Controller design for point-to-point trajectory generation through feedback linearization of nonlinear control systems

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In this paper some of the most common methods of nonlinear control are explored, firstly introducing the theorems and mathematical background upon which they are built, and then exploring their results. Using a solved example to guide us throughout the process and finally simulating the outcome of applying our solution to the system.

I. INTRODUCTION

Nonlinear control has special importance in modern technology. Nowadays, advanced technology demands sophisticated control laws to meet the desired specifications. Real-world systems are inherently nonlinear. Therefore, this highlights nonlinear control as an important topic in modern engineering [1]. Back in the 18th century, the first nonlinear controller was designed, capable of regulating the admission of steam into the cylinder. Although, there is no analytical work [2]. Following up the 19th century the works of Poincare [3]and Lyapunov (1892) [4] are the starters of the development of the nonlinear theory that we know today.

In this paper, we will apply nonlinear control to the rolling disk motion problem. Our main objective is through a diffeomorphism and a feedback law obtain the Brunovsky form of the system. The Brunovsky form allows us to implement a control law that can be easily modified. For the cases where it is impossible to apply a static feedback linearization, we will use prolongations, i.e add new variables to our system to obtain an overall linearizable system.

The paper will make a summary of the mathematics background that we will use. Then we will apply feedback linearization to our model (rolling disk) and finally, we will test our control law by several simulations, and results will be discussed.

II. MATHEMATICAL BACKGROUND

In this section we will provide a basic explanation of the mathematical tools we will use:

Definition 1. A control system $\dot{x} = f(x,u)$ is said to be controllable if for any two points x_o and x_f in m (The manifold where the system is defined), there exists a finite time T and a control function $u(t)$ defined on the interval [0,T] such that the solution of the system satisfies $x(0) = x_o$and $x(T) = x_f$.

Definition 2. Lie derivative of a function. Given a real valued function λ: U ⊂ $\mathbb{R}^n \mapsto \mathbb{R}^n$, and a vector field f: U ⊂ $\mathbb{R} \mapsto \mathbb{R}^n$, the derivative of λ along the vector field f, which is called the Lie derivative λ along f, is

$$L_f\lambda(x) =< d\lambda(x), f(x) >= \sum_{i=1}^{n} \frac{\partial \lambda}{\partial x_i} f_i(x)$$

Definition 3. Given two vector fields $f(x)$ and g(x), f, g: U ⊂ $\mathbb{R}^n \mapsto \mathbb{R}^n$, the lie bracket between them is another vector field defined as:

$$ad_f g = [f,g](x) = \frac{\partial g(x)}{\partial x} f(x) - \frac{\partial f(x)}{\partial x} g(x)$$

Definition 4. Given a set of k smooth vector fields, and $x \in \mathbb{R}^n$, the subspace of $\mathbb{R}^n$ defined by $\Delta(x) =< f_1(x), \ldots, f_k(x) >$ is called a smooth distribution. A distribution is said to be involutive if $[f,g] \in \Delta(x)$ and $f, g \in \Delta(x)$.

Static linearization feedback

In order to linearize a nonlinear control system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i$$

The following steps must be followed:

1. Construct de distributions $D_1, \ldots D_k$ with D_l:

$$D_l =< g_1, \ldots, g_m, ad_f g_1, \ldots, ad_f g_m, , ad_f^l g_1, \ldots, ad_f^l g_m >$$

Until the dimension of D_k equals n for some k

2. Check the involutivity of all the above distributions.

3. Define the indices:
$r_0 = d_0$, $r_1 = d_1 - d_0$, $r_i = d_i - d_{i-1}$, $\forall$ 1≤ i≤ k

Where d_i is the dimension of D_i. Define also $k_1 = \#\{r_i | r_i \geq 1\}$; ... ; $k_m = \#\{r_i | r_i \geq m\}$

4. Find a function $h_1(x)$ such that $dh_1(x) \perp D_{k_1-2}$, a function $h_2(x)$ such that $dh_2(x) \perp D_{k_2-2}$ and is differentially independent from $h_1(x)$, and go on until you find $\{h_1(x), \dots, h_i(x), \dots h_m(x)\}$ functions such that $dh_i(x) \perp D_{k_i-2}$ and are differentially independent from $h_1(x), \dots, h_{i-1}(x)$.

5. The change of variable is given by

$$z = \begin{pmatrix} h_1(x) \\ L_f h_1(x) \\ \vdots \\ L_f^{k_1-1} h_1(x) \\ \vdots \\ h_m(x) \\ L_f h_m(x) \\ \vdots \\ L_f^{k_m-1} h_m(x) \end{pmatrix}$$

And the feedback law is provided by

$$v = \begin{pmatrix} L_f^{k_1} h_1(x) \\ \vdots \\ L_f^{k_m} h_m(x) \end{pmatrix}$$

This linearization provides us with a change of variable that transforms the system into a Brunovsky canonical form:

$$\dot{z} = \hat{A} z + \sum_{i=1}^{m} \hat{g}_i(x) v_i$$

$$\hat{A} = \begin{pmatrix}
0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & & & & \vdots & \vdots & & & & \vdots & & \vdots & & & & \vdots \\
0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
* & * & * & \dots & * & * & * & * & \dots & * & \dots & * & * & * & \dots & * \\
0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & & & & \vdots & \vdots & & & & \vdots & & \vdots & & & & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\
* & * & * & \dots & * & * & * & * & \dots & * & \dots & * & * & * & \dots & * \\
\vdots & & & & \vdots & \vdots & & & & \vdots & & \vdots & & & & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\
\vdots & & & & \vdots & \vdots & & & & \vdots & & \vdots & & & & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\
* & * & * & \dots & * & * & * & * & \dots & * & \dots & * & * & * & \dots & *
\end{pmatrix}$$

The Brunovsky form can be solved employing a polynomial interpolator or by other ways.

Dynamic linearization feedback

When the system is not linearizable by static feedback (for example driftless, $f(x) = 0$, systems), it may be still possible to linearize using a prolongation. A prolongation of a system is given by:

$$\left\{ \begin{array}{l} \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \\ \dot{z}_1^i = \dot{z}_2^i \\ \vdots \\ \dot{z}_{k_i-1}^i = \dot{z}_{k_i}^i \\ \dot{z}_{k_i-1}^i = v_i \end{array} \right\}$$

Theorem 1: The two input driftless linearization system

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

Is linearizable by prolongations if and only if

1) $ad_{(ad^{k-1}{}_{g_2} g_1)} g_2 \in \Delta_k = < g_1, ad_{g_2} g_1, \dots, ad_{g_2}^k g_1 >$ $\forall k \in \{1, \dots, n-2\}$
2) $dim < g_1, ad_{g_2} g_1, \dots, ad_{g_2}^{n-2} g_1, g_2 >= n$

or

1) $ad_{(ad^{k-1}{}_{g_1} g_2)} g_1 \in \Delta_k = < g_2, ad_{g_1} g_2, \dots, ad_{g_1}^k g_2 >$ $\forall k \in \{1, \dots, n-2\}$
2) $dim < g_2, ad_{g_1} g_2, \dots, ad_{g_1}^{n-2} g_2, g_1 >= n$

Although there is a finite algorithm to determine if a system is linearizable by dynamic feedback and a condition to the particular case of two input driftless linearization system. There is no necessary and sufficient condition for dynamic linearization.

III. TRAJECTORY PLANNING AND CONTROL OF A ROLLING DISK

Our system is based on a thin flat disk rolling on a plane without slipping. As the system variables we have both "x" and "y" that characterize the position of the disk in the plane, the angle θ sets the direction to which the disk is pointing (also called azimuthal angle, measured from x) and the rolling angle π, that serves as an indicator of how many rolls the disk has performed (measured from the normal axis of the plane). Given those variables, the system reads the following way [5]**:**

Where:

$$\begin{cases} \dot{x}_1 = \rho\cos(x_3)u_1 \\ \dot{x}_2 = \rho\sin(x_3)u_1 \\ \dot{x}_3 = u_2 \\ \dot{x}_4 = u_1 \end{cases}$$

A first approach applying the linearization algorithm gives us a distribution that is not involutive. So the system is not linearizable by static feedback at first. As there are a couple of methods to linearize by dynamic feedback, we consider prolongations only. Apart from the general boundary on the number of prolongations needed to linearize a system, we can use a more precise theorem for our case that states whether driftless system with two inputs is linearizable by prolongations or not, previously introduced:

$$\Delta_0 =< \begin{pmatrix} \rho\cos(x_3) \\ \rho\sin(x_3) \\ 0 \\ 1 \end{pmatrix} > \qquad \Delta_1 =< \begin{pmatrix} \rho\cos(x_3) \\ \rho\sin(x_3) \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\rho\sin(x_3) \\ \rho\cos(x_3) \\ 0 \\ 0 \end{pmatrix} >$$

$$\Delta_2 =< \begin{pmatrix} \cos(x_3) \\ \sin(x_3) \\ 0 \\ 1/\rho \end{pmatrix}, \begin{pmatrix} -\sin(x_3) \\ \cos(x_3) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \cos(x_3) \\ \sin(x_3) \\ 0 \\ 0 \end{pmatrix} >$$

$$\Delta_3 =< \begin{pmatrix} \cos(x_3) \\ \sin(x_3) \\ 0 \\ 1/\rho \end{pmatrix}, \begin{pmatrix} -\sin(x_3) \\ \cos(x_3) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \cos(x_3) \\ \sin(x_3) \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} >$$

Then we know that a prolongation can make our system linear, but it is unknown which of the two variables need to be chosen and how many prolongations must be performed. As a first guess, choosing to prolongate u_1 ends in a non-involutive distribution, and it can be seen that no matter how many we perform on that same variable, it has the same outcome. Therefore, we try with u_2:

$$\begin{cases} \dot{x}_1 = \rho\cos(x_3)v_1 \\ \dot{x}_2 = \rho\sin(x_3)v_2 \\ \dot{x}_3 = x_5 \\ \dot{x}_4 = v_1 \\ \dot{x}_5 = x_6 \\ \dot{x}_6 = v_2 \end{cases}$$

The resulting system, after the addition of new variables, is linearizable by static feedback**,** so it is only a matter of calculus to find the change of variables that allows the Brunovsky form (for multi-input systems). The distributions are:

$$D_0 =< \begin{pmatrix} \rho\cos(x_3) \\ \rho\sin(x_3) \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} > \qquad D_1 =< \begin{pmatrix} \rho\cos(x_3) \\ \rho\sin(x_3) \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \sin(x_3) \\ -\cos(x_3) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} >$$

$$D_2 =< \begin{pmatrix} \cos(x_3) \\ \sin(x_3) \\ 0 \\ 1/\rho \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \sin(x_3) \\ -\cos(x_3) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \cos(x_3) \\ \sin(x_3) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, >$$

$$\begin{cases} r_0 = 2 \\ r_1 = 2 \\ r_2 = 2 \end{cases} \rightarrow \begin{cases} dh_1 \perp D_1 \\ dh_2 \perp D_1 \end{cases} \qquad dh \perp D_1 \rightarrow \begin{cases} \dfrac{\delta h}{\delta x_5} = 0 \\ \dfrac{\delta h}{\delta x_6} = 0 \\ \dfrac{\delta h}{\delta x_1}\cos(x_3) + \dfrac{\delta h}{\delta x_2}\sin(x_3) + \dfrac{1}{\rho}\dfrac{\delta h}{\delta x_4} = 0 \\ \dfrac{\delta h}{\delta x_1}\sin(x_3) + \dfrac{\delta h}{\delta x_2}\cos(x_3) = 0 \end{cases}$$

A solution to these equations can be:

$$\begin{cases} h_1(x) = x_3 \\ \dot{h}_1(x) = x_5 \\ \ddot{h}_1(x) = x_6 \\ h_2(x) = \cos(x_3)x_1 + \sin(x_3)x_2 - \rho x_4 \\ \dot{h}_2(x) = -\sin(x_3)x_1x_5 + x_5\cos(x_3)x_2 \\ \ddot{h}_2(x) = -\sin(x_3)[x_1x_6 + x_2x_5^2] + \cos(x_3)[x_2x_6 - x_1x_5^2] \end{cases}$$

and then, calculating the third derivative of both functions we compute the feedback law. It is then time to undo all the change of variables and perform a simulation on the dynamically extended system. Notice how the hyper-plane $x_5 = 0$ must be avoided as it leads to a singularity in the change of variables, seen before in the distribution.

$$\alpha = \begin{Bmatrix} \dddot{p}_1(t) \\ \dddot{p}_2(t) \end{Bmatrix}$$

$$\downarrow$$

$$v = \begin{Bmatrix} \frac{-1}{\rho x_5^2}(\dddot{p}_2(t) + \cos(x_3)x_5[3x_1x_6 - x_1x_5^2] + \sin(x_3)x_5[3x_2x_6 - x_1x_5^2] + \frac{-1}{\rho x_5^2}\dddot{p}_1(t)[x_1\sin(x_3) - x_2\cos(x_3)] \\ \dddot{p}_1(t) \end{Bmatrix}$$

IV. SIMULATION

Once we try random initial and final conditions and the system reaches the desired state in the desired time, we can conclude that our previous calculations are correct. As many conditions can be tried, always keeping in mind not to cross the $x_5 = 0$ condition. We have chosen a peculiar case of how by modifying the initial conditions, the system changes its trajectory to compensate said modification (different values of x_6). It can also be seen by the arrows that in some instants of time, the disk rotates in reverse. As the interpolating polynomial is not the only solution, we have tried a variation to prove that

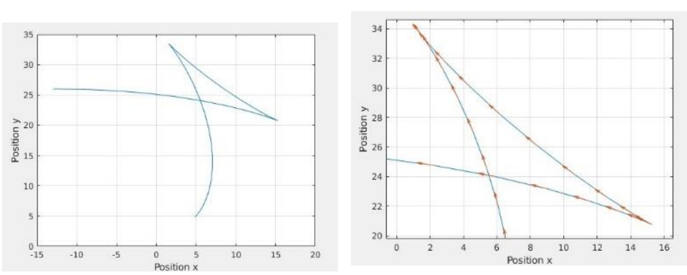

FIG. 1: x_0: $[5,5,\pi/4,0,3,15]$ to x_f: $[-13,26,\pi,40,3.-0.0042]$
First simulation of the disk trajectory on the plane

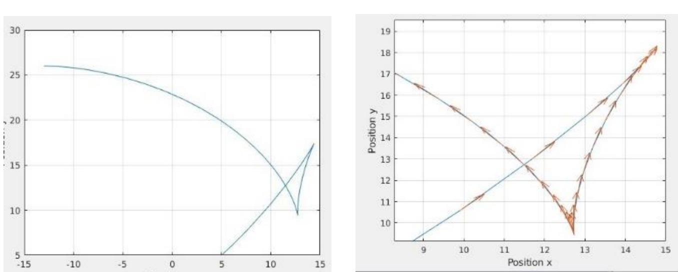

FIG. 2: x_0: $[5,5,\pi 4,0,3,-10]$ to x_f: $[-13,26,\pi,40,3.-0.0042]$
Second simulation of the disk trajectory on the XY plane

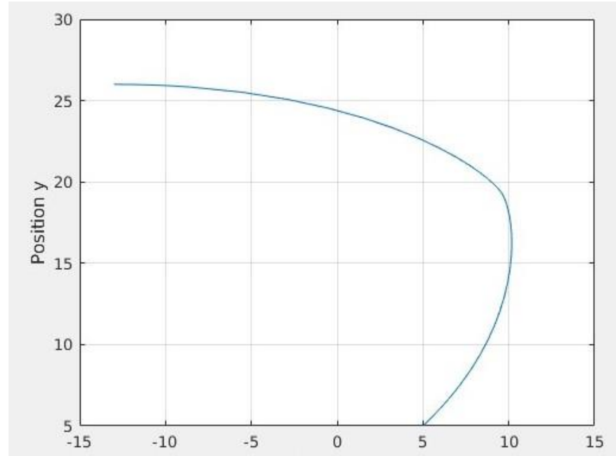

FIG 3: x_0: $[5,5,\pi/4,0,3,2]$ to x_f: $[-13,26,\pi,40,3,-0.0042]$
Third simulation of the disk trajectory on the XY plane

V. CONCLUSION

As we can see from the example, this is a very useful and precise tool to control a system, as we are not limited to linear systems there are infinite problems to explore. However, if these solutions were to be implemented in real life some modifications would need to be made, as our system is sensible to a change in parameters (for example, having a tolerance in r_0) or not being able to measure the state variables in order to compute the control at an instant or time (or the lack in accuracy). Nevertheless, there are some methods to reduce these errors introduced by lack of accuracy and could be applied to the system after the prolongations.

Having said that, it is also interesting to see which new variables need to be added for controllability (the prolongation) as physically one would think that only the second derivative, acceleration, is needed in order to define the system. That is indeed true when defining the motion of a system, but in order to control it we need to make use of powerful mathematical tools that are not limited by physics.

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