in \mathcal{C}(\bar{F})$ non-null such that $\mathcal{L}(u) = \lambda u$ on F and $\frac{\partial u}{\partial \mathbf{n}_F} = 0$ on $\delta(F)$. For this reason $\lambda(F, \delta(F), \emptyset)$ is usually denoted as $\lambda_N(F)$. Moreover throughout this section we will suppose that $|F| \geq 2$, since otherwise the problem becomes trivial.

From Corollary 3.3 and taking into account the bounds given in Theorem 4.2 we obtain lower and upper bounds for $\lambda_N(F)$ in terms of equilibrium measures of the associated weighted network $(\bar{\Gamma}_F, b, \nu)$.

Proposition 4.8 *Let (Γ, c, ν) be a weighted network and $F \subset V$ a proper subset. Then*

$$\min_{x \in F} \left\{ \frac{\text{vol}_\nu(F)}{\|\bar{\gamma}_x\|_{1,\nu}} \right\} \leq \lambda_N(F) \leq \min_{x \in F} \left\{ \frac{\text{vol}_\nu(F) \|\bar{\gamma}_x\|_{1,\nu}}{\text{vol}_\nu(F) \|\bar{\gamma}_x\|_{2,\nu}^2 - \|\bar{\gamma}_x\|_{1,\nu}^2} \right\},$$

where $\bar{\gamma}_x$ is the equilibrium measure for $F \setminus \{x\}$ in the weighted network $(\bar{\Gamma}_F, b, \nu)$.

Next we analyze the following non-trivial example. Consider T_k the infinite k -homogeneous tree rooted at o and $F = B_r(o)$. Then, the new graph consist on the finite k -homogeneous tree rooted at o and depth r , so $|F| = \frac{k(k-1)^r - 2}{k-2}$. In [2] it was proved that

$$\bar{\gamma}_y^F(x) = \frac{|F|}{2} d(x, y) + \left[\frac{|F|}{2} + \frac{1}{k-2} \right] (|y| - |x|) + \frac{1}{(k-2)^2} \left[(k-1)^{r+1-|y|} - (k-1)^{r+1-|x|} \right]$$

which implies that

$$\begin{aligned} \|\bar{\gamma}_y^F\|_1 &= |F| \left(|F| + \frac{2}{k-2} \right) |y| + \frac{2|F|}{(k-2)^2} (k-1)^{r+1-|y|} - \frac{|F|(k-1)^{r+1}}{(k-2)^2} \\ &\quad - \frac{k}{(k-2)^3} \left(r(k-1)^{r+1} - (r+1)(k-1)^r + 1 \right) - \frac{(k-1)^r}{(k-2)^2} (k-1+rk). \end{aligned}$$

This function attains its maximum value at any vertex y such that $|y| = r$ and hence

$$\min_{y \in F} \left\{ \frac{|F|}{\|\bar{\gamma}_y\|_1} \right\} = \frac{(k-2)^2(k(k-1)^r - 2)}{(k-1)^{2r}(kr(k-2) - (k-1)) + \mathcal{O}(k^{r+2})}.$$

On the other hand, if we take, $y = 0$, then we obtain

$$\frac{|F| \|\bar{\gamma}_o^F\|_1}{|F| \|\bar{\gamma}_o^F\|_2^2 - \|\bar{\gamma}_o^F\|_1^2} = \frac{k(k-2)^2(k-1)^{3r-1}(k(k-2)+1) + \mathcal{O}(k^{2r+4})}{(k-1)^{3r+1}(k(k-2)+2) + \mathcal{O}(k^{2r+3})}.$$

Definitely, we get

$$\lambda_N(F) \in \mathcal{O}(k) \quad \text{and} \quad \lambda_N^{-1}(F) \in \mathcal{O}(rk^{r-1}).$$

4.3 Dirichlet-Neumann eigenvalues

In this section we obtain bounds for the first Dirichlet-Neumann eigenvalue on a proper subset F with $\delta(F) = H_1 \cup H_2$, $H_1 \cap H_2 = \emptyset$ and $H_1, H_2 \neq \emptyset$. Recall that the Dirichlet-Neumann problem can be formulated as finding $\lambda \in \mathbb{R}$ and $u \in \mathcal{C}(F \cup H_1)$ non-null such that $\mathcal{L}(u) = \lambda u$ on F and $\frac{\partial u}{\partial \mathbf{n}_F} = 0$ on H_1 .

From Corollary 3.3 and tacking into account the bounds given in Theorem 4.5 we obtain lower and upper bounds for $\lambda(F, H_1, H_2)$ in terms of equilibrium measures of the associated weighted network $(\bar{\Gamma}_F, b, \nu)$.

Proposition 4.9 *Let (Γ, c, ν) be a weighted network, $F \subset V$ a proper subset and suppose that $\delta(F) = H_1 \cup H_2$ where $H_1 \cap H_2 = \emptyset$ and $H_1, H_2 \neq \emptyset$. Then*

$$\min_{x \in F} \left\{ \frac{1}{\bar{\gamma}^F(x)} \right\} \leq \lambda(F, H_1, H_2) \leq \frac{\|\bar{\gamma}^F\|_{1,\nu}}{\|\bar{\gamma}^F\|_{2,\nu}^2}$$

where $\bar{\gamma}^F$ is the equilibrium measure for F in the weighted network $(\bar{\Gamma}_F, b, \nu)$. Moreover, any of the above inequalities is an identity iff $\overset{\circ}{F} = \emptyset$ and $\int_{H_2} c(x, y) dy$ is a non-null multiple of ν on F .

We finish this section by analyzing the tightness of the bounds on some examples.

(i) Let K_n be the complete graph on n vertices, $F \subset V$ a proper set and H_1, H_2 a non trivial partition of $\delta(F)$. Then,

$$b(x, y) = \begin{cases} \frac{|F| + |H_1|}{|F|}, & \text{if } x, y \in F \text{ and } x \neq y, \\ 1, & \text{otherwise} \end{cases}$$

and therefore $\bar{\gamma}^F = \frac{1}{|H_2|} \chi_F$ which implies that $\lambda(F, H_1, H_2) = |H_2|$.

(ii) Let P_{n+2} be a path whose vertices are labeled as x_0, x_1, \dots, x_{n+1} . Consider the set $F = \{x_1, \dots, x_n\}$ and the Dirichlet-Neumann problem on F with boundary conditions $u(x_0) = 0$ and $u(x_{n+1}) = u(x_n)$. Then, the new network \bar{P}_{n+2} is a path on $n+1$ vertices with Dirichlet condition $u(x_0) = 0$ and therefore $\bar{\gamma}^F(x_i) = \frac{i(2n+1) - i^2}{2}$, $i = 0, \dots, n$. Moreover,

$$\|\bar{\gamma}^F\|_1 = \frac{1}{6}n(n+1)(2n+1) \quad \text{and} \quad \|\bar{\gamma}^F\|_2^2 = \frac{1}{30}n(n+1)(2n+1)(2n^2 + 2n + 1),$$

which implies

$$\frac{2}{n(n+1)} \leq \lambda(F, H_1, H_2) \leq \frac{5}{2n^2 + 2n + 1}.$$

On the other hand, in this case $\lambda(F, H_1, H_2) = 2 - 2 \cos\left(\frac{\pi}{2n+1}\right)$.

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