HAMILTONIAN SYSTEMS IN MULTISYMPLECTIC FIELD THEORIES

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Abstract

We consider Hamiltonian systems in first-order multisymplectic field theories. First we review the construction and properties of Hamiltonian systems in the so-called *restricted multimomentum bundle* using *Hamiltonian sections*, including the variational principle which leads to the Hamiltonian field equations. Then, we introduce Hamiltonian systems in the *extended multimomentum bundle*, in an analogous way to how these systems are defined in non-autonomous (symplectic) mechanics or in the so-called *extended* (symplectic) *formulation* of autonomous mechanics. The corresponding variational principle is also stated for these extended Hamiltonian systems and, after studying the geometric properties of these systems, we establish the relation between the extended and the restricted ones. These definitions and properties are also adapted to submanifolds of the multimomentum bundles in order to cover the case of almost-regular field theories.

Key words: *First order Field Theories, Hamiltonian systems, Fiber bundles, Multisymplectic manifolds.*

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1   Introduction

The Hamiltonian formalism of autonomous dynamical systems, and the study of Hamiltonian dynamical systems in general, is a fruitful subject in both applied mathematics and theoretical physics. From a generic point of view, the characteristics of these kinds of systems make them specially suitable for analyzing many of their properties; for instance: symmetries and related topics such as the existence of conservation laws and reduction, the integrability (including numerical methods), and the possible quantization of the system, which is based on the use of the Poisson bracket structure of this formalism. From the geometrical viewpoint, many of the characteristics of the autonomous Hamiltonian systems arise from the existence of a "natural" geometric structure with which the phase spaces of the systems is endowed: the *symplectic form* (a closed, nondegenerated two-form), which allows the construction of Poisson brackets. In this model, the dynamic information is carried out by the *Hamiltonian function*, which is not coupled to the geometry. Moreover, it is also important to point out the existence of dynamical Hamiltonian systems which have no Lagrangian counterpart (see an example in [44]).

When (first-order) field theories are considered, the usual way to work is with the Lagrangian formalism [1], [3], [9], [18], [15], [16], [35], [38], [43] because their Hamiltonian description presents different kinds of problems. First, several Hamiltonian models can be stated, and the equivalence among them is not always clear (see, for instance, [2], [12], [19], [21], [22], [23], [39], [42]). Furthermore, there are equivalent Lagrangian models with non-equivalent Hamiltonian descriptions [26], [27], [28]. Among the different geometrical descriptions to be considered for describing field theories, we focus our attention on the multisymplectic models [7], [20], [24], [25]; that is, when field theories are stated in the realm of multisymplectic manifolds, which are manifolds endowed with a closed and 1-nondegenerate k-form. In these models, this form plays the same role as symplectic forms in autonomous mechanics.

The aim of this paper is to generalize the concept of Hamiltonian system in autonomous mechanics to first-order multisymplectic field theories. Although there are canonical Hamiltonian models in non-autonomous mechanics [8], this is not the case in field theory. Hence, the first problem to be considered is the choice of a suitable multimomentum bundle to develop the formalism. The most reasonable and useful choice is to take the so-called *restricted multimomentum bundle*, where the Hamiltonian formalism has been extensively studied [6], [11], [31], [37]. Nevertheless, this bundle does not have a canonical multisymplectic form and the physical information (the "Hamiltonian") is used to construct the geometric structure. A way of overcoming this difficulty is to work in a greater dimensional manifold, the so-called *extended multimomentum bundle* which, since it is a vector subbundle of a multicotangent bundle, is endowed with a canonical multisymplectic form. In this manifold, Hamiltonian systems can be introduced as in autonomous mechanics, by using certain kinds of Hamiltonian multivector fields. The resultant *extended Hamiltonian formalism* is the generalization to field theories of the extended formalism for non-autonomous mechanical systems [29], [8] and, to our knowledge, it was introduced for the first time in field theories in [40]. The main goal of our work is to enlarge the definition of Hamiltonian system to this extended framework, generalizing the ideas of the above mentioned reference, and carrying out a deeper geometric study of these kinds of systems. One of the main results is that, to every Hamiltonian system in the extended multimomentum bundle, we can associate in a natural way a class of Hamiltonian systems in the restricted multimomentum bundle which are equivalent, and conversely. The solutions to the field equations in both models are also canonically related. In addition, the field equations for these kinds of systems can be derived from an appropriate variational principle which is also stated. The above results can be extended to the case of *non regular Hamiltonian systems*.

The paper is organized as follows: In Section 2 we review basic concepts and results, such as multivector fields and connections, multisymplectic manifolds and Hamiltonian multivector fields,

and the restricted and extended multimomentum bundles with their geometric structures. Section 3 is devoted to reviewing the definition and characteristics of Hamiltonian systems in the restricted multimomentum bundles; in particular, the definitions of Hamiltonian sections and densities, the variational principle which leads to Hamilton-De Donder-Weyl equations, and the use of multivector fields for writing these equations in a more suitable geometric way. Hamiltonian systems in the extended multimomentum bundle are introduced and studied in Section 4 and, in particular, their relation with those introduced in Section 3. Finally, in Section 5 we adapt the above definitions and results in order to consider Hamiltonian systems which are not defined everywhere in the restricted or the extended multimomentum bundles, but in certain submanifolds in one or the other of them: these are the here so-called *almost regular Hamiltonian systems*. As typical examples, in Section 6, we review the standard Hamiltonian formalism associated to a Lagrangian field theory, both in the regular and singular (almost-regular) cases, and the Hamiltonian formalisms of time-dependent dynamical systems in the extended and restricted phase space, which are recovered as a particular case of this theory.

All manifolds are real, paracompact, connected and C^∞. All maps are C^∞. Sum over crossed repeated indices is understood. Throughout this paper $\pi\colon E \to M$ will be a fiber bundle (dim $M = m$, dim $E = n + m$), where M is an oriented manifold with volume form $\omega \in \Omega^m(M)$, and (x^ν, y^A) (with $\nu = 1, \ldots, m$; $A = 1, \ldots, n$) will be natural local systems of coordinates in E adapted to the bundle, such that $\omega = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m \equiv \mathrm{d}^m x$.

2 Previous definitions and results

2.1 Multivector fields and connections

(See [10] for details).

Let $\mathcal{M}$ be a n-dimensional differentiable manifold. Sections of $\Lambda^m(\mathrm{T}\mathcal{M})$ are called *m-multivector fields* in $\mathcal{M}$ (they are the contravariant skew-symmetric tensors of order m in $\mathcal{M}$). We will denote by $\mathfrak{X}^m(\mathcal{M})$ the set of m-multivector fields in $\mathcal{M}$.

If $\mathcal{Y} \in \mathfrak{X}^m(\mathcal{M})$, for every $p \in \mathcal{M}$, there exists an open neighbourhood $U_p \subset \mathcal{M}$ and $Y_1, \ldots, Y_r \in \mathfrak{X}(U_p)$ such that $\mathcal{Y} \underset{U_p}{=} \sum_{1 \leq i_1 < \ldots < i_m \leq r} f^{i_1 \ldots i_m} Y_{i_1} \wedge \ldots \wedge Y_{i_m}$, with $f^{i_1 \ldots i_m} \in \mathrm{C}^\infty(U_p)$ and $m \leq r \leq \dim \mathcal{M}$. Then, $\mathcal{Y} \in \mathfrak{X}^m(\mathcal{M})$ is said to be *locally decomposable* if, for every $p \in \mathcal{M}$, there exists an open neighbourhood $U_p \subset \mathcal{M}$ and $Y_1, \ldots, Y_m \in \mathfrak{X}(U_p)$ such that $\mathcal{Y} \underset{U_p}{=} Y_1 \wedge \ldots \wedge Y_m$.

A non-vanishing m-multivector field $\mathcal{Y} \in \mathfrak{X}^m(\mathcal{M})$ and a m-dimensional distribution $D \subset \mathrm{T}\mathcal{M}$ are *locally associated* if there exists a connected open set $U \subseteq \mathcal{M}$ such that $\mathcal{Y}|_U$ is a section of $\Lambda^m D|_U$. If $\mathcal{Y}, \mathcal{Y}' \in \mathfrak{X}^m(\mathcal{M})$ are non-vanishing multivector fields locally associated with the same distribution D, on the same connected open set U, then there exists a non-vanishing function $f \in \mathrm{C}^\infty(U)$ such that $\mathcal{Y}' \underset{U}{=} f\mathcal{Y}$. This fact defines an equivalence relation in the set of non-vanishing m-multivector fields in $\mathcal{M}$, whose equivalence classes will be denoted by $\{\mathcal{Y}\}_U$. Then there is a one-to-one correspondence between the set of m-dimensional orientable distributions D in $\mathrm{T}\mathcal{M}$ and the set of the equivalence classes $\{\mathcal{Y}\}_\mathcal{M}$ of non-vanishing, locally decomposable m-multivector fields in $\mathcal{M}$.

If $\mathcal{Y} \in \mathfrak{X}^m(\mathcal{M})$ is non-vanishing and locally decomposable, and $U \subseteq \mathcal{M}$ is a connected open set, the distribution associated with the class $\{\mathcal{Y}\}_U$ is denoted by $\mathcal{D}_U(\mathcal{Y})$. If $U = \mathcal{M}$ we write $\mathcal{D}(\mathcal{Y})$.

A non-vanishing, locally decomposable multivector field $\mathcal{Y} \in \mathfrak{X}^m(\mathcal{M})$ is said to be *integrable*

(resp. *involutive*) if its associated distribution $\mathcal{D}_U(\mathcal{Y})$ is integrable (resp. involutive). Of course, if $\mathcal{Y} \in \mathfrak{X}^m(\mathcal{M})$ is integrable (resp. involutive), then so is every other in its equivalence class $\{\mathcal{Y}\}$, and all of them have the same integral manifolds. Moreover, *Frobenius theorem* allows us to say that a non-vanishing and locally decomposable multivector field is integrable if, and only if, it is involutive. Nevertheless, in many applications we have locally decomposable multivector fields $\mathcal{Y} \in \mathfrak{X}^m(\mathcal{M})$ which are not integrable in $\mathcal{M}$, but integrable in a submanifold of $\mathcal{M}$. A (local) algorithm for finding this submanifold has been developed [10].

The particular situation in which we are interested is the study of multivector fields in fiber bundles. If $\pi: \mathcal{M} \to M$ is a fiber bundle, we will be interested in the case where the integral manifolds of integrable multivector fields in $\mathcal{M}$ are sections of π. Thus, $\mathcal{Y} \in \mathfrak{X}^m(\mathcal{M})$ is said to be π-*transverse* if, at every point $y \in \mathcal{M}$, $(i(\mathcal{Y})(\pi^*\beta))_y \neq 0$, for every $\beta \in \Omega^m(M)$ with $\beta(\pi(y)) \neq 0$. Then, if $\mathcal{Y} \in \mathfrak{X}^m(\mathcal{M})$ is integrable, it is π-transverse if, and only if, its integral manifolds are local sections of $\pi: \mathcal{M} \to M$. In this case, if $\phi: U \subset M \to \mathcal{M}$ is a local section with $\phi(x) = y$ and $\phi(U)$ is the integral manifold of $\mathcal{Y}$ through y, then $\mathrm{T}_y(\mathrm{Im}\,\phi) = \mathcal{D}_y(\mathcal{Y})$.

Finally, it is clear that classes of locally decomposable and π-transverse multivector fields $\{\mathcal{Y}\} \subseteq \mathfrak{X}^m(\mathcal{M})$ are in one-to-one correspondence with orientable Ehresmann connection forms ∇ in $\pi: \mathcal{M} \to M$. This correspondence is characterized by the fact that the horizontal subbundle associated with ∇ is $\mathcal{D}(\mathcal{Y})$. In this correspondence, classes of integrable locally decomposable and π-transverse m multivector fields correspond to flat orientable Ehresmann connections.

2.2   Hamiltonian multivector fields in multisymplectic manifolds

(See [4] and [33] for details).

Let $\mathcal{M}$ be a n-dimensional differentiable manifold and $\Omega \in \Omega^{m+1}(\mathcal{M})$. The couple $(\mathcal{M}, \Omega)$ is said to be a *multisymplectic manifold* if Ω is closed and 1-nondegenerate; that is, for every $p \in \mathcal{M}$, and $X_p \in \mathrm{T}_p\mathcal{M}$, we have that $i(X_p)\Omega_p = 0$ if, and only if, $X_p = 0$.

If $(\mathcal{M}, \Omega)$ is a multisymplectic manifold, $\mathcal{X} \in \mathfrak{X}^k(\mathcal{M})$ is said to be a *Hamiltonian k-multivector field* if $i(\mathcal{X})\Omega$ is an exact $(m+1-k)$-form; that is, there exists $\zeta \in \Omega^{m-k}(\mathcal{M})$ such that

$$i(\mathcal{X})\Omega = \mathrm{d}\zeta \tag{1}$$

ζ is defined modulo closed $(m-k)$-forms. The class $\{\zeta\} \in \Omega^{m-k}(\mathcal{M})/Z^{m-k}(\mathcal{M})$ defined by ζ is called the *Hamiltonian* for $\mathcal{X}$, and every element in this class $\hat{\zeta} \in \{\zeta\}$ is said to be a *Hamiltonian form* for $\mathcal{X}$. Furthermore, $\mathcal{X}$ is said to be a *locally Hamiltonian k-multivector field* if $i(\mathcal{X})\Omega$ is a closed $(m+1-k)$-form. In this case, for every point $x \in \mathcal{M}$, there is an open neighbourhood $W \subset \mathcal{M}$ and $\zeta \in \Omega^{m-k}(W)$ such that

$$i(\mathcal{X})\Omega = \mathrm{d}\zeta \qquad (\text{on } W)$$

As above, changing $\mathcal{M}$ by W, we obtain the Hamiltonian for $\mathcal{X}$, $\{\zeta\} \in \Omega^{k-m-1}(W)/Z^{k-m-1}(W)$, and the *local Hamiltonian forms* for $\mathcal{X}$.

Conversely, $\zeta \in \Omega^k(\mathcal{M})$ (resp. $\zeta \in \Omega^k(W)$) is said to be a *Hamiltonian k-form* (resp. a *local Hamiltonian k-form*) if there exists a multivector field $\mathcal{X} \in \mathfrak{X}^{m-k}(\mathcal{M})$ (resp. $\mathcal{X} \in \mathfrak{X}^{m-k}(W)$) such that (1) holds (resp. on W). In particular, when $k = 0$, that is, if $\zeta \in \mathrm{C}^\infty(\mathcal{M})$), then the existence of Hamiltonian m-multivector fields for ζ is assured (see [4]).

2.3   Multimomentum bundles

(See, for instance, [12]).

Let $\pi\colon E \to M$ be the *configuration bundle* of a field theory, (with dim $M = m$, dim $E = n+m$). There are several multimomentum bundle structures associated with it.

First we have $\Lambda_2^m \mathrm{T}^* E$, which is the bundle of m-forms on E vanishing by the action of two π-vertical vector fields. Furthermore, if $J^1\pi \to E \to M$ denotes the *first-order jet bundle* over E, the set made of the affine maps from $J^1\pi$ to $\Lambda^m \mathrm{T}^* M$, denoted as $\mathrm{Aff}(J^1\pi, \Lambda^m \mathrm{T}^* M)$, is another bundle over E which is canonically diffeomorphic to $\Lambda_2^m \mathrm{T}^* E$ [6], [12]. We will denote

$$\mathcal{M}\pi \equiv \Lambda_2^m \mathrm{T}^* E \simeq \mathrm{Aff}(J^1\pi, \Lambda^m \mathrm{T}^* M)$$

It is called the *extended multimomentum bundle*, and its canonical submersions are denoted

$$\kappa\colon \mathcal{M}\pi \to E \quad ; \quad \bar{\kappa} = \pi \circ \kappa\colon \mathcal{M}\pi \to M$$

$\mathcal{M}\pi$ is a subbundle of $\Lambda^m \mathrm{T}^* E$, the multicotangent bundle of E of order m (the bundle of m-forms in E). Then $\mathcal{M}\pi$ is endowed with canonical forms. First we have the "tautological form" $\Theta \in \Omega^m(\mathcal{M}\pi)$ which is defined as follows: let $(x, \alpha) \in \Lambda_2^m \mathrm{T}^* E$, with $x \in E$ and $\alpha \in \Lambda_2^m \mathrm{T}_x^* E$; then, for every $X_1, \ldots, X_m \in \mathrm{T}_{(x,\alpha)}(\mathcal{M}\pi)$,

$$\Theta((x, \alpha); X_1, \ldots, X_m) := \alpha(x; \mathrm{T}_{(x,\alpha)}\kappa(X_1), \ldots, \mathrm{T}_{(q,\alpha)}\kappa(X_m))$$

Thus we define the multisymplectic form

$$\Omega := -\mathrm{d}\Theta \in \Omega^{m+1}(\mathcal{M}\pi)$$

They are known as the *multimomentum Liouville m and $(m+1)$-forms*.

We can introduce natural coordinates in $\mathcal{M}\pi$ adapted to the bundle $\pi\colon E \to M$, which are denoted by (x^ν, y^A, p_A^ν, p), and such that $\omega = \mathrm{d}^m x$. Then the local expressions of these forms are

$$\Theta = p_A^\nu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu + p\,\mathrm{d}^m x \quad , \quad \Omega = -\mathrm{d}p_A^\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - \mathrm{d}p \wedge \mathrm{d}^m x \tag{2}$$

(where $\mathrm{d}^{m-1}x_\nu := i\left(\frac{\partial}{\partial x^\nu}\right)\mathrm{d}^m x$).

Consider $\Lambda_1^m \mathrm{T}^* E \equiv \pi^* \Lambda^m \mathrm{T}^* M$, which is another bundle over E, whose sections are the π-semibasic m-forms on E, and denote by $J^1\pi^*$ the quotient $\Lambda_2^m \mathrm{T}^* E / \Lambda_1^m \mathrm{T}^* E \equiv \mathcal{M}\pi / \Lambda_1^m \mathrm{T}^* E$. We have the natural submersions

$$\tau\colon J^1\pi^* \to E \quad ; \quad \bar{\tau} = \pi \circ \tau\colon J^1\pi^* \to M$$

Furthermore, the natural submersion $\mu\colon \mathcal{M}\pi \to J^1\pi^*$ endows $\mathcal{M}\pi$ with the structure of an affine bundle over $J^1\pi^*$, with $\tau^* \Lambda_1^m \mathrm{T}^* E$ as the associated vector bundle. $J^1\pi^*$ is usually called the *restricted multimomentum bundle* associated with the bundle $\pi\colon E \to M$.

Natural coordinates in $J^1\pi^*$ (adapted to the bundle $\pi\colon E \to M$) are denoted by (x^ν, y^A, p_A^ν).

We have the diagram

Hamiltonian systems can be defined in $\mathcal{M}\pi$ or in $J^1\pi^*$. The construction of the Hamiltonian formalism in $J^1\pi^*$ was pionered in [6] (see also [11] and [12]), while a formulation in $\mathcal{M}\pi$ has been stated recently [40]. In the following sections we review the main concepts of the formalism in $J^1\pi^*$, and we make an extensive development of the formalism in $\mathcal{M}\pi$.

3   Hamiltonian systems in $J^1\pi^*$

First we consider the standard definition of Hamiltonian systems in field theory, which is stated using the restricted multimomentum bundle $J^1\pi^*$.

3.1   Restricted Hamiltonian systems

Definition 1 *Consider the bundle* $\bar\tau\colon J^1\pi^* \to M$.

1. *A section* $h\colon J^1\pi^* \to \mathcal{M}\pi$ *of the projection* μ *is called a* Hamiltonian section *of* μ.

2. *The differentiable forms*

$$\Theta_h := h^*\Theta \quad , \quad \Omega_h := -\mathrm{d}\Theta_h = h^*\Omega$$

   are called the Hamilton-Cartan m *and* $(m+1)$ *forms of* $J^1\pi^*$ *associated with the Hamiltonian section* h.

3. *The couple* $(J^1\pi^*, h)$ *is said to be a* restricted Hamiltonian system, *(or just a* Hamiltonian system*).*

In a local chart of natural coordinates, a Hamiltonian section is specified by a *local Hamiltonian function* $\mathrm{h} \in \mathrm{C}^\infty(U)$, $U \subset J^1\pi^*$, such that $h(x^\nu, y^A, p_A^\nu) \equiv (x^\nu, y^A, p_A^\nu, p = -\mathrm{h}(x^\gamma, y^B, p_B^\eta))$. The local expressions of the Hamilton-Cartan forms associated with h are

$$\Theta_h = p_A^\nu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - \mathrm{h}\,\mathrm{d}^m x \quad , \quad \Omega_h = -\mathrm{d}p_A^\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu + \mathrm{d}\mathrm{h} \wedge \mathrm{d}^m x \tag{3}$$

Remark 1 Notice that Ω_h is 1-nondegenerate; that is, a multisymplectic form (as a simple calculation in coordinates shows).

Hamiltonian sections can be obtained from connections. In fact, if we have a connection ∇ in $\pi\colon E \to M$, it induces a linear section $h^\nabla\colon J^1\pi^* \to \mathcal{M}\pi$ of μ [6]. Then, if Θ is the canonical m-form in $\Omega^m(\mathcal{M}\pi)$, the forms

$$\Theta_{h^\nabla} := h^{\nabla *}\Theta \in \Omega^m(J^1\pi^*) \quad , \quad \Omega_{h^\nabla} := -\mathrm{d}\Theta_{h^\nabla} \in \Omega^{m+1}(J^1\pi^*) \tag{4}$$

are the *Hamilton-Cartan m and $(m+1)$ forms* of $J^1\pi^*$ associated with the connection ∇. In a system of natural coordinates in $J^1\pi^*$, if $\nabla = \mathrm{d}x^\nu \otimes \left(\dfrac{\partial}{\partial x^\nu} + \Gamma_\nu^A \dfrac{\partial}{\partial y^A} \right)$ is the local expression of the connection ∇, the local expressions of these Hamilton-Cartan forms associated with ∇ are

$$\begin{aligned}
\Theta_{h^\nabla} &= p_A^\nu(\mathrm{d}y^A - \Gamma_\eta^A \mathrm{d}x^\eta) \wedge \mathrm{d}^{m-1}x_\nu = p_A^\nu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - p_A^\nu \Gamma_\nu^A \mathrm{d}^m x \\
\Omega_{h^\nabla} &= -\mathrm{d}p_A^\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu + \Gamma_\nu^A \mathrm{d}p_A^\nu \wedge \mathrm{d}^m x + p_A^\nu \mathrm{d}\Gamma_\nu^A \wedge \mathrm{d}^m x
\end{aligned}$$

Observe that a local Hamiltonian function associated with h^∇ is $\mathrm{h}^\nabla = p_A^\nu \Gamma_\nu^A$.

3.2 Variational principle and field equations

Now we establish the field equations for restricted Hamiltonian systems. They can be derived from a variational principle. In fact, first we state:

Definition 2 *Let $(J^1\pi^*, h)$ be a restricted Hamiltonian system. Let $\Gamma(M, J^1\pi^*)$ be the set of sections of $\bar{\tau}$. Consider the map*

$$
\mathbf{H} \ : \ \begin{array}{ccc} \Gamma(M, J^1\pi^*) & \longrightarrow & \mathbb{R} \\ \psi & \mapsto & \int_M \psi^* \Theta_h \end{array}
$$

(where the convergence of the integral is assumed). The variational problem for this restricted Hamiltonian system is the search for the critical (or stationary) sections of the functional $\mathbf{H}$, with respect to the variations of ψ given by $\psi_t = \sigma_t \circ \psi$, where $\{\sigma_t\}$ is the local one-parameter group of any compact-supported $Z \in \mathfrak{X}^{V(\bar{\tau})}(J^1\pi^*)$ (where $\mathfrak{X}^{V(\bar{\tau})}(J^1\pi^*)$ denotes the module of $\bar{\tau}$-vertical vector fields in $J^1\pi^*$), that is:

$$
\frac{d}{dt}\Big|_{t=0} \int_M \psi_t^* \Theta_h = 0
$$

This is the so-called Hamilton-Jacobi principle *of the Hamiltonian formalism.*

Then the following fundamental theorem is proven (see also [12]):

Theorem 1 *Let $(J^1\pi^*, h)$ be a restricted Hamiltonian system. The following assertions on a section $\psi \in \Gamma(M, J^1\pi^*)$ are equivalent:*

1. *ψ is a critical section for the variational problem posed by the Hamilton-Jacobi principle.*

2. *$\psi^* i(Z)\Omega_h = 0$, for every $Z \in \mathfrak{X}^{V(\bar{\tau})}(J^1\pi^*)$.*

3. *$\psi^* i(X)\Omega_h = 0$, for every $X \in \mathfrak{X}(J^1\pi^*)$.*

4. *If $(U; x^\nu, y^A, p_A^\nu)$ is a natural system of coordinates in $J^1\pi^*$, then ψ satisfies the following system of equations in U*

$$
\frac{\partial(y^A \circ \psi)}{\partial x^\nu} = \frac{\partial \mathrm{h}}{\partial p_A^\nu} \circ \psi \equiv \frac{\partial \mathrm{h}}{\partial p_A^\nu}\Big|_\psi \quad ; \quad \frac{\partial(p_A^\nu \circ \psi)}{\partial x^\nu} = -\frac{\partial \mathrm{h}}{\partial y^A} \circ \psi \equiv -\frac{\partial \mathrm{h}}{\partial y^A}\Big|_\psi \tag{5}
$$

where h is a local Hamiltonian function associated with h. They are known as the Hamilton-De Donder-Weyl equations *of the restricted Hamiltonian system.*

(*Proof*) (1 $\iff$ 2) We assume that ∂U is a $(m-1)$-dimensional manifold and that $\bar{\tau}(supp(Z)) \subset U$, for every compact-supported $Z \in \mathfrak{X}^{V(\bar{\tau})}(J^1\pi^*)$. Then

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \int_U \psi_t^* \Theta_h &= \frac{d}{dt}\Big|_{t=0} \int_U \psi^*(\sigma_t^* \Theta_h) = \int_U \psi^* \left(\lim_{t \to 0} \frac{\sigma_t^* \Theta_h - \Theta_h}{t} \right) \\
&= \int_U \psi^*(\mathrm{L}(Z)\Theta_h) = \int_U \psi^*(i(Z)\mathrm{d}\Theta_h + \mathrm{d}\,i(Z)\Theta_h) \\
&= -\int_U \psi^*(i(Z)\Omega_h - \mathrm{d}\,i(Z)\Theta_h) = -\int_U \psi^*(i(Z)\Omega_h) + \int_U \mathrm{d}[\psi^*(i(Z)\Theta_h)] \\
&= -\int_U \psi^*(i(Z)\Omega_h) + \int_{\partial U} \psi^*(i(Z)\Theta_h) = -\int_U \psi^*(i(Z)\Omega_h)
\end{aligned}
$$

(as a consequence of Stoke's theorem and the hypothesis made on the supports of the vertical fields). Thus, by the fundamental theorem of the variational calculus we conclude that $\frac{d}{dt}\Big|_{t=0}\int_U \psi_t^*\Theta_h = 0$ $\Leftrightarrow \psi^*(i(Z)\Omega_h) = 0$, for every compact-supported $Z \in \mathfrak{X}^{V(\bar\tau)}(J^1\pi^*)$. However, as compact-supported vector fields generate locally the $C^\infty(J^1\pi^*)$-module of vector fields in $J^1\pi^*$, it follows that the last equality holds for every $Z \in \mathfrak{X}^{V(\bar\tau)}(J^1\pi^*)$.

$(2 \iff 3)$ If $p \in \mathrm{Im}\,\psi$, then $\mathrm{T}_p J^1\pi^* = \mathrm{V}_p(\bar\tau) \oplus \mathrm{T}_p(\mathrm{Im}\,\psi)$. So if $X \in \mathfrak{X}(J^1\pi^*)$, then

$$X_p = (X_p - \mathrm{T}_p(\psi \circ \bar\tau)(X_p)) + \mathrm{T}_p(\psi \circ \bar\tau)(X_p) \equiv X_p^V + X_p^\psi$$

and therefore

$$\psi^*(i(X)\Omega_h) = \psi^*(i(X^V)\Omega_h) + \psi^*(i(X^\psi)\Omega_h) = \psi^*(i(X^\psi)\Omega_h) = 0$$

since $\psi^*(i(X^V)\Omega_h) = 0$ by the above item, and furthermore, $X_p^\psi \in \mathrm{T}_p(\mathrm{Im}\,\psi)$, and $\dim\,(\mathrm{Im}\,\psi) = m$, being $\Omega_h \in \Omega^{m+1}(J^1\pi^*)$. Hence we conclude that $\psi^*(i(X)\Omega_h) = 0$, for every $X \in \mathfrak{X}(J^1\pi^*)$. The converse is proved reversing this reasoning.

$(3 \Leftrightarrow 4)$ If $X = \alpha^\nu \frac{\partial}{\partial x^\nu} + \beta^A \frac{\partial}{\partial y^A} + \gamma_A^\nu \frac{\partial}{\partial p_A^\nu} \in \mathfrak{X}(J^1\pi^*)$, taking into account the local expression (3) of Ω_h, we have

$$
\begin{aligned}
i(X)\Omega_h \;=\;& (-1)^\eta \alpha^\eta \left(\mathrm{d}p_A^\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-2}x_{\eta\nu} - \frac{\partial \mathrm{h}}{\partial p_A^\nu}\mathrm{d}p_A^\nu \wedge \mathrm{d}^{m-1}x_\eta \right) \\
+\;& \beta^A \left(\mathrm{d}p_A^\nu \wedge \mathrm{d}^{m-1}x_\nu + \frac{\partial \mathrm{h}}{\partial y^A}\mathrm{d}^m x \right) + \gamma_A^\nu \left(-\mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu + \frac{\partial \mathrm{h}}{\partial p_A^\nu} \wedge \mathrm{d}^m x \right)
\end{aligned}
$$

but if $\psi = (x^\nu, y^A(x^\eta), p_A^\nu(x^\eta))$, then

$$
\begin{aligned}
\psi^* i(X)\Omega_h \;=\;& (-1)^{\eta+\nu}\alpha^\eta \left(\frac{\partial(y^A \circ \psi)}{\partial x^\nu} - \frac{\partial \mathrm{h}}{\partial p_A^\nu}\Big|_\psi \right) \frac{\partial(p_A^\nu \circ \psi)}{\partial x^\eta}\mathrm{d}^m x \\
+\;& \beta^A \left(\frac{\partial(p_A^\nu \circ \psi)}{\partial x^\nu} + \frac{\partial \mathrm{h}}{\partial y^A}\Big|_\psi \right) \mathrm{d}^m x + \gamma_A^\nu \left(-\frac{\partial(y^A \circ \psi)}{\partial x^\nu} + \frac{\partial \mathrm{h}}{\partial p_A^\nu}\Big|_\psi \right) \mathrm{d}^m x
\end{aligned}
$$

and, as this holds for every $X \in \mathfrak{X}(J^1\pi^*)$, we conclude that $\psi^* i(X)\Omega_h = 0$ if, and only if, the Hamilton-De Donder-Weyl equations hold for ψ. ∎

Remark 2 It is important to point out that equations (5) are not covariant, since the Hamiltonian function h is defined only locally, and hence it is not intrinsically defined. In order to write a set of covariant Hamiltonian equations we must use a global Hamiltonian function, that is, a *Hamiltonian density* (see [6] and [12] for comments on this subject).

Observe also that the solution to these equations is not unique.

3.3   Hamiltonian equations for multivector fields

(See [10] and [13] for more details).

Let $(J^1\pi^*, h)$ be a restricted Hamiltonian system. The problem of finding critical sections solutions of the Hamilton-Jacobi principle can be formulated equivalently as follows: to find a distribution D of $\mathrm{T}(J^1\pi^*)$ satisfying that:

- D is m-dimensional.

- D is $\bar{\tau}$-transverse.

- D is integrable (that is, *involutive*).

- The integral manifolds of D are the critical sections of the Hamilton-Jacobi principle.

However, as explained in Section 2.1, these kinds of distributions are associated with classes of integrable (i.e., non-vanishing, locally decomposable and involutive) $\bar{\tau}$-transverse multivector fields in $J^1\pi^*$. The local expression in natural coordinates of an element of one of these classes is

$$\mathcal{X} = \bigwedge_{\nu=1}^{m} f \left(\frac{\partial}{\partial x^\nu} + F_\nu^A \frac{\partial}{\partial y^A} + G_{A\nu}^\rho \frac{\partial}{\partial p_A^\rho} \right) \tag{6}$$

where $f \in \mathrm{C}^\infty(J^1\pi^*)$ is a non-vanishing function.

Therefore, the problem posed by the Hamilton-Jacobi principle can be stated in the following way [11], [33]:

Theorem 2 *Let $(J^1\pi^*, h)$ be a restricted Hamiltonian system, and $\{\mathcal{X}\} \subset \mathfrak{X}^m(J^1\pi^*)$ a class of integrable and $\bar{\tau}$-transverse multivector fields. Then, the integral manifolds of $\{\mathcal{X}\}$ are critical section for the variational problem posed by the Hamilton-Jacobi principle if, and only if,*

$$i(\mathcal{X}_h)\Omega_h = 0 \quad , \quad \text{for every } \mathcal{X}_h \in \{\mathcal{X}_h\} \tag{7}$$

Remark 3 The $\bar{\tau}$-transversality condition for multivector fields solution to (7) can be stated by demanding that $i(\mathcal{X}_h)(\bar{\tau}^*\omega) \neq 0$. In particular, if we take $i(\mathcal{X}_h)(\bar{\tau}^*\omega) = 1$ we are choosing a representative of the class of $\bar{\tau}$-transverse multivector fields solution to (7). (This is equivalent to putting $f = 1$ in the local expression (6)).

Thus, the problem posed in Definition 2 is equivalent to looking for a multivector field $\mathcal{X}_h \in \mathfrak{X}^m(J^1\pi^*)$ such that:

1. $i(\mathcal{X}_h)\Omega_h = 0$.

2. $i(\mathcal{X}_h)(\bar{\tau}^*\omega) = 1$.

3. $\mathcal{X}_h$ is integrable.

From the conditions 1 and 2, using the local expressions (3) of Ω_h and (6) for $\mathcal{X}_h$, we obtain that $f = 1$ and

$$F_\nu^A = \frac{\partial \mathrm{h}}{\partial p_A^\nu} \quad ; \quad G_{A\nu}^\nu = -\frac{\partial \mathrm{h}}{\partial y^A}$$

and, if $\psi(x) = (x^\nu, y^A(x^\gamma), p_A^\nu(x^\gamma))$ must be an integral section of $\mathcal{X}_h$, then

$$\frac{\partial(y^A \circ \psi)}{\partial x^\nu} = F_\nu^A \circ \psi \quad ; \quad \frac{\partial(p_A^\rho \circ \psi)}{\partial x^\nu} = G_{A\nu}^\rho \circ \psi$$

Thus the Hamilton-De Donder-Weyl equations (5) for ψ are recovered from (7).

Remark 4 Classes of locally decomposable and $\bar{\tau}$-transverse multivector fields are in one-to one correspondence with connections in the bundle $\bar{\tau} \colon J^1\pi^* \to M$ (see Section 2.1). Then, it can be proven [11] that the condition stated in Theorem 2 is equivalent to finding an integrable connection ∇_h in $J^1\pi^* \to M$ satisfying the equation

$$i(\nabla_h)\Omega_h = (m-1)\Omega_h$$

whose integral sections are the critical sections of the Hamilton-Jacobi problem. Of course, ∇_h is the connection associated to the class $\{\mathcal{X}_h\}$ solution to (7), and $\mathcal{X}_h$ is integrable if, and only if, the curvature of ∇_h vanishes everywhere.

The expression of ∇_h in coordinates is

$$\nabla_h = \mathrm{d}x^\nu \otimes \left(\frac{\partial}{\partial x^\nu} + F_\nu^A \frac{\partial}{\partial y^A} + G_{A\nu}^\rho \frac{\partial}{\partial p_A^\rho} \right)$$

Definition 3 $\mathcal{X}_h \in \mathfrak{X}^m(J^1\pi^*)$ *will be called a* Hamilton-De Donder-Weyl (HDW) *multivector field for the system* $(J^1\pi^*, h)$ *if it is $\bar{\tau}$-transverse, locally decomposable and verifies the equation* $i(\mathcal{X}_h)\Omega_h = 0$. *Then, the associated connection* ∇_h *is called a* Hamilton-De Donder-Weyl (HDW) *connection for* $(J^1\pi^*, h)$.

For restricted Hamiltonian systems, the existence of Hamilton-De Donder Weyl multivector fields or connections is guaranteed, although they are not necessarily integrable [10], [13]:

Theorem 3 (Existence and local multiplicity of HDW-multivector fields): *Let* $(J^1\pi^*, h)$ *be a restricted Hamiltonian system. Then there exist classes of HDW-multivector fields* $\{\mathcal{X}_h\}$. *In a local system the above solutions depend on* $n(m^2 - 1)$ *arbitrary functions.*

Remark 5 In order to find a class of integrable HDW-multivector fields (if it exists) we must impose that $\mathcal{X}_h$ verify the integrability condition: the curvature of ∇_h vanishes everywhere. Hence the number of arbitrary functions will in general be less than $n(m^2-1)$. If this integrable multivector field does not exist, we can eventually select some particular HDW-multivector field solution, and apply an integrability algorithm in order to find a submanifold $\mathcal{I} \hookrightarrow J^1\pi^*$ (if it exists), where this multivector field is integrable (and tangent to $\mathcal{I}$).

4   Hamiltonian systems in $\mathcal{M}\pi$

Now we introduce Hamiltonian systems in the extended multimomentum bundle $\mathcal{M}\pi$, and we study their relation with those defined in the above section.

4.1   Extended Hamiltonian systems

Now we have the multisymplectic manifold $(\mathcal{M}\pi, \Omega)$, and we are interested in defining Hamiltonian systems on this manifold which are suitable for describing Hamiltonian field theories. Thus we must consider Hamiltonian or locally Hamiltonian m-multivector fields and forms of a particular kind. In particular, bearing in mind the requirements in Remark 3, we can state:

Definition 4 *The triple* $(\mathcal{M}\pi, \Omega, \alpha)$ *is said to be an* extended Hamiltonian system *if:*

1. $\alpha \in Z^1(\mathcal{M}\pi)$ *(it is a closed 1-form).*

2. *There exists a locally decomposable multivector field $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi)$ satisfying that*

$$i(\mathcal{X}_\alpha)\Omega = (-1)^{m+1}\alpha \quad , \quad i(\mathcal{X}_\alpha)(\bar{\kappa}^*\omega) = 1 \quad (\bar{\kappa}\text{-transversality}) \tag{8}$$

If α is an exact form, then $(\mathcal{M}\pi, \Omega, \alpha)$ is an extended global Hamiltonian system. *In this case, there exist $H \in C^\infty(\mathcal{M}\pi)$ such that $\alpha = dH$, which are called* Hamiltonian functions *of the system. (For an extended Hamiltonian system, these functions exist only locally, and they are called* local Hamiltonian functions*).*

The condition that α is closed plays a crucial role (see Proposition 2 and Section 4.2). The factor $(-1)^{m+1}$ in the definition will be justified later (see Proposition 1 and Remark 7).

Observe that, if $(\mathcal{M}\pi, \Omega, \alpha)$ is an extended global Hamiltonian system, giving a Hamiltonian function H is equivalent to giving a Hamiltonian density $\tilde{\mathcal{H}} \equiv H(\bar{\kappa}^*\omega) \in \Omega^m(\mathcal{M}\pi)$.

In natural coordinates of $\mathcal{M}\pi$, the most general expression for a locally decomposable multivector field $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi$ is

$$\mathcal{X}_\alpha = \bigwedge_{\nu=1}^m \tilde{f}\left(\frac{\partial}{\partial x^\nu} + \tilde{F}_\nu^A \frac{\partial}{\partial y^A} + \tilde{G}_{A\nu}^\rho \frac{\partial}{\partial p_A^\rho} + \tilde{g}_\nu \frac{\partial}{\partial p}\right) \tag{9}$$

where $\tilde{f} \in C^\infty(\mathcal{M}\pi)$ is a non-vanishing function which is equal to 1 if the equation $i(\mathcal{X}_\alpha)(\bar{\kappa}^*\omega) = 1$ holds.

Remark 6 In addition, bearing in mind Remark 5, the integrability of $\mathcal{X}_\alpha$ must be imposed. Then all the multivector fields in the integrable class $\{\mathcal{X}_\alpha\}$ have the same integral sections.

A first important observation is that not every closed form $\alpha \in \Omega^m(\mathcal{M}\pi)$ defines an extended Hamiltonian system. In fact:

Proposition 1 *If $(\mathcal{M}\pi, \Omega, \alpha)$ is an extended Hamiltonian system, then $i(Y)\alpha \neq 0$, for every μ-vertical vector field $Y \in \mathfrak{X}^{V(\mu)}(\mathcal{M}\pi)$, $Y \neq 0$. In particular, for every system of natural coordinates $(x^\nu, y^A, p_A^\nu, p))$ in $\mathcal{M}\pi$ adapted to the bundle $\pi\colon E \to M$ (with $\omega = d^m x$),*

$$i\left(\frac{\partial}{\partial p}\right)\alpha = 1 \tag{10}$$

(*Proof*) In order to prove this, we use natural coordinates of $\mathcal{M}\pi$. The local expression of Ω is given in (2), and a μ-vertical vector field is locally given by $Y = f\dfrac{\partial}{\partial p}$. Then, if $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi)$ is a multivector field solution to the equations (8), we have

$$
\begin{aligned}
i(Y)\alpha &= (-1)^{m+1}\, i(Y)\, i(\mathcal{X}_\alpha)\Omega = (-1)^{m+1}(-1)^m\, i(\mathcal{X}_\alpha)\, i(Y)\Omega \\
&= -i(\mathcal{X}_\alpha)\, i\left(f\frac{\partial}{\partial p}\right)[-dp \wedge d^m x - dp_A^\nu \wedge dy^A \wedge d^{m-1}x_\nu] \\
&= -i(\mathcal{X}_\alpha)\, i\left(f\frac{\partial}{\partial p}\right)[-dp \wedge d^m x] = f\, i(\mathcal{X}_\alpha)d^m x = f
\end{aligned}
$$

and, as $Y \neq 0 \Leftrightarrow f \neq 0$, the first result holds. In particular, taking $f = 1$, the expression (10) is reached. ∎

As a consequence of this result we have:

Proposition 2 *If $(\mathcal{M}\pi, \Omega, \alpha)$ is an extended Hamiltonian system, locally $\alpha = \mathrm{d}p + \beta$, where β is a closed and μ-basic local 1-form in $\mathcal{M}\pi$.*

(*Proof*)   As a consequence of (10), $\alpha = \mathrm{d}p + \beta$ locally, where β is a μ-semibasic local 1-form. But, as α is closed, so is β. Hence, for every $Y \in \mathfrak{X}^{\mathrm{V}(\mu)}(\mathcal{M}\pi)$, we have that $\mathrm{L}(Y)\beta = i(Y)\mathrm{d}\beta + \mathrm{d}\,i(Y)\beta = 0$, and β is μ-basic. ■

Therefore, by Poincaré's lemma, on an open set $U \subset \mathcal{M}\pi$ α has necessarily the following coordinate expression

$$\alpha = \mathrm{d}p + \mathrm{d}\tilde{\mathrm{h}}(x^\nu, y^A, p^\nu_A) \tag{11}$$

where $\tilde{\mathrm{h}} = \mu^*\mathrm{h}$, for some $\mathrm{h} \in \mathrm{C}^\infty(\mu(U))$. Then, if H is a (local) Hamiltonian function for α; that is, such that $\alpha = \mathrm{dH}$ (at least locally), we have that (see also [40])

$$\mathrm{H} = p + \tilde{\mathrm{h}}(x^\nu, y^A, p^\nu_A) \tag{12}$$

where $\tilde{\mathrm{h}}(x^\nu, y^A, p^\nu_A)$ is determined up to a constant.

Conversely, every closed form $\alpha \in \Omega^1(\mathcal{M}\pi)$ satisfying the above condition defines an extended Hamiltonian system since, in an analogous way to Theorem 3, we can prove:

Theorem 4 *Let $\alpha \in Z^1(\mathcal{M}\pi)$ satisfying the condition stated in Propositions 1 and 2. Then there exist locally decomposable multivector fields $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi)$ (not necessarily integrable) satisfying equations (8) (and hence $(\mathcal{M}\pi, \Omega, \alpha)$ is an extended Hamiltonian system). In a local system the above solutions depend on $n(m^2 - 1)$ arbitrary functions.*

(*Proof*)   We use the local expressions (2), (11) and (9) for Ω, α and $\mathcal{X}_\alpha$ respectively. Then $i(\mathcal{X}_\alpha)(\bar{\kappa}^*\omega) = 1$ leads to $\tilde{f} = 1$. Furthermore, from $i(\mathcal{X}_\alpha)\Omega = (-1)^{m+1}\alpha$ we obtain that the equality for the coefficients on $\mathrm{d}p^\nu_A$ leads to

$$\tilde{F}^A_\nu = \frac{\partial \mathrm{H}}{\partial p^\nu_A} = \frac{\partial \tilde{\mathrm{h}}}{\partial p^\nu_A} \qquad \text{(for every } A, \nu) \tag{13}$$

For the coefficients on $\mathrm{d}y^A$ we have

$$\tilde{G}^\nu_{A\nu} = -\frac{\partial \mathrm{H}}{\partial y^A} = -\frac{\partial \tilde{\mathrm{h}}}{\partial y^A} \qquad (A = 1, \dots, n) \tag{14}$$

and for the coefficients on $\mathrm{d}x^\nu$, using these results, we obtain

$$\begin{aligned}
\tilde{g}_\nu &= -\frac{\partial \mathrm{H}}{\partial x^\nu} + \tilde{F}^A_\nu \tilde{G}^\eta_{A\eta} - \tilde{F}^A_\eta \tilde{G}^\eta_{A\nu} = -\frac{\partial \mathrm{H}}{\partial x^\nu} + \frac{\partial \mathrm{H}}{\partial p^\nu_A}\tilde{G}^\eta_{A\eta} - \frac{\partial \mathrm{H}}{\partial p^\eta_A}\tilde{G}^\eta_{A\nu} \\
&= -\frac{\partial \tilde{\mathrm{h}}}{\partial x^\nu} + \frac{\partial \tilde{\mathrm{h}}}{\partial p^\nu_A}\tilde{G}^\eta_{A\eta} - \frac{\partial \tilde{\mathrm{h}}}{\partial p^\eta_A}\tilde{G}^\eta_{A\nu} \qquad (A = 1, \dots, n;\ \eta \neq \nu)
\end{aligned} \tag{15}$$

where the coefficients $G^\rho_{A\nu}$ are related by the equations (14). Finally, the coefficient on $\mathrm{d}p$ are identical, taking into account the above results.

Thus, equations (13) make a system of nm linear equations which determines univocally the functions $\tilde{F}^A_\nu$, equations (14) are a compatible system of n linear equations on the nm^2 functions $\tilde{G}^\eta_{A\nu}$, and equations (15) make a system of m linear equations which determines univocally the functions $\tilde{g}_\nu$. In this way, solutions to equations (8) are determined locally from the relations (13) and (15), and through the n independent linear equations (14). Therefore, there are $n(m^2 - 1)$

arbitrary functions. These results assure the local existence of $\mathcal{X}_\alpha$. The global solutions are obtained using a partition of unity subordinated to a covering of $\mathcal{M}\pi$ made of natural charts. ∎

(A further local analysis of these multivector fields solution and other additional details can be found in [11] and [40]).

Remark 7 With regard to this result, it is important to point out that, if $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi)$ is a solution (not necessarily integrable) to the equations (8), then every multivector field $\mathcal{X}'_\alpha \in \{\mathcal{X}_\alpha\}$; that is, such that $\mathcal{X}'_\alpha = \tilde{f}\mathcal{X}_\alpha$ (where $\tilde{f} \in C^\infty(\mathcal{M}\pi)$ is non-vanishing) is a solution to the equations

$$i(\mathcal{X}'_\alpha)\Omega = \tilde{f}(-1)^{m+1}\alpha \quad , \quad i(\mathcal{X}'_\alpha)(\bar{\kappa}^*\omega) = \tilde{f} \quad (\bar{\kappa}\text{-transversality})$$

In particular, if we have a 1-form $\alpha = \mathrm{d}H$ (locally), with $0 \neq \dfrac{\partial H}{\partial p} \neq 1$, then the $\bar{\kappa}$-transversality condition must be stated as $i(\mathcal{X}_\alpha)(\bar{\kappa}^*\omega) = -\dfrac{\partial H}{\partial p}$, and the solutions $\mathcal{X}_\alpha$ to the equation $i(\mathcal{X}_\alpha)\Omega = (-1)^{m+1}\alpha$ have the local expression (9) with $\tilde{f} = -\dfrac{\partial H}{\partial p}$, and the other coefficients being solutions to the system of equations

$$\tilde{f}\tilde{F}^A_\nu = \frac{\partial H}{\partial p^\nu_A} \quad , \quad \tilde{f}\tilde{G}^\nu_{A\nu} = -\frac{\partial H}{\partial y^A} \quad , \quad \tilde{f}\tilde{g}_\nu = -\frac{\partial H}{\partial x^\nu} + \frac{\partial H}{\partial p^\nu_A}\tilde{G}^\eta_{A\eta} - \frac{\partial H}{\partial p^\eta_A}\tilde{G}^\eta_{A\nu} \quad (\eta \neq \nu)$$

Thus, in an analogous way to restricted Hamiltonian systems in $J^1\pi^*$, we define:

Definition 5 $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi)$ *will be called an* extended Hamilton-De Donder-Weyl multivector field *for the system* $(\mathcal{M}\pi, \Omega, \alpha)$ *if it is $\bar{\kappa}$-transverse, locally decomposable and verifies the equation* $i(\mathcal{X}_\alpha)\Omega = (-1)^{m+1}\alpha$. *Then, the associated connection* ∇_α *in the bundle* $\bar{\kappa}\colon \mathcal{M}\pi \to M$ *is called an* extended Hamilton-De Donder-Weyl connection *for* $(\mathcal{M}\pi, \Omega, \alpha)$.

Now, if $\{X_\alpha\}$ is integrable and $\tilde{\psi}(x) = (x^\nu, y^A(x^\gamma), p^\nu_A(x^\gamma), p(x^\gamma))$ must be an integral section of $\mathcal{X}_\alpha$ then

$$\frac{\partial(y^A \circ \tilde{\psi})}{\partial x^\nu} = \tilde{F}^A_\nu \circ \tilde{\psi} \quad , \quad \frac{\partial(p^\rho_A \circ \tilde{\psi})}{\partial x^\nu} = \tilde{G}^\rho_{A\nu} \circ \tilde{\psi} \quad , \quad \frac{\partial(p \circ \tilde{\psi})}{\partial x^\nu} = \tilde{g}_\nu \circ \tilde{\psi} \tag{16}$$

so equations (13), (14) and (15) give PDE's for $\tilde{\psi}$. In particular, the Hamilton-De Donder-Weyl equations (5) are recovered from (13) and (14).

As for restricted Hamiltonian systems, in order to find a class of integrable extended HDW-multivector fields (if it exists) we must impose that $\mathcal{X}_\alpha$ verify the integrability condition, that is, that the curvature of ∇_α vanishes everywhere, and thus the number of arbitrary functions will in general be less than $n(m^2 - 1)$. Just as in that situation, we cannot assure the existence of an integrable solution. If it does not exist, we can eventually select some particular extended HDW-multivector field solution, and apply an integrability algorithm in order to find a submanifold of $\mathcal{M}\pi$ (if it exists), where this multivector field is integrable (and tangent to it).

4.2 Geometric properties of extended Hamiltonian systems

Most of the properties of the extended Hamiltonian systems are based in the following general results:

Lemma 1 *Let $\mu\colon \mathcal{M} \to \mathcal{F}$ be a surjective submersion, with $\dim \mathcal{M} = \dim \mathcal{F} + r$. Consider $\alpha_1, \ldots, \alpha_r \in \Omega^1(\mathcal{M})$ such that $\alpha \equiv \alpha_1 \wedge \ldots \wedge \alpha_r$ is a closed r-form, and $\alpha(\mathbf{p}) \neq 0$, for every $\mathbf{p} \in \mathcal{M}$. Finally, let $\{\alpha\}^0 := \{Z \in \mathfrak{X}(\mathcal{M}) \,|\, i(Z)\alpha = 0\}$ be the annihilator of α. Therefore:*

1. $\{\alpha\}^0 := \{Z \in \mathfrak{X}(\mathcal{M}) \,|\, i(Z)\alpha_i = 0, \ \forall i = 1 \ldots r\}$.

2. *$\{\alpha\}^0$ generates an involutive distribution in $\mathcal{M}$ of corank equal to r, which is called the characteristic distribution of α, and is denoted $\mathcal{D}_\alpha$.*

   If, in addition, the condition $i(Y)\alpha \neq 0$ holds, for every $Y \in \mathfrak{X}^{\mathrm{V}(\mu)}(\mathcal{M})$, then:

3. *$\mathcal{D}_\alpha$ is a μ-transverse distribution.*

4. *The integral submanifolds of $\mathcal{D}_\alpha$ are r-codimensional and μ-transverse local submanifolds of $\mathcal{M}$.*

5. *$\mathrm{T}_{\mathbf{p}}\mathcal{M} = \mathrm{V}_{\mathbf{p}}(\mu) \oplus \{\alpha\}^0_{\mathbf{p}}$, for every $\mathbf{p} \in \mathcal{M}$.*

6. *If S is an integral submanifold of $\mathcal{D}_\alpha$, then $\mu|_S\colon S \to \mathcal{F}$ is a local diffeomorphism.*

7. *For every integral submanifold S of $\mathcal{D}_\alpha$, and $\mathbf{p} \in S$, there exists $W \subset \mathcal{M}$, with $\mathbf{p} \in W$, such that $h = (\mu|_{W \cap S})^{-1}$ is a local section of μ defined on $\mu(W \cap S)$ (which is an open set of $\mathcal{F}$).*

(*Proof*)   First, observe that, for every $\mathbf{p} \in \mathcal{M}$, $\alpha(\mathbf{p}) \neq 0$ implies that $\alpha_i(\mathbf{p})$, for every $i = 1, \ldots, r$, are linearly independent, then

$$0 = i(Z)\alpha = \sum_{i=1}^{r} (-1)^{i-1} i(Z)\alpha_i (\alpha_1 \wedge \ldots \wedge \alpha_{i-1} \wedge \alpha_{i+1} \wedge \ldots \wedge \alpha_r) \Leftrightarrow i(Z)\alpha_i = 0$$

hence the statement in the item 1 holds and, as a consequence, we conclude that $\{\alpha\}^0$ generates a distribution in $\mathcal{M}$ of rank equal to $\dim \mathcal{F}$.

Furthermore, if α is closed, for every $Z_1, Z_2 \in \{\alpha\}^0$, we obtain that $[Z_1, Z_2] \in \{\alpha\}^0$ because

$$i([Z_1, Z_2])\alpha = \mathrm{L}(Z_1)\, i(Z_2)\alpha - i(Z_2)\, \mathrm{L}(Z_1)\alpha = -\, i(Z_2)[i(Z_1)\mathrm{d}\alpha - \mathrm{d}\, i(Z_1)\alpha] = 0$$

Then $\mathcal{D}_\alpha$ is involutive.

The other properties follow straighforwardly from these results and the condition $i(Y)\alpha \neq 0$, for every $Y \in \mathfrak{X}^{\mathrm{V}(\mu)}(\mathcal{M})$. ∎

Now, from this lemma we have that:

Proposition 3 *If $(\mathcal{M}\pi, \Omega, \alpha)$ is an extended Hamiltonian system, then:*

1. *$\mathcal{D}_\alpha$ is a μ-transverse involutive distribution of corank equal to 1.*

2. *The integral submanifolds S of $\mathcal{D}_\alpha$ are 1-codimensional and μ-transverse local submanifolds of $\mathcal{M}\pi$. (We denote by $\jmath_S\colon S \hookrightarrow \mathcal{M}\pi$ the natural embedding).*

3. *For every $\mathbf{p} \in \mathcal{M}\pi$, we have that $\mathrm{T}_{\mathbf{p}}\mathcal{M}\pi = \mathrm{V}_{\mathbf{p}}(\mu) \oplus (\mathcal{D}_\alpha)_{\mathbf{p}}$, and thus, in this way, α defines an integrable connection in the affine bundle $\mu\colon \mathcal{M}\pi \to J^1\pi^*$.*

4. *If S is an integral submanifold of $\mathcal{D}_\alpha$, then $\mu|_S\colon S \to J^1\pi^*$ is a local diffeomorphism.*

5. *For every integral submanifold S of $\mathcal{D}_\alpha$, and $\mathbf{p} \in S$, there exists $W \subset \mathcal{M}\pi$, with $\mathbf{p} \in W$, such that $h = (\mu|_{W \cap S})^{-1}$ is a local Hamiltonian section of μ defined on $\mu(W \cap S)$.*

Remark 8 Observe that, if $(\mathcal{M}\pi, \Omega, \alpha)$ is an extended Hamiltonian system, as $\alpha = \mathrm{d}H$ (locally), every local Hamiltonian function H is a constraint defining locally the integral submanifolds of $\mathcal{D}_\alpha$. Thus, bearing in mind the coordinate expression (12), the local Hamiltonian sections associated with these submanifolds are locally expressed as

$$h(x^\nu, y^A, p_A^\nu) = (x^\nu, y^A, p_A^\nu, p = -\mathrm{h}(x^\gamma, y^B, p_B^\eta))$$

where $\mu^* \mathrm{h} = \tilde{\mathrm{h}}$.

A relevant question is under what conditions the existence of global Hamiltonian sections is assured. The answer is given by the following:

Proposition 4 *Let $(\mathcal{M}\pi, \Omega, \alpha)$ be an extended global Hamiltonian system. If there is a Hamiltonian function $H \in C^\infty(\mathcal{M}\pi)$, and $k \in \mathbb{R}$, such that $\mu(H^{-1}(k)) = J^1\pi^*$, then there exists a global Hamiltonian section $h \in \Gamma(J^1\pi^*, \mathcal{M}\pi)$.*

(*Proof*)   In order to construct h, we prove that, for every $q \in J^1\pi^*$, we have that $\mu^{-1}(q) \cap S_k$ contains only one point. Let $(U; x^\nu, y^A, p_A^\nu, p)$ be a local chart in $\mathcal{M}\pi$, with $q \in \mu(U)$. By Proposition 2 we have that $H|_U = p + \mu^* \mathrm{h}$, for $\mathrm{h} \in C^\infty(\mu(U))$. If $\mathbf{p} \in \mu^{-1}(q) \cap S_k$ we have that

$$k = H(\mathbf{p}) = p(\mathbf{p}) - (\mu^* \mathrm{h})(\mathbf{p} = p(\mathbf{p}) - \mathrm{h}(\mu(\mathbf{p}))$$

then $p(\mathbf{p})$ is determined by q, and $\mathbf{p}$ is unique. This allows to define a global section $h \colon J^1\pi^* \to \mathcal{M}\pi$ by $h(q) := \mu^{-1}(q) \cap S_k$, for every $q \in J^1\pi^*$, which obviously does not depend on the local charts considered.   ∎

Observe tha, if the first de Rahm chomology group $H^1(\mathcal{M}\pi) = 0$, then every extended Hamiltonian system is a global one, but this does not assure the existence of global Hamiltonian sections, as we have shown.

In addition, we have:

Proposition 5 *Given an extended Hamiltonian system $(\mathcal{M}\pi, \Omega, \alpha)$, every extended HDW multivector field $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi)$ for the system $(\mathcal{M}\pi, \Omega, \alpha)$ is tangent to every integral submanifold of $\mathcal{D}_\alpha$. As a consequence, if $\mathcal{X}_\alpha$ is integrable, then its integral sections are contained in the integral submanifolds of $\mathcal{D}_\alpha$.*

(*Proof*)   By definition, an extended HDW multivector field is locally decomposable, so locally $\mathcal{X}_\alpha = X_1 \wedge \ldots \wedge X_m$, with $X_1, \ldots, X_m \in \mathfrak{X}(\mathcal{M}\pi)$. Then $\mathcal{X}_\alpha$ is tangent to every integral submanifold S of $\mathcal{D}_\alpha$ if, and only if, X_ν are tangent to S, for every $i = 1, \ldots, m$. But, as $\mathcal{D}_\alpha$ is the characteristic distribution of α, this is equivalent to $j_S^* i(X_\nu)\alpha = 0$, and this is true because

$$i(X_\nu)\alpha = i(X_\nu)\, i(\mathcal{X}_\alpha)\Omega = i(X_\nu)\, i(X_1 \wedge \ldots \wedge X_m)\Omega = 0$$

The last consequence is immediate.   ∎

Remark 9 Observe that, if $\mathcal{X}_\alpha = X_1 \wedge \ldots \wedge X_m$ locally, using the local expressions (9) and (11) and equations (13) and (14), the conditions $i(X_\nu)\alpha = 0$ lead to

$$
\begin{aligned}
0 &= \frac{\partial \tilde{\mathrm{h}}}{\partial x^\nu} + \tilde{F}^A_\nu \frac{\partial \tilde{\mathrm{h}}}{\partial y^A} + \tilde{G}^\rho_{A\nu} \frac{\partial \tilde{\mathrm{h}}}{\partial p^\rho_A} + \tilde{g}_\nu \\
&= \frac{\partial \tilde{\mathrm{h}}}{\partial x^\nu} - \frac{\partial \tilde{\mathrm{h}}}{\partial p^\rho_A} \tilde{G}^\rho_{A\rho} + \tilde{G}^\rho_{A\nu} \frac{\partial \tilde{\mathrm{h}}}{\partial p^\rho_A} + \tilde{g}_\nu \\
&= \frac{\partial \tilde{\mathrm{h}}}{\partial x^\nu} - \frac{\partial \tilde{\mathrm{h}}}{\partial p^\rho_A} (\tilde{G}^\eta_{A\eta} + \tilde{G}^\nu_{A\nu}) + \tilde{G}^\eta_{A\nu} \frac{\partial \tilde{\mathrm{h}}}{\partial p^\eta_A} + \tilde{G}^\nu_{A\nu} \frac{\partial \tilde{\mathrm{h}}}{\partial p^\nu_A} + \tilde{g}_\nu \\
&= \frac{\partial \tilde{\mathrm{h}}}{\partial x^\nu} - \frac{\partial \tilde{\mathrm{h}}}{\partial p^\nu_A} \tilde{G}^\eta_{A\eta} + \frac{\partial \tilde{\mathrm{h}}}{\partial p^\eta_A} \tilde{G}^\eta_{A\nu} + \tilde{g}_\nu \qquad (\rho = 1, \ldots, m; \ \nu \text{ fixed}, \ \eta \neq \nu)
\end{aligned}
$$

which are the equations (15). So these equations are consistency conditions. (See also the comment in Remark 10).

Finally, we have the following result:

Proposition 6 *The integral submanifolds of $\mathcal{D}_\alpha$ are m-coisotropic submanifolds of $(\mathcal{M}\pi, \Omega)$.*

(*Proof*)   Let S be an integral submanifold of $\mathcal{D}_\alpha$. First remember that, for every $\mathbf{p} \in S$, the *m-orthogonal multisymplectic complement* of S at $\mathbf{p}$ is the vector space

$$
\mathrm{T}_\mathbf{p} S^{\perp, m} := \{ X_\mathbf{p} \in \mathrm{T}_\mathbf{p} \mathcal{M}\pi \mid i(X_\mathbf{p} \wedge \mathcal{X}_\mathbf{p})\Omega_\mathbf{p} = 0, \text{ for every } \mathcal{X}_\mathbf{p} \in \bigwedge^m \mathrm{T}_\mathbf{p} S \equiv \bigwedge^m (\mathcal{D}_\alpha)_\mathbf{p} \}
$$

and S is said to be a *m-coisotropic submanifold* of $(\mathcal{M}\pi, \Omega)$ if $\mathrm{T}_\mathbf{p} S^{\perp, m} \subset \mathrm{T}_\mathbf{p} S$ [5], [33]. Then, for every $X_\mathbf{p} \in \mathrm{T}_\mathbf{p} S^{\perp, m}$, if $\mathcal{X}_\alpha$ is a HDW-multivector field for the extended Hamiltonian system $(\mathcal{M}\pi, \Omega, \alpha)$, as $(\mathcal{X}_\alpha)_\mathbf{p} \in \Lambda^m \mathrm{T}_\mathbf{p} S$, by Proposition 5 we have

$$
0 = i(X_\mathbf{p}) \, i((\mathcal{X}_\alpha)_\mathbf{p})\Omega_\mathbf{p} = i(X_\mathbf{p})\alpha_\mathbf{p}
$$

therefore $X_\mathbf{p} \in (\mathcal{D}_\alpha)_\mathbf{p} = \mathrm{T}_\mathbf{p} S$. ∎

This statement generalizes a well-known result in time-dependent mechanics (see the example in section 6.2): considering the line bundle $\mu : \mathrm{T}^*Q \times \mathrm{T}^*\mathbb{R} \to \mathrm{T}^*Q \times \mathbb{R}$, the zero section gives a canonical coisotropic embedding of the submanifold $\mathrm{T}^*Q \times \mathbb{R}$ into the symplectic manifold $\mathrm{T}^*(Q \times \mathbb{R}) \simeq \mathrm{T}^*Q \times \mathrm{T}^*\mathbb{R}$. Furthermore, in field theories, every maximal integral submanifold S of $\mathcal{D}_\alpha$ gives a local m-coisotropic embedding of $U \subset \mu(S) \subset J^1\pi^*$ into $\mathcal{M}\pi$, given by $(\mu|_S)^{-1}$, which is obviously not canonical.

4.3   Relation between extended and restricted Hamiltonian systems

Now we can establish the relation between extended and restricted Hamiltonian systems in $J^1\pi^*$. Taking into account the considerations made in the above section, we can state:

Theorem 5 *Let $(\mathcal{M}\pi, \Omega, \alpha)$ be an extended global Hamiltonian system, and $(J^1\pi^*, h)$ a restricted Hamiltonian system such that $\operatorname{Im} h = S$ is an integral submanifold of $\mathcal{D}_\alpha$. For every $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi)$ solution to the equations (8):*

$$
i(\mathcal{X}_\alpha)\Omega = (-1)^{m+1}\alpha \quad , \quad i(\mathcal{X}_\alpha)(\bar{\kappa}^*\omega) = 1
$$

there exists $\mathcal{X}_h \in \mathfrak{X}^m(J^1\pi^*)$ which is h-related with $\mathcal{X}_\alpha$ and is a solution to the equations

$$i(\mathcal{X}_h)\Omega_h = 0 \quad , \quad i(\mathcal{X}_h)(\bar{\tau}^*\omega) = 1$$

(i.e., satisfying the conditions 1 and 2 in Remark 3).

Furthermore, if $\mathcal{X}_\alpha$ is integrable, then $\mathcal{X}_h$ is integrable too, and the integral sections of $\mathcal{X}_h$ are recovered from those of $\mathcal{X}_\alpha$ as follows: if $\tilde{\psi}\colon M \to \mathcal{M}\pi$ is an integral section of $\mathcal{X}_\alpha$, then $\psi = \mu \circ \tilde{\psi}\colon M \to J^1\pi^*$ is an integral section of $\mathcal{X}_h$.

(*Proof*) Given $S = \operatorname{Im} h$, let $\jmath_S\colon S \hookrightarrow \mathcal{M}\pi$ be the natural embedding, and $h_S\colon J^1\pi^* \to S$ the diffeomorphism between $J^1\pi^*$ and $\operatorname{Im} h$, then $h = \jmath_S \circ h_S$.

If $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi)$ is a solution to the equations (8), by Proposition 5, it is tangent to S, then there exists $\mathcal{X}_S \in \mathfrak{X}^m(S)$ such that $\Lambda^m \jmath_{S*} \mathcal{X}_S = \mathcal{X}_\alpha|_S$. Let $\mathcal{X}_h \in \mathfrak{X}^m(J^1\pi^*)$ defined by $\mathcal{X}_h = \Lambda^m h_{S*}^{-1} \mathcal{X}_S$. Therefore, from the equation $i(\mathcal{X}_\alpha)\Omega = (-1)^{m+1}\alpha$ and the condition $\jmath_S^*\alpha = 0$ (which holds because S is an integral submanifold of $\mathcal{D}_\alpha$), we obtain

$$\begin{aligned} 0 &= h^*[i(\mathcal{X}_\alpha)\Omega - (-1)^{m+1}\alpha] = (\jmath_S \circ h_S)^*[i(\mathcal{X}_\alpha)\Omega - (-1)^{m+1}\alpha] = (h_S^* \circ \jmath_S^*)[i(\mathcal{X}_\alpha)\Omega - (-1)^{m+1}\alpha] \\ &= h_S^*[i(\mathcal{X}_S)\jmath_S^*\Omega - (-1)^{m+1}\jmath_S^*\alpha] = i(\mathcal{X}_h)(h_S^* \circ \jmath_S^*)\Omega = i(\mathcal{X}_h)h^*\Omega = i(\mathcal{X}_h)\Omega_h \end{aligned}$$

Furthermore, bearing in mind that $\mu \circ h = \operatorname{Id}_{J^1\pi^*}$, we have that

$$i(\mathcal{X}_h)(\bar{\tau}^*\omega) = i(\mathcal{X}_h)[(\bar{\kappa} \circ h)^*\omega] = h^*[i(\mathcal{X}_\alpha)(\bar{\kappa}^*\omega)]$$

and, if $i(\mathcal{X}_\alpha)(\bar{\kappa}^*\omega) = 1$, this equality holds, in particular, at the points of the image of h, therefore $i(\mathcal{X}_h)(\bar{\tau}^*\omega) = 1$. Then $\mathcal{X}_h$ is the desired multivector field, since $\mathcal{X}_h|_S = \mathcal{X}_S = \Lambda_*^m \mathcal{X}_h$.

Finally, if $\mathcal{X}_\alpha$ is integrable, as it is tangent to S, the integral sections of $\mathcal{X}_\alpha$ passing through any point of S remain in S, and hence they are the integral sections of $\mathcal{X}_S$, so $\mathcal{X}_\alpha$ is integrable and, as a consequence, $\mathcal{X}_h$ is integrable too. ∎

All of these properties lead to establish the following:

Definition 6 *Given an extended global Hamiltonian system $(\mathcal{M}\pi, \Omega, \alpha)$, and considering all the Hamiltonian sections $h\colon J^1\pi^* \to \mathcal{M}\pi$ such that $\operatorname{Im} h$ are integral submanifolds of $\mathcal{D}_\alpha$, we have a family $\{(J^1\pi^*, h)\}_\alpha$, which will be called the* class of restricted Hamiltonian systems associated with $(\mathcal{M}\pi, \Omega, \alpha)$.

As it is obvious, in general, the above result holds only locally.

The following result show how to obtain extended Hamiltonian systems from restricted Hamiltonian ones, at least locally. In fact:

Proposition 7 *Given a restricted Hamiltonian system $(J^1\pi^*, h)$, let $\jmath_S\colon S = \operatorname{Im} h \hookrightarrow \mathcal{M}\pi$ be the natural embedding. Then, there exists a unique local form $\alpha \in \Omega^1(\mathcal{M}\pi)$ such that:*

1. *$\alpha \in Z^1(\mathcal{M}\pi)$ (it is a closed form).*

2. *$\jmath_S^*\alpha = 0$.*

3. *$i(Y)\alpha \neq 0$, for every non-vanishing $Y \in \mathfrak{X}^{\mathrm{V}(\mu)}(\mathcal{M}\pi)$ and, in particular, such that $i\left(\dfrac{\partial}{\partial p}\right)\alpha = 1$, for every system of natural coordinates (x^ν, y^A, p_A^ν, p) in $\mathcal{M}\pi$, adapted to the bundle $\pi\colon E \to M$ (with $\omega = \mathrm{d}^m x$).*

(*Proof*)   Suppose that there exist α, α' satisfying the above conditions. Taking into account the comments after Proposition 1, we have that, locally in $U \subset \mathcal{M}\pi$, $\alpha = \mathrm{d}p + \beta$ and $\alpha' = \mathrm{d}p + \beta'$, where $\beta = \mu^* \bar{\beta}$, $\beta' = \mu^* \bar{\beta}'$, with $\bar{\beta}, \bar{\beta}' \in B^1(\mu(U))$ (they are exact 1-forms). From condition 2 in the statement we have that $\jmath_S^* \alpha = \jmath_S^* \alpha'$; hence

$$0 = \jmath_S^*(\alpha - \alpha') \iff 0 = \jmath_S^*(\beta - \beta') = (\mu \circ \jmath_S)^*(\bar{\beta} - \bar{\beta}')$$
$$\implies 0 = (\mu \circ \jmath_S \circ h_S)^*(\bar{\beta} - \bar{\beta}') \implies \bar{\beta} - \bar{\beta}' = 0 \iff \bar{\beta} = \bar{\beta}' \iff \bar{\alpha} = \bar{\alpha}'$$

since $\mu \circ \jmath_S \circ h_S = \mu \circ h = \mathrm{Id}_{\mu(U)}$. This proves the uniqueness.

The existence is trivial since, locally, every section h of μ is given by a function $\mathrm{h} \in C^\infty(\mu(U))$ such that $p = -\mathrm{h}(x^\nu, y^A, p_A^\nu)$. Hence $\alpha|_{\mu(U)} = \mathrm{d}p + \mathrm{d}(\mu^* \mathrm{h}) \equiv \mathrm{d}p + \mathrm{d}\tilde{\mathrm{h}}$.   ∎

Definition 7 *Given a restricted Hamiltonian system* $(J^1\pi^*, h)$, *let* $\alpha \in \Omega^1(\mathcal{M}\pi)$ *be the local form satisfying the conditions in the above proposition. The couple* $(\mathcal{M}\pi, \alpha)$ *will be called the (local) extended Hamiltonian system associated with* $(J^1\pi^*, h)$.

As a consequence of the last proposition, if $\alpha = \mathrm{d}p + \mu^*\bar{\beta}$, there exists a class $\{\mathrm{h}\} \in C^\infty(\mu(U))/\mathbb{R}$, such that $\bar{\beta} = \mathrm{dh}$, where h is a representative of this class. Then:

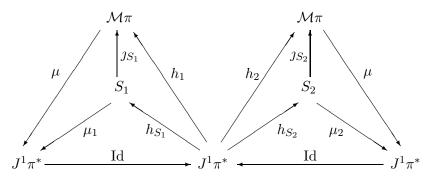**Corollary 1** *Let* α *be the unique local 1-form verifying the conditions of Proposition 7, associated with a section* h. *Consider its characteristic distribution* $\mathcal{D}_\alpha$, *and let* $\{h\}_\alpha$ *the family of local sections of* μ *such that* $\mathrm{Im}\, h \equiv S$ *are local integral submanifolds of* $\mathcal{D}_\alpha$. *Then, for every* $h' \in \{h\}_\alpha$, *we have that* $\mathrm{Im}\, h'$ *is locally a level set of the function* $\mathrm{H} = p + \mu^* \mathrm{h} \equiv p + \tilde{\mathrm{h}}$.

(*Proof*)   If $S = \mathrm{Im}\, h'$ then, for every $\mathrm{p} \in S$, we have $\mathrm{T}_\mathrm{p} S = (\mathcal{D}_\alpha)_\mathrm{p}$, which is equivalent to $\mathrm{d}(p + \mu^* \mathrm{h})|_{\mathrm{T}_\mathrm{p} S} = 0$, and this holds if, and only if, $\mathrm{H}|_S \equiv (p + \mu^* \mathrm{h})|_S = ctn$.   ∎

Bearing in mind these considerations, we can finally prove that:

Proposition 8 *Let* $(\mathcal{M}\pi, \Omega, \alpha)$ *be an extended Hamiltonian system, and* $\{(J^1\pi^*, h)\}_\alpha$ *the class of restricted Hamiltonian systems associated with* $(\mathcal{M}\pi, \Omega, \alpha)$. *Consider the submanifolds* $\{S_h = \mathrm{Im}\, h\}$, *for every Hamiltonian section* h *in this class, and let* $\jmath_{S_h} \colon S_h \hookrightarrow \mathcal{M}\pi$ *be the natural embeddings. Then the submanifolds* $(S_h, \jmath_{S_h}^* \Omega)$ *are multisymplectomorphic.*

(*Proof*)   Let $h_1, h_2 \in \{h\}$ and $S_1 = \mathrm{Im}\, h_1$, $S_2 = \mathrm{Im}\, h_2$. We have the diagram

Denote $\Omega_1 = \jmath_{S_1}^* \Omega$, $\Omega_2 = \jmath_{S_2}^* \Omega$. As a consequence of the above corollary, if $\Omega_{h_1} = h_1^* \Omega$, $\Omega_{h_2} = h_2^* \Omega$, we have that $\Omega_{h_1} = \Omega_{h_2}$. But

$$\Omega_{h_1} = \Omega_{h_2} \iff (\jmath_{S_1} \circ h_{S_1})^* \Omega = (\jmath_{S_2} \circ h_{S_2})^* \Omega \iff h_{S_1}^* \Omega_1 = h_{S_2}^* \Omega_2$$

Then, the map $\Phi := h_{S_2} \circ \mu_1 \colon S_1 \to S_2$ is a multisymplectomorphism. In fact, it is obviously a diffeomorphism, and

$$\Phi^*\Omega_2 = (h_{S_2} \circ \mu_1)^*\Omega_2 = \mu_1^* h_{S_2}^* \Omega_2 = \mu_1^* h_{S_1}^* \Omega_1 = (h_{S_1} \circ \mu_1)^*\Omega_1 = \Omega_1$$

$\blacksquare$

As an immediate consequence of this, if $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi)$ is a solution to the equations (8), the multivector fields $\mathcal{X}_{S_h} \in \mathfrak{X}^m(S_h)$ such that $\Lambda^m(\jmath_{S_h})_*\mathcal{X}_{S_h} = \mathcal{X}_\alpha|_S$, for every submanifold S_h of this family (see the proof of Theorem 5), are related by these multisymplectomorphisms.

4.4   Variational principle and field equations

As in the case of restricted Hamiltonian systems, the field equations for extended Hamiltonian systems can be derived from a suitable variational principle.

First, denote by $\mathfrak{X}_\alpha(\mathcal{M}\pi)$ the set of vector fields $Z \in \mathfrak{X}(\mathcal{M}\pi)$ which are sections of the subbundle $\mathcal{D}_\alpha$ of $\mathrm{T}\mathcal{M}\pi$, that is, satisfying that $i(Z)\alpha = 0$ (and hence, they are tangent to all the integral submanifolds of $\mathcal{D}_\alpha$). Let $\mathfrak{X}_\alpha^{\mathrm{V}(\bar\kappa)}(\mathcal{M}\pi) \subset \mathfrak{X}_\alpha(\mathcal{M}\pi)$ be those which are also $\bar\kappa$-vertical.

Furthermore, as we have seen in previous sections, the image of the sections $\tilde\psi \colon M \to \mathcal{M}\pi$, which are solutions to the extended field equations, must be in the integral submanifolds of the characteristic distribution $\mathcal{D}_\alpha$; that is, they are also integral submanifolds, and hence $\jmath_{\tilde\psi}^*\alpha = 0$ (where $\jmath_{\tilde\psi} \colon \operatorname{Im}\tilde\psi \hookrightarrow \mathcal{M}\pi$ is the natural embedding). We will denote by $\Gamma_\alpha(M, \mathcal{M}\pi)$ the set of sections of $\bar\kappa$ satisfying that $\jmath_{\tilde\psi}^*\alpha = 0$.

Taking all of this into account, we can state the following:

Definition 8 *Let $(\mathcal{M}\pi, \Omega, \alpha)$ be an extended Hamiltonian system. Consider the map*

$$\begin{array}{rccc} \tilde{\mathbf{H}}_\alpha & : & \Gamma_\alpha(M, \mathcal{M}\pi) & \longrightarrow & \mathbb{R} \\ & & \tilde\psi & \mapsto & \int_U \tilde\psi^*\Theta \end{array}$$

(where the convergence of the integral is assumed). The variational problem *for this extended Hamiltonian system is the search for the critical (or stationary) sections of the functional $\tilde{\mathbf{H}}_\alpha$, with respect to the variations of $\tilde\psi \in \Gamma_\alpha(M, \mathcal{M}\pi)$ given by $\tilde\psi_t = \sigma_t \circ \tilde\psi$, where $\{\sigma_t\}$ is the local one-parameter group of any compact-supported vector field $Z \in \mathfrak{X}_\alpha^{\mathrm{V}(\bar\kappa)}(\mathcal{M}\pi)$, that is*

$$\frac{d}{dt}\bigg|_{t=0} \int_U \tilde\psi_t^*\Theta = 0$$

This is the extended Hamilton-Jacobi principle.

Observe that, as α is closed, the variation of the set $\Gamma_\alpha(M, \mathcal{M}\pi)$ is stable under the action of $\mathfrak{X}_\alpha^{\mathrm{V}(\bar\kappa)}(\mathcal{M}\pi)$. In fact; being α closed, for every $Z \in \mathfrak{X}_\alpha^{\mathrm{V}(\bar\kappa)}(\mathcal{M}\pi)$, we have that $\mathrm{L}(Z)\alpha = i(Z)\mathrm{d}\alpha + \mathrm{d}\,i(Z)\alpha = 0$, that is, $\sigma_t^*\alpha = \alpha$. Hence, if $\tilde\psi \in \Gamma_\alpha(M, \mathcal{M}\pi)$, we obtain

$$\tilde\psi_t^*\alpha = (\sigma_t \circ \tilde\psi)^*\alpha = \tilde\psi^*\sigma_t^*\alpha = \tilde\psi^*\alpha = 0$$

Then we have the following fundamental theorems:

Theorem 6 *Let $(\mathcal{M}\pi, \Omega, \alpha)$ be an extended Hamiltonian system. The following assertions on a section $\tilde\psi \in \Gamma_\alpha(M, \mathcal{M}\pi)$ are equivalent:*

1. $\tilde{\psi}$ *is a critical section for the variational problem posed by the extended Hamilton-Jacobi principle.*

2. $\tilde{\psi}^* \, i(Z)\Omega = 0$, *for every* $Z \in \mathfrak{X}_\alpha^{\mathrm{V}(\bar{\kappa})}(\mathcal{M}\pi)$.

3. $\tilde{\psi}^* \, i(X)\Omega = 0$, *for every* $X \in \mathfrak{X}_\alpha(\mathcal{M}\pi)$.

4. *If* $(U; x^\nu, y^A, p_A^\nu, p)$ *is a natural system of coordinates in* $\mathcal{M}\pi$, *then* $\tilde{\psi}$ *satisfies the following system of equations in* U

$$\frac{\partial(y^A \circ \tilde{\psi})}{\partial x^\nu} = \frac{\partial\tilde{\mathrm{h}}}{\partial p_A^\nu}\circ\tilde{\psi} \quad , \quad \frac{\partial(p_A^\nu \circ \tilde{\psi})}{\partial x^\nu} = -\frac{\partial\tilde{\mathrm{h}}}{\partial y^A}\circ\tilde{\psi} \quad , \quad \frac{\partial(p \circ \tilde{\psi})}{\partial x^\nu} = -\frac{\partial(\tilde{\mathrm{h}} \circ \tilde{\psi})}{\partial x^\nu} \tag{17}$$

where $\tilde{\mathrm{h}} = \mu^*\mathrm{h}$, *for some* $\mathrm{h} \in \mathrm{C}^\infty(\mu(U))$, *is any function such that* $\alpha|_U = \mathrm{d}p + \mathrm{d}\tilde{\mathrm{h}}(x^\nu, y^A, p_A^\nu)$. *These are the* extended Hamilton-De Donder-Weyl equations *of the extended Hamiltonian system.*

(*Proof*)   (1 $\Longleftrightarrow$ 2)   We assume that ∂U is a $(m-1)$-dimensional manifold and that $\bar{\kappa}(supp\,(Z)) \subset U$, for every compact-supported $Z \in \mathfrak{X}_\alpha^{\mathrm{V}(\bar{\kappa})}(\mathcal{M}\pi)$. Then

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \int_U \tilde{\psi}_t^* \Theta &= \frac{d}{dt}\Big|_{t=0} \int_U \tilde{\psi}^*(\sigma_t^* \Theta) = \int_U \tilde{\psi}^*\left(\lim_{t\to 0}\frac{\sigma_t^*\Theta - \Theta}{t}\right)\\
&= \int_U \tilde{\psi}^*(\mathrm{L}(Z)\Theta) = \int_U \tilde{\psi}^*(i(Z)\mathrm{d}\Theta + \mathrm{d}\,i(Z)\Theta)\\
&= -\int_U \tilde{\psi}^*(i(Z)\Omega - \mathrm{d}\,i(Z)\Theta) = -\int_U \tilde{\psi}^*(i(Z)\Omega) + \int_U \mathrm{d}[\tilde{\psi}^*(i(Z)\Theta)]\\
&= -\int_U \tilde{\psi}^*(i(Z)\Omega) + \int_{\partial U} \tilde{\psi}^*(i(Z)\Theta) = -\int_U \tilde{\psi}^*(i(Z)\Omega)
\end{aligned}$$

(as a consequence of Stoke's theorem and the hypothesis made on the supports of the vertical fields). Thus, by the fundamental theorem of the variational calculus we conclude that $\frac{d}{dt}\Big|_{t=0}\int_U \psi_t^* \Theta = 0$ $\Leftrightarrow \tilde{\psi}^*(i(Z)\Omega) = 0$, for every compact-supported $Z \in \mathfrak{X}_\alpha^{\mathrm{V}(\bar{\kappa})}(\mathcal{M}\pi)$. But, as compact-supported vector fields generate locally the $\mathrm{C}^\infty(\mathcal{M}\pi)$-module of vector fields in $\mathcal{M}\pi$, it follows that the last equality holds for every $Z \in \mathfrak{X}_\alpha^{\mathrm{V}(\bar{\kappa})}(\mathcal{M}\pi)$.

(2 $\Longleftrightarrow$ 3)   If $\mathbf{p} \in \mathrm{Im}\,\tilde{\psi}$, and S is the integral submanifold of $\mathcal{D}_\alpha$ passing through $\mathbf{p}$, then

$$(\mathcal{D}_\alpha)_\mathbf{p} = [\mathrm{V}_\mathbf{p}(\bar{\kappa}) \cap (\mathcal{D}_\alpha)_\mathbf{p}] \oplus \mathrm{T}_\mathbf{p}(\mathrm{Im}\,\tilde{\psi})$$

So, for every $X \in \mathfrak{X}_\alpha(\mathcal{M}\pi)$,

$$X_\mathbf{p} = (X_\mathbf{p} - \mathrm{T}_\mathbf{p}(\tilde{\psi}\circ\bar{\kappa})(X_\mathbf{p})) + \mathrm{T}_\mathbf{p}(\tilde{\psi}\circ\bar{\kappa})(X_\mathbf{p}) \equiv X_\mathbf{p}^V + X_\mathbf{p}^{\tilde{\psi}}$$

and therefore

$$\tilde{\psi}^*(i(X)\Omega) = \tilde{\psi}^*(i(X^V)\Omega) + \tilde{\psi}^*(i(X^{\tilde{\psi}})\Omega) = \tilde{\psi}^*(i(X^{\tilde{\psi}})\Omega) = 0$$

since $\tilde{\psi}^*(i(X^V)\Omega) = 0$ by the above item, and furthermore, $X_\mathbf{p}^{\tilde{\psi}} \in \mathrm{T}_\mathbf{p}(\mathrm{Im}\,\tilde{\psi})$, and $\dim(\mathrm{Im}\,\tilde{\psi}) = m$, being $\Omega \in \Omega^{m+1}(\mathcal{M}\pi)$. Hence we conclude that $\tilde{\psi}^*(i(X)\Omega) = 0$, for every $X \in \mathfrak{X}_\alpha(\mathcal{M}\pi)$. The converse is proved reversing this reasoning.

(3 $\Longleftrightarrow$ 4)   The local expression of any $X \in \mathfrak{X}_\alpha(\mathcal{M}\pi)$ is

$$X = \lambda^\nu\frac{\partial}{\partial x^\nu} + \beta^A\frac{\partial}{\partial y^A} + \gamma_A^\nu\frac{\partial}{\partial p_A^\nu} - \left(\alpha^\eta\frac{\partial\tilde{\mathrm{h}}}{\partial x^\eta} + \beta^B\frac{\partial\tilde{\mathrm{h}}}{\partial y^B} + \gamma_B^\eta\frac{\partial\tilde{\mathrm{h}}}{\partial p_B^\eta}\right)\frac{\partial}{\partial p}$$

then, taking into account the local expression (2) of Ω, if $\tilde{\psi} = (x^\nu, y^A(x^\eta), p_A^\nu(x^\eta), p(x^\eta))$, we obtain

$$
\tilde{\psi}^* \, i(X)\Omega = \lambda^\eta \left(\frac{\partial(p \circ \tilde{\psi})}{\partial x^\nu} + \frac{\partial \tilde{h}}{\partial x^\nu}\bigg|_{\tilde{\psi}} - \sum_{\eta \neq \nu} \left(\frac{\partial \tilde{h}}{\partial p_A^\nu}\bigg|_{\tilde{\psi}} \frac{\partial(p_A^\eta \circ \tilde{\psi})}{\partial x^\eta} - \frac{\partial \tilde{h}}{\partial p_A^\eta}\bigg|_{\tilde{\psi}} \frac{\partial(p_A^\eta \circ \tilde{\psi})}{\partial x^\nu} \right) \right) \mathrm{d}^m x
$$

$$
+ \beta^A \left(\frac{\partial(p_A^\nu \circ \tilde{\psi})}{\partial x^\nu} + \frac{\partial \tilde{h}}{\partial y^A}\bigg|_{\tilde{\psi}} \right) \mathrm{d}^m x + \gamma_A^\nu \left(-\frac{\partial(y^A \circ \tilde{\psi})}{\partial x^\nu} + \frac{\partial \tilde{h}}{\partial p_A^\nu}\bigg|_{\tilde{\psi}} \right) \mathrm{d}^m x
$$

and, as this holds for every $X \in \mathfrak{X}_\alpha(\mathcal{M}\pi)$ (i.e., for every $\lambda^\eta, \beta^A, \gamma_A^\nu$), we conclude that $\psi^* \, i(X)\Omega_h = 0$ if, and only if,

$$
\frac{\partial(y^A \circ \tilde{\psi})}{\partial x^\nu} = \frac{\partial \tilde{h}}{\partial p_A^\nu}\bigg|_{\tilde{\psi}} \quad , \quad \frac{\partial(p_A^\nu \circ \tilde{\psi})}{\partial x^\nu} = -\frac{\partial \tilde{h}}{\partial y^A}\bigg|_{\tilde{\psi}} \tag{18}
$$

$$
\frac{\partial(p \circ \tilde{\psi})}{\partial x^\nu} = -\frac{\partial \tilde{h}}{\partial x^\nu}\bigg|_{\tilde{\psi}} + \sum_{\eta \neq \nu} \left(\frac{\partial \tilde{h}}{\partial p_A^\nu}\bigg|_{\tilde{\psi}} \frac{\partial(p_A^\eta \circ \tilde{\psi})}{\partial x^\eta} - \frac{\partial \tilde{h}}{\partial p_A^\eta}\bigg|_{\tilde{\psi}} \frac{\partial(p_A^\eta \circ \tilde{\psi})}{\partial x^\nu} \right) \tag{19}
$$

and using equations (18) in (19) we obtain

$$
\frac{\partial(p \circ \tilde{\psi})}{\partial x^\nu} = -\frac{\partial \tilde{h}}{\partial x^\nu}\bigg|_{\tilde{\psi}} + \frac{\partial \tilde{h}}{\partial p_A^\nu}\bigg|_{\tilde{\psi}} \left(\frac{\partial(p_A^\delta \circ \tilde{\psi})}{\partial x^\delta} - \frac{\partial(p_A^\nu \circ \tilde{\psi})}{\partial x^\nu} \right) - \left(\frac{\partial \tilde{h}}{\partial p_A^\delta}\bigg|_{\tilde{\psi}} \frac{\partial(p_A^\delta \circ \tilde{\psi})}{\partial x^\nu} - \frac{\partial \tilde{h}}{\partial p_A^\nu}\bigg|_{\tilde{\psi}} \frac{\partial(p_A^\nu \circ \tilde{\psi})}{\partial x^\nu} \right)
$$

$$
= -\frac{\partial \tilde{h}}{\partial x^\nu}\bigg|_{\tilde{\psi}} - \frac{\partial \tilde{h}}{\partial y^A}\bigg|_{\tilde{\psi}} \frac{\partial(y^A \circ \tilde{\psi})}{\partial x^\nu} - \frac{\partial \tilde{h}}{\partial p_A^\delta}\bigg|_{\tilde{\psi}} \frac{\partial(p_A^\delta \circ \tilde{\psi})}{\partial x^\nu}
$$

$$
= -\frac{\partial(\tilde{h} \circ \tilde{\psi})}{\partial x^\nu} \qquad (\delta = 1, \ldots, m \; ; \; \nu \text{ fixed})
$$

∎

Remark 10 It is important to point out that the last group of equations (17) are consistency conditions with respect to the hypothesis made on the sections $\tilde{\psi}$. In fact, this group of equations leads to $p \circ \tilde{\psi} = -\tilde{h} \circ \tilde{\psi} + ctn.$ or, what means the same thing, $\tilde{\psi} \in \Gamma_\alpha(M, \mathcal{M}\pi)$. (See also the comment in Remark 9). The rest of equations (17) are just the Hamilton-De Donder-Weyl equations (5) of the restricted case, since the local expressions of the functions $\tilde{h}$ and h are the same.

Theorem 7 *Let $(\mathcal{M}\pi, \Omega, \alpha)$ be an extended Hamiltonian system, and $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M}\pi)$ an integrable multivector field verifying the condition $i(\mathcal{X})(\bar{\kappa}^*\omega) = 1$. Then, the integral manifolds of $\mathcal{X}$ are critical section for the variational problem posed by the extended Hamilton-Jacobi principle if, and only if, $\mathcal{X}$ satisfies the condition $i(\mathcal{X})\Omega = (-1)^{m+1}\alpha$.*

(*Proof*)   ($\Longleftarrow$)   Let S be an integral submanifold of $\mathcal{X}$. By the $\bar{\kappa}$-transversality condition $i(\mathcal{X})(\bar{\kappa}^*\omega) = 1$, S is locally a section of $\bar{\kappa}$. Then, for every $\mathbf{p} \in S$, there are an open set $U \subset M$, with $\bar{\kappa}(\mathbf{p})$, and a local section $\tilde{\psi} \colon U \subset M \to \mathcal{M}\pi$ of $\bar{\kappa}$, such that $\mathrm{Im}\,\tilde{\psi} = S|_{\bar{\kappa}^{-1}(U)}$). Now, let $q \in U$, and $u_1, \ldots, u_m \in \mathrm{T}_q M$, with $i(u_1 \wedge \ldots \wedge u_m)(\omega(\bar{\kappa}(\mathbf{p}))) = 1$. Then, there exists $\lambda \in \mathbb{R}$ such that

$$
\tilde{\psi}_*(u_1 \wedge \ldots \wedge u_m) = \lambda \mathcal{X}_{(\tilde{\psi}(q))}
$$

therefore

$$
i(\tilde{\psi}_*(u_1 \wedge \ldots \wedge u_m))(\Omega(\tilde{\psi}(q))) = \lambda(-1)^{m+1}\alpha(\tilde{\psi}(q))
$$

Thus, for every $X \in \mathfrak{X}_\alpha(\mathcal{M}\pi)$ we obtain that

$$
i(X_{(\tilde{\psi}(q))}) \, i(\tilde{\psi}_*(u_1 \wedge \ldots \wedge u_m)(\Omega(\tilde{\psi}(q))) = 0
$$

hence $\tilde\psi^* \, i(X)\Omega = 0$, for every $X \in \mathfrak{X}_\alpha(\mathcal{M}\pi)$, and $\tilde\psi$ is a critical section by the third item of the last Theorem.

($\Longrightarrow$) Let $\mathbf{p} \in \mathcal{M}\pi$, by the hypothesis there exists a section $\tilde\psi \colon M \to \mathcal{M}\pi$ such that

1. $\tilde\psi(\mathbf{p}) = \bar\kappa(\mathbf{p})$.

2. $\tilde\psi$ is a critical section for the extended Hamilton-Jacobi variational problem, that is, $\tilde\psi^* \, i(X)\Omega = 0$, for every $X \in \mathfrak{X}_\alpha(\mathcal{M}\pi)$.

3. $\operatorname{Im} \tilde\psi$ is an integral submanifold of $\mathcal{X}$.

Now, let $u_1, \ldots, u_m \in \mathrm{T}_{\bar\kappa(\mathbf{p})}M$, with $i(u_1 \wedge \ldots \wedge u_m)\omega(\bar\kappa(\mathbf{p}))) = 1$. Then, there exists $\lambda \in \mathbb{R}$ such that

$$\tilde\psi_*(u_1 \wedge \ldots \wedge u_m) = \lambda \mathcal{X}_\mathbf{p}$$

but the condition imposed to $u_1, \ldots, u_m$ leads to $\lambda = 1$. Therefore

$$i(\tilde\psi_*(u_1 \wedge \ldots \wedge u_m))(\Omega(\mathbf{p})) = i(\mathcal{X}_\mathbf{p})(\Omega(\mathbf{p}))$$

Thus, for every $X \in \mathfrak{X}_\alpha(\mathcal{M}\pi)$, as $\tilde\psi$ is a critical section, we obtain that

$$i(X_\mathbf{p}) \, i(\mathcal{X}_\mathbf{p})(\Omega(\mathbf{p})) = 0$$

and hence $i(X) \, i(\mathcal{X})\Omega = 0$, for every $X \in \mathfrak{X}_\alpha(\mathcal{M}\pi)$. This implies that $i(\mathcal{X})\Omega = f\alpha$, for some non-vanishing $f \in \mathrm{C}^\infty(\mathcal{M}\pi)$. Nevertheless, as $(\mathcal{M}\pi, \Omega, \alpha)$ is an extended Hamiltonian system, in any local chart we have that $\alpha = \mathrm{d}p + \mathrm{d}\tilde{\mathrm{h}}(x^\nu, y^A, p_A^\nu)$, which by the condition $i(\mathcal{X})(\bar\kappa^*\omega) = 1$, and bearing in mind the local expression of Ω, leads to $f = (-1)^{m+1}$. So the result holds. ∎

Observe that the extended Hamilton-De Donder-Weyl equations (17) can also be obtained as a consequence of this last theorem, taking into account equations (13), (14), (15) and (16).

5   Almost-regular Hamiltonian systems

There are many interesting cases in Hamiltonian field theories where the Hamiltonian field equations are established not in $J^1\pi^*$, but rather in a submanifold of $J^1\pi^*$ (for instance, when considering the Hamiltonian formalism associated with a singular Lagrangian). Next we consider this kind of systems in $J^1\pi^*$, as well as in $\mathcal{M}\pi$.

5.1   Restricted almost-regular Hamiltonian systems

Definition 9 *A* restricted almost-regular Hamiltonian system *is a triple* $(J^1\pi^*, \mathcal{P}, h_\mathcal{P})$, *where:*
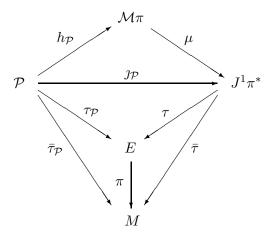
1. $\mathcal{P}$ *is a submanifold with* $\dim \mathcal{P} > n + m$, *and such that, if* $\jmath_\mathcal{P} \colon \mathcal{P} \hookrightarrow J^1\pi^*$ *denotes the natural embedding, the map* $\tau_\mathcal{P} = \tau \circ \jmath_\mathcal{P} \colon \mathcal{P} \to E$ *is a surjective submersion (and hence, so is the map* $\bar\tau_\mathcal{P} = \bar\tau \circ \jmath_\mathcal{P} = \pi \circ \tau_\mathcal{P} \colon \mathcal{P} \to M$ *).*

2. $h_\mathcal{P} \colon \mathcal{P} \to \mathcal{M}\pi$ *satisfies that* $\mu \circ h_\mathcal{P} = \jmath_\mathcal{P}$, *and it is called a* Hamiltonian section *of* μ *on* $\mathcal{P}$.

Then, the differentiable forms

$$\Theta_{h_\mathcal{P}} := h_\mathcal{P}^*\Theta \quad , \quad \Omega_{h_\mathcal{P}} := -\mathrm{d}\Theta_{h_\mathcal{P}} = h_\mathcal{P}^*\Omega$$

are the Hamilton-Cartan m *and* $(m+1)$ *forms on* $\mathcal{P}$ *associated with the Hamiltonian section* $h_\mathcal{P}$.

We have the diagram

Remark 11 Notice that $\Omega_{h_{\mathcal{P}}}$ is, in general, a 1-degenerate form and hence it is premultisymplectic. This is the main difference with the regular case.

Furthermore, if we make the additional assumption that $\mathcal{P} \to E$ is a fiber bundle, the Hamilton-Jacobi variational principle of Definition 2 can be stated in the same way, now using sections of $\bar{\tau}_{\mathcal{P}} \colon \mathcal{P} \to M$, and the form $\Theta_{h_{\mathcal{P}}}$. So we look for sections $\psi_{\mathcal{P}} \in \Gamma(M, \mathcal{P})$ which are stationary with respect to the variations given by $\psi_t = \sigma_t \circ \psi_{\mathcal{P}}$, where $\{\sigma_t\}$ is a local one-parameter group of any compact-supported $\bar{\tau}_{\mathcal{P}}$-vertical vector field $Z_{\mathcal{P}} \in \mathfrak{X}(\mathcal{P})$; i.e., such that

$$\frac{d}{dt}\Big|_{t=0} \int_M \psi_t^* \Theta_{h_{\mathcal{P}}} = 0$$

Then these critical sections will be characterized by the condition (analogous to Theorem 1)

$$\psi_{\mathcal{P}}^* \, i(X_{\mathcal{P}})\Omega_{h_{\mathcal{P}}} = 0 \quad ; \quad \text{for every } X_{\mathcal{P}} \in \mathfrak{X}(\mathcal{P})$$

And, as in the case of restricted Hamiltonian systems (Theorem 2), we have that:

Theorem 8 *The critical sections of the Hamilton-Jacobi principle are the integral sections* $\psi_{\mathcal{P}} \in \Gamma(M, \mathcal{P})$ *of a class of integrable and $\bar{\tau}_{\mathcal{P}}$-transverse multivector fields* $\{\mathcal{X}_{h_{\mathcal{P}}}\} \subset \mathfrak{X}^m(\mathcal{P})$ *satisfying that*

$$i(\mathcal{X}_{h_{\mathcal{P}}})\Omega_{h_{\mathcal{P}}} = 0 \quad , \quad \text{for every } \mathcal{X}_{h_{\mathcal{P}}} \in \{\mathcal{X}_{h_{\mathcal{P}}}\}$$

or equivalently, the integral sections of an integrable multivector field $\mathcal{X}_{h_{\mathcal{P}}} \in \mathfrak{X}^m(\mathcal{P})$ *such that:*

1. $i(\mathcal{X}_{h_{\mathcal{P}}})\Omega_{h_{\mathcal{P}}} = 0$.

2. $i(\mathcal{X}_{h_{\mathcal{P}}})(\bar{\tau}_{\mathcal{P}}^*\omega) = 1$.

A multivector field $\mathcal{X}_{h_{\mathcal{P}}} \in \mathfrak{X}^m(\mathcal{P})$ will be called a *Hamilton-De Donder-Weyl multivector field* for the system $(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})$ if it is $\bar{\tau}_{\mathcal{P}}$-transverse, locally decomposable and verifies the equation $i(\mathcal{X}_{h_{\mathcal{P}}})\Omega_{h_{\mathcal{P}}} = 0$. Then, the associated connection $\nabla_{h_{\mathcal{P}}}$, which is a connection along the submanifold $\mathcal{P}$ (see [30], [31] and [34]), is called a *Hamilton-De Donder-Weyl connection* for $(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})$, and satisfies the equation

$$i(\nabla_{h_{\mathcal{P}}})\Omega_{h_{\mathcal{P}}} = (m-1)\Omega_{h_{\mathcal{P}}}$$

Remark 12 It should be noted that, as $\Omega_{h_{\mathcal{P}}}$ can be 1-degenerate, the existence of the corresponding Hamilton-De Donder-Weyl multivector fields for $(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})$ is in general not assured except perhaps on some submanifold S of $\mathcal{P}$, where the solution is not unique. A geometric algorithm for determining this submanifold S has been developed [32].

5.2 Extended almost-regular Hamiltonian systems

Definition 10 *An* extended almost-regular Hamiltonian system *is a triple* $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$, *such that:*

1. *$\tilde{\mathcal{P}}$ is a submanifold of $\mathcal{M}\pi$ and, if $\jmath_{\tilde{\mathcal{P}}} \colon \tilde{\mathcal{P}} \hookrightarrow \mathcal{M}\pi$ denotes the natural embedding, then:*

    (a) *$\kappa_{\tilde{\mathcal{P}}} = \kappa \circ \jmath_{\tilde{\mathcal{P}}} \colon \tilde{\mathcal{P}} \to E$ is a surjective submersion (and hence, so is the map $\bar{\kappa}_{\tilde{\mathcal{P}}} = \bar{\kappa} \circ \jmath_{\tilde{\mathcal{P}}} = \pi \circ \kappa_{\tilde{\mathcal{P}}} \colon \tilde{\mathcal{P}} \to M$).*

    (b) *$(\mu \circ \jmath_{\tilde{\mathcal{P}}})(\tilde{\mathcal{P}}) \equiv \mathcal{P}$ is a submanifold of $J^1\pi^*$.*

    (c) *$\tilde{\mathcal{P}} = \mathcal{M}\pi|_{\mu(\tilde{\mathcal{P}})}$; that is, for every $\mathbf{p} \in \tilde{\mathcal{P}}$ we have that $\mu^{-1}(\mu(\mathbf{p})) = \mathbf{p} + \Lambda_1^m \mathrm{T}^*_{\kappa(\mathbf{p})} E \subset \tilde{\mathcal{P}}$.*

2. *$\alpha_{\tilde{\mathcal{P}}} \in Z^1(\tilde{\mathcal{P}})$ (it is a closed 1-form in $\tilde{\mathcal{P}}$).*

3. *There exists a locally decomposable multivector field $\mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}} \in \mathfrak{X}^m(\tilde{\mathcal{P}})$ satisfying that*
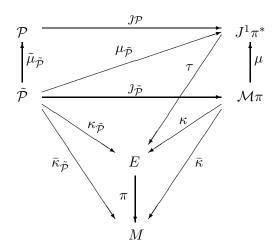
$$i(\mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}})\Omega_{\tilde{\mathcal{P}}} = (-1)^{m+1}\alpha_{\tilde{\mathcal{P}}} \quad , \quad i(\mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}})(\bar{\kappa}_{\tilde{\mathcal{P}}}^*\omega) = 1 \quad (\bar{\kappa}_{\tilde{\mathcal{P}}}\text{-transversality}) \tag{20}$$

where $\Omega_{\tilde{\mathcal{P}}} = \jmath_{\tilde{\mathcal{P}}}^*\Omega$.

If $\alpha_{\tilde{\mathcal{P}}}$ is an exact form, then $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ is an extended almost-regular global Hamiltonian system. *In this case there exist functions $\mathrm{H}_{\tilde{\mathcal{P}}} \in \mathrm{C}^\infty(\tilde{\mathcal{P}})$, which are called* Hamiltonian functions *of the system, such that $\alpha_{\tilde{\mathcal{P}}} = \mathrm{dH}_{\tilde{\mathcal{P}}}$. (For an extended Hamiltonian system, these functions exist only locally, and they are called* local Hamiltonian functions*).*

Remark 13 As straighforward consequences of this definition we have that:

- The condition (1.c) of Definition 10 imply, in particular, that $\dim \tilde{\mathcal{P}} > \dim E + 1$. Furthermore, it means that $\tilde{\mathcal{P}}$ is the union of fibers of μ.

- $\kappa \circ \jmath_{\tilde{\mathcal{P}}}$ is a surjective submersion if, and only if, so is $\tau \circ \mu \circ \jmath_{\tilde{\mathcal{P}}}$. This means that $\mathcal{P} \equiv \mathrm{Im}\,(\mu \circ \jmath_{\tilde{\mathcal{P}}})$ is a submanifold verifying the conditions stated in the first item of definition 9, and such that $\dim \mathcal{P} = \dim \tilde{\mathcal{P}} - 1$, as a consequence of the properties given in item 1 of Definition 10. This submanifold is diffeomorphic to $\tilde{\mathcal{P}}/\Lambda_1^m \mathrm{T}^* E$.

Denoting $\mu_{\tilde{\mathcal{P}}} = \mu \circ \jmath_{\tilde{\mathcal{P}}} \colon \tilde{\mathcal{P}} \to J^1\pi^*$, and $\tilde{\mu}_{\tilde{\mathcal{P}}} \colon \tilde{\mathcal{P}} \to \mathcal{P}$ its restriction to the image (that is, such that $\mu_{\tilde{\mathcal{P}}} = \jmath_{\mathcal{P}} \circ \tilde{\mu}_{\tilde{\mathcal{P}}}$), we have the diagram

Remark 14 In addition, as for extended Hamiltonian systems (see Remarks 5 and 6), the integrability of $\mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}}$ is not assured, so it must be imposed. Then all the multivector fields in the integrable class $\{\mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}}\}$ have the same integral sections.

As in Propositions 1 and 2, we have that:

Proposition 9 *If $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ is an extended almost-regular Hamiltonian system, then $i(Y_{\tilde{\mathcal{P}}})\alpha_{\tilde{\mathcal{P}}} \neq 0$, for every non-vanishing $\mu_{\tilde{\mathcal{P}}}$-vertical vector field $Y_{\tilde{\mathcal{P}}} \in \mathfrak{X}^{\mathrm{V}(\mu_{\tilde{\mathcal{P}}})}(\tilde{\mathcal{P}})$. In particular, for every system of natural coordinates in $\tilde{\mathcal{P}}$ adapted to the bundle $\pi\colon E \to M$ (with $\omega = \mathrm{d}^m x$),*

$$i\left(\frac{\partial}{\partial p}\right)\alpha_{\tilde{\mathcal{P}}} = 1$$

(*Proof*)   As a consequence of the condition (1.c) of Definition 10, we have the local expression

$$\Omega_{\tilde{\mathcal{P}}} = \jmath_{\tilde{\mathcal{P}}}^*\Omega = \jmath_{\tilde{\mathcal{P}}}^*(-\mathrm{d}p_A^\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - \mathrm{d}p \wedge \mathrm{d}^m x) = \jmath_{\tilde{\mathcal{P}}}^*(-\mathrm{d}p_A^\nu) \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - \mathrm{d}p \wedge \mathrm{d}^m x$$

and $Y_{\tilde{\mathcal{P}}} = f\dfrac{\partial}{\partial p}$, for every μ-vertical vector field in $\tilde{\mathcal{P}}$. Therefore, the proof follows the same pattern as in the proof of Proposition 1.

The last part of the proof is a consequence of the condition (1.c) given in Definition 10, from which we have that every system of natural coordinates in $\tilde{\mathcal{P}}$ adapted to the bundle $\pi\colon E \to M$ contains the coordinate p of the fibers of μ, and the coordinates (x^ν) in E. This happens because $\tilde{\mathcal{P}}$ reduces only degrees of freedom in the coordinates p_A^ν of $\mathcal{M}\pi$. ∎

Proposition 10 *If $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ is an extended Hamiltonian system, locally $\alpha_{\tilde{\mathcal{P}}} = \mathrm{d}p + \beta_{\tilde{\mathcal{P}}}$, where $\beta_{\tilde{\mathcal{P}}}$ is a closed and $\tilde{\mu}_{\tilde{\mathcal{P}}}$-basic local 1-form in $\tilde{\mathcal{P}}$.*

A multivector field $\mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}} \in \mathfrak{X}^m(\tilde{\mathcal{P}})$ will be called an *extended Hamilton-De Donder-Weyl multivector field* for the system $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ if it is $\bar{\kappa}_{\tilde{\mathcal{P}}}$-transverse, locally decomposable and verifies the equation $i(\mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}})\Omega_{\tilde{\mathcal{P}}} = (-1)^{m+1}\alpha_{\tilde{\mathcal{P}}}$. Then, the associated connection $\nabla_{\alpha_{\tilde{\mathcal{P}}}}$ is called a *Hamilton-De Donder-Weyl connection* for $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$,

Remark 15 Notice that $\Omega_{\tilde{\mathcal{P}}}$ is usually a 1-degenerate form and hence premultisymplectic.

As a consequence, the existence of extended Hamilton-De Donder-Weyl multivector fields for $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ is not assured, except perhaps on some submanifold $\tilde{S}$ of $\mathcal{P}$, where the solution is not unique.

5.3   Geometric properties of extended almost-regular Hamiltonian systems. Variational principle

Let $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ be an extended almost-regular Hamiltonian system, and the submanifold $\mu_{\tilde{\mathcal{P}}}(\tilde{\mathcal{P}}) \equiv \mathcal{P}$. As for the general case, we can define the *characteristic distribution* $\mathcal{D}_{\alpha_{\tilde{\mathcal{P}}}}$ of $\alpha_{\tilde{\mathcal{P}}}$. Then, following the same pattern as in the proofs of the propositions and theorems given in Section 4.2 we can prove that:

Proposition 11   *1. $\mathcal{D}_{\alpha_{\tilde{\mathcal{P}}}}$ is an involutive and $\mu_{\tilde{\mathcal{P}}}$-transverse distribution of corank equal to 1.*

2. *The integral submanifolds of $\mathcal{D}_{\alpha_{\tilde{\mathcal{P}}}}$ are $\mu_{\tilde{\mathcal{P}}}$-transverse submanifolds of $\mathcal{M}\pi$, with dimension equal to $\dim \tilde{\mathcal{P}} - 1$. (We denote by $\tilde{\jmath}_S \colon S \hookrightarrow \tilde{\mathcal{P}}$ the natural embedding).*

3. *For every $\mathbf{p} \in \tilde{\mathcal{P}}$, we have that $T_{\mathbf{p}}\tilde{\mathcal{P}} = V_{\mathbf{p}}(\mu_{\tilde{\mathcal{P}}}) \oplus (\mathcal{D}_{\alpha_{\tilde{\mathcal{P}}}})_{\mathbf{p}}$, and thus, in this way, $\alpha_{\tilde{\mathcal{P}}}$ defines a connection in the bundle $\mu_{\tilde{\mathcal{P}}} \colon \tilde{\mathcal{P}} \to \mathcal{P}$.*

4. *If S is an integral submanifold of $\mathcal{D}_{\alpha_{\tilde{\mathcal{P}}}}$, then $\tilde{\mu}_{\tilde{\mathcal{P}}}|_S \colon S \to \mathcal{P}$ is a local diffeomorphism.*

5. *For every integral submanifold S of $\mathcal{D}_{\alpha_{\tilde{\mathcal{P}}}}$, and $\mathbf{p} \in S$, there exists $W \subset \tilde{\mathcal{P}}$, with $\mathbf{p} \in W$, such that $h = (\mu|_{W \cap S})^{-1}$ is a local Hamiltonian section of $\tilde{\mu}_{\tilde{\mathcal{P}}}$ defined on $\tilde{\mu}_{\tilde{\mathcal{P}}}(W \cap S)$.*

If $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ is an extended almost-regular Hamiltonian system, as $\alpha_{\tilde{\mathcal{P}}} = \mathrm{dH}_{\tilde{\mathcal{P}}}$ (locally), every local Hamiltonian function $\mathrm{H}_{\tilde{\mathcal{P}}}$ is a constraint defining the local integral submanifolds of $\mathcal{D}_{\tilde{\mathcal{P}}}$. If $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ is an extended almost-regular global Hamiltonian system, the Hamiltonian functions $\mathrm{H}_{\tilde{\mathcal{P}}}$ are globally defined, and we have:

Proposition 12 *Let $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ be an extended almost-regular global Hamiltonian system. If there is a global Hamiltonian function $\mathrm{H}_{\tilde{\mathcal{P}}} \in C^{\infty}(\tilde{\mathcal{P}})$, and $k \in \mathbb{R}$, such that $\mu(\mathrm{H}_{\tilde{\mathcal{P}}}^{-1}(k)) = \mathcal{P}$, then there exists a global Hamiltonian section $h_{\tilde{\mathcal{P}}} \in \Gamma(\mathcal{P}, \tilde{\mathcal{P}})$.*

Proposition 13 *Given an extended almost-regular Hamiltonian system $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$, every extended HDW multivector field $\mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}} \in \mathfrak{X}^m(\tilde{\mathcal{P}})$ for the system $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ is tangent to every integral submanifold of $\mathcal{D}_{\alpha_{\tilde{\mathcal{P}}}}$.*

At this point, the extended Hamilton-Jacobi variational principle of Definition 8 can be stated in the same way, now using sections $\psi_{\tilde{\mathcal{P}}}$ of $\bar{\kappa}_{\tilde{\mathcal{P}}} \colon \tilde{\mathcal{P}} \to M$, satisfying that $\jmath_{\psi_{\tilde{\mathcal{P}}}}^* \alpha_{\tilde{\mathcal{P}}} = 0$ (where $\jmath_{\psi_{\tilde{\mathcal{P}}}} \colon \operatorname{Im}\psi_{\tilde{\mathcal{P}}} \hookrightarrow \tilde{\mathcal{P}}$ denotes the natural embedding). Thus, using the notation introduced in section 4.4, we look for sections $\psi_{\tilde{\mathcal{P}}} \in \Gamma_{\alpha_{\tilde{\mathcal{P}}}}(M, \tilde{\mathcal{P}})$ which are stationary with respect to the variations given by $\tilde{\psi}_t = \sigma_t \circ \psi_{\tilde{\mathcal{P}}}$, where $\{\sigma_t\}$ is a local one-parameter group of every compact-supported $Z \in \mathfrak{X}_{\alpha_{\tilde{\mathcal{P}}}}^{V(\bar{\kappa}_{\tilde{\mathcal{P}}})}(\tilde{\mathcal{P}})$; that is

$$\frac{d}{dt}\Big|_{t=0} \int_U \tilde{\psi}_t^* \Theta_{\tilde{\mathcal{P}}} = 0$$

And then the statements analogous to Theorems 6 and 7 can be established and proven in the present case.

5.4   Relation between extended and restricted almost-regular Hamiltonian systems

Finally, we study the relation between extended and restricted almost-regular Hamiltonian systems. (The proofs of the following propositions and theorems are analogous to those in Section 4.3).

First, bearing in mind Remark 13, we have:

Theorem 9 *Let $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ be an extended global Hamiltonian system, and $(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})$ a restricted Hamiltonian system such that $\dim \tilde{\mathcal{P}} = \dim \mathcal{P} + 1$, and $\operatorname{Im} h_{\mathcal{P}} = S$ is an integral submanifold of $\mathcal{D}_{\alpha_{\tilde{\mathcal{P}}}}$. Then, for every $\mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}} \in \mathfrak{X}^m(\tilde{\mathcal{P}})$ solution to the equations:*

$$i(\mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}})\Omega_{\tilde{\mathcal{P}}} = (-1)^{m+1}\alpha_{\tilde{\mathcal{P}}} \quad , \quad i(\mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}})(\bar{\kappa}_{\tilde{\mathcal{P}}}^* \omega) = 1$$

(i.e., an extended HDW multivector field for $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$) there exists $\mathcal{X}_{h_{\mathcal{P}}} \in \mathfrak{X}^m(\mathcal{P})$ which is $h_{\mathcal{P}}$-related with $\mathcal{X}_\alpha$ and is a solution to the equations

$$i(\mathcal{X}_{h_{\mathcal{P}}})\Omega_{h_{\mathcal{P}}} = 0 \quad , \quad i(\mathcal{X}_{h_{\mathcal{P}}})(\bar{\tau}_{\mathcal{P}}^*\omega) = 1$$

(i.e., a HDW multivector field for $(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})$). Furthermore, if $\mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}}$ is integrable, then $\mathcal{X}_{h_{\mathcal{P}}}$ is integrable too.

As a consequence, the following definition can be established:

Definition 11 *Given an extended almost-regular global Hamiltonian system* $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$, *and considering all the Hamiltonian sections* $h_{\mathcal{P}} \colon \mathcal{P} \to \mathcal{M}\pi$ *such that* $\mathrm{Im}\, h_{\mathcal{P}}$ *are integral submanifolds of* $\mathcal{D}_{\alpha_{\mathcal{P}}}$, *we have a family* $\{(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})\}_{\alpha_{\tilde{\mathcal{P}}}}$, *which will be called the* class of restricted almost-regular *Hamiltonian systems associated with* $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$.
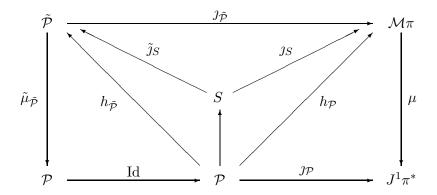
Remark 16 Observe that, for every Hamiltonian section $h_{\mathcal{P}}$ in this class, $\mathrm{Im}\, h_{\mathcal{P}}$ is a submanifold of $\tilde{\mathcal{P}}$. Therefore we have induced a Hamiltonian section $h_{\tilde{\mathcal{P}}} \colon \mathcal{P} \to \tilde{\mathcal{P}}$ of $\tilde{\mu}_{\tilde{\mathcal{P}}}$ such that $h_{\mathcal{P}} = \jmath_{\tilde{\mathcal{P}}} \circ h_{\tilde{\mathcal{P}}}$.

Proposition 14 *Let* $(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})$ *be a restricted almost-regular Hamiltonian system.*

1. *There exits a unique submanifold* $\tilde{\mathcal{P}}$ *of* $\mathcal{M}\pi$ *satisfying the conditions of Definition 10, and such that* $\mu_{\tilde{\mathcal{P}}}(\tilde{\mathcal{P}}) = \mathcal{P}$.

2. *Consider the submanifold* $S = \mathrm{Im}\, h_{\mathcal{P}}$ *and the natural embedding* $\tilde{\jmath}_S \colon S = \mathrm{Im}\, h_{\mathcal{P}} \hookrightarrow \tilde{\mathcal{P}}$. *Then, there exists a unique local form* $\alpha_{\tilde{\mathcal{P}}} \in \Omega^1(\tilde{\mathcal{P}})$ *such that:*

   (a) $\alpha_{\tilde{\mathcal{P}}} \in Z^1(\tilde{\mathcal{P}})$ *(it is a closed form).*
   (b) $\tilde{\jmath}_S^* \alpha_{\tilde{\mathcal{P}}} = 0$.
   (c) $i(Y_{\tilde{\mathcal{P}}})\alpha_{\tilde{\mathcal{P}}} \neq 0$, *for every non-vanishing* $Y_{\tilde{\mathcal{P}}} \in \mathfrak{X}^{\mathrm{V}(\mu_{\tilde{\mathcal{P}}})}(\tilde{\mathcal{P}})$ *and, in particular, such that* $i\left(\dfrac{\partial}{\partial p}\right)\alpha_{\tilde{\mathcal{P}}} = 1$, *for every system of natural coordinates in* $\tilde{\mathcal{P}}$, *adapted to the bundle* $\pi \colon E \to M$ *(with* $\omega = \mathrm{d}^m x$*).*

(*Proof*)   The existence and uniqueness of the submanifold $\tilde{\mathcal{P}}$ is assured, since it is made of all the fibers of μ at every point of $\mathcal{P}$. The rest of the proof is like in Proposition 7.   ∎

So we have the diagram

Bearing in mind Remark 16, we can also state:

Corollary 2 *Let $(\tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ be the couple associated with a given restricted almost-regular Hamiltonian system $(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})$ by the Proposition 14. Consider the characteristic distribution $\mathcal{D}_{\alpha_{\tilde{\mathcal{P}}}}$, and let $\{h\}_{\alpha_{\tilde{\mathcal{P}}}}$ be the family of local sections of $\tilde{\mu}_{\tilde{\mathcal{P}}}$ such that $\operatorname{Im} h \equiv S$ are local integral submanifolds of $\mathcal{D}_{\alpha_{\tilde{\mathcal{P}}}}$. Then, for every $\tilde{h}'_{\mathcal{P}} \in \{h\}_{\alpha_{\tilde{\mathcal{P}}}}$, we have that $\operatorname{Im} \tilde{h}'_{\mathcal{P}}$ is locally a level set of a function $\mathrm{H}_{\tilde{\mathcal{P}}}$ such that $\alpha_{\tilde{\mathcal{P}}}|_S = (\mathrm{dH}_{\tilde{\mathcal{P}}})|_S$, locally.*

Definition 12 *Given a restricted almost-regular Hamiltonian system $(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})$, let $(\tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ be the couple associated with $(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})$ by the Proposition 14. The triple $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ will be called the (local) extended almost-regular Hamiltonian system associated with $(J^1\pi^*, h)$.*

Proposition 15 *Let $\{(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})\}$ be the class of restricted almost-regular Hamiltonian systems associated with an extended almost-regular Hamiltonian system $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$. Consider the submanifolds $\{S_{h_{\mathcal{P}}} = \operatorname{Im} h_{\mathcal{P}}\}$, for every Hamiltonian section h in this class, and let $\jmath_{S_{h_{\mathcal{P}}}} : S_{h_{\mathcal{P}}} \hookrightarrow \tilde{\mathcal{P}}$ be the natural embeddings. Then the submanifolds $\{S = \operatorname{Im} h_{\mathcal{P}}\}$, for every Hamiltonian section $h_{\mathcal{P}}$ in this class are premultisymplectomorphic.*

As a consequence of this, if $\mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}} \in \mathfrak{X}^m(\tilde{\mathcal{P}})$ is a solution to the equations (20), the multivector fields $\mathcal{X}_{S_{h_{\mathcal{P}}}} \in \mathfrak{X}^m(S_{h_{\mathcal{P}}})$ such that $\Lambda^m(\jmath_{S_{h_{\mathcal{P}}}})_* \mathcal{X}_{S_{h_{\mathcal{P}}}} = \mathcal{X}_{\alpha_{\tilde{\mathcal{P}}}}|_{S_{h_{\mathcal{P}}}}$, for every submanifold $S_{h_{\mathcal{P}}}$ of this family, are related by these presymplectomorphisms.

6 Examples

6.1 Restricted Hamiltonian system associated with a Lagrangian system

A particular but relevant case concerns (first-order) Lagrangian field theories and their Hamiltonian counterparts.

In field theory, a *Lagrangian system* is a couple $(J^1\pi, \Omega_{\mathcal{L}})$, where $J^1\pi$ is the first-order jet bundle of $\pi\colon E \to M$, and $\Omega_{\mathcal{L}} \in \Omega^{m+1}(J^1\pi)$ is the *Poincaré-Cartan $(m+1)$-form* associated with the Lagrangian density $\mathcal{L}$ describing the system ($\mathcal{L}$ is a $\bar{\pi}^1$-semibasic m-form on $J^1\pi$, which is usually written as $\mathcal{L} = \mathcal{L}\bar{\pi}^{1*}\eta \equiv \mathcal{L}\omega$, where $\mathcal{L} \in \mathrm{C}^\infty(J^1\pi)$ is the *Lagrangian function* associated with $\mathcal{L}$ and ω). The Lagrangian system is *regular* if $\Omega_{\mathcal{L}}$ is 1-nondegenerate; elsewhere it is called *singular*.

The *extended Legendre map* associated with $\mathcal{L}$, $\widetilde{\mathcal{FL}}\colon J^1\pi \to \mathcal{M}\pi$, is defined by

$$(\widetilde{\mathcal{FL}}(\bar{y}))(Z_1, \ldots, Z_m) := (\Theta_{\mathcal{L}})_{\bar{y}}(\bar{Z}_1, \ldots, \bar{Z}_m)$$

where $Z_1, \ldots, Z_m \in \mathrm{T}_{\pi^1(\bar{y})}E$, and $\bar{Z}_1, \ldots, \bar{Z}_m \in \mathrm{T}_{\bar{y}}J^1\pi$ are such that $\mathrm{T}_{\bar{y}}\pi^1\bar{Z}_{\alpha_{\tilde{\mathcal{P}}}} = Z_{\alpha_{\tilde{\mathcal{P}}}}$. ($\widetilde{\mathcal{FL}}$ can also be defined as the "first order vertical Taylor approximation to $\mathcal{L}$" [6]). We have that $\widetilde{\mathcal{FL}}^*\Omega = \Omega_{\mathcal{L}}$. If $(x^\alpha, y^A, v^A_\alpha)$ is a natural chart of coordinates in $J^1\pi$ (adapted to the bundle structure, and such that $\omega = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m \equiv \mathrm{d}x^m$) the local expressions of these maps are

$$\widetilde{\mathcal{FL}}^*x^\nu = x^\nu \quad , \quad \widetilde{\mathcal{FL}}^*y^A = y^A \quad , \quad \widetilde{\mathcal{FL}}^*p^\nu_A = \frac{\partial \mathcal{L}}{\partial v^A_\nu} \quad , \quad \widetilde{\mathcal{FL}}^*p = \mathcal{L} - v^A_\nu \frac{\partial \mathcal{L}}{\partial v^A_\nu}$$

$$\mathcal{FL}^*x^\nu = x^\nu \quad , \quad \mathcal{FL}^*y^A = y^A \quad , \quad \mathcal{FL}^*p^\nu_A = \frac{\partial \mathcal{L}}{\partial v^A_\nu}$$

Using the natural projection $\mu\colon \mathcal{M}\pi \to J^1\pi^*$, we define the *restricted Legendre map* associated with $\mathcal{L}$ as $\mathcal{FL} := \mu \circ \widetilde{\mathcal{FL}}$.

Then, $(J^1\pi, \Omega_{\mathcal{L}})$ is a *regular* Lagrangian system if $\mathcal{FL}$ is a local diffeomorphism (this definition is equivalent to that given above). Elsewhere $(J^1\pi, \Omega_{\mathcal{L}})$ is a *singular* Lagrangian system. As a particular case, $(J^1\pi, \Omega_{\mathcal{L}})$ is a *hyper-regular* Lagrangian system if $\mathcal{FL}$ is a global diffeomorphism. Finally, a singular Lagrangian system $(J^1\pi, \Omega_{\mathcal{L}})$ is *almost-regular* if: $\mathcal{P} := \mathcal{FL}(J^1\pi)$ is a closed submanifold of $J^1\pi^*$, $\mathcal{FL}$ is a submersion onto its image, and for every $\bar{y} \in J^1\pi$, the fibers $\mathcal{FL}^{-1}(\mathcal{FL}(\bar{y}))$ are connected submanifolds of J^1E.

If $(J^1\pi, \Omega_{\mathcal{L}})$ is a hyper-regular Lagrangian system, then $\widetilde{\mathcal{FL}}(J^1\pi)$ is a 1-codimensional imbedded submanifold of $\mathcal{M}\pi$, which is transverse to the projection μ, and is diffeomorphic to $J^1\pi^*$. This diffeomorphism is μ^{-1}, when μ is restricted to $\widetilde{\mathcal{FL}}(J^1\pi)$, and also coincides with the map $h := \widetilde{\mathcal{FL}} \circ \mathcal{FL}^{-1}$, when it is restricted onto its image (which is just $\widetilde{\mathcal{FL}}(J^1\pi)$). This map h is the Hamiltonian section needed to construct the *restricted Hamiltonian system associated with* $(J^1\pi, \Omega_{\mathcal{L}})$. In other words, the Hamiltonian section h is given by the image of the extended Legendre map.

Using charts of natural coordinates in $J^1\pi^*$ and $\mathcal{M}\pi$, we obtain that the local Hamiltonian function h representing this Hamiltonian section is
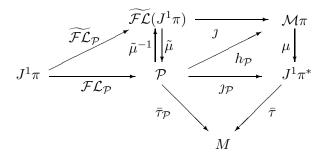
$$\mathrm{h}(x^\nu, y^A, p_A^\nu) = (\mathcal{FL}^{-1})^* \left(v_\nu^A \frac{\partial \pounds}{\partial v_\nu^A} - \pounds \right) = p_A^\nu (\mathcal{FL}^{-1})^* v_\nu^A - \mathcal{FL}^{-1*} \pounds$$

Of course, if $(\mathcal{M}\pi, \Omega, \alpha)$ is any extended Hamiltonian system associated with $(J^1\pi^*, h)$, then $\widetilde{\mathcal{FL}}(J^1\pi)$ is an integral submanifold of the characteristic distribution of α.

In an analogous way, if $(J^1\pi, \Omega_{\mathcal{L}})$ is an almost-regular Lagrangian system, and the submanifold $\jmath_{\tilde{\mathcal{P}}} \colon \mathcal{P} \hookrightarrow J^1\pi^*$ is a fiber bundle over E and M, the μ-transverse submanifold $\jmath \colon \widetilde{\mathcal{FL}}(J^1\pi) \hookrightarrow \mathcal{M}\pi$ is diffeomorphic to $\mathcal{P}$. This diffeomorphism $\tilde{\mu} \colon \widetilde{\mathcal{FL}}(J^1\pi) \to \mathcal{P}$ is just the restriction of the projection μ to $\widetilde{\mathcal{FL}}(J^1\pi)$. Then, taking the Hamiltonian section $h_{\mathcal{P}} := \jmath \circ \tilde{\mu}^{-1}$, we define the Hamilton-Cartan forms

$$\Theta_{h_{\mathcal{P}}} = h_{\mathcal{P}}^* \Theta \qquad \Omega_{h_{\mathcal{P}}} = h_{\mathcal{P}}^* \Omega$$

which verify that $\mathcal{FL}_{\mathcal{P}}^* \Theta_h^0 = \Theta_{\mathcal{L}}$ and $\mathcal{FL}_{\mathcal{P}}^* \Omega_h^0 = \Omega_{\mathcal{L}}$ (where $\mathcal{FL}_{\mathcal{P}}$ is the restriction map of $\mathcal{FL}$ onto $\mathcal{P}$). Once again, this Hamiltonian section $h_{\mathcal{P}}$ is given by the image of the extended Legendre map. Then $(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})$ is the *Hamiltonian system* associated with the almost-regular Lagrangian system $(J^1\pi, \Omega_{\mathcal{L}})$, and we have the following diagram

The construction of the (local) extended almost-regular Hamiltonian system associated with $(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})$ can be made by following the procedure described in section 5.4. Of course, if $(\mathcal{M}\pi, \tilde{\mathcal{P}}, \alpha_{\tilde{\mathcal{P}}})$ is the extended Hamiltonian system associated with $(J^1\pi^*, \mathcal{P}, h_{\mathcal{P}})$, then $\widetilde{\mathcal{FL}}(J^1\pi)$ is an integral submanifold of the characteristic distribution of $\alpha_{\tilde{\mathcal{P}}}$.

6.2   Non-autonomous dynamical systems

Another relevant example consists in showing how the so-called *extended formalism of time-dependent mechanics* (see [17], [29], [36], [41], [45]) can be recovered from this more general frame-

work.

The starting point consists in giving the configuration bundle, which for a large class of non-autonomous dynamical systems can be taken to be $\pi\colon E \equiv Q \times \mathbb{R} \to \mathbb{R}$, where Q is a n-dimensional differentiable manifold endowed with local coordinates (q^i), and $\mathbb{R}$ has as a global coordinate t. The *extended* and *restricted momentum phase spaces* are

$$\mathcal{M}\pi \simeq \mathrm{T}^*E \equiv \mathrm{T}^*(Q \times \mathbb{R}) \simeq \mathrm{T}^*Q \times \mathbb{R} \times \mathbb{R}^* \quad , \quad J^1\pi^* \simeq \mathrm{T}^*Q \times \mathbb{R}$$

Then, the following projections can be defined

$$pr_1\colon \mathcal{M}\pi \to \mathrm{T}^*Q \quad , \quad \mu\colon \mathcal{M}\pi \to \mathrm{T}^*Q \times \mathbb{R}$$
$$pr_2\colon \mathcal{M}\pi \to \mathbb{R} \times \mathbb{R}^* \quad , \quad p\colon \mathcal{M}\pi \to \mathbb{R}^*$$

If $\Omega_Q \in Z^2(\mathrm{T}^*Q)$ and $\Omega_{\mathbb{R}} \in Z^2(\mathbb{R} \times \mathbb{R}^*)$ denote the natural symplectic forms of T^*Q and $\mathbb{R} \times \mathbb{R}^*$, then the natural symplectic structure of $\mathcal{M}\pi$ is just

$$\Omega = pr_1^*\Omega_Q + pr_2^*\Omega_{\mathbb{R}}.$$

Then, we define the so-called *extended time-dependent Hamiltonian function*

$$\mathrm{H} := \mu^*\mathrm{h} + p \in \mathrm{C}^\infty(\mathrm{T}^*(Q \times \mathbb{R}))$$

where the dynamical information is given by the "time-dependent Hamiltonian function" $\mathrm{h} \in \mathrm{C}^\infty(\mathrm{T}^*Q \times \mathbb{R})$.

Now we have that $(\mathrm{T}^*(Q \times \mathbb{R}), \Omega, \alpha)$, with $\alpha = \mathrm{dH}$, is an *extended global Hamiltonian system*, and then the equations of motion are

$$i(X_{\mathrm{H}})\Omega = \mathrm{dH} \quad , \quad i(X_{\mathrm{H}})\mathrm{d}t = 1 \qquad \text{with } X_{\mathrm{H}} \in \mathfrak{X}(\mathrm{T}^*(Q \times \mathbb{R})) \tag{21}$$

In order to analyze the information given by this equation, we take a local chart of coordinates (q^i, p_i, t, p) in $\mathrm{T}^*(Q \times \mathbb{R})$, and one can check that the unique solution to these equations is

$$
\begin{aligned}
X_{\mathrm{H}} &= \frac{\partial \mathrm{H}}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial \mathrm{H}}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial}{\partial t} - \frac{\partial \mathrm{H}}{\partial t}\frac{\partial}{\partial p} \\
&= \frac{\partial(\mu^*\mathrm{h})}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial(\mu^*\mathrm{h})}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial}{\partial t} - \frac{\partial(\mu^*\mathrm{h})}{\partial t}\frac{\partial}{\partial p}
\end{aligned}
\tag{22}
$$

If $\tilde{\psi}(t) = (q^i(t), p_i(t), t, p(t))$ denote the integral curves of this vector field, the last expression leads to the following system of extended Hamiltonian equations

$$\frac{d(q^i \circ \tilde{\psi})}{dt} = \frac{\partial(\mu^*\mathrm{h})}{\partial p_i} \circ \tilde{\psi} \quad , \quad \frac{d(p_i \circ \tilde{\psi})}{dt} = -\frac{\partial(\mu^*\mathrm{h})}{\partial q^i} \circ \tilde{\psi} \quad , \quad \frac{d(p \circ \tilde{\psi})}{dt} = -\frac{\partial(\mu^*\mathrm{h})}{\partial t} \circ \tilde{\psi} \tag{23}$$

Observe that the last equation corresponds to the last group of equations (17) in the general case of field theories. In fact, using the other Hamilton equations we get

$$
\begin{aligned}
\frac{d(p \circ \tilde{\psi})}{dt} &= -\frac{(\mu^*\mathrm{h} \circ \psi)}{dt} = -\frac{\partial(\mu^*\mathrm{h})}{\partial t}\bigg|_{\tilde{\psi}} - \frac{\partial(\mu^*\mathrm{h})}{\partial q^i}\bigg|_{\tilde{\psi}}\frac{(q^i \circ \tilde{\psi})}{t} - \frac{\partial(\mu^*\mathrm{h})}{\partial p_i}\bigg|_{\tilde{\psi}}\frac{(p_i \circ \tilde{\psi})}{t} \\
&= -\frac{\partial(\mu^*\mathrm{h})}{\partial t}\bigg|_{\tilde{\psi}} - \frac{\partial(\mu^*\mathrm{h})}{\partial q^i}\bigg|_{\tilde{\psi}}\frac{\partial(\mu^*\mathrm{h})}{\partial p_i}\bigg|_{\tilde{\psi}} + \frac{\partial(\mu^*\mathrm{h})}{\partial p_i}\bigg|_{\tilde{\psi}}\frac{\partial(\mu^*\mathrm{h})}{\partial q^i}\bigg|_{\tilde{\psi}} = -\frac{\partial(\mu^*\mathrm{h})}{\partial q^i}\bigg|_{\tilde{\psi}}
\end{aligned}
$$

However, as the physical states are the points of $\mathrm{T}^*Q \times \mathbb{R}$ and not those of $\mathrm{T}^*(Q \times \mathbb{R})$, the vector field which gives the real dynamical evolution is not X_{H}, but another one in $\mathrm{T}^*Q \times \mathbb{R}$ which,

as X_{H} is μ-projectable, is just $\mu_* X_{\mathrm{H}} = X_h \in \mathfrak{X}(\mathrm{T}^*Q \times \mathbb{R})$, that is, in local coordinates (q^i, p_i, t) of $\mathrm{T}^*Q \times \mathbb{R}$,

$$X_h = \frac{\partial \mathrm{h}}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial \mathrm{h}}{\partial q^i}\frac{\partial}{\partial p_i} + \frac{\partial}{\partial t} \tag{24}$$

Thus, the integral curves $\psi(t) = (q^i(t), p_i(t), t)$ of X_h are the μ-projection of those of X_H, and they are solutions to the system of Hamilton equations

$$\frac{d(q^i \circ \psi)}{dt} = \frac{\partial h}{\partial p_i} \circ \psi \quad , \quad \frac{d(p_i \circ \psi)}{dt} = -\frac{\partial h}{\partial q^i} \circ \psi$$

This result can also be obtained by considering the class of restricted Hamiltonian systems associated with $(\mathrm{T}^*(Q \times \mathbb{R}), \Omega, \mathrm{dH})$. In fact, $\mathrm{T}^*(Q \times \mathbb{R})$ is foliated by the family of hypersurfaces of $\mathrm{T}^*(Q \times \mathbb{R})$ where the extended Hamiltonian function is constant; that is,

$$S := \{\mathbf{p} \in \mathrm{T}^*(Q \times \mathbb{R}) \mid \mathrm{H}(\mathbf{p}) = r \ (ctn.)\}$$

which are the integral submanifolds of the characteristic distribution of $\alpha = \mathrm{dH}$. Thus, every S is defined in $\mathrm{T}^*(Q \times \mathbb{R})$ by the constraint $\zeta := \mathrm{H} - r$, and the vector field given in (22), which is the solution to (21), is tangent to all of these submanifolds. Then, taking the global Hamiltonian sections

$$h \colon (q^i, p_i, t) \mapsto (q^i, p_i, t, p = r - \mu^* \mathrm{h})$$

we can construct the restricted Hamiltonian systems $(\mathrm{T}^*Q \times \mathbb{R}, h)$ associated with $(\mathrm{T}^*(Q \times \mathbb{R}), \Omega, \mathrm{dH})$. Therefore (24) is the solution to the equations

$$i(X_h)\Omega_h = 0 \quad , \quad i(X_h)\mathrm{d}t = 1 \qquad \text{with } X_{\mathrm{H}} \in \mathfrak{X}(\mathrm{T}^*(Q \times \mathbb{R}))$$

where

$$\Omega_h = h^*\Omega = \omega_Q - \mathrm{dh} \wedge \mathrm{d}t \in \Omega^2(\mathrm{T}^*Q \times \mathbb{R})$$

The dynamics on each one of these restricted Hamiltonian systems is associated to a given constant value of the extended Hamiltonian. Observe also that, on every submanifol S, the global coordinate p is identified with the physical energy by means of the time-dependent Hamiltonian function $\mu^*\mathrm{h}$, and hence the last equation (23) shows the known fact that the energy is not conserved on the dynamical trajectories of time-dependent systems.

In this way, we have also recovered one of the standard Hamiltonian formalisms for time-dependent systems (see [8]).

7   Conclusions and outlook

We have introduced an alternative way of defining Hamiltonian systems in first-order field theory.

The usual way consists in working in the restricted multimomentum bundle $J^1\pi^*$, which is the natural multimomentum phase space for field theories, but $J^1\pi^*$ has no a natural multisymplectic structure. Thus, in order to define restricted Hamiltonian systems we use Hamiltonian sections $h \colon J^1\pi^* \to \mathcal{M}\pi$, which carry the 'physical information' and allow us to pull-back the natural multisymplectic structure of $\mathcal{M}\pi$ to $J^1\pi^*$. In this way we obtain the Hamilton-Cartan form $\Omega_h \in \Omega^{m+1}(J^1\pi^*)$, and then the Hamiltonian field equations can be derived from the Hamilton-Jacobi variational principle. As a consequence, both the geometry and the 'physical information' are coupled in the non-canonical multisymplectic form Ω_h.

The alternative consists in working directly in the extended multimomentum bundle $\mathcal{M}\pi$, which is endowed with a canonical multisymplectic structure $\Omega \in \Omega^{m+1}(\mathcal{M}\pi)$. Then we define extended

Hamiltonian systems as a triple $(\mathcal{M}\pi, \Omega, \alpha)$, where $\alpha \in Z^1(\mathcal{M}\pi)$ is a μ-transverse closed form, and the Hamiltonian equation is $i(\mathcal{X})\Omega = (-1)^{m+1}\alpha$, with $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M}\pi)$. Thus, in these models, the geometry Ω and the 'physical information' α are not coupled, and (geometric) field equations can be expressed in an analogous way to those of mechanical autonomous Hamiltonian systems.

The characteristic distribution $\mathcal{D}_\alpha$ associated with α, being involutive, have 1-codimensional and μ-transverse integrable submanifolds of $\mathcal{M}\pi$, where the sections solution to the field equations are contained. These integrable submanifolds can be locally identified with local sections of the affine bundle $\mu\colon \mathcal{M}\pi \to J^1\pi^*$. Each one of them allows us to define locally a restricted Hamiltonian system, although all those associated with the same form α are, in fact, multisymplectomorphic. The conditions for the existence of global Hamiltonian sections have been also analyzed. Conversely, every restricted Hamiltonian system is associated with an extended Hamiltonian system (at least locally).

In addition, the extended Hamiltonian field equations can be obtained from an *extended Hamilton-Jacobi variational principle*, stated on the set of sections of the bundle $\bar{\kappa}\colon \mathcal{M}\pi \to M$, which are integral sections of the characteristic distribution of α, taking the variations given by the set of the $\bar{\kappa}$-vertical vector fields incident to α. In fact, a part of the local system of differential equations for the critical sections of an extended Hamiltonian system is the same as for the associated restricted Hamiltonian system. Nevertheless, there is another part of the whole system of differential equations which leads to the condition that the critical sections must also be integral submanifolds of the characteristic distribution $\mathcal{D}_\alpha$.

Restricted and extended Hamiltonian systems for submanifolds of $J^1\pi^*$ and $\mathcal{M}\pi$ (satisfying suitable conditions) have been defined in order to include the almost-regular field theories in this picture. Their properties are analogous to the former case.

The extended Hamiltonian formalism has already been used for defining Poisson brackets in field theories [14]. It could provide new insights into some classical problems, such as: reduction of multisymplectic Hamiltonian systems with symmetry, integrability, and quantization of multisymplectic Hamiltonian field theories.

Acknowledgments

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