Power Factor Compensation of Electrical Circuits: A Control Theory Viewpoint

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Abstract— **The purpose of this paper is to bring to the attention of the control community some of the aspects of the practically important, and mathematically challenging, power factor compensation problem. Our main contribution is identifying the key role played by *cyclo–dissipativity* in the solution of the problem. Namely, we prove that a necessary condition for a (shunt) compensator to improve the power transfer is that the load satisfies a given cyclo–dissipativity property, which naturally leads to a formulation of the compensation problem as one of cyclo–dissipasivation. Cyclo–dissipative systems exhibit a net absorption of (abstract) energy only along *closed paths*, while a dissipative system cannot create energy *for all* trajectories, henceforth, this concept generalizes the one of passivation.**

I. Introduction to the Power Factor Compensation Problem

We consider the classical scenario of energy transfer from an q–phase ac generator to a load as depicted in Fig. 1. The voltage and current of the source are denoted by the column vectors $\boldsymbol{v}_s(t), \boldsymbol{i}_s(t) \in \mathbb{R}^q$ and the load is described by a (possibly nonlinear and time varying) q–port system Σ. We make the following assumptions:

Assumption A.1 All the signals in the system are periodic with fundamental period T and belong to the space

$$\mathcal{L}_2[0,T) := \{\boldsymbol{x} : [0,T) \to \mathbb{R}^q \mid \|\boldsymbol{x}\|^2 := \frac{1}{T}\int_0^T |\boldsymbol{x}(\tau)|^2 d\tau < \infty\},$$

where $\|\cdot\|$ is called the *rms value* of $\boldsymbol{x}$ and $|\cdot|$ is the Euclidean norm.

Assumption A.2 The source is ideal, in the sense that $\boldsymbol{v}_s(t)$ remains unchanged for all loads $\boldsymbol{\Sigma}$.

Assumption A.1 captures the practically reasonable scenario that the system operates in a periodic, though not necessarily sinusoidal, steady state regime. This is the case of the vast majority of applications of interest for the problem at hand. Assumption A.2 is tantamount to saying that the source has no impedance and is justified by the fact that most ac apparatus operate at a given voltage, with the actual drained current being specified by the load.

The presence of distorted signals, in this case the current $\boldsymbol{i}_s(t)$, has the deleterious effect of reducing the power transmission efficiency. Let us discuss how this happens. The rated power of the source is the product of its maximum deliverable rms voltage and current. On the other hand, the *average (or active) power* delivered by the source is defined as

$$P := < \boldsymbol{v}_s, \boldsymbol{i}_s >, \tag{1}$$

where $< \boldsymbol{v}_s, \boldsymbol{i}_s > := \frac{1}{T}\int_0^T \boldsymbol{v}_s^\top(t) \; \boldsymbol{i}_s(t) \; dt$ denotes the inner product in $\mathcal{L}_2[0,T)$. From (1) and the Cauchy–Schwarz inequality [1] we have

$$P \leq \|\boldsymbol{v}_s\|\|\boldsymbol{i}_s\| =: S,$$

where we have defined the *apparent power* S. From the inequality above we conclude that, under Assumption A.2, S is the highest average power delivered to the load among all loads that have the same rms current $\|\boldsymbol{i}_s\|$. The identity holds if and only if $\boldsymbol{v}_s(t) = \mathbf{R}\boldsymbol{i}_s(t)$ for some unitary matrix $\mathbf{R} \in \mathbb{R}^{q \times q}$, that is $\mathbf{R}^\top\mathbf{R} = I_q$. If this is not the case $P < S$ and compensation schemes are introduced to reduce this mismatch. That is, to maximize the ratio $\frac{P}{S}$—that is called the *power factor* (PF) [2].

A typical compensation configuration is shown in Fig. 2 where, to preserve the rated voltage at the load terminals the compensator $\boldsymbol{\Sigma_c}$ is placed in shunt. Also, to avoid power dissipation, $\boldsymbol{\Sigma_c}$ is restricted to be lossless, that is,

$$< \boldsymbol{v}, \boldsymbol{i}_c > = 0, \tag{2}$$

where $\boldsymbol{i}_c(t)$ is the compensator current and we notice that $\boldsymbol{v}_s(t) = \boldsymbol{v}(t)$. Given these restrictions, and under the standing Assumption A.2, the problem of maximizing $\frac{P}{S}$ admits an equivalent reformulation, which has a simple geometric interpretation and an explicit solution. First, referring to Fig. 2 we notice that the compensator losslessness condition (2) translates into

$$< \boldsymbol{v}, \boldsymbol{i}_s > = < \boldsymbol{v}, \boldsymbol{i} > . \tag{3}$$

Second, if $\boldsymbol{\Sigma}$ and $\boldsymbol{v}(t)$ are fixed then $\boldsymbol{i}(t)$, and consequently P, are fixed. Hence:

maximizing $\frac{P}{S}$ with lossless compensators is equivalent to minimizing $\|\boldsymbol{i}_s\|$ subject to the constraint (3).

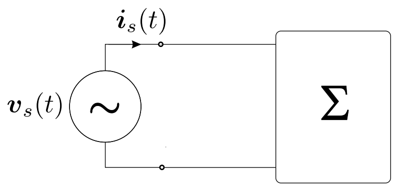

Fig. 1. Circuit schematic of a polyphase ac system.

This work has been done in the context of the European sponsored project GeoPlex with reference code IST-2001-34166. Further information is available at http://www.geoplex.cc. The work of Robert Griñó was supported in part by the *Ministerio de Educación y Ciencia (MEC)* under project DPI2004-06871-C02-02.

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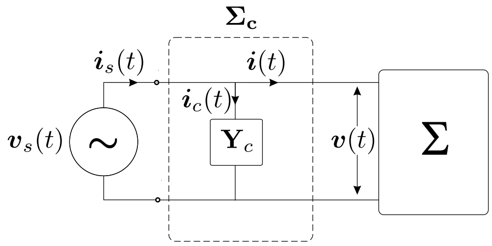

Fig. 2. Typical compensation configuration.

The optimization problems above are rather unprecise and considerations on the actuator are required for their clean formulation. Two types of compensator devices are available in practice, circuits with energy–storing components or regulated current sources—also called "with or without energy storage" in the circuits literature.

II. Lossless Shunt Compensation: A Cyclo–Dissipativity Condition

In this section we consider the use of lossless shunt elements for PF compensation and assume that $\mathbf{\Sigma_c}$ can be represented with its admittance operator $\mathbf{Y}_c : \boldsymbol{v} \to \boldsymbol{i}_c$. From the Projection Theorem [1] we have

Fact 1 Fix $\boldsymbol{v}(t), \boldsymbol{i}(t)$. Among all currents $\boldsymbol{i}_s(t)$ that satisfy the constraint (3), the one with minimal rms value $\|\boldsymbol{i}_s\|$ is given by

$$\boldsymbol{i}_s^\star(t) = \frac{<\boldsymbol{i}, \boldsymbol{v}>}{\|\boldsymbol{v}\|^2}\boldsymbol{v}(t), \tag{4}$$

which is known in the literature as *Fryze's current* [3].

Hence, from Fact 1 and $\boldsymbol{i}_c(t) = \boldsymbol{i}_s(t) - \boldsymbol{i}(t)$ it is clear that any lossless operator $\mathbf{Y}_c$ that solves

$$\frac{<\boldsymbol{i}, \boldsymbol{v}>}{\|\boldsymbol{v}\|^2}\boldsymbol{v}(t) - \boldsymbol{i}(t) = (\mathbf{Y}_c\boldsymbol{v})(t),$$

for the given $\boldsymbol{i}(t), \boldsymbol{v}(t)$, is optimal. In spite of its apparent simplicity, it is not clear to these authors how to use this "data–interpolation" relationship in a practically meaningful way.

From the previous Section we have that the PF compensation problem is mathematically equivalent to the problem of minimization of $\|\boldsymbol{i}_s\|$ subject to the constraint (3). From

$$\|\boldsymbol{i}_s\|^2 = \|\boldsymbol{i}\|^2 + \|\boldsymbol{i}_c\|^2 + 2 <\boldsymbol{i}_c, \boldsymbol{i}>,$$

it is clear that a *necessary condition* to reduce the rms value of $\boldsymbol{i}_s(t)$ is

$$<\boldsymbol{i}_c, \boldsymbol{i}> < 0. \tag{5}$$

It turns our that the latter condition and the restriction of compensator losslessness can be nicely captured using the concept of *cyclo–dissipativity* and its associated *abstract energy*[1]. In our previous work [9]–[12] we used the more restrictive notion of passivity and the actual electric and magnetic energies that, unfortunately, impose extremely conservative conditions. Roughly speaking, the key advantages of cyclo–dissipativity are that it restricts the set of inputs of interest to those that generate periodic solutions—a feature that is intrinsic in PF compensation problems—it furthermore deals with "abstract" energies.

Definition 1 *[13] Assume a dynamical system, with input $\boldsymbol{u} \in \mathcal{L}_2[0,T)$ and output $\boldsymbol{y} \in \mathcal{L}_2[0,T)$, admits a state–space representation with state vector $\boldsymbol{x} \in X$. The system is cyclo–dissipative with respect to the supply rate $w(\boldsymbol{u}, \boldsymbol{y})$, where $w : \mathcal{L}_2[0,T) \times \mathcal{L}_2[0,T) \to \mathbb{R}$, if and only if*

$$\int_0^T w(\boldsymbol{u}(t), \boldsymbol{y}(t))dt \geq 0$$

for all $\boldsymbol{u} : [0,T) \to \mathcal{L}_2[0,T)$ such that $\boldsymbol{x}(T) = \boldsymbol{x}_0$ where $\boldsymbol{x}(0) = \boldsymbol{x}_0$. It is said to be cyclo–lossless if the inequality holds with identity.

In words, a system is cyclo–dissipative when it cannot create (abstract) energy over closed paths in the state–space. It might, however, produce energy along some initial portion of such a trajectory; if so, it would not be dissipative. On the other hand, every dissipative system is cyclo–dissipative. As an example, (possibly nonlinear) RLC circuits with input and output their port currents and voltages, respectively, are cyclo–dissipative with supply rate $w(\boldsymbol{u}, \boldsymbol{y}) = \boldsymbol{u}^\top \boldsymbol{y}$ provided that all resistances are passive[2]. Notice that we do not assume the inductors and capacitors are passive—that is, that their stored energy is non–negative—if so, the circuit is in addition passive.

It has been shown in [13] that, similarly to dissipative systems, one can use storage functions and dissipation inequalities to characterize cyclo–dissipativity provided we eliminate the restriction that the functions—called virtual storage functions—be *non–negative* and they are only required to be bounded (from above and below).

Theorem 1 *[13] A system with state representation is cyclo–dissipative iff, for all $\boldsymbol{x} \in X$ which are both controllable and reachable, there exists a virtual storage function $\phi : X \to \mathbb{R}$. That is, a function that satisfies*

$$\phi(\boldsymbol{x}_0) + \int_0^T w(\boldsymbol{u}(t), \boldsymbol{y}(t))dt \geq \phi(\boldsymbol{x}_1)$$

for all $\boldsymbol{u} \in \mathcal{L}_2[0,T)$ such that $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\boldsymbol{x}(T) = \boldsymbol{x}_1$.

We are in position to formulate the PF compensation problem in terms of cyclo–dissipativity—a paradigm that we propose for future study.

Definition 2 (Cyclo–dissipasivation) *Consider the system of Fig. 2. Assume the load $\mathbf{\Sigma}$ is described by its admittance operator $\mathbf{Y} : \boldsymbol{v} \to \boldsymbol{i}$ and it admits a state–space representation. Find a compensator with admittance $\mathbf{Y}_c : \boldsymbol{v} \to \boldsymbol{i}_c$ and a state–space representation, such that*

(i) *$\mathbf{Y}_c$ is cyclo–lossless with supply rate $\boldsymbol{v}^\top \boldsymbol{i}_c$,*
(ii) *The overall system with input $\boldsymbol{v}$ and output col$(\boldsymbol{i}, \boldsymbol{i}_c)$ is cyclo–dissipative with supply rate $-\boldsymbol{i}^\top \boldsymbol{i}_c$.*

A first step toward the solution of this general synthesis problem is to consider that the (lossless) *compensator is given*

[1]Dissipativity theory has been extensively investigated by the control community in the last few years, see the books [4]–[6] or the recent survey papers [7], [8] for an extensive list of references.

[2]This fact that can be easily proven using Tellegen's Theorem [17].

and we investigate under which conditions *on the load* the cyclo–dissipativity property is satisfied, that is, we want to characterize classes of loads for which it is possible to improve the PF with a given compensator.

III. Compensation with LTI Capacitors and Inductors and Load Cyclo–Dissipativity

Let us consider first capacitive compensation, for which we have, $\mathbf{Y}_c = \mathbf{C}_c p$, where $\mathbf{C}_c = \{C_{ij}\} \in \mathbb{R}^{q\times q}, C_{ij} \geq 0$ is the capacitance matrix and $p = \frac{d}{dt}$, then the necessary condition for PF compensation (5) becomes

$$< \boldsymbol{i}, \mathbf{C}_c \dot{\boldsymbol{v}} > \leq 0 \quad \Leftrightarrow \quad < \dot{\hat{\boldsymbol{i}}}, \mathbf{C}_c \boldsymbol{v} > \geq 0, \tag{6}$$

where we have used the property $< \dot{\boldsymbol{x}}, \boldsymbol{y} > = - < \boldsymbol{x}, \dot{\boldsymbol{y}} >$, which holds for all periodic signals $\boldsymbol{x}, \boldsymbol{y}$. Let us assume that the load admits a state representation. If the voltage sources are in series with inductors the elements of $\boldsymbol{i}$ qualify as state variables, that is, $\boldsymbol{x} = \mathrm{col}(\boldsymbol{i}, \chi)$, with $\chi \in \mathbb{R}^{n-q}$ denoting the remaining state variables. The dynamics of the load is then described by

$$\dot{\boldsymbol{x}} = \mathbf{f}(\boldsymbol{x}, \boldsymbol{v}) = \begin{bmatrix} \mathbf{f}_i(\boldsymbol{x}, \boldsymbol{v}) \\ \mathbf{f}_\chi(\boldsymbol{x}, \boldsymbol{v}) \end{bmatrix}, \tag{7}$$

where $\mathbf{f}_i(\boldsymbol{x}, \boldsymbol{v}) \in \mathbb{R}^q$. We can state the following:

Fact 2 Consider the nonlinear polyphase load $\boldsymbol{\Sigma}$, with port variables $(\boldsymbol{v}, \boldsymbol{i})$, and dynamics (7). If the PF can be improved (for all periodic, possibly non–sinusoidal, $\boldsymbol{v}_s$) with shunt LTI capacitors then there exists a matrix $\mathbf{C}_c = \{C_{ij}\} \in \mathbb{R}^{q\times q}, C_{ij} \geq 0$ such that the system with output $\boldsymbol{y} = \mathbf{C}_c^\top \mathbf{f}_i(\boldsymbol{x}, \boldsymbol{v})$, is cyclo–dissipative with supply rate $\boldsymbol{y}^\top \boldsymbol{v}$. That is, the system is cyclo–passive.

The fact above indicates that cyclo–dissipative loads (in the sense defined above) constitute an extension, to the nonlinear non–sinusoidal polyphase case, of the so–called *inductive* loads. This characterization takes a particularly simple form for *single–phase* loads, i.e., (6) is equivalent to

$$< \dot{\hat{i}}, v > \geq 0, \tag{8}$$

establishing that:

If the PF can be improved (for all, possibly non–sinusoidal, v_s) with a shunt LTI capacitor then the single–phase load (possibly nonlinear) is cyclo–dissipative (with supply rate $v\dot{\hat{i}}$).

For inductor compensators $\mathbf{Y}_c = \mathbf{L}_c^{-1}\frac{1}{p}$, with $\mathbf{L}_c = \{L_{ij}\} \in \mathbb{R}^{q\times q}, L_{ij} \geq 0$ the inductance matrix, and the inequality of interest is $< \boldsymbol{i}, \mathbf{L}_c^{-1} \int \boldsymbol{v} > \leq 0 \quad \Leftrightarrow \quad < \int \boldsymbol{i}, \mathbf{L}_c^{-1} \boldsymbol{v} > \geq 0$,, that admits also a cyclo–dissipativity interpretation—provided the current sources have capacitors in parallel. As before, for $q = 1$ the condition becomes $< \int i, v > \geq 0$.

For brevity, we restrict in the sequel to capacitor compensation—a scenario which is very common since loads are typically assumed to be dominantly inductive—for *single–phase* loads, that is $q = 1$. Two questions arise immediately:

Q1 How can we characterize cyclo–dissipative loads? (that is, loads for which (8) holds)

Q2 If PF improvement is possible, what is the optimal value of the capacitance?

A solution to the second question is straightforward and well known in the circuits community. Indeed, $\|i_s\|^2$ in this case takes the form

$$\|i_s\|^2 = \|i\|^2 - 2C_c < \dot{\hat{i}}, v > + C_c^2 \|\dot{v}\|^2,$$

which is a quadratic equation in the unknown C_c and achieves its minimum at

$$C_c^\star = \frac{< \dot{\hat{i}}, v >}{\|\dot{v}\|^2}. \tag{9}$$

See also [3] for the polyphase case and some illustrative examples. Similar optimization problems for other reactive circuit topologies have been studied in the circuits literature. See [14] for an extensive treatment of the topic. However, there seems to be many open problems, for instance in [15], [16], it is shown that for parallel RL circuits the optimal solution corresponds to a negative inductance and a switched series LC circuit is proposed as an alternative option. To the best of our knowledge, no systematic study of this kind of optimization problem—that would lead, among other things, to a better understanding of admissible topologies and suboptimal solutions—has been carried out.

To explore question Q1, consider the following lemma

Lemma 1 *Assume a (possibly nonlinear) RLC circuit, then the cyclo–dissipativity of the circuit is* independent *of the average steady–state behavior of the resistors. Moreover, inner products for the resistive elements, either voltage or current–controlled, are zero. Henceforth,*

$$< \dot{\hat{i}}, v > = < \boldsymbol{v}_L, \dot{\hat{\boldsymbol{i}}}_L > - < \dot{\boldsymbol{v}}_C, \boldsymbol{i}_C > . \tag{10}$$

where the sub–indices L and C stand from the voltages and currents at inductive and capacitive elements, respectively.

Even though resistors voltages and currents do not explicitly appear in the average supply rate (10) it is clear that they play a role in the overall voltage and current distributions. To unveil the role of the resistors on the cyclo–dissipativity property we define the function

$$q(t) := \boldsymbol{v}_L^\top(t) \dot{\hat{\boldsymbol{i}}}_L(t) - \dot{\boldsymbol{v}}_C^\top(t) \boldsymbol{i}_C(t),$$

in view of (10), we have $\frac{1}{T}\int_0^T q(t)dt = < \dot{\hat{i}}, v >$. With a slight modification to the construction proposed in [18] we can prove the following interesting result.

Fact 3 Assume the inductors and capacitors are linear and passive and the circuit is topologically complete[3]. Then

$$\dot{q}(t) = -2 \begin{bmatrix} \dot{\hat{\boldsymbol{i}}}_L^\top & \dot{\boldsymbol{v}}_C^\top \end{bmatrix} \begin{bmatrix} \nabla^2 G(\boldsymbol{i}_L) & \boldsymbol{\Gamma} \\ \boldsymbol{\Gamma}^\top & -\nabla^2 F(\boldsymbol{v}_C) \end{bmatrix} \begin{bmatrix} \dot{\hat{\boldsymbol{i}}}_L \\ \dot{\boldsymbol{v}}_C \end{bmatrix},$$

where $\boldsymbol{\Gamma} \in \mathbb{R}^{n_L \times n_C}$ is a constant matrix (with elements $(1, 0, -1)$) determined by the interconnection between the inductors and capacitors. $G(\boldsymbol{i}_L)$ and $F(\boldsymbol{v}_C)$ are the content and

[3] basically, this means that all current–controlled and voltage–controlled resistors are in series with inductors and in parallel with capacitors, respectively.

the co–content of the current–controlled resistors and voltage–controlled resistors defined in [17].

Even though we have shown above that the resistors do not intervene in the inner product $<\ \dot{\hat{i}}, v\ >$ we see that they contribute with sign–definite quadratic terms in the time evolution of $q(t)$[4]. As pointed out in [18] the content and co–content functions can be modified adding current and voltage sources, which suggests a procedure to regulate $q(t)$ and henceforth modify the circuit cyclo–dissipativity. Current investigation is under way to further analyze the properties of this differential equation in some circuit configurations of interest for PF compensation.

Fact 4 The cyclo–dissipativity condition $<\ \dot{\hat{\boldsymbol{i}}}, \boldsymbol{v}\ >\geq\ 0$ has a clear energy interpretation for the case of linear L, C, but possibly nonlinear resistors elements. We do not pursue any further this topic here and only mention that for general nonlinear RLC circuits [18]:

magnetic energy "much larger" than electrical energy $\Rightarrow$
cyclo–dissipativity with supply rate $\dot{\hat{\boldsymbol{i}}}^\top \boldsymbol{v}$,

electrical energy "much larger" than magnetic energy $\Rightarrow$
cyclo–dissipativity with supply rate $\boldsymbol{i}^\top \dot{\boldsymbol{v}}$

.

IV. Examples

In this section we present a series of examples that illustrate some of the points discussed in the paper. Example 1, taken from [19], contains a nonlinear RLC circuit which—under some conditions on the parameters—satisfies the cyclo–dissipativity property (8) needed for capacitor PF improvement. The circuit of Example 2 satisfies both cyclo–dissipativity properties, (8) and (III), hence the PF can be improved with either a shunt inductor or a shunt capacitor. At the other extreme, the circuit of Example 3 does not satisfy either one of the cyclo–dissipativity properties. Finally, Example 4 (taken from [20]) presents a circuit, without energy–storing elements, whose PF can be improved with a capacitor. Hence establishing that cyclo–dissipativity does not imply an order relation between the stored energies. In the last three examples we compare the result of PF compensation based on cyclo–dissipativity, with the well–known technique of Budeanu, that we can summarize as follows. The following definition of reactive power was introduced by Budeanu [21]

$$Q_B := \sum_{n=1}^{N} \hat{V}(n)\hat{I}(n)\sin\phi(n), \qquad (11)$$

where $\hat{V}(n)$ and $\hat{I}(n)$ are the rms values of the n–harmonic of the voltage and current respectively, with $N \in \mathbb{Z}_+$ the number of harmonics of interest, and $\phi(n)$ is the phase angle difference of the n–th harmonic. This definition is an attempt to generalize, to the case of multi–frequencies, the classical definition of reactive power for a single harmonic [22] given by

$$Q := VI\sin\phi,$$

with V, I the rms values of the voltage and current signal respectively, and ϕ the phase angle difference.

[4]A similar quadratic form, but with a more complicated "matrix", is obtained for the case of *nonlinear inductors and capacitors* [18].

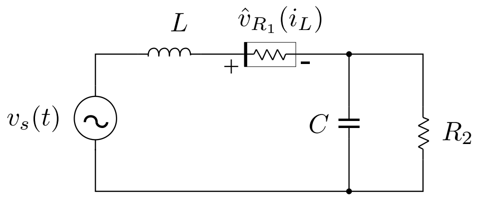

Fig. 3. Nonlinear RLC circuit.

Since the Fourier transform is a linear operator, it follows from Generalized Tellegen's Theorem [17] that Q_B in (11) obeys power conservation, henceforth, the Q_B's of different branch elements of a circuit add up. This suggests a compensation procedure based on the addition of an inductor or a capacitor depending on whether $Q_B > 0$ or $Q_B < 0$, respectively.

Even though the deficiency of this approach has been widely documented in the circuits literature, see for instance, [20], [23], Budeanu's technique enjoys a widespread popularity. (The problem with Budeanu's reactive power definition essentially boils down to the fact that negative and positive Q_B's add up only if they correspond to the same frequency.)

A. Nonlinear RLC circuit

Consider the nonlinear RLC circuit depicted in Fig. 3 with linear elements L, C and R_2 and a nonlinear (current–controlled) resistor R_1 with characteristic function $\hat{v}_{R_1}(i_L)$. A state–space representation of the circuit is given by

$$\begin{aligned} -L\,\dot{\hat{i}}_L &= \hat{v}_{R_1}(i_L) + v_C - v_S, \\ C\dot{v}_C &= i_L - \frac{v_C}{R_2}. \end{aligned}$$

Take the virtual storage function candidate $\phi(i_L, v_C) = \int_0^{i_L} \hat{v}_{R_1}(i_L')di_L' + \frac{R_2}{2}\left[\left(i_L - \frac{1}{R_2}v_C\right)^2 + i_L^2\right]$, which is obtained following the constructive procedure proposed in [24]. It can be shown that the time derivative of ϕ is obtained as

$$\dot{\phi}(i_L, v_C) = \begin{bmatrix} \dot{\hat{i}}_L \\ \dot{v}_C \end{bmatrix}^\top \mathbf{A} \begin{bmatrix} \dot{\hat{i}}_L \\ \dot{v}_C \end{bmatrix} + \dot{\hat{i}}_L v_S, \qquad (12)$$

with

$$\mathbf{A} = \begin{bmatrix} -L & 2R_2C \\ 0 & -C \end{bmatrix}.$$

Notice that the symmetric part of the matrix $\mathbf{A}$ is negative semi–definite under the condition

$$R_2 \leq \sqrt{\frac{L}{C}}. \qquad (13)$$

Thus, from (12) and Theorem 1, we can conclude that the circuit is *cyclo–dissipative* with respect to the supply rate $\dot{\hat{i}}_L v_S$ if L, C, R_2 satisfy (13). Moreover, if R_1 is passive, that is, if $\hat{v}_{R_1}(i_L)$ is a first–third quadrant function, then $\phi \geq 0$ and qualifies as a storage function, proving that the circuit is, in addition, *dissipative* with supply rate $\dot{\hat{i}}_L v_S$.

B. LTI series RLC with two cyclo–dissipativity properties

Consider the LTI series RLC circuit of Fig. 4, supplied with a periodic voltage source

$$v_s(t) = 360\sqrt{2}\sin(\omega_0 t) + 144\sqrt{2}\sin(3\omega_0 t) + 42\sqrt{2}\sin(5\omega_0 t),$$

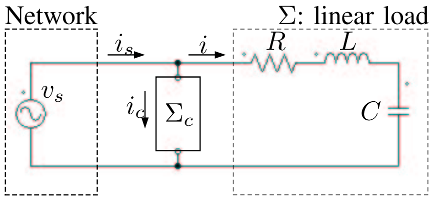

Fig. 4. LTI series RLC load.

TABLE I

VALUES OF THE LOAD VARIABLES FOR THE LTI SERIES RLC CIRCUIT.

Var.	Uncomp.	Shunt cap.	Shunt ind.	Comp. Budeanu (Shunt ind.)
$\|v\|$	390 V	390 V	390 V	390 V
$\|i\|$	5.72 A	5.52 A	5.28 A	5.4 A
P	491.76 W	491.76 W	491.76 W	491.76 W
S	2233.04 VA	2155.09 VA	2059.51 VA	2107.44 VA
PF	0.2202	0.2281	0.2393	0.2333

where $\omega_0 = 100\pi$ rad/s and $R = 15\ \Omega$, $L = 0.0796$ H and $C = 0.0212$ mF.

For this circuit we have $< v, \hat{\hat{i}} >= 282804.23$, hence the PF can be compensated using a shunt capacitor. The optimal capacitor, given by (9), is $C_c^\star = 7.952\ \mu$F, yielding an improved power factor PF $=$ 0.2281. Interestingly, the cyclo–dissipativity condition $< v, \int i > \geq 0$ is also satisfied, in fact, $< v, \int i >= 2.59$. Hence, the compensator system could be a shunt inductor with optimal value

$$L_c^\star = \frac{-\| \int v\|^2}{< \int v, i >} = 0.5161 \text{ H},$$

yielding an improved power factor PF $= 0.2393$.

Budeanu's reactive power (11) yields $Q_B = -392.66$ VAr, a negative value that suggests that the load is capacitive. However, as shown above the power factor can be improved either using a shunt capacitor or a shunt inductor, so the load has no special character: it is not inductive nor capacitive. Table I summarizes the different values of the load variables, for the three compensation schemes illustrated in this example.

C. LTI series-parallel RLC load

The circuit depicted in Fig. 5 with

$$v_s(t) = 220\sqrt{2}\sin(\omega_0 t) + 70\sqrt{2}\sin(3\omega_0 t),$$

$\omega_0 = 100\pi$ rad/s, and $R = 10\ \Omega$, $L = 0.2$ H and $C = 0.04$ mF does not satisfy either one of the cyclo–dissipativity conditions, hence cannot be compensated with a capacitor or an inductor.

The cyclo–dissipativity conditions take the values $< v, \hat{\hat{i}} >= -84524.79$ and $< \int v, i >= 0.3629$, therefore the power factor *cannot be increased* using a capacitor or an inductor. On the other hand, Budeanu's reactive power is $Q_B = 18.26$ VAr

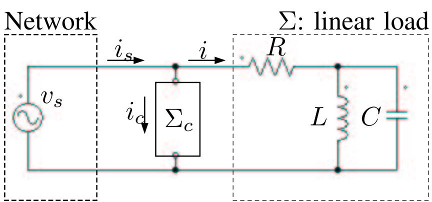

Fig. 5. LTI series-parallel RLC load.

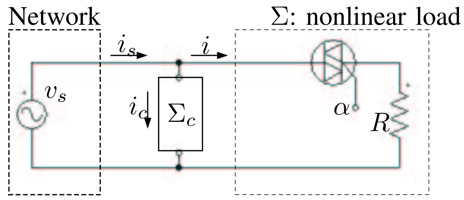

Fig. 6. Circuit with a TRIAC controlled resistive load.

suggesting that the PF can be compensated with an capacitor $C_B = 0.9211\ \mu$F. However, we can see in Table II that the PF is degraded with the shunt capacitor, as it was predicted by the cyclo–dissipativity analysis.

D. Circuit with a TRIAC controlled resistive load

This example shows that the implication that cyclo–dissipativity implies some kind of order relation between the stored energies is not true. Take the nonlinear circuit of Fig. 6 with a TRIAC controlled purely resistive load ($R = 10\ \Omega$) operating in two different regimes: a purely sinusoidal and a periodic source with fundamental frequency and third harmonic. In both cases $\omega_0 = 100\pi$ rad/s and the TRIAC firing angle is $\alpha = \pi/2$. The corresponding voltage v_S and currents i_S for the sinusoidal and periodic cases, without compensation, are depicted in Fig. 7.

Sinusoidal source: For $v_s(t) = 220\sqrt{2}\sin(\omega_0 t)$ the cyclo–dissipativity condition is $< v, \hat{\hat{i}} >= 484000$, which means that a shunt capacitor can be used to improve the PF. The optimal capacitor is obtained as $C_\star = 0.101$ mF, yielding an improved PF of PF $= 0.7919$.

In this *single–frequency* example Budeanu's analysis is consistent with the (necessary) cyclo–dissipativity condition ($< v, \hat{\hat{i}} >= 484000$) for capacitive compensation, and both yield the same optimal capacitor compensator $C_B = C_\star = 0.101$ mF. Table III shows the variables values at the load terminals.

Periodic source: For $v_s(t) = 220\sqrt{2}\sin(\omega_0 t) + 50\sqrt{2}\sin(3\omega_0 t)$ we obtain $< v, \hat{\hat{i}} >= 289000$, hence the load can be compensated with a capacitor whose optimal value is $C_\star = 0.0413$ mF, yielding PF $= 0.7258$. It is important to remark that if Budeanu's reactive power ($Q_B = 1567.14$ VAr) is used to design the capacitor the results would be: $C_B = 0.08923$ mF and PF $= 0.7014$. So, the capacitor doubles the previous value and the power factor is worse than in the no-compensated case. Table IV summarizes the results obtained for this example.

TABLE II

LOAD VARIABLES FOR THE LTI SERIES-PARALLEL RLC CIRCUIT.

Variables	Without comp.	Comp. Budeanu
$\|v\|$	230.86 V	230.86 V
$\|i\|$	2.28 A	2.31 A
P	52.0 W	52.0 W
S	526.3 VA	534.46 VA
PF	0.0987	0.0972

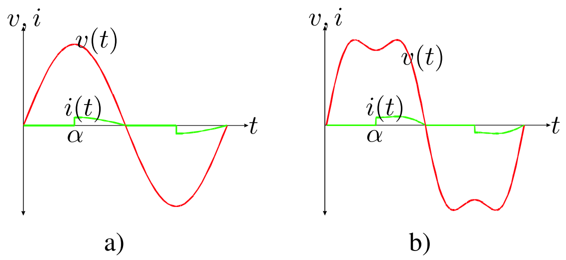

Fig. 7. Voltage and current waveforms in the (uncompensated) circuit with the TRIAC controlled resistive load.

TABLE III

LOAD VARIABLES FOR THE SINUSOIDAL CASE IN THE TRIAC EXAMPLE.

Variables	Without comp.	Comp. Cyclo–diss.	Comp. Budeanu
$\|\|v\|\|$	220 V	220 V	220 V
$\|\|i\|\|$	15.55 A	13.89 A	13.89 A
P	2420.0 W	2420.0 W	2420.0 W
S	3422.39 VA	3056.02 VA	3056.02 VA
PF	0.7071	0.7919	0.7919

V. CONCLUDING REMARKS

The main contribution of the paper is the proof that a certain cyclo–dissipativity property of the compensated load, namely (5), is *necessary for PF improvement*. This important observation suggests an analysis and compensator design framework based on cyclo–dissipativity, which is a natural alternative candidate to replace (standard) dissipativity for applications where we are interested in inducing periodic orbits, instead of stabilizing equilibria.

Although the framework applies for general polyphase—possibly unbalanced—circuits, for the sake of clarity, we have presented in some detail the problem of PF compensation with LTI capacitors or inductors of single phase loads only. It is our belief that the full power of the proposed approach will become evident for polyphase unbalanced loads with (possibly nonlinear) general lossless compensators, where the existing solutions are far from satisfactory [3]. It is not clear at this point whether this, unquestionably elegant, setting will yield practical solutions. The main obstacle been that in this application very little is known about the nature of the load—a piece of information that is essential for a successful design. Also, it would be highly desirable to formulate a clear parameter minimization problem, in particular, for other circuit topologies and poly–phase loads, where questions like compensator circuit complexity, existence of the optimal solution and sub–optimality could be addressed.

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TABLE IV

LOAD VARIABLES FOR THE PERIODIC CASE IN THE TRIAC EXAMPLE.

Variables	Without comp.	Comp. Cyclo–diss	Comp. Budeanu
$\|\|v\|\|$	225.61 V	225.61 V	225.61 V
$\|\|i\|\|$	15.95 A	15.57 A	16.08 A
P	2545 W	2545 W	2545 W
S	3599.17 VA	3513.76 VA	3628.36 VA
PF	0.7071	0.7258	0.7014

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