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http://dx.doi.org/10.1007/s10659-023-10020-1.


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A SECOND GRADIENT THEORY OF THERMOVISCOELASTICITY

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ABSTRACT. In this paper we present a linear theory of thermoviscoelastic materials of Kelvin-Voigt type, where the constitutive equations depend on the first and second temperature gradients. We use some established results in strain gradient theory and Green-Naghdi thermomechanics. Alongside double force per unit area we introduce the entropy flux tensor. First, we establish the basic equations of the theory by using the balance of energy, the Clausius-Duhem inequality and invariance requirements under superposed rigid body motions. It is shown that the stress tensors depend on the first and second temperature gradients. The boundary-initial-value problems are formulated. We present a uniqueness result in the dynamic theory of thermoviscoelastic materials. We also establish an existence result and prove the analyticity of the semigroup, which implies the exponential decay of the solutions and their impossibility of localization.

KEYWORDS: Stress tensors dependent on temperature gradients, Thermoviscoelasticity, Dissipation inequality, Boundary conditions, Existence and uniqueness, A fourth-order heat equation.

MATHEMATICS SUBJECT CLASSIFICATION 2010: 35L56, 74A15, 74A20, 74A30, 74D05, 74H25, 74H45.

1. Introduction

In the last years the theories of non-simple materials have been the object of intensive research. The origin of the theory of non-simple elastic materials goes back to the works of Toupin [1] and Mindlin [2]. The equations and the boundary conditions of the nonlinear strain gradient theory of elastic solids were first established in [1]. The linear theory has been investigated by Mindlin
and Eshel [3]. Non-local theories have gained fundamental importance in various research fields. The strain gradient theory of elasticity is suitable for investigating problems related to size effects and nanotechnology [4]. Green and Naghdi [5,6] developed a thermomechanical theory of deformable continua that relies on an entropy balance law rather than on an entropy inequality. The linear theory of thermoelasticity based on the new entropy balance law has been established in [7]. Various works have been devoted to the study of non-simple thermoelastic materials (see, e.g., [8–14] and the references therein). The interest in introducing temperature gradient effects in thermomechanics was motivated in [11]. Some recent results in the study of viscoelastic materials were presented in [15–18].

In the first part of this paper we extend the strain gradient theory of thermoelastic materials to the case when the time derivative of the strain tensors and the time derivative of the first and second gradient of thermal displacement are included in the set of independent constitutive variables. We use some established results in strain gradient theory [1, 2] and Green-Naghdi thermomechanics [5–7]. However, as in [15], instead of using the entropy balance law we use an entropy production inequality. Analogous to the introduction of the hyperstress tensor we introduce an entropy flux tensor. We present the restrictions imposed on the constitutive equations by the entropy production inequality. In Section 2 we establish the equations of a linear theory of thermoviscoelastic materials of Kelvin-Voigt type. We obtain constitutive equations that depend on the second temperature gradient. The boundary-initial-value problems are formulated in Section 3. Section 4 is devoted to homogeneous and isotropic solids. We present the field equations and show that the present theory leads to a fourth order equation for temperature. Section 5 concerns a uniqueness result for basic boundary-initial-value problems. In Section 6 we provide an existence result under the assumption that an appropriate quadratic form is strictly positive definite. With the same hypotheses, in Section 7 we prove that the semigroup is analytic. Finally, in Section 8 we summarize the main results.

2. Basic equations

In this section we establish a linear theory of non-simple thermoviscoelastic solids. We consider a body that at time $t_0$ occupies the bounded region $B$ with Lipschitz boundary $\partial B$. As in [1,2], we assume that the boundary consists of the union of a finite number of smooth surfaces, smooth curves (edges) and points (corners). We denote by $C$ the union of the edges. The motion of the body is referred to the reference configuration $B$ and to a fixed system of rectangular Cartesian axes $Ox_j, \ (j = 1, 2, 3)$. Throughout we employ standard indicial notations: Greek subscripts are understood to range over the integers (1, 2), whereas Latin subscripts (unless otherwise specified) range over (1, 2, 3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. In what follows, we use a superposed dot to denote partial differentiation with respect to the time $t$.

We consider an arbitrary region $\mathcal{P}$ at time $t$, bounded by a surface $\partial \mathcal{P}$, and suppose that $P$ is the corresponding region at time $t_0$, bounded by the surface $\partial P$. We postulate an entropy production inequality in the form [5,19],

$$\int_P \rho \dot{\eta} dv - \int_P \rho s dv - \int_{\partial P} \sigma da \geq 0,$$

for all regions $P$ of $B$ and every time. In (2.1) we have used the following notations: $\rho$ is the reference mass density, $\eta$ is the entropy per unit mass, $\sigma$ is the internal flux of entropy per unit
A second gradient theory of thermoviscoelasticity

area, and $s$ is the external rate of supply of entropy per unit mass [7]. Following [20], from (2.1) we get

\begin{equation}
\sigma = \Lambda_j n_j,
\end{equation}

where $\Lambda_j$ is the entropy flux vector [5] and $n_j$ is the outward unit normal of $\partial P$. If we use (2.2) and the divergence theorem, then from (2.1) we obtain the local form of the second law of thermodynamics

\begin{equation}
\rho \dot{\eta} - \Lambda_j n_j - \rho s \geq 0.
\end{equation}

Following Green and Naghdi [5, 7] we introduce the thermal displacement $\alpha$ by

\begin{equation}
\dot{\alpha} = \theta,
\end{equation}

where $\theta$ is the absolute temperature that is assumed to be positive. Based on the results presented in [1, 2, 7, 21] we postulate the conservation of energy, for every regular region $P$ of $B$ and every time, in the form

\begin{equation}
\int_P \rho \dot{e} + \rho \dot{\alpha} \dot{\alpha} = \int_P \left[ t_i \dot{u}_i + \mu_{ij} \dot{u}_i \dot{u}_j + \sigma \dot{\alpha} + H_j \dot{\alpha}_j \right] da,
\end{equation}

where $u_i$ is the displacement vector, $e$ is the internal energy per unit mass, $f_i$ is the body force per unit mass, $t_i$ is a part of the stress vector associated with the surface $\partial P$ but measured per unit area of $\partial P$, $\mu_{ij}$ is the dipolar surface force [21] and $H_j$ is the monopolar entropy flux per unit area. Mindlin [2] called $\mu_{ij}$ the double force per unit area. From (2.5) we obtain the balance of linear momentum [20]. Using the well-known method, we find

\begin{equation}
t_i = t_{ji} n_j,
\end{equation}

and

\begin{equation}
t_{ji,j} + \rho f_i = \rho \ddot{u}_i,
\end{equation}

where $t_{ij}$ is the stress tensor. With the help of (2.6) and (2.7) we can write the relation (2.5) in the form

\begin{equation}
\int_P \rho \dot{e} = \int_P \left[ t_{ji,j} \dot{u}_{ij} + (\Lambda_j \dot{\alpha})_j + \rho s \dot{\alpha} \right] da + \int_{\partial P} \left[ \mu_{ji} \dot{u}_{ij} + H_j \dot{\alpha}_j \right] da.
\end{equation}

We apply this equation to a region which in the reference state was a tetrahedron bounded by coordinate planes through the point $x$ and by a plane whose unit normal is $n_j$, to obtain

\begin{equation}
(\mu_{ji} - \mu_{kji} n_k) \hat{u}_{ij} + (H_j - H_{kj} n_k) \hat{\alpha}_j = 0,
\end{equation}

where $\mu_{ijk}$ is the dipolar stress tensor (or hyperstress tensor [1]) and $H_{ij}$ is entropy flux tensor. If we take into account (2.2) and (2.9), then from (2.8) we find the local form of energy balance

\begin{equation}
\rho \dot{e} = \tau_{ji} \dot{u}_{ij} + \mu_{ijk} \dot{u}_k \dot{u}_{ij} + \Pi_j \dot{\alpha}_{ij} + H_{ij} \dot{\alpha}_{ij} + (\rho s + \Lambda_{ij,j}) \dot{\alpha},
\end{equation}

where we have used the notations

\begin{equation}
\tau_{ji} = t_{ji} + \mu_{kji,k}, \quad \Pi_i = \Lambda_i + H_{ki,k}.
\end{equation}

We now consider motions of the body which are such that the velocities differ from those of the given motion only by a superposed uniform rigid body angular velocity, and assume that $\rho, \dot{e}, \tau_{ji}, \mu_{ijk}, \Pi_j, H_{ij}, s$ and $\Lambda_j$ are unaltered by such motions. It is a simple matter to see that from (2.10) we obtain [1, 20]

\begin{equation}
\tau_{ij} = \tau_{ji}.
\end{equation}
Then, the relation (2.10) becomes
\[
\rho \dot{e} = \tau_{ij}\dot{e}_{ij} + \mu_{ijk}\dot{\kappa}_{ijk} + \Pi_j\dot{\alpha}_{,j} + H_{ij}\dot{\alpha}_{,ij} + (\rho s + \Lambda_{ij,j})\dot{\alpha},
\]
where \(e_{ij}\) and \(\kappa_{ijk}\) are the strain tensors from the linear theory of strain gradient elasticity [3]
\[
e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \kappa_{ijk} = u_{k,ij}.
\]
If we introduce the Helmholtz free energy
\[
\psi = e - \dot{\alpha}\eta,
\]
then the equation of energy (2.13) becomes
\[
\rho(\dot{\psi} + \ddot{\alpha}\eta) = \tau_{ij}\dot{e}_{ij} + \mu_{ijk}\dot{\kappa}_{ijk} + \Pi_j\dot{\alpha}_{,j} + H_{ij}\dot{\alpha}_{,ij} + (\rho s + \Lambda_{ij,j})\dot{\alpha}.
\]
We assumed that the absolute temperature \(\dot{\alpha}\) is positive so that from (2.3) we get
\[
\rho\dot{\eta} - (\rho s + \Lambda_{ij,j})\dot{\alpha} \geq 0.
\]
Combining (2.16) and (2.17), we arrive at the local dissipation inequality
\[
\tau_{ij}\dot{e}_{ij} + \mu_{ijk}\dot{\kappa}_{ijk} + \Pi_j\dot{\alpha}_{,j} + H_{ij}\dot{\alpha}_{,ij} - \rho(\dot{\psi} + \ddot{\alpha}\eta) \geq 0.
\]
In what follows we consider homogeneous thermoviscoelastic materials of Kelvin-Voigt type for which the following constitutive equations hold
\[
\psi = \hat{\psi}(A), \quad \tau_{ij} = \hat{\tau}_{ij}(A), \quad \mu_{ijk} = \hat{\mu}_{ijk}(A), \quad \Pi_j = \hat{\Pi}_j(A), \quad \eta = \hat{\eta}(A),
\]
where
\[
A = (e_{ij}, \kappa_{ijk}, \theta, \alpha_{,j}, \dot{e}_{ij}, \dot{\kappa}_{ijk}, \dot{\alpha}_{,j}).
\]
We note that in [7] the authors established a theory of thermoelasticity without energy dissipation where the independent constitutive variables are \(e_{ij}, \theta\) and \(\alpha_{,j}\). In view of (2.20), the inequality (2.18) leads to
\[
\left(\tau_{ij} - \frac{\partial U}{\partial \dot{e}_{ij}}\right)\dot{e}_{ij} + \left(\mu_{ijk} - \frac{\partial U}{\partial \dot{\kappa}_{ijk}}\right)\dot{\kappa}_{ijk} + \left(\Pi_j - \frac{\partial U}{\partial \dot{\alpha}_{,j}}\right)\dot{\alpha}_{,j} + \left(H_{ij} - \frac{\partial U}{\partial \dot{\alpha}_{,ij}}\right)\dot{\alpha}_{,ij}
\]
\[- \left(\rho\dot{\eta} + \frac{\partial U}{\partial \dot{\alpha}}\right)\dot{\alpha} - \frac{\partial U}{\partial \dot{\alpha}_{,i}}\dot{\alpha}_{,i} - \frac{\partial U}{\partial \dot{\alpha}_{,ij}}\dot{\alpha}_{,ij} - \frac{\partial U}{\partial \dot{e}_{ij}}\dot{e}_{ij} - \frac{\partial U}{\partial \dot{\kappa}_{ijk}}\dot{\kappa}_{ijk} \geq 0,
\]
where we have used the notation
\[
U = \rho\hat{\psi}.
\]
In absence of internal constraints, from (2.21) we get
\[
U = \hat{U}(e_{ij}, \kappa_{ijk}, \dot{\alpha}, \alpha_{,i}, \alpha_{,ij}),
\]
and
\[
\rho\dot{\eta} = -\frac{\partial U}{\partial \dot{\alpha}}.
\]
If we introduce the functions \(\tau^*_{ij}, \mu^*_{ijk}, \Pi^*_j\) and \(H^*_{ij}\) by
\[
\tau_{ij} = \frac{\partial U}{\partial \dot{e}_{ij}} + \tau^*_{ij}, \quad \mu_{ijk} = \frac{\partial U}{\partial \dot{\kappa}_{ijk}} + \mu^*_{ijk}, \quad \Pi_j = \frac{\partial U}{\partial \dot{\alpha}_{,j}} + \Pi^*_j, \quad H_{ij} = \frac{\partial U}{\partial \dot{\alpha}_{,ij}} + H^*_{ij},
\]
(2.25)
then the inequality (2.21) can be be written in the form
\begin{equation}
\tau^*_{ij} \dot{e}_{ij} + \mu^*_{ijk} \dot{k}_{ijk} + \Pi^*_{ij} \dot{\alpha}_{ij} + H^*_{ij} \dot{\alpha}_{ij} \geq 0.
\end{equation}
Clearly, the functions $\tau^*_{ij}$, $\mu^*_{ijk}$, $\Pi^*_{ij}$ and $H^*_{ij}$ depend on the elements of the set $A$. It follows from (2.26) that these functions must vanish when all the variables $\dot{e}_{ij}$, $\dot{k}_{ijk}$, $\dot{\alpha}_{ij}$ and $\dot{\alpha}_{ij}$ are equal to zero,
\begin{equation}
\tau^*_{ij}(A_0) = 0, \mu^*_{ijk}(A_0) = 0, \Pi^*_{ij}(A_0) = 0, H^*_{ij}(A_0) = 0,
\end{equation}
where
\begin{equation}
A_0 = (e_{ij}, \kappa_{ijk}, \dot{\alpha}, \alpha_{ij}, 0, 0, 0, 0).
\end{equation}
With the help of (2.25) we find that the equation (2.16) becomes
\begin{equation}
\rho \theta \dot{\eta} = \tau^*_{ij} \dot{e}_{ij} + \mu^*_{ijk} \dot{k}_{ijk} + \Pi^*_{ij} \dot{\alpha}_{ij} + H^*_{ij} \dot{\alpha}_{ij} + \Lambda_{ij} \dot{\alpha} + \rho s \dot{\alpha}.
\end{equation}
Following [5] we introduce the notations
\begin{equation}
\alpha(x, t_0) = \alpha_0, \dot{\alpha}(x, t_0) = T_0,
\end{equation}
and assume that $\alpha_0$ and $T_0$ are prescribed constants. If we denote
\begin{equation}
T = \theta - T_0 \text{ and } \varphi = \int_{t_0}^t T dt,
\end{equation}
then from (2.4) and (2.31) we get
\begin{equation}
\alpha = \varphi + (t - t_0)T_0 + \alpha_0, \dot{\varphi} = T, \alpha_{ij} = \varphi_{ij}.
\end{equation}
In the linear theory we assume that $u_i = \varepsilon u_i'$, $\varphi = \varepsilon \varphi'$, where $\varepsilon$ is a constant small enough for squares and higher powers to be neglected, and $u_i'$ and $\varphi'$ are independent of $\varepsilon$. In this case, the independent constitutive variables of the set $A$ become $e_{ij}$, $\kappa_{ijk}$, $\dot{\varphi}_{ij}$, $\varphi_{ij}$, $\dot{\alpha}_{ij}$, $\kappa_{ijk}$, $\dot{\varphi}_{ij}$ and $\dot{\varphi}_{ij}$. We assume that the functions $\tau^*_{ij}$, $\mu^*_{ijk}$, $\Pi^*_{ij}$, $H^*_{ij}$ and $\eta$ vanish in the reference configuration. In the linear theory of centrosymmetric solids, we have
\begin{equation}
2U = A_{ijrs} e_{ij} e_{rs} + B_{ijklpq} \kappa_{ijkl} \kappa_{pq} + 2 C_{ijrs} e_{ij} \varphi_{rs} + D_{ijrs} \dot{\varphi}_{ij} \varphi_{rs} + 2 E_{ijrs} \kappa_{ijr} \varphi_{s} + K_{ij} \dot{\varphi}_{ij} \varphi_{ij} - \alpha \dot{\varphi}^2 - 2 \beta_{ij} e_{ij} \dot{\varphi}^2 - 2 \gamma_{ij} e_{ij} \dot{\varphi},
\end{equation}
where the constitutive coefficients have the following symmetries
\begin{equation}
A_{ijrs} = A_{jirs} = A_{rsij}, B_{ijklpq} = B_{pqijkl} = B_{jiklpq},
\end{equation}
\begin{equation}
C_{ijrs} = C_{jirs} = C_{ijsr}, D_{ijrs} = D_{jirs} = D_{rsij}, E_{ijrs} = E_{jirs},
\end{equation}
\begin{equation}
K_{ij} = K_{ji}, b_{ij} = b_{ji}, c_{ij} = c_{ji}.
\end{equation}
If we take into account (2.27) and (2.32), then we find
\begin{equation}
\tau^*_{ij} = A^*_{ijrs} \dot{e}_{rs} + C^*_{ijrs} T_{rs}, \mu^*_{ijk} = B^*_{ijklpq} \kappa_{pq} + E^*_{ijkp} T_{kp},
\end{equation}
\begin{equation}
\Pi^*_{ij} = F^*_{pqrs} \kappa_{pq} + K^*_{ij} T_{rs}, H^*_{ij} = G^*_{r} \dot{e}_{rs} + D^*_{ijrs} T_{rs}.
\end{equation}
where $A^*_{ijrs}$, $B^*_{ijklpq}$, $C^*_{ijkl}$, $D^*_{ijrs}$, $E^*_{ijkp}$, $F^*_{ijrs}$, $G^*_{ijrs}$ and $K^*_{ij}$ are constitutive coefficients. In view of (2.35), the dissipation inequality (2.26) takes the form
\begin{equation}
\Gamma \geq 0,
\end{equation}
where
\begin{equation}
\Gamma = A^*_{ijrs} \dot{e}_{ij} \dot{e}_{rs} + B^*_{ijklpq} \kappa_{ijkl} \kappa_{pq} + K^*_{ij} \varphi^*_{ij} \varphi^*_{ij} + D^*_{ijrs} \dot{\varphi}_{ij} \dot{\varphi}_{rs} + (C^*_{ijrs} + G^*_{ijrs}) \dot{e}_{ij} \dot{\varphi}_{rs} + (E^*_{ijkp} + F^*_{ijkp}) \kappa_{ijkl} \dot{\varphi}_{ij} \dot{\varphi}_{rs}.
We assume that the parts of $\tau^*_ij$, $\mu^*_ijk$, $\Pi^*_i$ and $H^*_ij$ that do not contribute to dissipation are zero. Thus, we have

$$
\begin{align*}
A^*_{ijrs} &= A^*_{jirs} = A^*_{rsij}, \quad B^*_{pqrijk} = B^*_{pqrijk}, \\
C^*_{ijrs} &= C^*_{jirs} = C^*_{rsij}, \quad K^*_i = K^*_{ji}, \quad D^*_ijrs = D^*_jirs = D^*_rsij, \\
E^*_{ijkp} &= E^*_{jikp}, \quad F^*_{ijkp} = F^*_{jikp}, \quad G^*_{ijrs} = G^*_{jirs} = G^*_{rsij}.
\end{align*}
$$

(2.38)

Let us note that in this theory the stress tensors depend on the temperature gradients. From (2.25), (2.33) and (2.34) we find the following constitutive equations of the linear theory

$$
\begin{align*}
\tau_{ij} &= A_{ijrs} \varepsilon_{rs} + C_{ijrs} \varphi_{,rs} - b_{ij} \dot{\varphi} + \tau_{ij}^*, \\
\mu_{ijk} &= B_{ijkpqr} \kappa_{pqr} + E_{ijks} \varphi_{,s} + \mu_{ijk}^*, \\
\rho n &= b_{ij} e_{ij} + c_{ij} \dot{\varphi}_{,ij} + a \dot{\varphi}, \\
\Pi_i &= E_{pqri} \kappa_{pqr} + K_{ij} \varphi_{,j} + \Pi_i^*, \\
H_{ij} &= C_{rsij} \varepsilon_{rs} + D_{ijrs} \varphi_{,rs} - c_{ij} \dot{\varphi} + H_{ij}^*.
\end{align*}
$$

(2.39)

where $\tau_{ij}^*, \mu_{ijk}^*, \Pi_i^*$ and $H_{ij}^*$ are given by (2.35). In the linear theory, the equation of energy (2.29) reduces to

$$
\rho \ddot{\varphi} = \Lambda_{j,j} + \rho s.
$$

(2.40)

If we use (2.11), then the equations of motion (2.7) become

$$
\tau_{ji,j} - \mu_{kji,kj} + \rho f_i = \rho \ddot{u}_i,
$$

(2.41)

and the equation of energy (2.40) reduces to

$$
\Pi_{j,j} - H_{k,j,kj} + \rho s = \rho \ddot{\varphi}.
$$

(2.42)

The linear theory is characterized by the following equations: equations of motion (2.41), equation of energy (2.42), constitutive equations (2.39) and (2.35), and the geometrical equations (2.14). To these equations we have to add the boundary and initial conditions.

Equations (2.41) and (2.42) can be expressed as:

$$
\begin{align*}
\rho \ddot{u}_i &= (A_{ijrs} u_{rs} + C_{ijrs} \varphi_{,rs} - b_{ij} \dot{\varphi} + A^*_{ijrs} \ddot{u}_{rs} + C^*_{ijrs} \dot{\varphi}_{,rs})_{,j} \\
&\quad - (B_{kjispq} u_{rpq} + E_{kjis} \dot{\varphi}_{,s} + B^*_{kjispq} \ddot{u}_{rpq} + E^*_{kjis} \dot{\varphi}_{,s})_{,kj} + \rho f_i, \\
a \ddot{\varphi} &= (E_{pqri} u_{pq} + K_{ij} \dot{\varphi}_{,j} + F^*_{pqri} \ddot{u}_{pq} + K^*_{ij} \dot{\varphi}_{,j})_{,i} \\
&\quad - (C_{rsij} u_{rs} + D_{ijrs} \varphi_{,rs} + G^*_{rsij} \ddot{u}_{i,j} + D^*_{ijrs} \dot{\varphi}_{,rs})_{,ij} - b_{ij} \dddot{u}_{i,j} - c_{ij} \dot{\varphi}_{,ij} + (c_{ij} \dot{\varphi})_{,ij} + \rho s.
\end{align*}
$$

(2.43)

3. Boundary-initial-value problems

The boundary conditions in the strain gradient theory of elasticity were established by Toupin [1]. In this section we present boundary conditions in the second gradient theory of thermoviscoelasticity. In view of (2.2), (2.6), (2.9) and (2.11), the surface integral from (2.5) becomes

$$
J = \int_{\partial P} (\tau_{ki} \dddot{u}_i + \mu_{kji} \dddot{u}_{i,j} + \sigma \dddot{\varphi} + H_j \dddot{\varphi}_{,j}) \, da
$$

(3.1)

$$
= \int_{\partial P} [(\tau_{ki} - \mu_{ski,s}) \dddot{u}_i + \mu_{kji} \dddot{u}_{i,j} + (\Pi_k - H_jk_{ki,j}) \dddot{\varphi} + H_{ki} \dddot{\varphi}_{,j}] \, nk \, da.
$$
Here, \( u_{i,j} \) is not independent of \( u_i \) on \( \partial B \). Following the procedure of Toupin [1] we can express \( J \) in the form

\[
(3.2) \quad J = \int_{\partial B} (P_i \dot{u}_i + R_i D \dot{u}_i + \Sigma \dot{\varphi} + \Omega D \dot{\varphi}) da + \int_C (Z_i \dot{u}_i + Y \dot{\varphi}) ds,
\]

where \( Df = f_j n_j \) and

\[
(3.3) \quad P_i = (r_{ij} - \mu_{kji,k}) n_j - D_j (\mu_{kji,n} k + (D_j n_j) \mu_{pqi} n_p n_q),
\]

\[
R_i = \mu_{rsi} n_r n_s, Z_i = <\mu_{pqi} n_p y_q >,
\]

\[
\Sigma = (\Pi_j - H_{kij,k}) n_j + (D_j n_j) H_{kpi} n_p n_q - D_j (H_{kji} n_k),
\]

\[
\Omega = H_{ij} n_i n_j, Y = <H_{ki} n_k y_j >, y_j = \varepsilon_{irj} n_i n_j s_r.
\]

In (3.3), \( D_i \) is the surface gradient, \( D_i = (\delta_{ij} - n_i n_j) \partial / \partial x_j \), \( \varepsilon_{ijk} \) is the alternating symbol, \( s_j \) are the components of the unit vector tangent to \( C \), and \( <f> \) denotes the difference of limits of \( f \) from the both sides of \( C \).

The first boundary-initial-value problem is characterized by the boundary conditions

\[
(3.4) \quad u_i = \tilde{u}_i, D u_i = \tilde{w}_i, \varphi = \tilde{\varphi}, D \varphi = \tilde{\phi} \quad \text{on} \quad \partial B \times (t_0, t_1),
\]

where \( \tilde{u}_i, \tilde{w}_i, \tilde{\varphi} \) and \( \tilde{\phi} \) are prescribed functions and \( t_1 \) is some instant that may be infinite.

In the second boundary-initial-value problem, the boundary conditions are (see [1])

\[
(3.5) \quad P_i = \bar{P}_i, R_i = \bar{R}_i, \Sigma = \bar{\Sigma}, \Omega = \bar{\Omega} \quad \text{on} \quad \partial B \times (t_0, t_1),
\]

\[
Z_i = \bar{Z}_i, Y = \bar{Y} \quad \text{on} \quad C \times (t_0, t_1),
\]

where \( \bar{P}_i, \bar{R}_i, \bar{\Sigma}, \bar{\Omega}, \bar{Z}_i \) and \( \bar{Y} \) are given functions.

The initial conditions are

\[
(3.6) \quad u_i(x, 0) = u_i^0(x), \dot{u}_i(x, 0) = v_i^0(x), \varphi(x, 0) = \psi^0(x), \dot{\varphi}(x, 0) = \chi^0(x), \quad x \in \bar{B},
\]

where the functions \( u_i^0, v_i^0, \psi^0 \) and \( \chi^0 \) are prescribed. Throughout this paper we assume that: (i) \( \rho \) is a strictly positive function on \( \bar{B} \); (ii) \( f_j \) and \( s \) are continuous on \( \bar{B} \times (t_0, t_1); \) (iii) \( u_i^0, v_i^0, \psi^0 \) and \( \chi^0 \) are continuous on \( \bar{B} \); (iv) \( \tilde{u}_i \) and \( \tilde{\varphi} \) are continuous on \( \partial B \times (t_0, t_1); \) (v) \( \tilde{w}_i, \tilde{\varphi}, \tilde{P}_i, \tilde{R}_i, \Sigma \) and \( \tilde{\Omega} \) are continuous in time and piecewise regular on \( \partial B \times (t_0, t_1); \) (vi) \( \tilde{Z}_i \) and \( \tilde{Y} \) are continuous in time and piecewise regular on \( C \times (t_0, t_1); \) (vii) the constitutive coefficients are functions of class \( C^2 \) on \( B \) that satisfy the relations (2.34), (2.36) and (2.38).

4. ISOTROPIC MATERIALS

In this section we present the constitutive equations for isotropic and homogeneous solids and express the basic equations in terms of the functions \( u_j \) and \( \varphi \). By taking into account the results from [2,3], we find that for isotropic bodies the function \( U \) has the form

\[
(4.1) \quad U = \frac{1}{2} \left[ \lambda e_{rr} e_{ss} + d_1 (\Delta \varphi)^2 + d_2 \varphi_{ij} \varphi_{ij} + k \varphi_{j} \varphi_{,j} - a \varphi^2 \right] + \mu e_{ij} e_{ij} + \alpha_1 \kappa_{ijk} \kappa_{kij} + \alpha_2 \kappa_{ijij} \kappa_{irr} + \alpha_3 \kappa_{iiir} \kappa_{jjir} + \alpha_4 \kappa_{ik} \kappa_{iik} + \alpha_5 \kappa_{ijk} \kappa_{jki} + \beta_1 \epsilon_{rr} \Delta \varphi + \beta_2 e_{ij} \varphi_{,ij} - be_{rr} \varphi + \gamma_1 \kappa_{ss} \varphi_{,k} + 2 \gamma_2 \kappa_{iss} \varphi_{,i} - c \varphi \Delta \varphi,
\]
where $\Delta$ is the Laplacian and $\lambda, \mu, \alpha_i, (j = 1, 2, \ldots, 5), \beta_\rho, \gamma_\rho, d_\rho, a, b, c$ and $k$ are constitutive coefficients. The constitutive equations of the dissipative variables $\tau_{ij}^*, \mu_{ij}^*, \Pi_i^*$ and $H_{ij}^*$ become

$$\tau_{ij}^* = \lambda^* \varepsilon_{rr} \delta_{ij} + 2\mu^* \varepsilon_{ij} + \beta_1^* \delta_{ij} \Delta T + 2\beta_2^* T_{ij},$$

$$\mu_{ij}^* = \frac{1}{2} \alpha_1^*(\kappa_{rr} \delta_{ij} + 2\kappa_{r} \delta_{ij} + \kappa_{rr} \delta_{ik}) + \alpha_2^*(\kappa_{rr} \delta_{ij} + \kappa_{rr} \delta_{ik}),$$

$$\Pi_i^* = f_i^* \kappa_{ss} + 2f_2^* \kappa_{ss} + k^* T_i,$$

$$H_{ij}^* = g_{ij}^* \delta_{ij} \varepsilon_{rr} + 2g_{ij}^* \varepsilon_{ij} + d_{i}^* \delta_{ij} \Delta T + d_{ij}^* T_{ij}.$$  

(4.2)

Here, $\delta_{ij}$ is the Kronecker's delta and $\lambda^*, \mu^*, \alpha_j^*, (j = 1, 2, \ldots, 5), \beta_\alpha^*, \gamma_\alpha, d_\alpha^*, f_\alpha^*, g_\alpha^*$ and $k^*$ are constitutive coefficients.

It follows from (2.25), (4.1) and (4.2) that the constitutive equations for isotropic materials are given by

$$\tau_{ij} = \lambda_0 e_{rr} \delta_{ij} + 2\mu_0 \varepsilon_{ij} + \beta_1^0 \delta_{ij} \varphi + 2\beta_2^0 \varphi_{,ij} - b T \delta_{ij},$$

$$\mu_{ij} = \frac{1}{2} \alpha_1^0 (\kappa_{rr} \delta_{ij} + 2\kappa_{r} \delta_{ij} + \kappa_{rr} \delta_{ik}) + \alpha_2^0 (\kappa_{rr} \delta_{ij} + \kappa_{rr} \delta_{ik}),$$

$$\Pi_i = \xi_1^0 \kappa_{si} + 2\xi_2^0 \kappa_{ss} + k^0 \varphi_{,i},$$

$$H_{ij} = \zeta_1^0 \delta_{ij} \varepsilon_{rr} + 2\zeta_2^0 \varepsilon_{ij} + d_{i1}^0 \delta_{ij} \Delta \varphi + d_{ij}^0 \varphi_{,ij} - c T \delta_{ij},$$

where we have used the notations

$$\lambda_0 = \lambda + \lambda^* \frac{\partial}{\partial t}, \mu_0 = \mu + \mu^* \frac{\partial}{\partial t}, \alpha_j^0 = \alpha_j + \alpha_j^* \frac{\partial}{\partial t}, \beta_1^0 = \beta_1 + \beta_1^* \frac{\partial}{\partial t},$$

$$\beta_2^0 = \beta_2 + \beta_2^* \frac{\partial}{\partial t}, \gamma_0 = \gamma + \gamma^* \frac{\partial}{\partial t}, \zeta_1^0 = \zeta_1 + f_\alpha^* \frac{\partial}{\partial t},$$

$$k_0 = k + k^* \frac{\partial}{\partial t}, \xi_1^0 = \xi_1 + g_\alpha^* \frac{\partial}{\partial t},$$

(4.4)

From (2.41), (2.42) and (4.3) we obtain the following system of partial differential equations for the functions $\varphi(i)$ and $\varphi(i)$,

$$\left(\mu_0 - \nu_1 \Delta\right) u_{ij} + (\lambda_0 + \mu_0 - \nu_2 \Delta) u_{ij} + (\beta - \gamma) \Delta \varphi_{,ij} - b \varphi_{,ij} + \rho f_i - \rho \ddot{u}_i,$$

$$d \Delta \varphi - k_0 \Delta \varphi + (\zeta - \xi) \Delta u_{ij} + a \varphi_{,ij} + b \dot{u}_{ij} = \rho \dot{s},$$

(4.5)

where

$$\beta = \beta_0 + 2\beta_2^0, \gamma = \gamma_0 + 2\gamma_0^0, d = d_0 + d_2^0, \xi = \xi_0 + 2\xi_0^0, \nu_1 = 2(\alpha_3^0 + \alpha_3^0), \nu_2 = 2(\alpha_1^0 + \alpha_0^0 + \alpha_3^0), \zeta = \zeta_0 + 2\zeta_0^0.$$

(4.6)

If we use (2.32), then from (4.5) we obtain the following equations involving $T$ and $u_{ij}$,

$$d \Delta T - k_0 \Delta T + (\zeta - \xi) \Delta \dot{u}_{ij} + \alpha \ddot{T} + b \dot{u}_{ij} = \rho \dot{s}.$$
5. A uniqueness result

This section is devoted to a uniqueness theorem in the dynamic theory of thermoviscoelastic solids. We introduce the notations

\[
2V = A_{ijrs}e_i e_j e_r e_s + B_{ijkpqr}k_{ijk}k_{pqr} + 2C_{ijrs}e_i e_j \varphi_{rs} \\
+ D_{ijrs} \varphi_{ij} \varphi_{rs} + 2E_{ijks}k_{ijk} \varphi_s + K_{ij} \varphi_i \varphi_j,
\]

(5.1)

\[
E = \frac{1}{2} \int_B (\rho \dot{u}_i \dot{u}_i + a \dot{\varphi}^2 + 2V) \, dv.
\]

**Theorem 5.1.** Assume that:

(i) \( \rho \) and \( a \) are strictly positive;

(ii) the constitutive coefficients satisfy the symmetry relations (2.34), (2.38) and the inequality (2.36);

(iii) \( V \) is a positive semi-definite quadratic form.

Then, the boundary-initial-value problems have at most one solution.

**Proof.** In view of (2.33)–(2.35), (2.38), (2.39) and (5.1) we find that

\[
\tau_{ij} e_{ij} + \mu_{ijk} k_{ijk} + \rho \dot{\varphi} + \Pi_j \dot{\varphi}_j + H_{ij} \dot{\varphi}_{,ij} = \frac{1}{\partial t} \left( 2V + a \dot{\varphi}^2 \right) + \Gamma.
\]

(5.2)

On the other hand, in view of (2.14), (2.41) and (2.42) we obtain

\[
\tau_{ij} e_{ij} + \mu_{ijk} k_{ijk} + \rho \dot{\varphi} + \Pi_j \dot{\varphi}_j + H_{ij} \dot{\varphi}_{,ij} = \left[ (\tau_{jk} - \mu_{jik}) \dot{u}_k + \mu_{ijk} \dot{u}_{i,k} + \Pi_j \dot{\varphi} + H_{ij} \dot{\varphi}_{,i} - H_{ji} \dot{\varphi}_{,j} \right] + \rho (f_j \dot{u}_j + s \dot{\varphi}) - \rho \ddot{u}_i \dot{u}_i.
\]

(5.3)

By using (2.11), (5.2) and (5.3) we get

\[
\Gamma + \frac{1}{\partial t} \left( 2V + a \dot{\varphi}^2 \right) = \left[ t_{jk} \dot{u}_k + \mu_{jki} \dot{u}_{i,k} + \Lambda_j \dot{\varphi} + H_{kj} \dot{\varphi}_{,k} \right] + \rho (f_j \dot{u}_j + s \dot{\varphi}) - \rho \ddot{u}_i \dot{u}_i.
\]

(5.4)

If we integrate the relation (5.4) over \( B \) and use the relations (2.2), (2.6), (2.9) and (5.1), then we arrive at

\[
\dot{E} + \int_B \Gamma \, dv = \int_{\partial B} (t_i \dot{u}_i + \mu_{ji} \dot{u}_{i,j} + \sigma \dot{\varphi} + H_{ij} \dot{\varphi}_{,j}) \, da + \int_B \rho (f_j \dot{u}_j + s \dot{\varphi}) \, dv.
\]

(5.5)

In view of (3.2) we obtain

\[
\dot{E} + \int_B \Gamma \, dv = \int_{\partial B} (P_i \dot{u}_i + R_i D \dot{u}_i + \Sigma \dot{\varphi} + \Omega D \dot{\varphi}) \, da + \int_C (Z_i \dot{u}_i + Y \dot{\varphi}) \, ds + \int_B \rho (f_j \dot{u}_j + s \dot{\varphi}) \, dv.
\]

(5.6)

Suppose that there are two solutions \( \{ u_i^{(\alpha)}, \varphi^{(\alpha)} \} \), \( \alpha = 1, 2 \). Then their difference \( \{ u_i, \varphi \} = u_i^{(1)} - u_i^{(2)}, \varphi = \varphi^{(1)} - \varphi^{(2)} \), corresponds to null data. The functions \( E \) and \( \Gamma \) associated to the difference \( \{ u_i, \varphi \} \) are denoted by \( \dot{E} \) and \( \dot{\Gamma} \), respectively. From (5.6) we get

\[
\dot{E} + \int_B \dot{\Gamma} \, dv = 0.
\]

(5.7)

In view of (2.36) we find that \( \dot{E} \leq 0 \) on \( [0, t_1] \) so that \( \dot{E}(t) \leq \dot{E}(0), t \in [0, t_1] \). From (5.1) and the initial conditions we get \( \dot{E}(0) = 0 \). By using the hypotheses of the theorem we find that \( \dot{E}(t) = 0, t \in [0, t_1] \), and \( \overline{u}_i = 0, \overline{\varphi} = 0 \) on \( B \times [0, t_1] \). Taking into account that \( \overline{u}_i \) and \( \overline{\varphi} \) vanish initially, we conclude that the difference of the two solutions is equal to zero. \( \square \)
This uniqueness result can be extended to unbounded domains.

6. Existence

We provide now an existence of solutions theorem for the problem determined by system (2.43) with the initial conditions given in (3.6) and the following boundary conditions:

\[ u_i = Du_i = \varphi = D\varphi = 0 \text{ in } \partial B \times I. \]

We assume again that \( \rho \) and \( a \) are strictly positive, that the constitutive coefficients satisfy the relations (2.34), (2.38) and the inequality (2.36), and that \( V \) is strictly positive definite. Notice that these assumptions are the same as those of Theorem 5.1 but now we impose the positivity of the quadratic form \( V \).

First of all we reformulate the problem. Let us consider the Hilbert space \( H \) given by

\[ H = W_0^{2,2}(B) \times L^2(B) \times W_0^{2,2}(B) \times L^2(B), \]

where \( W_0^{2,2}(B) \) and \( L^2(B) \) are the usual real Sobolev spaces, and bold uppercase letters mean the three dimensional space obtained from the triple cartesian product of the corresponding one-dimensional initial space.

We denote the elements of \( H \) by \( U = (u, v, \varphi, \chi) \). Taking into account (5.1), we consider in \( H \) the inner product associated with the norm

\[ ||U||^2 = \frac{1}{2} \int_B (\rho v_i v_i + a \chi^2 + 2V) \, dv. \]

Notice that, in view of the above assumptions, this norm is equivalent to the usual one in the Hilbert space \( H \).

We define the matrix operator \( A \) such that

\[ AU = \begin{pmatrix} v \\ M \\ \chi \\ N \end{pmatrix}, \]

where

\[ M = \rho^{-1} \left[ (A_{ijrs}u_{r,s} + C_{ijrs}\varphi_{r,s} - b_{ij}\chi + A^*_{ijrs}\chi_{r,s})_{,ij} \\
- (B_{kijpq}u_{r,pq} + E_{kijpq}\varphi_{r,pq} + B^*_{kijpq}v_{r,pq} + E^*_{kijpq}\chi_{r,pq})_{,kij} \right] \]

\[ N = a^{-1} \left[ (E_{pqri}u_{r,pq} + K_{ij}\varphi_{r,i} + F_{pqri}v_{r,pq} + K^*_{ij}\chi_{r,i})_{,ij} \\
- (C_{rski}u_{r,s} + D_{ijrs}\varphi_{r,s} + G_{rski}v_{r,s} + D^*_{ijrs}\chi_{r,s})_{,ij} - b_{ij}v_{i,j} - c_{ij}\xi_{i,j} + (c_{ij}\xi)_{,ij} \right] \]

Notice that the domain of the operator is given by the elements of the Hilbert space such that \( v \in W_0^{2,2}(B), M \in L^2(B), \chi \in W_0^{2,2}(B) \) and \( N \in L^2(B) \). Notice also that the domain of the operator is a dense subspace of \( H \).

With the above notation, we can rewrite system (2.43) and the initial conditions (3.6) in the following way:

\[ \frac{dU}{dt} = AU + F(t), \quad U(0) = (u^0, v^0, \varphi^0, \chi^0), \]

where \( F = (0, f, 0, a^{-1}\rho s) \).
The next step consists in proving that the operator $A$ defines a contractive semigroup of linear operators and, as a consequence, the existence of solutions and their continuous dependence with respect to the initial data. To do so, we state and prove two lemmas.

**Lemma 6.1.** Let $\mathcal{U}$ be an element of the domain of $A$. Then, $\langle A\mathcal{U},\mathcal{U} \rangle \leq 0$.

**Proof.** It is enough to apply the definition of the operator: taking into account the boundary conditions, the use of the divergence theorem proves the inequality. In fact, we can see that

$$
\langle A\mathcal{U},\mathcal{U} \rangle = -\frac{1}{2} \int_B \left( A^*_{ijrs}u_{r,s}v_{i,j} + C^*_{ijrs}\chi_{r,s}u_{i,j} + B^*_{kji\ell pqr}v_{i,jk\ell}v_{r,pq} + E^*_{kjis}\chi_{s}v_{i,kj} \\
+ F^*_{pqri}v_{r,pq}c_{i,j} + K^*_{ij}\chi_{i,j} + G^*_{r\ell s}v_{i,j}\chi_{r,s} + D^*_{\ell ijrs}\chi_{r,ij} \right) \, dv \leq 0.
$$

**Lemma 6.2.** Zero belongs to the resolvent of $A$ or, simply, $0 \in \rho(A)$.

**Proof.** We have to prove that for any $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$, there exists $\mathcal{U} \in \mathcal{D}(A)$ such that $A\mathcal{U} = F$. It is clear that $\mathbf{v} = f_1$ and $\chi = f_2$. Therefore, the system of equations reduces to

$$
(A_{ijrs}u_{r,s} + C_{ijrs}\varphi_{r,s})_{i,j} - (B_{kji\ell pqr}u_{r,pq} + E_{kjis}\varphi_{s})_{k,j} \\
= \rho f_{i1} + b_{ij}f_{3,j} - (A^*_{ijrs}f_{1,r,s} - C^*_{ijrs}\varphi_{r,s})_{i,j} + (B^*_{kji\ell pqr}f_{1,r,pq} + E^*_{kjis}\varphi_{s})_{k,j} \\
(E_{pqri}u_{r,pq} + K_{ij}\varphi_{i,j})_{i,j} = 0.
$$

The Lax-Milgram lemma (see [22], for example) allows us to solve this system of equations. Let us define the bilinear form

$$
B[\mathcal{U},\mathcal{U}^*] = \int_B \left( A_{ijrs}u_{i,j}u_{r,s}^* + B_{ik\ell pqr}u_{k,ij}u_{r,pq}^* + C_{ijrs}(u_{i,j}\varphi_{r,s}^* + u_{i,j}^*\varphi_{r,s}) \\
+ E_{ikr}(u_{k,ij}\varphi_{i,j}^* + u_{k,ij}^*\varphi_{r,s}) + K_{ij}\varphi_{i,j}^* + G_{r\ell s}(\varphi_{i,j}^*)_{i,j} + D_{kji\ell s}(\varphi_{r,s}^*)_{k,j} \right) \, dv.
$$

This bilinear form is bounded on $W^{1,2}_0(B) \times W^{1,2}_0(B)$ and, as we are supposing that $V$ is positive definite, $B$ is coercive. On the other hand,

$$
\rho f_{i1} + b_{ij}f_{3,j} - A^*_{ijrs}f_{1,r,s} - C^*_{ijrs}\varphi_{r,s} - B^*_{kji\ell pqr}f_{1,r,pq}^* - E^*_{kjis}\varphi_{s}^* \in W^{-1,2}_0(B)
$$

and

$$
a f_{i1} - (F_{pqri}f_{1,r,pq} + K_{ij}\varphi_{i,j})_{i,j} + (G_{r\ell s}(\varphi_{i,j}^*)_{i,j} + D_{kji\ell s}(\varphi_{r,s}^*)_{k,j} + b_{ij}f_{1,i,j} + c_{ij}f_{3,i,j} - (c_{ij}f_{3})_{i,j} \in W^{-1,2}_0(B).
$$

Using therefore the Lax-Milgram lemma, we conclude the existence of solutions. Moreover, the solution $(u, v, \varphi, \chi)$ satisfies the inequality

$$
\|(u, v, \varphi, \chi)\| \leq K\|(f_1, f_2, f_3, f_4)\|,
$$

where $K$ is a positive real number independent of the solution.

In view of the previous lemmas, we conclude that the operator $A$ generates a contractive semigroup in $\mathcal{H}$, or stated formally:

**Theorem 6.3.** The operator $A$ generates a contractive semigroup in the Hilbert space $\mathcal{H}$.

As a consequence, we obtain the following result.
Theorem 6.4. Suppose that \((f, s)\) are continuous in \(W^{2,2}(B)\) and smooth on \(L^2(B)\), and that \(U_0\) belongs to the domain of the operator, \(\mathcal{D}(A)\). Then, there exists a solution \(U\) to the problem which is smooth in the Hilbert space and takes values in \(\mathcal{D}(A)\).

As a corollary, we know that the solutions satisfy the inequality
\[
\|U(t)\| \leq \|U(0)\| + \int_0^t \|(f(\tau), s(\tau))\|_{L^2} \, d\tau,
\]
because they are defined by means of a semigroup of contractions.

The above inequality gives the continuous dependence of the solutions with respect to the initial data and supply terms. This fact assures that the problem is well posed in the sense of Hadamard.

7. Analyticity of the semigroup

In the above section we have worked with real Sobolev spaces. It is clear that the same arguments can be done if we consider spaces with complex values. In this section we consider a complex Sobolev space. We assume again the hypotheses of the previous section and, to avoid any misunderstanding, we emphasize that we are supposing that \(V\) and \(\Gamma\) are strictly positive definite.

We suppose also that there are no supply terms. Under these assumptions, we prove that the semigroup of contractions generated by \(e^{At}\) is analytic. As a consequence, we obtain the exponential decay of the solutions and their impossibility of localization.

The analyticity will be proved if we can show that \(e^{At}\) satisfies the following result (see the book of Liu and Zheng [23]).

**Theorem 7.1.** Let \(S(t) = e^{At}\) be a \(C_0\)-semigroup of contractions in a Hilbert space \(\mathcal{H}\). If \(i\mathbb{R} \subset \rho(A)\), therefore, \(S(t)\) is analytic if and only if
\[
\lim_{\beta \to \infty} \|\beta(i\beta I - A)^{-1}\|_{\mathcal{H}} < \infty.
\]

**Proof.** First of all, we prove that \(i\mathbb{R} \subset \rho(A)\).

Let us suppose that there exists an imaginary number at the spectrum. We will see that this is not possible. Let \(i\beta \neq 0\) be an element of the spectrum with \(|\beta| < \infty\). Then, there exist a sequence of real numbers \(\beta_n\) converging to \(\beta\) and a sequence of unit norm vectors \(U_n = (u_n, v_n, \varphi_n, \chi_n) \in \mathcal{D}(A)\) such that
\[
\|(i\beta_n I - A)U_n\|_{\mathcal{H}} \to 0.
\]

This condition implies that
\[
\begin{align*}
i\beta_n u_n - v_n &\to 0 \text{ in } W^{2,2} \\
i\beta_n v_n - M_n &\to 0 \text{ in } L^2 \\
i\beta_n \varphi_n - \chi_n &\to 0 \text{ in } W^{2,2} \\
i\beta_n \chi_n - N_n &\to 0 \text{ in } L^2
\end{align*}
\]
Here \(M_n\) and \(N_n\) are defined in (6.3).

From the assumption that \(\Gamma\) is positive definite and taking the real part of \(\langle A U_n, U_n \rangle\), we find that \(v_n \to 0\) in \(W^{2,2}\) and \(\chi_n \to 0\) in \(W^{2,2}\). Then, it is clear that \(u_n \to 0\) and \(\varphi_n \to 0\) in \(W^{2,2}\) and in \(W^{2,2}\), respectively. These facts prove that \(U_n\) cannot be of unit norm.
We now focus on the second condition of the theorem. Suppose that $\beta_n \to \infty$. Let us divide the four expressions stated above by $\beta_n$. We get

\begin{align}
(7.6) \quad & iu_n - \beta_n^{-1}v_n \to 0 \text{ in } W^{2,2}
(7.7) \quad & iv_n - \beta_n^{-1}M_n \to 0 \text{ in } L^2
(7.8) \quad & i\varphi_n - \beta_n^{-1}\chi_n \to 0 \text{ in } W^{2,2}
(7.9) \quad & i\chi_n - \beta_n^{-1}N_n \to 0 \text{ in } L^2
\end{align}

Notice that

$$\Re\langle (iI - \beta_n^{-1}A)\mathcal{U}_n, \mathcal{U}_n \rangle = -\beta_n^{-1}\langle A\mathcal{U}_n, \mathcal{U}_n \rangle,$$

and, hence, $\beta_n^{-1/2}v_n \to 0$ and $\beta_n^{-1/2}\chi_n \to 0$ in $W^{2,2}$ and in $W^{2,2}$, respectively. These conditions imply that $u_n \to 0$ and $\varphi_n \to 0$ in $W^{2,2}$ and in $W^{2,2}$, respectively.

Now, let us multiply (7.7) by $v_n$ to obtain that $v_n \to 0$ in $L^2$. Similarly, from (7.9) we get that $\chi_n \to 0$ in $L^2$. Again, these facts imply that $\mathcal{U}_n$ cannot be of unit norm.

In fact, the proof of this second condition implies that the first is also satisfied, since the same argument can be made when $\beta_n$ tends to a finite number.

\[\square\]

The previous results lead to the following theorem.

**Theorem 7.2.** The semigroup of contractions generated by $e^{At}$ is analytic.

As a corollary we obtain that the solutions decay exponentially as time goes to infinity and also their impossibility of localization. The last statement means that the only solution which can be identically null in an open interval of time is the null solution.

8. Conclusions

The paper is concerned with the second gradient theory of thermoviscoelastic materials of Kelvin-Voigt type. The results presented in this paper can be summarized as follows:

1. We extend the strain gradient theory of thermoelastic materials to the case when the time derivative of the strain tensors and the time derivative of the first and second gradients of the thermal displacement are included in the set of independent constitutive variables.
2. We introduce the entropy flux tensor and present the restrictions imposed on the constitutive equations by an entropy production inequality.
3. We present the field equations and show that the theory leads to a fourth order equation for temperature.
4. We establish a uniqueness result for basic boundary-initial-value problems.
5. We establish that the problem is well posed in the sense of Hadamard.
6. Finally, we prove the analyticity of the solutions when the energy and the dissipation are both strictly positive definite. This fact implies the exponential decay of the solutions and their impossibility of localization.
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Funding

The investigation reported in this paper is partially supported by the project “Análisis matemático aplicado a la termomecánica” (PID2019-105118GB-I00), of the Spanish Ministry of Science, Innovation and Universities and FEDER, “A way to make Europe”.

Declarations

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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