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Reconstruction of sampled surfaces with boundary via Morse theory

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Abstract

We study the perception problem for garments (e.g. a pair of pants) using tools from computational topology: the identification of their geometry and position from point-cloud samples, as obtained e.g. with 3D scanners. We present a reconstruction algorithm based on Morse theory that proceeds directly from the point-cloud to obtain a cellular decomposition of the surface derived via a Morse function. No intermediate triangulation or local implicit equations are used, avoiding reconstruction-induced artifices. The results are a piecewise parametrization of the surface as a union of Morse cells, suitable for tasks such as noise-filtering or mesh-independent reparametrization, and a cell complex of small rank determining the surface topology. This algorithm can be applied to smooth surfaces with or without boundary, embedded in an ambient space of any dimension.

Keywords: computational topology; Morse functions; surface reconstruction; point-clouds.

1. Introduction

Robotic manipulation of textiles in a domestic environment is a very relevant problem because of the everyday presence of cloth in human activities; with promising applications ranging from folding clothes to making beds [DSP*16]. To succeed in this endeavor, the robot needs to "understand" the textile it is trying to manipulate [Jim17]. When an arbitrary and unknown garment is presented before the robot, a point-sample of it can be obtained through the use of depth cameras or 3D scanners. Nevertheless, this sample of points will in general have no known structure. This is where *reconstructing* and *recognizing* the textile from the point-cloud (i.e. to parametrize and deduce its topology) becomes of great importance if one wishes the robot to manipulate the garment [YVK21]; e.g. using physical cloth models [CAACnT22]. Naturally, reconstruction of a surface in space from a sample of points on it is a question to which considerable attention has been devoted in the areas of Computational Geometry and Computer Graphics (see [Dey06, HWW*22] for a variety of methods). Nevertheless, to our knowledge almost all reconstruction algorithms focus on triangulations, implicit functions, and piecewise parametrizations, but disregard a direct topological study of the point-cloud.

In this work, we present a reconstruction algorithm that proceeds directly from a point-cloud to obtain a cellular decomposition of the cloth surface: a global piecewise parametrization of the surface is found, with a small number of pieces. From the cellular decomposition, the topology of the surface can be deduced immediately. Our method can be applied to surfaces with or without boundary, embedded in an ambient space of any dimension. Moreover, the algorithm is robust: it always produces a surface, and it captures the topological features of the sampled surface with a size greater than the average distance between sample points. Our algorithm was first sketched by the authors in [ACAC*22] for surfaces without boundary. Here we expand on that work to treat the case with boundary, give more details on how to compute Morse cells and their attachment maps and reconstruct two challenging novel point-clouds of garments with boundary.

2. Related work

Since its beginnings, Differential Topology has tackled the piecewise parametrization problem for manifolds through Morse functions. A smooth map $f : M \to \mathbb{R}$ defined on a compact manifold without boundary is *Morse* if it has only finitely many critical points, and at all of these the Hessian $H(f)$ is nondegenerate. Classical Morse theory (see [Hir76]) shows that a generic Morse function f induces, through its gradient flow, a decomposition of the manifold M *as a CW complex* (see [Mun84]): each critical point of f, together with its unstable manifold for the vector field $-\nabla f$, forms a cell which is topologically a ball, whose boundary attaches to lower-dimensional cells. A global piecewise decomposition of M is achieved, and a Morse-Smale complex, with the critical points of f as a basis, giving the singular homology of M.

The success of Morse theory comes from the fact that Morse functions, and the Morse-Smale transversality conditions required for the above analysis, are generic among smooth maps from M to $\mathbb{R}$. For instance, the height function in a random direction in $\mathbb{R}^N$ has probability 1 of being a Morse-Smale function. Morse theory also extends to manifolds with boundary via stratified spaces [GM88]. Applying Morse theoretical ideas directly to the sample point-cloud of a surface S was first suggested by [GSZ08, ZSG09], who propose an algorithm for point clouds with a known, homogeneous density of sampling. Later, in [CRRC13] a Morse decomposition scheme from point-clouds sampling manifolds without boundary of any dimension is proposed. All these works, however, stop short of questions such as cell parametrization or attachment maps, which are relevant to robotic applications where point-clouds of textiles may need to be filtered and down-sampled in order to e.g. be simulated. We use the gradient flows of [GSZ08, ZSG09] as the starting point, but then detect critical points and their Morse cells differently, proposing a new procedure based on studying the level sections of these flows.

3. Local structure of the point-cloud

Before we can define a Morse function on a given point-cloud X of a surface with boundary S, we need to give it a local structure:

Neighbors identification: the first step is the identification of a set of neighbors of each point v in the cloud $X \subset \mathbb{R}^N$. We merge two classical criteria, and declare two points as neighbors when (i) each point is among the k-nearest neighbors of the other and (ii) their Voronoi cells in the decomposition of the ambient space $\mathbb{R}^N$ induced by X are adjoining. The relationship of neighborhood is made symmetric by reciprocating neighboring relationships where needed. The neighbors of v will be denoted by $\mathrm{Neigh}(v)$.

Tangent space estimation: this task is performed as usual by finding the regression plane in the least squares sense: given the point v and all its neighbors $\mathrm{Neigh}(v) = \{v_1, \dots v_k\}$ we compute the *singular value decomposition* of the matrix with vectors $\vec{vv_i}$ as rows.

Boundary recognition: in order to detect boundary points robustly, we propose the following novel graph-theoretical approach: let $v \in X$, and $T_v S$ the estimated tangent plane at v. We follow the following steps:

1. Given $\{v_1, \dots v_k\}$ the neighbors of $v_0 = v$ we project them on the planes $T_{v_j} S$ for $j = 0, \dots k$. These projections will be denoted by $\pi_j(v_i)$.
2. We create a plane graph G_j with the projected points for every $j = 0, \dots k$, where we add an edge between $\pi_j(v_i)$ and $\pi_j(v_l)$ only when v_i, v_l are themselves neighbors in the cloud and $i, l \neq 0$.
3. We declare the vertex v as a boundary point only when none of the plane graphs G_j enclose $\pi_j(v)$ for every projection to $T_{v_j} S$.

4. Morse function and Morse cells

We define $f : X \to \mathbb{R}$ as the height function $f(v) = \langle v, \hat{n} \rangle$ where $\hat{n}$ is a fixed unitary direction (several random $\hat{n}$ are tried, and the one with a smaller set of local maxima and minima is selected; randomness assures that the $f(v)$ are all different). We define the **upward flow** of f as the function $\mathrm{Up} : X \to X$ such that

$$\mathrm{Up}(v) = \mathrm{argmax}_{v_k \in \mathrm{Neigh}(v),\, f(v_k) > f(v)} \frac{f(v_k) - f(v)}{||v_k - v||}. \qquad (1)$$

Analogously, we define the **downward flow** $\mathrm{Down} : X \to X$. Then, when $f(v) > f(v_k)$ for every $k \in \mathrm{Neigh}(v)$, we have a local maximum and define $\mathrm{Up}(v) = v$. Likewise, when $f(v) < f(v_k)$ for every $k \in \mathrm{Neigh}(v)$, we have a local minimum and write $\mathrm{Down}(v) = v$. Since we have identified the boundary curves of X, we can define the two flows $\partial\mathrm{Down}(\cdot)$ and $\partial\mathrm{Up}(\cdot)$ analogously (considering that each boundary point has two natural neighbors in ∂X) so that they preserve the boundaries. Then we can easily compute boundary maxima and minima (which will not be in general local extrema of the full cloud X).

4.1. Hyperplane sections and critical points

In order to compute all critical points of f we perform n level set intersections at equispaced levels $c_i = c_0 + i \cdot h$ ranging from $c_n = \max f(X)$ to $c_0 = \min f(X)$. We intersect the realization in $\mathbb{R}^N$ of the oriented graph G_{down}, with vertices X and edges given by $\mathrm{Down}(\cdot)$ (i.e. two vertices $v, w \in X$ share an edge if $w = \mathrm{Down}(v)$) and the boundary curves, with the (hyper)planes $H_c = \{p \in \mathbb{R}^N : p \cdot \hat{n} - c = 0\}$. These intersections $\Gamma(c) = G_{\mathrm{down}} \cap H_c$ are (hyper)plane point-samples of one-dimensional curves (possibly with boundary, i.e. intervals) that we can reconstruct and parametrize. Changes in the number or topological type of connected components of these level sets tell us that the level c_i has crossed a critical point of f (see Figure 1).

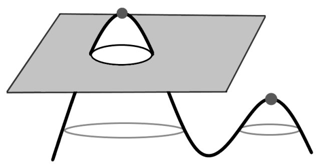

Figure 1: *When a local maximum appears, a new connected component* $\mathbb{S}^1$ *appears in* $\Gamma(c)$.

Since we are assuming that the underlying surface S of the point-cloud X has a boundary ∂S, the reconstructed curves $\Gamma(c)$ are homeomorphic to a union of $\mathbb{S}^1$ and closed intervals $[0, 1]$. These intervals always join two points of ∂S, introducing a bordism equivalence relation in $\partial S \cap H_c$. We will make a distinction between (local) minima (resp. maxima) of the whole point-cloud, and boundary minima (resp. maxima) which are points that are critical when we restrict f to the boundary curves, but not in the whole surface. In Figure 2 we can see a summary of the generic (i.e. non-degenerate) level set transformations when the level crosses that of a particular type of critical point.

4.2. Identification of Morse cells

Now we explain how to identify all the Morse cells that give us a cellular decomposition of the surface. 0-cells correspond to local minima or to boundary minima of f: they are the fixed points of $\mathrm{Down}(\cdot)$ and $\partial\mathrm{Down}(\cdot)$ respectively. In the case of 2-cells, there is one for each local maximum of f (the fixed points of $\mathrm{Up}(\cdot)$). To deduce which points of the cloud correspond to each 2-cell, we flow up every point $v \in X$ by $\mathrm{Up}(\cdot)$ and see at which maximum it ends up. Finally, there are three cases that generate the 1-cells of the skeleton: (i) *boundaries:* when S has a non-empty boundary, the boundary curves are also part of the 1-skeleton; (ii) *saddle points:* there are one-dimensional curves that go from one local minimum m_1 or boundary point to another (not necessarily distinct) local minimum m_2 or boundary point passing through the saddle point

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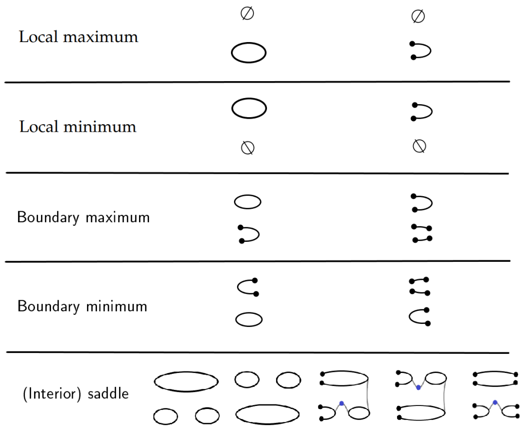

Figure 2: *Different local level set transformations: the reconstructed curves* $\Gamma(c)$ *are homeomorphic to a union of* $\mathbb{S}^1$ *and closed intervals* $[0,1]$.

(these are detected by finding two pair of points in a level set that can be paired by proximity, such that the flow $\text{Down}(\cdot)$ changes the pairing in the next level); and finally (iii) *boundary minima:* there are 1-cells connecting a boundary minimum to a local minimum or boundary point of the cloud. In order to compute this 1-cell we simply flow down every boundary minimum using $\text{Down}(\cdot)$. When 1-cells introduced by saddle points or boundary minima end at points of the boundary ∂S not previously labeled as 0-cells, we introduce the point where they meet to the 0-skeleton and divide the 1-cell into two.

4.3. Attachment maps of the Morse cells

Now that we have the skeleton of S, i.e. the $0,1,2$-cells, we encounter the problem of figuring out the attachment maps between the cells. We choose a simple (i.e. without self-intersections) closed curve of neighbors around each maximum and flow it by $\text{Down}(\cdot)$ until it reaches the 1-cells. We do this in a way such that when a point of the curve has neighbors that are in the 1-cells, we stop following the flow $\text{Down}(\cdot)$ and match the point in question to the point in the 1-cell with results in the most negative downwards slope. If the local maximum is located at ∂S, we consider a curve around the maximum $\hat{m}$, meaning that we take an arc of the boundary centered at $\hat{m}$ and complete it with neighboring interior nodes so that we obtain a closed simple curve. We now have a matching between the initial crown and the 1-cells (although not a bijection). But this is enough to deduce which 1-cells are the boundary of the 2-cell (only the 1-cells that are reached), in which order (this can be deduced since the curve is parametrized), the orientation (again thanks to the parametrization) and how many times they appear (once or twice, depending on the matching).

5. Results

In this section, we reconstruct two different surfaces with boundary, one is synthetic whereas the other is a real scan of an actual textile.

Ellipsoidal vest. Figure 3 shows our algorithm applied to a sample of 36 000 points from a vest embedded in $\mathbb{R}^3$. The cloud was obtained by cutting out parts of an ellipsoid in order to obtain a surface with the same topology as an open vest. Therefore, it has positive Gaussian curvature everywhere and very large boundary curves. After detecting and parametrizing the boundary successfully, the algorithm correctly detects 2 local maxima, 2 boundary maxima, 1 local minima and 3 boundary minima for the height function depicted in the figure. All critical points are located at the boundary. Moreover, two new points (shown in purple) are added as 0-cells where the 1-cells meet each other or the boundary curves. Then, a decomposition of the surface into 16 Morse cells is found (five 0-cells, nine 1-cells and two 2-cells). In order to deduce how the two 2-cells attach and which 1-cells are their boundary, we apply the curve flow explained in Section 4.3. This process in action for one of the 2-cells can be visualized at `https://youtu.be/8cgr54oRf6w`. Thus, we deduce how the cells attach with each other (e.g. the 1-cell number 5 appears on both 2-cells and it is precisely one of the curves where they attach, see Figure 3). From this, we recover the entire topology of the vest.

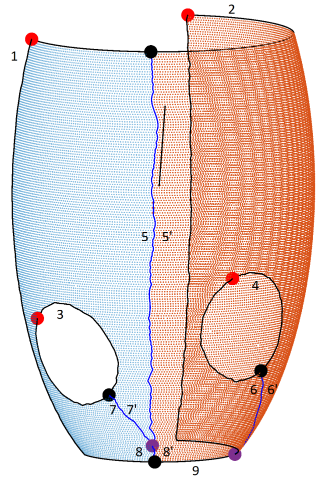

Figure 3: *A sampled vest: the black line is the direction of the height function; maxima are painted in red, minima in black;* 1*-cells corresponding to boundary minima are outlined in blue and the boundary curves in black. The 2 purple points where the* 1*-cells meet each other or the boundary curves are added to the decomposition. The numbers correspond to the different formal* 1*-cells that, when identified (e.g. 7 with 7'), reconstruct the entire surface from 2 pieces homeomorhic to disks.*

3D-scan of pants. Figure 4 shows our algorithm applied to a 3D scan of a pair of real jeans. The scan was made by a *Artec Eva* professional handheld 3D scanner while the garment was worn. In order to apply our algorithm and to reduce irrelevant details and noise, we down-sample the initial cloud of more than 500 000 points to 11 000 by using a *box-grid filter*. Still, the cloud has a lot of detail (e.g. wrinkles) that cause the proliferation of critical points and hence of Morse-cells. After computing for each point its neighbors as explained in Section 3, we apply a *discrete-curvature filter* to the point-cloud. This means that we substitute each point v for a weighted average of its position and the location of its neighbors. When applied a small number of times this filter defines a bijection between the original cloud and the filtered one, which preserves the topology of the underlying surface. Hence, the decomposition we

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find for the filtered case is still valid and topologically accurate for the original cloud, which is the one shown in Figure 4.

The algorithm correctly detects 1 local maxima, 2 boundary maxima, 2 local minima, 1 boundary minima and 1 saddle point for the height function depicted in Figure 4. A decomposition of the surface into 8 Morse cells is found (three 0-cells: the minima and a point added because the original 1-cells intersect, six 1-cells and one 2-cell, for the local maximum). On the bottom of Figure 4 we display the level set curves $\Gamma(c) = G_{\text{down}} \cap H_c$ (see Section 4.1). Notice that for this point-cloud we encounter several of the possible local transformations of level-set curves for surfaces with boundary shown in Figure 2.

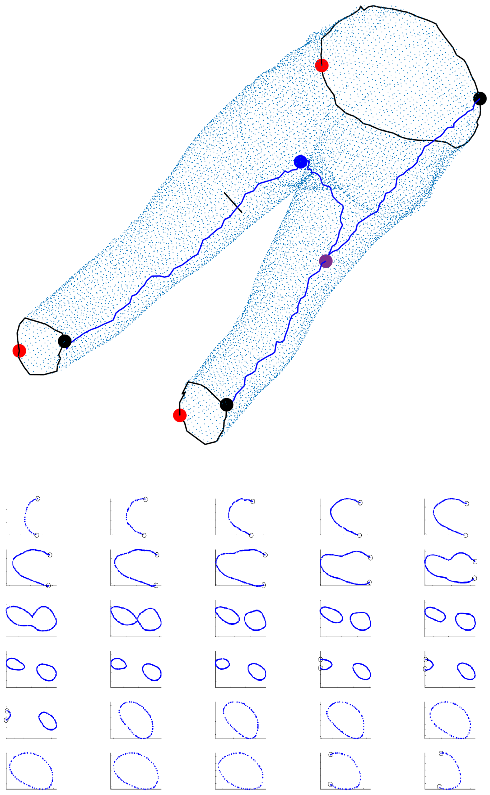

Figure 4: *A 3D-scan of real pants: the black line is the direction of the height function; maxima are painted red, minima in black, saddle points are painted blue; 1-cells are outlined in blue and the boundary curves in black. The purple point where the 1-cells meet each other is added to the decomposition. On the bottom, we plot the level-set curves obtained by intersecting the surface with planes perpendicular to the height function.*

6. Conclusions and further work

In this work, we have presented an algorithm that successfully reconstructs surfaces with boundary by finding a Morse cellular decomposition from a cloud of sampled points on it. It can be applied to surfaces with any topology, and correctly determines it. We reconstructed with our method two different example surfaces with boundary –a *vest* and a pair of *pants*– which posed various challenges. For instance, the *pants* were obtained as a 3D-scan of a real pair of jeans and thus its point-cloud presented wrinkles, noise and an irregular density distribution of points. For both surfaces a global piece-wise decomposition was found, with a very small number of pieces. From the cellular decomposition, the topology of the surfaces can be deduced immediately. We intend to test the reconstruction algorithm with more real point-clouds of various textiles. This would allow the creation of a reconstructed data-base of textiles that could be later simulated with any cloth model. Furthermore, we expect to extend the point-cloud reconstruction algorithm for surfaces to higher dimensional manifolds by iterating the hyperplane sections, reconstructing the manifold from lower dimensional slices.

7. Acknowledgments

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