Ejection-collision orbits in the Elliptic Restricted Three-Body Problem.

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Abstract

This master’s thesis consists in the study of the Elliptical Restricted Three-Body Problem, known as Elliptical RTBP. In particular, a first approach will be obtaining a compact formulation, through some changes of variables, in order to derive an adimensionalised and synodical system of equations. This will be done also for the Circular RTBP, in order to see the similarities and dissimilarities of both systems. After this, we will perform a regularization of both systems, since we want to avoid the singularities that appear when we are close to collision, which is known as the Levi-Civita regularization. We will also see some properties of the system with and without regularization. Finally, in the numerical part of this work, we will introduce the concept of $n$-ejection-collision orbits and a characterization of them, and we will show an algorithm for finding them, and how sensitive this system is to small variations of the various parameters that we have. We will also do a first exploration of the $n$-ejection-collision orbits, studying the sensitiveness of this system and the bifurcations that will appear.

Keywords

Elliptical RTBP, Levi-Civita regularization, ejection-collision orbits.
1. Introduction

Along this master’s thesis, we will focus on the so-called Restricted Three-Body Problem (RTBP), which is one of the richest and most interesting problems in the field of celestial mechanics. This problem consists in understanding the motion of a small mass in the gravitational field created by two larger masses, while disregarding the gravitational influence of the small mass itself. Depending on the shape of the orbits that these bodies follow, we can be on a circular or elliptical framework (the circular one is easier to work with and to comprehend, but in real life this does not happen, since all the examples that we know about the motion of planets or asteroids follow elliptical orbits, although their eccentricity is usually small).

The main objective of this work is to give a clear and organized view of the Elliptical RTBP formulation, and then perform a numerical analysis of a particular type of orbits, the so-called ejection-collision orbits (we are going to introduce a formal definition in the third chapter but, for now, it is sufficient for our purpose to say that an ejection-collision or ECOs, are the orbits that a particle follows when is ejected from one of the primaries, and after some path, whatever it is, returns to the same primary from which it was ejected).

In recent years, significant attention has been drawn to the study of ejection-collision orbits within the restricted Three-Body Problem, due to their relevance in various celestial scenarios, such as satellite dynamics and the motion of asteroids. Some of the results that have been obtained, for the 1-ECOs, are the following: concerning the planar RTBP, in [5] the authors proved that there exist four families of ECOs, for any value for the mass parameter \( \mu \), when \( C \), the Jacobi constant, is big enough. Regarding the spatial RTBP, in [8] it was proven the existence of ECOs, when \( \mu \) is small enough. There are also analytical results concerning ECOs in other problems: in [1], the authors analysed the behavior of the binary collision in a Planar Restricted \((n+1)\)-Body Problem, and they also proved that there exists four ECOs for values of \( C \), the Jacobi constant, big enough. In the Hill’s problem, there exists four families of ECOs when \( C \) is big enough. In a Restricted Four-Body Problem it has also been proved a similar result: in [11] the authors proved the existence of four ECOs, for \( C \) big enough and for any values of the mass parameter, \( \mu \in (0, 1/2] \). All the results that we have just enumerated are regarding the existence of ECOs, but there also analytical results regarding the existence of \( n \)-ECOs: in [10] the authors proved that for every value of \( \mu \) and \( n \), there are exactly four families of \( n \)-ECOs when varying \( C \).

Moreover, similar models can be applied to other kind of situations, not only to celestial mechanics. For example, in [2] the authors studied the phase space structure of the hydrogen atom in a circularly polarized microwave field, since it can be modeled as a perturbation of the Kepler problem. In the same framework we have [14], where the author focuses on the so-called to and fro motion, which can be thought as ECOs, and also a computation and continuation of families of ECOs is done. Recently this topic has been deeply studied, and in [3] the authors work with the Hamiltonian of the system, which depends on a parameter \( K > 0 \), and they prove analytically that, for big enough value of \( C \), there exist 2 families of \( n \)-ECOs for all \( K \). Also, an analysis of the bifurcations and periodic and quasi-periodic ECOs is done.

This work is divided into three parts: the first one will consist in the presentation of the problem, and then perform some changes of variables, in order to adimensionalize the system and also to obtain a coordinate system that rotates together with the primaries. In this way, we will obtain a compact version of the equations of motion. We will follow this approach first for the circular case, and then for the elliptical one, since in this way we will be able to see that both will be pretty similar, at least in the structure of
the system, although we also will see that the elliptical case will present more difficulties (we will need to
deal with the eccentricity, the true anomaly and the effect of the pulsating coordinates and, also we will
obtain a non-autonomous system). Finally, we will comment on some properties of both systems, such as
symmetries, equilibrium points and a parameter that will play an important role: the Jacobi Constant.

On the second part of this work, we will perform a regularization of both systems, the circular and the
elliptical. We will need this due to the kind of numerical approach that we will do: since we want to work
with ejection-collision orbits, we need to compute orbits that start at ejection, therefore they will begin
at one of the primaries, where we have singularities. The method that we have chosen is the Levi-Civita
regularization, and consists of two changes of variables, one for the positions and the other for the time.
As we have said, we will perform this regularization for both problems: first for the circular and then for
the elliptical (we will see that the steps are the same, and we only need to be careful at some point). Also,
we will present an explicit expression for the equations of motion and we will see some properties of this
new system, in a way analogous to the first chapter.

The last chapter will be dedicated to the numerical results: first of all, we will introduce and define the
concept of ejection-collision orbits (ECOs) and $n$-ejection-collision orbits ($n$-ECOs), and we will present
a suitable algorithm to compute them (for this, we will use some characterization for these orbits, and
also some considerations regarding the initial conditions that we must use). Once this is done, we will be
able to begin the study of the 1-ECOs, taking into account the different parameters that we have in the
problem, and how each one affects the dynamics of the system. We will also see some ”strange” effects
(such as loops and collision with the other primary), and why they appear. Although this part is the most
important of this chapter, we will also do an initial study of the effects of the parameters on the $n$-ECOs.
It is important to comment that, unlike in the circular case, in the elliptical case there are practically no
analytical results regarding the 1-ECOs and none regarding $n$-ECOs, $n > 1$ (in this chapter we will also
make a brief review of the articles that exist for now).

In summary, this study represents a comprehensive exploration of ejection-collision orbits within the elliptical
restricted three-body problem. By developing an adimensionalized and synodical formulation, employing
regularization techniques, and conducting numerical investigations, we would like to be able to contribute,
as far as possible, to the knowledge of this problem within the field of celestial mechanics.
2. The planar RTBP

2.1 The planar and circular RTBP

First of all, we are going to introduce the equations of motion for the planar and circular RTBP, in both, sidereal (inertial system) and synodic (rotating system). Also, we will derive the dimensionless system.

In order to state the problem, we need to define first the involved elements. We are going to consider three bodies (in a celestial sense) as

- The first body, $P_1$, with mass $m_1$, following a circular orbit with radius $R_1$, and the second body, $P_2$, with mass $m_2$, following a circular orbit with radius $R_2$ (both bodies follow circular orbits with a common center of masses, and we can consider that $m_1 > m_2$). From now on, we will call them primaries.

- The third body, $P_3$, which is considered massless.

The interaction between these three bodies is the following: On one hand, the primaries follow circular orbits around their respective center of masses, which is located at the origin, while facing the gravitational influence of each other. On the other hand, the third body moves on the same plane as the primaries, where its motion is affected by the primaries influence, but not influencing them at all, since it has no mass.

![Figure 1: Circular restricted three-body problem](image)

The circular Restricted Three-Body Problem consists in obtaining the motion of the third body (taking into account the above considerations).
Equations of motion in a sidereal coordinate system

Now, we can translate this situation by using physical laws, in concrete, by means of the gravitational (lhs) and the centrifugal forces (rhs) laws, obtaining

\[ G \frac{m_1 m_2}{d^2} = m_2 R_2 n^2 = m_1 R_1 n^2, \]  

(1)

where \( G \) is the universal gravitation constant, \( d = \text{dist}(P_1, P_2) \) is the distance between the two primaries, \( n \) is the mean motion, and \( R_1 \) and \( R_2 \) are the corresponding radius as we can observe in Fig. 1.

Using also the center of mass law, we can isolate \( R_1 \) and \( R_2 \), leading to

\[ R_2 = \frac{m_1 d}{M}, \quad \text{and} \quad R_1 = \frac{m_2 d}{M}, \quad \text{with} \quad M = m_1 + m_2. \]  

(2)

Now, having in mind the picture, we have that we can describe in the so-called sidereal system, or inertial coordinate system \((X, Y)\), the distances between the third body and the primaries by

\[ d_1 = d(P_1, P_3) = \sqrt{(X - X_1)^2 + (Y - Y_1)^2}, \]

\[ d_2 = d(P_2, P_3) = \sqrt{(X - X_2)^2 + (Y - Y_2)^2}, \]  

(3)

where the coordinates of \( P_1 \) and \( P_2 \), that depend on time (we are going to denote the time by \( t^* \) since later on we will use a different one) are

\[ X_1 = R_1 \cos nt^*, \quad Y_1 = R_1 \sin nt^*, \]

\[ X_2 = -R_2 \cos nt^*, \quad Y_2 = -R_2 \sin nt^*. \]  

(4)

Denoting by \( \tilde{V} \) the potential function, given by

\[ \tilde{V} = G \left( \frac{m_1}{d_1} + \frac{m_2}{d_2} \right), \]  

(5)

we can write the equations of motion of \( P_3 \) in the sidereal coordinate system, that are given by

\[ \frac{d^2 X}{d t^{*2}} = -\frac{\partial \tilde{V}}{\partial X}, \]

\[ \frac{d^2 Y}{d t^{*2}} = -\frac{\partial \tilde{V}}{\partial Y}, \]  

(6)

or, which is the same

\[ \frac{d^2 X}{d t^{*2}} = -G \left( \frac{m_1 (X - R_1 \cos nt^*)}{d_1^2} + \frac{m_2 (X + R_2 \cos nt^*)}{d_2^2} \right), \]

\[ \frac{d^2 Y}{d t^{*2}} = -G \left( \frac{m_1 (Y - R_1 \sin nt^*)}{d_1^2} + \frac{m_2 (Y + R_2 \sin nt^*)}{d_2^2} \right). \]  

(7)

Although this is the system of equations that describes our problem, we will make some changes in order to obtain a new system easier to work with. This is going to consist in adimensionalising the system, and then transforming it onto a moving coordinate system, in which the coordinates axis move with the primaries.
Dimensionless coordinates

The main goal of this step is to obtain a system that, at least as much as possible, does not depend on the physical parameters that we have seen when we defined the problem. We will see at the end that the only parameter that will remain will be $\mu$, which has to do with the primaries’ mass.

The first step is then to define the new variables that we are going to use, all of them dimensionless

$$
\bar{x} = \frac{X}{d}, \quad \mu_1 = \frac{m_1}{M} = \frac{R_2}{d}, \quad \bar{r}_1 = \frac{d_1}{d},
$$

$$
\bar{y} = \frac{Y}{d}, \quad \mu_2 = \frac{m_2}{M} = \frac{R_1}{d}, \quad \bar{r}_2 = \frac{d_2}{d},
$$

and, for the time

$$
t = nt^*. \tag{9}
$$

With this, we have the following expression for the radius

$$
\bar{r}_1 = (\bar{x} - \mu_2 \cos t)^2 + (\bar{y} - \mu_2 \sin t)^2,
$$

$$
\bar{r}_2 = (\bar{x} + \mu_1 \cos t)^2 + (\bar{y} + \mu_1 \sin t)^2, \tag{10}
$$

and introducing the new function $V$, which is simply

$$
V = \frac{\bar{V}}{d^2 n^2} = \frac{\mu_1}{\bar{r}_1} + \frac{\mu_2}{\bar{r}_2}, \tag{11}
$$

it is easy to see that the system (7) is transformed to

$$
d^2\bar{x} = \frac{\partial F}{\partial \bar{x}} = -\left( \frac{\mu_1}{\bar{r}_1} \bar{x} - \mu_2 \cos t \frac{\bar{r}_1}{\bar{r}_2} + \mu_2 \bar{x} + \mu_1 \cos t \right),
$$

$$
d^2\bar{y} = \frac{\partial F}{\partial \bar{y}} = -\left( \frac{\mu_1}{\bar{r}_1} \bar{y} - \mu_2 \sin t \frac{\bar{r}_1}{\bar{r}_2} + \mu_2 \bar{y} + \mu_1 \sin t \right). \tag{12}
$$

Remark: As we said before, this system only depends on one parameter: in fact we have $\mu_1$ and $\mu_2$, but remember that $M = m_1 + m_2$, so $\mu_1 + \mu_2 = \frac{m_1}{M} + \frac{m_2}{M} = 1$. Indeed, if we know one of them, we can have the other, so only one parameter is needed. Because of this, from now on we are going to use that $\mu_1 = 1 - \mu$ and $\mu_2 = \mu$, with $\mu \in (0, 1)$, that corresponds to the situation where the bigger primary is located at the right side of the origin, at the point $(\mu_1, 0)$, and the other is located in $(-\mu_2, 0)$.

Equations of motion in a synodic dimensionless coordinate system

Let us start with the transformation to a synodic coordinate system, in order to get rid of the dependence on time. For this, we are going to use a rotation transformation of the coordinates as follows

$$
\bar{x} = x \cos t - y \sin t,
$$

$$
\bar{y} = x \sin t + y \cos t. \tag{13}
$$

Using also that $\mu_1 = 1 - \mu$ and $\mu_2 = \mu$, substituting in (10), the new radius are

$$
\bar{r}_1 = (x - \mu)^2 + y^2,
$$

$$
\bar{r}_2 = (x - \mu + 1)^2 + y^2. \tag{14}
$$
Now, if we use this change of variables in the sidereal and adimensional system (12), it follows that

\[
\begin{align*}
\frac{d^2x}{dt^2} - 2\frac{dy}{dt} &= x - \frac{1}{r_1^3}(1 - \mu)(x - \mu) - \frac{\mu(x - \mu + 1)}{r_1^3}, \\
\frac{d^2y}{dt^2} + 2\frac{dx}{dt} &= y \left(1 - \frac{1 - \mu}{r_2^3} - \frac{\mu}{r_2^3}\right).
\end{align*}
\]

(15)

And in a similar way as in the other systems, if we introduce the potential function \(\Omega\)

\[
\Omega = \frac{1}{2} [(1 - \mu) r_1^2 + \mu r_2^2] + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2},
\]

(16)

then we can rewrite the system in the most simplified way

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= \Omega_x, \\
\ddot{y} + 2\dot{x} &= \Omega_y,
\end{align*}
\]

(17)

where \((\cdot)\) denotes the derivative with respect to the time \(t\), and \(\Omega_x\) and \(\Omega_y\) represents the derivative of \(\Omega\) with respect to \(x\) and \(y\), respectively.

Now we have obtained the dimensionless system that we were looking for, so, before continuing with the elliptical case, we are going to present some useful properties (since it is not the objective of this work, we are not going to prove them, but they can be consulted in [21]).

**Property 1: Symmetry**

The system (17) has the following symmetry

\[
(t, x, y, \dot{x}, \dot{y}) \rightarrow (-t, x, -y, -\dot{x}, \dot{y}).
\]

(18)

In Figure 2 we have an example of an orbit and its corresponding symmetric orbit. Indeed, it is enough to compute one of the orbits forward (backward) in time, and then integrating the same initial point backward (forward) in time.

![Symmetrical orbits for \(\mu = 9.53881e - 4\)](image-url)
Property 2: Equilibrium points

The system (17) has five equilibrium points, known as the Lagrangian points (as shown in Figure 3), which are divided in two groups:

- The collinear points \(L_1, L_2, L_3\), located in the \(x\) axis, where \(x_{L_2}(\mu) < \mu - 1 < x_{L_1}(\mu) < \mu < x_{L_3}(\mu)\).
- The triangular points \(L_4, L_5\) that are the vertex of an equilateral triangle, where the other two vertices are the primaries.

\[\text{Figure 3: Equilibrium points for } \mu = 0.4. \text{ In blue the triangular points, in green the collinear ones.}\]

The exact value of the Lagrangian points can be calculated numerically, making use of the Euler’s quintic equations, and also it is easy to see the stability of these points through the eigenvalues of jacobian matrix of the system (17) evaluated on the equilibrium points, where the jacobian matrix is

\[
DF(x, y) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\Omega_{xx} & \Omega_{xy} & 0 & 2 \\
\Omega_{xy} & \Omega_{yy} & -2 & 0
\end{pmatrix}
\]

Then, computing this matrix, and evaluating it in the equilibrium points, it is easy to check that (see Chapter 5 of [21]):

- \(L_i, \ i = 1, 2, 3\) are a center \(\times\) saddle point.
- For \(L_i, \ i = 4, 5\) we have some possibilities, depending on the value of \(\mu\):
  - If \(\mu \in (0, \mu_R)\), they are center \(\times\) center,
  - If \(\mu = \mu_R\), they are a degenerate center \(\times\) center,
  - If \(\mu \in (\mu_R, 0.5)\), they are a complex saddle,

where \(\mu_R = \frac{1}{2}(1 - \frac{\sqrt{69}}{9})\) is known as the Routh value of the mass parameter.
**Property 3: Jacobi constant and Jacobi integral**

By means of the transformation to the new dimensionless synodical system coordinate, we have obtained a system (17) that is autonomous, so it is a good point since we have avoided the time-dependency on the equations. But, as a counterpart, we have now a new term that involves the first derivative. We may ask ourselves if this change is worth it, which in fact is, because we have a first integral, $\Omega$. And not only we have the integral, but also we can obtain a relation that will play an important role. In order to find it, we only have to multiply the first equation by $\dot{x}$ and add it to the second one multiplied by $\dot{y}$. Integrating the resulting equation with respect to the time, we obtain

$$
\frac{1}{2} (\dot{x}^2 + \dot{y}^2) = \int_{t_0}^{t} (\Omega_x dx + \Omega_y dy) = \Omega - \frac{C}{2},
$$

where it is usual to call them as the Jacobi integral (for the integral) and the Jacobi constant (for the constant $C$).

**Property 4: Hill's region and relation with the Jacobi Constant**

The Hill’s region is a set that contains only the points where the motion is possible. Recalling the expression for the Jacobi integral that we have just seen, we have that $2\Omega - C = \dot{x}^2 + \dot{y}^2 \geq 0$, so, then, we define the Hill’s region as

$$
R(C) = \{ (x, y) \in \mathbb{R}^2 \text{ such that } 2\Omega \geq C \}. \tag{21}
$$

Depending on the value of the Jacobi Constant, we have different possible situations. In particular, we will be interested in the $C$ values for the equilibrium points $L_i$, $i = 1, ..., 5$, that we will denote by $C_{L_i}$. We do this because an interesting property of this problem is that the topology of the Hill’s region is different depending on the value of this constant: we can have a situation where the motion can happen in all the plane or only around one of them (see Figure 4, where the blue region is where the motion is not allowed).

![Figure 4: Hill’s region for $\mu = 0.2$, and for $C < C_{L_2}$, $C = C_{L_2}$, $C_{L_2} < C < C_{L_1}$, $C = C_{L_1}$ and $C > C_{L_1}$, respectively. The red stars are $P_2$ (left) and $P_1$ (right).](image)

In the circular case, if we want to compute ECOs, it is usual to work with $C > C_{L_1}$ (last plot in Figure 4), because this allows us to ensure that nothing rare happens, since the motion in this region is only allowed
Ejection-collision orbits in the ERTBP

on a neighbourhood of each primary. The relationship between the value of $\mu$ and the value of $C$, for the point $x_{L1}$, can be seen in Figure 5. As we will see later, in the elliptical system we will not be able to talk about Hill’s region, so, in general, this choice has not the same consequence. But, when the eccentricity is very small, an elliptical orbit is almost a circular one, so in these situations the above consideration about the choice of $C$ will be useful.

![Figure 5: Relation between $\mu$ and $C_{L1}$](image)

Once we have seen the different equations of motion, depending on the coordinate system used, and the relevant properties of the circular case, we are going to repeat the procedure in a similar way, but for the elliptical case. This approach makes sense, since the steps are going to be the same, but now things are going to complicate due to the fact that we will have new parameters (such as the eccentricity and the true anomaly) and also the resultant system will be non-autonomous.
2.2 The planar and elliptic RTBP

As we have said, the scheme for the elliptical case is practically the same. We have the massless third body, and the two primaries $P_1, P_2$ with a common center of masses. But now, we have that the two primaries move following elliptical orbits, with semi-axis $a_1$ and $a_2$, respectively, and where $e$ is the common eccentricity, $a$ is the common semi-major axis, and with $f$ as the true anomaly of $P_1$.

![Elliptical restricted three-body problem]

Before starting the transformations for finding the adimensional synodical coordinates, we are going to see a few properties that we will need. They come from the two body problem (2BP), which has been studied deeply, and also the solution and the dynamics are well known (see the chapter related to the two body problem in [4]).

Proposition 2.1. The expression for the solution of the 2BP is

$$r(f) = \frac{a(1 - e^2)}{1 + e \cos f}.$$  (22)

Proposition 2.2. Principle of conservation of angular momentum (for two bodies)

$$\frac{df}{dt^*} r^2 = \sqrt{a(1 - e^2)G(m_1 + m_2)},$$  (23)

and its derivative

$$\frac{d}{dt^*} \left( \frac{df}{dt^*} r^2 \right) = 0 \rightarrow \frac{d^2 f}{dt^* dt^*} r + 2 \frac{df}{dt^*} \frac{dr}{dt^*} = 0.$$  (24)

Proposition 2.3. The equation of motion for the 2BP is

$$\frac{d^2 r}{dt^* dt^*} - r \left( \frac{df}{dt^*} \right)^2 = -\frac{G(m_1 + m_2)}{r^2}.$$  (25)

And we can rewrite this by using the conservation of angular momentum (23), which leads to

$$\frac{d^2 r}{dt^* dt^*} - r \left( \frac{df}{dt^*} \right)^2 = -\frac{r^2}{a(1 - e^2)} \left( \frac{df}{dt^*} \right)^2.$$  (26)
Proposition 2.4. If we have an ellipse, then the semi-major axis verifies \( p = a(1 - e^2) \), where \( p \) is called the ellipse parameter.

Also, we have the elliptical orbits that follow \( P_1 \) and \( P_2 \), with semi-major axis \( a_1 \) and \( a_2 \), and we have also the elliptical orbit of \( P_2 \), where the focus is \( P_1 \), and the semi-major axis is \( a = a_1 + a_2 \), where \( a_1 = \mu a \) and \( a_2 = (1 - \mu)a \). Recall that the positions of the primaries are \((\mu, 0)\) and \((1 - \mu, 0)\).

Once this is done, let us start with the equations of motion.

Equations of motion

Regarding the equations on the sidereal coordinates, we have that the resultant system has the same structure as in the circular case, the only thing that changes is the expression for the primaries position, since now we have elliptical orbits and not circular ones. Then, if we denote the positions and the radius by

\[
P_1 = (X_1, Y_1) = (R_1 \cos f, R_1 \sin f),
\]

\[
P_2 = (X_2, Y_2) = (-R_2 \cos f, -R_2 \sin f),
\]

\[
D_1 = \sqrt{(X - R_1 \cos f)^2 + (Y - R_1 \sin f)^2},
\]

\[
D_2 = \sqrt{(X + R_2 \cos f)^2 + (Y + R_2 \sin f)^2},
\]

where

\[
R_i = \frac{a_i(1 - e^2)}{1 + e \cos f} = \frac{p_i}{1 + e \cos f}, \quad i = 1, 2
\]

and, if we compute \( R_1 + R_2 \), we recover the expression (22).

Then, the equations of motion in a sidereal coordinates are

\[
\frac{d^2X}{dt^2} = -G \left( \frac{m_1(X - X_1)}{D_1^3} + \frac{m_2(X + X_2)}{D_2^3} \right),
\]

\[
\frac{d^2Y}{dt^2} = -G \left( \frac{m_1(Y - Y_1)}{D_1^3} + \frac{m_2(Y + Y_2)}{D_2^3} \right).
\]

Now we want to obtain this model in synodical coordinates, in order to obtain a system where the axes rotate together with the primaries. But this time, the axes will rotate with respect to the true anomaly, and not with respect to the time \( t^* \), so we need the relationship between them. Recall that this is what we had in (23):

\[
\frac{df}{dt^*} = \frac{\sqrt{G(m_1 + m_2)}}{(a(1 - e^2))^{3/2}}(1 + e \cos f)^2.
\]

And the change of variables that we are going to use is

\[
X = x \cos f - y \sin f,
\]

\[
Y = x \sin f + y \cos f.
\]
First observation is related with the primaries positions. If we apply this change to the primaries’ positions, and having in mind that now they are fixed in the $x$ axis, therefore the second component is zero, we obtain:

$$P_1 = (x_1, y_1) = \left( \frac{p_1}{1 + e \cos f}, 0 \right),$$

$$P_2 = (x_2, y_2) = \left( \frac{-p_2}{1 + e \cos f}, 0 \right).$$  \hfill (32)

And we also have

$$d_1 = \sqrt{(x - x_1)^2 + y^2},$$

$$d_2 = \sqrt{(x - x_2)^2 + y^2}.$$  \hfill (33)

Now we are going to apply the change of variables (31) to the sidereal coordinate system (29), but since there are a lot of computations, we are going to do this in two steps, and only for the first equation (the other one comes in an analogous way). Let us focus first in the lhs of the equations:

$$\frac{d^2 X}{dt^2} = \frac{d}{dt} \left[ \frac{d}{dt^*} (x \cos f - y \sin f) \right]$$

$$= \frac{d}{dt^*} \left[ \frac{dx}{dt^*} \cos f - x \frac{df}{dt^*} \sin f - \frac{dy}{dt^*} \sin f - y \frac{df}{dt^*} \cos f \right]$$

$$= \left[ \frac{d^2 x}{dt^2} - \left( \frac{df}{dt^*} \right)^2 x - 2 \frac{df}{dt^*} \frac{dy}{dt^*} - \frac{d^3 f}{dt^3} y \right] \cos f + \left[ -\frac{d^2 y}{dt^2} + \left( \frac{df}{dt^*} \right)^2 y - 2 \frac{df}{dt^*} \frac{dx}{dt^*} - \frac{d^3 f}{dt^3} x \right] \sin f,$$

and applying the same change to the rhs of the equations, we have

$$\frac{d^2 X}{dt^2} = -G \left[ m_1 \frac{(x - x_1) \cos f - y \sin f}{d_1^3} + m_2 \frac{(x - x_2) \cos f - y \sin f}{d_2^3} \right].$$  \hfill (35)

When we repeat the same procedure to the second equation of the system, regarding $\frac{d^2 Y}{dt^2}$, we obtain the same equalities. Finally, when we join the expressions obtained in (34) and (35), and setting $G = 1$, we have the equations of motion in a synodical coordinate system:

$$\frac{d^2 x}{dt^2} - 2 \frac{df}{dt^*} \frac{dy}{dt^*} = \left( \frac{df}{dt^*} \right)^2 x + \frac{d^2 f}{dt^3} y - m_1 \frac{(x - x_1)}{d_1^3} - m_2 \frac{(x - x_2)}{d_2^3},$$

$$\frac{d^2 y}{dt^2} + 2 \frac{df}{dt^*} \frac{dx}{dt^*} = \left( \frac{df}{dt^*} \right)^2 y - \frac{d^2 f}{dt^3} x - m_1 \frac{y}{d_1^3} - m_2 \frac{y}{d_2^3}.$$  \hfill (36)

The last step consists in adimensionalising the system, which can be done by means of the followings transformations

$$\xi = \frac{x}{r}, \quad \mu_1 = \frac{m_1}{M} = 1 - \mu,$$

$$\eta = \frac{y}{r}, \quad \mu_2 = \frac{m_2}{M} = \mu.$$  \hfill (37)
where \( r \) is the expression that we have seen in (22), \( \mu \in (0, 1) \) and also we are going to change the temporal variable by

\[
\frac{d}{dt^*} = \frac{df}{dt} \frac{d}{dt^* dt^*}.
\] (38)

Applying these changes to the synodical system (36) is not conceptually difficult, but involves many computations, so in order to perform this in a clear way, we are going to do it by little steps. First we are going to compute each term of the lhs (again, only for the first equation, the other one comes in a similar way). For the first one we have

\[
\frac{d^2 x}{dt^{*2}} = \frac{d}{dt^*} \left( \frac{d}{dt^*} r \xi \right) = \frac{d}{dt^*} \left( \frac{dr}{dt^*} \xi + r \frac{d\xi}{dt^*} \right) = \frac{d^2 r}{dt^*^2} \xi + 2 \frac{dr}{dt^*} \frac{d\xi}{dt^*} + r \frac{d^2 \xi}{dt^*^2}
\]

\[
= \frac{d^2 r}{dt^*^2} \xi + \frac{2dr df d\xi}{dt^*^2 df} + r \left( \frac{d^2 f}{dt^*^2 df} + \left( \frac{df}{dt^*} \right)^2 \frac{d^2 \xi}{dt^*^2 df} \right),
\] (39)

where in the last equality we have used the change (38), but only in the derivatives involving \( \xi \), since in the derivatives involving \( r \) we will use the properties that we have seen, and we are not interested in applying this change. Regarding the second term we have

\[
\frac{df}{dt^*} \frac{dy}{dt^*} = 2 \frac{df}{dt^*} \frac{df}{dt^* df} \frac{d}{dt^*} (r\eta) = 2 \left( \frac{df}{dt^*} \right)^2 \left( \frac{dr}{dt^*} + r \frac{d\eta}{df} \right).
\] (40)

When we put together these expressions with the two first terms on the rhs of (36), we obtain

\[
r \left( \frac{d^2 f}{dt^*^2 df} + \left( \frac{df}{dt^*} \right)^2 \frac{d^2 \xi}{dt^*^2 df} \right) + \frac{2dr df d\xi}{dt^*^2 df} + \frac{d^2 r}{dt^*^2} \xi - 2 \left( \frac{df}{dt^*} \right)^2 \left( \frac{dr}{dt^*} + r \frac{d\eta}{df} \right) - \left( \frac{df}{dt^*} \right)^2 r \xi - \frac{d^2 f}{dt^*^2} r\eta
\]

\[
= r \left( \frac{df}{dt^*} \right)^2 \left( \frac{d^2 \xi}{dt^*^2 df} - 2 \frac{d\eta}{df} \right) + \xi \left( \frac{d^2 r}{dt^*^2} - r \left( \frac{df}{dt^*} \right)^2 \right) + \left( \frac{df}{dt^*} \right)^2 \left( \frac{d\xi}{dt^*} - \eta \right) \left( \frac{d^2 f}{dt^*^2} + 2 \frac{dr df}{dt^*^2} \right),
\] (41)

where in the last equality we have only rearranged the terms (in a few moments we will see that this will make the computations easier). Dividing by \( r \left( \frac{df}{dt^*} \right)^2 \) we have

\[
\frac{d^2 \xi}{df^2} - 2 \frac{d\eta}{df} + \xi \frac{r}{r} \left( \frac{df}{dt^*} \right)^{-2} \frac{d^2 r}{dt^*^2} - r \left( \frac{df}{dt^*} \right)^{-2} \left( \frac{df}{dt^*} \right)^2 + \frac{1}{r} \left( \frac{df}{dt^*} \right)^{-2} \left( \frac{df}{dt^*} - \eta \right) \left( r \frac{d^2 f}{dt^*^2} + 2 \frac{dr df}{dt^*^2} \right)
\]

\[
= \left(\right) \frac{d^2 \xi}{df^2} - 2 \frac{d\eta}{df} + \frac{\xi}{r} \left( \frac{df}{dt^*} \right)^{-2} \left( - \frac{r^2}{a(1 - e^2)} \left( \frac{df}{dt^*} \right)^2 \right)
\]

\[
= \left(\right) \frac{d^2 \xi}{df^2} - 2 \frac{d\eta}{df} - \frac{\xi}{1 + e \cos f},
\] (42)

where we have used the properties that we have presented at the start of this chapter: \( \left(\right) \) is due to we can change the expression on the first parenthesis using (26), and the last parenthesis vanishes by (24). Also, in \( \left(\right) \), we have used (22).
Now we are going to apply the change to the remaining two terms in the rhs of (36), but first we need to apply it to each primary’s position and radius:

\[
\begin{align*}
\xi_1 &= \frac{x_1}{r} = \frac{p_1}{a(1-e^2)} = \frac{a_1}{a} = \mu, & \eta_1 &= 0, \\
\xi_2 &= \frac{x_2}{r} = -\frac{p_2}{a(1-e^2)} = -\frac{a_2}{a} = \mu - 1, & \eta_2 &= 0,
\end{align*}
\] (43)

In the last two equalities for \(\xi_i\), \(i = 1, 2\), we just used Prop 2.4.

\[
\begin{align*}
& r_1 := \sqrt{(\xi - \xi_1)^2 + \eta^2}, & d_1^2 = (x - x_1)^2 + y^2 = r^2 ((\xi - \xi_1)^2 + \eta^2) = r_1^2, \\
& r_2 := \sqrt{(\xi - \xi_2)^2 + \eta^2}, & d_2^2 = (x - x_2)^2 + y^2 = r^2 ((\xi - \xi_2)^2 + \eta^2) = r_2^2.
\end{align*}
\] (44)

Then, with this, we have that

\[
\frac{d^2 X}{dt^2} = -m_1 \frac{x-x_1}{d_1^3} - m_2 \frac{x-x_2}{d_2^3} = -M \left( (1 - \mu) \frac{\xi r - \xi_1 r}{r_1^3} + \mu \frac{\xi r - \xi_2 r}{r_2^3} \right)
\] (45)

Now we have transformed all the terms. Finally, if we put together this last expression (divided by \(r \left( \frac{df}{dt} \right)^2\) because we have done this with the lhs) with (42), we have

\[
\begin{align*}
\frac{d^2 \xi}{df^2} - 2 \frac{d \eta}{df} - \frac{\xi}{1 + e \cos f} &= \frac{M}{r^3} \left( \frac{df}{dt} \right)^2 \left( (1 - \mu)\frac{(\xi - \mu)}{r_1^3} + \frac{\mu(\xi - \mu + 1)}{r_2^3} \right) \\
&= -\frac{r}{a(1-e^2)} \left( (1 - \mu)\frac{(\xi - \mu)}{r_1^3} + \frac{\mu(\xi - \mu + 1)}{r_2^3} \right) \\
&= -\frac{1}{1 + e \cos f} \left( (1 - \mu)\frac{(\xi - \mu)}{r_1^3} + \frac{\mu(\xi - \mu + 1)}{r_2^3} \right),
\end{align*}
\] (46)

where we have used (23) in the second equality, and (22) in the third one. Then, this expression can be written as

\[
\frac{d^2 \xi}{df^2} - 2 \frac{d \eta}{df} = \frac{1}{1 + e \cos f} \left( \xi - \frac{(1 - \mu)(\xi - \mu)}{r_1^3} - \frac{\mu(\xi - \mu + 1)}{r_2^3} \right).
\] (47)

Also, repeating the same steps with the second equation of the synodic system (36), we end up with a similar expression

\[
\frac{d^2 \eta}{df^2} + 2 \frac{d \xi}{df} = \frac{\eta}{1 + e \cos f} \left( 1 - \frac{(1 - \mu)(\xi - \mu)}{r_1^3} - \frac{\mu(\xi - \mu + 1)}{r_2^3} \right).
\] (48)

Finally, if we introduce the following potential function

\[
\omega = \frac{\Omega}{1 + e \cos f},
\] (49)

where \(\Omega\) is the potential defined in (16), and denoting \(\cdot := \frac{d}{df}\) (let us use this notation although its meaning is not the same as in the circular case), \(\omega_\xi = \frac{\partial \omega}{\partial \xi}\) and \(\omega_\eta = \frac{\partial \omega}{\partial \eta}\), we obtain the simplest version of the equations of motion for the elliptical restricted three-body problem (it is interesting to note that the
Ejection-collision orbits in the ERTBP

expression is the same as in the circular case, at least in the structure: the differences are hidden, since we have a dependency of \( f \) on \( \omega \), and therefore this system is now non-autonomous).

\[
\begin{align*}
\ddot{\xi} - 2\dot{\eta} &= \omega \xi, \\
\dot{\eta} + 2\dot{\xi} &= \omega \eta.
\end{align*}
\] (50)

Before finishing this chapter, we introduce some properties of this system, as we had done in the circular one.

**Property 1: Symmetry**

As in the circular case, now the system (50) has the following symmetry

\[
(f, \xi, \eta, \dot{\xi}, \dot{\eta}) \rightarrow (-f, \xi, -\eta, -\dot{\xi}, \dot{\eta}),
\] (51)

and this system is also periodic in the variable \( f \), with period \( 2\pi \).

**Property 2: Equilibrium points**

Since the system (50) presents the exact form as the circular system (17), the equilibrium points are found exactly in the same way, therefore they are the same. Regarding their stability, we have that the collinear points are still unstable, but for the triangular points we have that the stability changes depending on the effect of the eccentricity and the value of the mass parameter \( \mu \) (see Chapter 10 of [21]).

**Property 3: Jacobian integral and Jacobi Constant**

If we apply the same procedure to find the Jacobi integral that we have seen in (20), we end up with

\[
\dot{\xi}^2 + \dot{\eta}^2 = 2 \int (\omega \xi d\xi + \omega \eta d\eta).
\] (52)

But this time we have that the terms that are integrated are not the differential of \( \omega \), since this function depends not only on the coordinates \((\xi, \eta)\), but also on the true anomaly \( f \). Having in mind an alternative expression for this integral

\[
2 \int (d\omega - \omega df) = 2\omega - 2 \int \omega df - C,
\] (53)

then we can rewrite the equation (52) as

\[
(\dot{\xi}')^2 + (\dot{\eta}')^2 = 2\omega - 2e \int_0^f \frac{\Omega \sin f}{(1 + e \cos f)^2} df - C,
\] (54)

which is the Jacobi integral.

We can express this equality in a more compact way, as

\[
\dot{\xi}^2 + \dot{\eta}^2 = 2(V - Z),
\] (55)

where \( V \) is simply

\[
V = \omega - \frac{C}{2} = \frac{\Omega}{1 + e \cos f} - \frac{C}{2},
\] (56)
and $Z$ denotes the integral

$$Z = e \int_{F_0}^{f} \frac{\Omega \sin f}{(1 + e \cos f)^2} df.$$  \hspace{1cm} (57)

Regarding Hill’s region, now we cannot do an analogy, since this time if we try to find the zero velocity curves (i.e., imposing $\dot{\xi} = \dot{\eta} = 0$), we notice that

$$2\omega - C = 2\frac{\Omega}{(1 + e \cos f)} - C = 0 \quad \rightarrow \quad 2\Omega - C(1 + e \cos f) = 0,$$  \hspace{1cm} (58)

then, in the elliptical case, it makes no sense to talk about the Hill’s region, since we will obtain a variable topology for a fixed value of $C$.\footnote{17}
3. The Levi-Civita regularization

The main objective of this work is to find and study (numerically) the so-called ejection-collision orbits or ECOs. In the next chapter we will present a formal definition and properties, but now it is enough to present a sketch of the situation: these orbits consist in the motion of the third body, that is ejected from one of the primaries, then follows some path, whatever it is, and finally collides with the primary (no matter if it is the one at which the motion started or the other). At this point, it is important to recall the expression for $\Omega$ (16), since we have that the domain of this function is \( \{(x, y) \in \mathbb{R}^2 : r_1 \neq 0, r_2 \neq 0\} \). It is interesting to note this, because we want to study the motion that starts in one of the primaries, and after some time, returns to it, so if we work with the formulation that we presented until now, we are not going to be able to do anything because of the singularities. For this reason, we need to apply a regularization to this problem. There are many options, but we choose the Levi-Civita regularization (see [7] for the regularization of the circular case), because of the effectiveness of this method and also because it is not conceptually difficult, and understanding how it works will help us to understand also how the regularization in the spatial case works, which is in fact a generalization of this method (this was done by Kustaanheimo-Stiefel in [19]). The regularization for the elliptical RTBP was also done by Szebehely and Giacaglia in [20].

The Levi-Civita regularization works for problems that can be written as

\[ \ddot{x} + B\dot{x} = \nabla_x \Omega(x), \quad x \in \mathbb{R}^2, \tag{59} \]

where $B$ is

\[ B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \tag{60} \]

and $\Omega$ is a gravitational potential, at which the singularity that we want to regularize is at the point $x = p$. From now, let us use a matrix notation for simplicity.

First of all, we will show how to apply this regularization to the circular case, and in the end we will apply this to the elliptical problem. This regularization consists in applying two changes of variables, one for the coordinates and another for the time. Let us start with the change of coordinates: we define $w = (u, v) \in \mathbb{R}^2$, $r = (w, w)$, where $(\cdot, \cdot)$ denotes the scalar product and $p$ is the point we want to regularize. Then, the change is

\[ x = L(w)w + p, \tag{61} \]

with

\[ L(w) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}. \tag{62} \]

Then, since we will use this change in the equation (59), we are interested in the first and second derivatives with respect to the new time $s$ that we will present in a few moments, $' = d/ds$, so it is easy to check that

\[ x' = L(w)w' + L(w')w = 2L(w)w', \]
\[ x'' = 2L(w)w'' + 2L(w')w', \tag{63} \]

and also

\[ r' = 2(w, w'). \tag{64} \]
Regarding the change of time, let us set \( dt = ards \), where \( a \) is a constant and \( \dot{'} \) denotes the derivative with respect to \( s \). Thus, if we compute the first and second derivative

\[
\dot{x} = \frac{dx}{dt} = \frac{dx}{ds} \frac{ds}{dt} = \frac{x'}{ar},
\]

\[
\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{ds} \frac{ds}{dt} = \frac{d}{ds} \left( \frac{x'}{ar} \right) \frac{1}{ar} = \frac{1}{a^2 r} \left( rx'' - r'x' \right),
\]

and applying this to the problem (59), and multiplying by \( a^2 r^3 \), we obtain

\[
rx'' - r'x' + ar^2 Bx' = a^2 r^3 \nabla_x \Omega(x).
\]

Then, substituting in this equation the expressions obtained in (61) and (64), we have

\[
(w, w) \left[ 2L(w)w'' + 2L(w')L(w) - 4(w, w')L(w)w' + 2a(w, w)^2 BL(w)w' \right] = a^2 (w, w)^3 \nabla_x \Omega.
\]

We can rewrite the second and third term by using the following property

\[
2(w, w)L(w')w' - 4(w, w')L(w)w' = -2(w', w')L(w)w,
\]

which leads to

\[
2(w, w)L(w)w'' - 2(w', w')L(w)w + 2a(w, w)^2 BL(w)w' = a^2 (w, w)^3 \nabla_x \Omega.
\]

Now, we multiply this equation by \( \frac{1}{2}L^{-1}(w) = \frac{1}{2(w, w)}LT(w) \) and divide by \( (w, w) \):

\[
w'' - \frac{(w', w')}{(w, w)}w + aLT(w)BL(w)w' = \frac{a^2}{2} (w, w)LT(w) \nabla_x \Omega.
\]

Now it is time to apply the change of coordinates to the gradient, so using that \( \nabla_w \Omega = 2LT(w) \nabla_x \Omega \), we have

\[
w'' - \frac{(w', w')}{(w, w)}w + aLT(w)BL(w)w' = \frac{a^2}{4} (w, w) \nabla_w \Omega.
\]

Using again this change, now applied to the constant of motion, we obtain

\[
2\Omega - C = \langle \dot{x}, \dot{x} \rangle = \frac{4}{a^2 (w, w)^2} (L(w)w', L(w)w') = \frac{4(w', w')}{a^2 (w, w)},
\]

which is pretty similar to the second term of (71), so we have

\[
w'' - \frac{a^2}{4} (2\Omega - C)w + aLT(w)BL(w)w' = \frac{a^2}{4} (w, w) \nabla_w \Omega,
\]

or, equivalently

\[
w'' + aLT(w)BL(w)w' = 4 [(2\Omega - C)w + (w, w) \nabla_w \Omega].
\]

Now, if we denote \( U = \Omega - \frac{C}{4} \), we can see that the two terms in the rhs are the expression for the gradient

\[
\nabla_w [(w, w)U] = 2wU + (w, w) \nabla_w U = 2 \left( \Omega - \frac{C}{2} \right) w + (w, w) \nabla_w \Omega.
\]
Finally, using this, and setting the values for the constant $a = 4$, we have that the final expression for this regularization is
\[
w'' + 4L^T(w)BL(w)w' = \nabla_w [4(w, w) U].
\]
(76)

Let us write this expression in an explicit way. Since we want to regularize the position of the first primary, which recall that is located at $p = (\mu, 0)$, the new coordinates will be
\[
\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
u - v \\
v \\
u \\
v
\end{pmatrix} + \begin{pmatrix}
\mu \\
0
\end{pmatrix},
\]
(77)
which leads to
\[
x = \mu + u^2 - v^2,
\]
\[
y = 2uv.
\]
(78)

And for the new time, we have
\[
\frac{dt}{ds} = 4(w, w) = 4(u^2 + v^2).
\]
(79)

And, computing all the terms in (76) and using these previous changes, we will end up with the regularization of the equations of motion for the circular RTBP (17) for the first primary $P_1 = (\mu, 0)$:
\[
u'' - 8(u^2 + v^2)v' = \frac{\partial}{\partial u} (4U(u^2 + v^2)),
\]
\[
v'' + 8(u^2 + v^2)u' = \frac{\partial}{\partial v} (4U(u^2 + v^2)),
\]
(80)
or, equivalently
\[
u'' - 8(u^2 + v^2)v' = \left(\mu + 4\mu u^2 + 3(u^2 + v^2)^2 + \frac{2\mu}{\mu} \frac{u}{r_2} - \frac{2\mu (u^2 + v^2)(u^2 + v^2 + 1)}{r_2^2} - C\right) 4u,
\]
\[
v'' + 8(u^2 + v^2)u' = \left(\mu - 4\mu v^2 + 3(u^2 + v^2)^2 + \frac{2\mu}{\mu} \frac{v}{r_2} - \frac{2\mu (u^2 + v^2)(u^2 + v^2 - 1)}{r_2^2} - C\right) 4v.
\]
(81)

Recall that $U = \Omega - C/2$, and after applying the change of variables, this expression results in
\[
U = \Omega - \frac{C}{2} = \frac{1}{2} [(1 - \mu)(u^2 + v^2)^2 + \mu r_2^2] + \frac{1 - \mu}{u^2 + v^2} + \frac{\mu}{r_2} - \frac{C}{2},
\]
(82)
where
\[
r_2 = \sqrt{(1 + u^2 - v^2)^2 + 4u^2v^2}.
\]
(83)

It is easy to see that now the system (81) has no singularities except in the position of the second primary, at the point $P_2 = (\mu - 1, 0)$, because at this point $r_2 = 0$. If we want to regularize this position, we only have to use this same formulation, but using $p = (\mu - 1, 0)$ when we define the change of variables (77). Note that with this method, we are able to regularize only one of the singularities.
Regularization of the elliptical RTBP

In order to obtain the regularized system for the elliptical case, we have to do almost the same steps. Regarding the changes of variables we still have that

\[
\xi = u^2 - v^2 + \mu, \\
\eta = 2uv, \\
\frac{df}{d\tau} = 4(u^2 + v^2).
\]  

But now we need to be careful, since in the step (72), when the constant of motion is used, we need to change it by the expression for the elliptical case. Recall that \( \dot{\xi}^2 + \dot{\eta}^2 = 2(V - Z) \), so the expression for the regularization (76) now becomes

\[
w'' + 4L^T (w) BL(w) w' = 4 \nabla_w [(w, w) V] - 8wZ,  
\]

which leads, after some computations, to the equations of motion in the Levi-Civita transformation

\[
u'' - 8(u^2 + v^2)v' = \frac{\partial}{\partial u} [4(u^2 + v^2)V] - 8uZ, \\
v'' + 8(u^2 + v^2)u' = \frac{\partial}{\partial v} [4(u^2 + v^2)V] - 8vZ,
\]

where now the integral (57) is regularized by introducing the change of time, which leads to

\[
Z = e \int_{\tau(0)}^{\tau} \frac{\sin f(\tau)}{(1 + e \cos f(\tau))^2} \left( \frac{\Omega df}{d\tau} \right) d\tau, 
\]

and also we have that we can recover the time \( f \) by \( f(\tau) \), that is defined as

\[
f(\tau) = \int_{\tau(0)}^{\tau} \frac{df}{d\tau} d\tau = \int_{\tau(0)}^{\tau} 4(u^2 + v^2) d\tau. 
\]

We have just seen the regularized equations of motion, which is a system of odes of second order. But since we are going to perform some simulations, we will need to transform it to a system of first order. Since we have two equations of second order, \textit{a priori} our new system should have four equations, but we also have two integrals included in the system, so we finally are going to need six equations, the first two regarding \( u' \) and \( v' \), and also the following equations

\[
u'' = \frac{1}{1 + e \cos f} \left[ 4\mu + 16\mu u^2 + 12(u^2 + v^2)^2 + \frac{8\mu}{r_2} - \frac{8\mu(u^2 + v^2)(u^2 + v^2 + 1)}{r_2^3} \right] v \\
- 4Cu + 8(u^2 + v^2) v' - 8uZ, \\
u'' = \frac{1}{1 + e \cos f} \left[ 4\mu - 16\mu v^2 + 12(u^2 + v^2)^2 + \frac{8\mu}{r_2} - \frac{8\mu(u^2 + v^2)(u^2 + v^2 - 1)}{r_2^3} \right] u \\
- 4Cv - 8(u^2 + v^2) u' - 8vZ, \\
Z' = e \frac{\sin f}{(1 + e \cos f)^2} 4\Omega, \\
f' = 4(u^2 + v^2). 
\]
Now our function $\Omega$ is regularized at the first singularity thanks to now is multiplied by $(u^2 + v^2)$:

$$\tilde{\Omega} = \Omega(u^2 + v^2) = \frac{1}{2}(u^2 + v^2)\left((1 - \mu)(u^2 + v^2)^2 + \mu r_2^2\right) + (1 - \mu) + \frac{\mu (u^2 + v^2)}{r_2}, \quad (90)$$

and

$$r_2 = \sqrt{(1 + u^2 - v^2)^2 + 4u^2v^2}. \quad (91)$$

This system is evaluated at time $[0, \tau_{\text{fin}}]$ and as for initial condition for the true anomaly, we have that $f(\tau_0) = f_0$.

Now, we are going to see some interesting properties of this system, as we have done until now.

**Property 1: Symmetry**

We have now two symmetries, one is the one that we have seen in the non-regularized system

$$(\tau, u, v, u', v', f) \rightarrow (-\tau, u, -v, -u', v', -f), \quad (92)$$

and the other comes from the fact that the Levi-Civita regularization duplicates the configuration space

$$(\tau, u, v, u', v', f) \rightarrow (-\tau, -u, v, u', -v', -f). \quad (93)$$

**Property 2: Equilibrium points**

Since now the configuration space is duplicated, we have that each equilibrium point appears also two times, as we can see in Figure 7.

![Figure 7: Primaries' position and equilibrium points in Levi Civita coordinates. In blue the triangular points, in green the triangular ones.](image-url)
Property 3: Regularized form of the invariant relation

If we apply the change of variables of the regularization to the expression for the Jacobi integral in (55), we end up with the regularized form for the invariant expression, which is

\[(u')^2 + (v')^2 = \frac{df}{d\tau}2(V - Z) = 8(u^2 + v^2)(V - Z),\]  

(94)

which now does not have the singularity in \(P_1\).

With this, we can obtain a condition for the velocities in the position of the first primary, i.e., for \(u = 0, v = 0\):

\[(u')^2 + (v')^2 = \frac{8(1 - \mu)}{1 + e \cos \phi_0},\]  

(95)

where the integral \(Z\) vanishes because we are in the initial time, and also because it is multiplied by \((u^2 + v^2)\) which is equal to 0 in our initial point.

Summary

Using the change (61), and also differentiating in order to obtain the derivatives (recall that we have denoted \((') = \frac{d}{d\tau}\) and \((\cdot) = \frac{d}{d\tau}\), we have the two possibilities:

- If we have a point \((u, v)\) and the velocities \((u', v')\) in the Levi-Civita coordinates, then for recovering the synodical coordinates we just need to use

\[
\begin{align*}
\xi &= u^2 - v^2 + \mu, & \dot{\xi} &= \frac{uu' - vv'}{2(u^2 + v^2)}, \\
\eta &= 2uv, & \dot{\eta} &= \frac{u'v + uv'}{2(u^2 + v^2)}. 
\end{align*}
\]  

(96)

- If we have a point \((\xi, \eta)\) and the velocities \((\dot{\xi}, \dot{\eta})\) in the synodical coordinates, then for obtaining the Levi-Civita variables we have to compute

\[
\begin{align*}
u &= \frac{\pm \sqrt{(\xi - \mu) + \sqrt{(\xi - \mu)^2 + \eta^2}}}{\sqrt{2}}, & u' &= 2(u\dot{\xi} + v\dot{\eta}), \\
v &= \frac{\pm \sqrt{2(\eta) + \sqrt{(\xi - \mu)^2 + \eta^2}}}{\eta}, & v' &= 2(u\dot{\eta} - v\dot{\xi}).
\end{align*}
\]  

(97)
4. Ejection-Collision orbits and numerical results

Once we have finished with the theoretical part about the equations of motion and the regularization, the next step is to define which is going to be our approach. There is a great variety and richness in the dynamics of this system, and we have many different options. But in this work, we are going to focus on a specific type of orbits, the ones called ECOs:

**Definition.** An orbit is an $n$-ejection-collision orbit ($n$-ECO), when the body is ejected from one of the primaries, and after some path, collides with the same primary in the $n$-th minimum distance with respect to it.

As we mentioned briefly before, this is the reason why we have introduced the regularization, since in all the formulations that we have seen (at least the not regularized ones), we have the values of $r_1$ and $r_2$ in the denominator, thus we will face with singularities when we try to study this kind of orbits (recall that in the ejection and in the collision we have that the corresponding radius is zero).

In Figure 8 we have some examples of how these orbits look like. As we can appreciate, these orbits only collide with the primary in the $n$ minimum of the distance.

![Figure 8: 1, 2, 3-ECO, respectively, with parameters $\mu = 0.5$, $f_0 = 0$, $C = 5$ and $e = 0.1$.](image)

But, how can we compute these orbits? There are many methods to find them, but we will use the one that has to be with the value of the angular momentum, which is based on the following characterization:

**Proposition 4.1.** If we start from ejection, i.e., at the point $w = (u, v) = (0, 0)$, then we will have a collision iff

1. $|w|^2 = u^2 + v^2$ is a minimum.
2. The angular momentum, $M_{LC}(w) = uv' - u'v$, is zero.
3. $u^{'2} + v^{'2} \neq 0$.

**Proof.** $\Rightarrow$ In a collision we have that $u = v = 0$, then is trivial to check the conditions.

$\Leftarrow$ By (i), since we have a minimum, $uu' + vv' = 0$. Let us assume that $v' \neq 0$ (the other case is analogous). Using (ii) we have that $u = \frac{uv'}{v'}$. Then, $\frac{uv^{'2}}{v'} + vv' = 0 \Rightarrow v(u^{'2} + v^{'2}) = 0$. By (iii), $u^{'2} + v^{'2} \neq 0$, therefore
Then, our goal will be to find the initial conditions such that we have an ECO, i.e., we need to find \((u, v, u', v')\) satisfying the properties that we have just seen. Since we are starting on the first primary, our initial positions will be \((u_0, v_0) = (0, 0)\), so we have to play with the angle at which the body is ejected. In order to do this, we can parameterize the velocities as 
\[
  u'_0 = b \cos \theta \\
  v'_0 = b \sin \theta
\]
for \(\theta \in [0, 2\pi]\). But recall that we have seen a condition for the velocities at the first primary in (95), that gives us the concrete expression for \(b\)
\[
  b := \sqrt{\frac{8(1 - \mu)}{1 + e \cos f_0}}, \tag{98}
\]
and also, due to the duplication of the space that comes with the LC regularization, we have that is enough to work with half the plane, so \(\theta \in [0, \pi]\). Moreover, we have that at the initial time, the integral (57) is equal to zero. So, in order to find the \(n\)-ECOs, we will set as initial conditions
\[
  (u_0, v_0, u'_0, v'_0, Z_0, f_0) = (0, 0, b \cos \theta, b \sin \theta, 0, f_0). \tag{99}
\]
Then, the first step is to integrate the Elliptical system, in the LC formulation, up to the Poincaré section that corresponds to the one where we have a local minimum in the distance between the body and the first primary (we impose this because we need it by the characterization that we have seen), which is
\[
  \Sigma_n := \{(u, v) \in \mathbb{R}^2 : uu' + vv' = 0, (u')^2 + (v')^2 + uu'' + vv'' > 0\}, \tag{100}
\]
where the subindex \(n\) means which minimum we are going to consider, i.e., for the 1-ECOs we will use \(\Sigma_1\), and for the \(n\)-ECOs, \(\Sigma_n\). By doing this, we can obtain the angular momentum for each corresponding value of \(\theta\), so the next step is to find for which angles we have a zero angular momentum on the intersection with \(\Sigma_1\). In Figure 9 we can observe that, for the parameters \(\mu = 0.5, f_0 = 0, C = 5\) and \(e = 0.1\), we have that there are four crossings with the \(x\) axis, so we will have that, for \(i = 1, \ldots, 4\), \(\exists \theta_i\), for which we will have a 1-ECOs.

![Figure 9: Values for the angular momentum at the crossing with \(\Sigma_1\). In circles the values where \(M_{LC} = 0\). Parameters: \(\mu = 0.5, f_0 = 0, C = 5\) and \(e = 0.1\)](image)

Once we have a mesh with the values for the angular momentum, we only need to apply the bisection method in every section where the sign changes. With this, we have obtained the values for \(\theta_i, i = 1, \ldots, 4\).
Finally, integrating the system up to the Poincaré section $\Sigma_1$ and using with initial conditions these angles, we have obtained the four 1-ECOs, as we can see in Figure 10 (be aware that if we do not specify otherwise, the plots will be in synodical coordinates, by using the changes that we have defined before (96)).

![Figure 10: The four 1-ECOs, $\sigma_i$, corresponding to the orbit with initial condition $\theta_i$, $i = 1, \ldots, 4$, that we have in Figure 9. Parameters: $\mu = 0.5$, $f_0 = 0$, $C = 5$ and $e = 0.1$.](image)

In an analogous way, we can compute the initial condition for the angles where we have $n$-ECOs, simply using $\Sigma_n$ instead of $\Sigma_1$. To sum up, the steps that we have followed in this method are

i. We set the initial conditions with the form that we had in (99).

ii. We establish the mesh, as refined as we want, for $\theta \in [0, \pi]$.

iii. We compute the angular momentum for every value of $\theta$ in the mesh, in the intersection with $\Sigma_n$, imposing the conditions that are necessary to have a collision (Prop 4.1).

iv. Once we have a mesh with the values for the angular momentum, we want to find where it is zero. We use as initial conditions for the bisection method the ones in which we have a change of sign.

v. Finally, we obtain a set of $k$ values in which we have a $n$-ECOs. These values are the angles $\theta_i$, for $i = 1, \ldots, k$ that we have to use on the initial condition (99). Integrating the system up to the cross with the Poincaré section, $\Sigma_n$, we obtain the $n$-ECOs.

**Study of 1-ECOs**

Now that we have an algorithm for computing such orbits, we are ready to play a little. But before, we would like to summarize (as possible) the results that we have until now, regarding this kind of orbits. In the circular case, although we are not going to focus on this problem, there were many results. In [15] a first approach to the study of the $n$-ECOs was done, exploring the bifurcations that appear. In [16] the authors proved that for $n$ fixed, and a value for the Jacobi constant big enough, there exists a $\mu$ sufficiently small for which there exist exactly four $n$-ECOs, which are also well characterized. Later on, this result was improved, showing that, for every value of $\mu$, and for $n \geq 1$, there exists a value big enough for the Jacobi constant, that depends of $\mu$ and $n$, for which there are exactly four families of $n$-ECOs when varying $C$. This proof was done using the Levi-Civita’s regularization and then perturbing the system (see [10]). There is also an analogous study of the 1-ECOs, but in the spatial circular RTBP, using the McGehee
regularization (see [13]).

Unfortunately, in the elliptical RTBP there are only a few results. Up to know, it has been proven that, when \( \mu \) is small enough, then there exist 1-ECOs (see [9]). Later on, this result was improved, proving that for small enough value of the mass parameter \( \mu \) and also small eccentricity \( e \), if a sufficiently big value of the Jacobi Constant is fixed, then there exist four continuous families of initial conditions for 1-ECOs (see [18]). To sum up, in the elliptical RTBP, up to now, we only have analytical results regarding 1-ECOs, and also only for values small enough of the mass parameter and the eccentricity.

So, our first goal is to check numerically this last result, and, if possible, going further. Then, let us start with a small value for \( \mu \) and \( e \). In Figure 11 we can see that, for \( C \in [10, 30] \) we have exactly four families of initial conditions for the angles, as we expected by the analytical results.

![Figure 11: Initial conditions for \( \theta \) for the four families of 1-ECOs, where \( e = 0.01, \mu = 0.01, f_0 = 0 \) and \( C \in [10, 30] \).](image)

In this simulation, we have a small eccentricity, which means that our primaries follow almost circular orbits, so the initial value that we choose for \( f_0 \) is practically negligible since it has no sense in considering the true anomaly on a circular framework. We can also observe in Figure 11 that the families of initial conditions for the angles tend to the values \( 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4} \), respectively, when \( C \) tends to infinity.

As we commented, the analytical result in [18] has been proven for small enough values for \( e \) and \( \mu \), so we wanted to see numerically if we can “relax” this condition a little and still have four families of initial conditions for \( \theta \). In Figure 12 and Figure 13 we have increased these values (in the three first we have \( \mu \) fixed, and different values for \( e \), and in the three second, \( e \) is fixed and we vary \( \mu \)) and, as we can observe, these four families persist also for \( e \) and \( \mu \) not so small.
Ejection-collision orbits in the ERTBP

Figure 12: Initial conditions for $\theta$ for the four families of 1-ECOs, where $C \in [10, 30]$, $f_0 = 0$, $\mu = 0.01$ and $e = 0.1, 0.3, 0.5$, from left to right.

Figure 13: Initial conditions for $\theta$ for the four families of 1-ECOs, where $C \in [10, 30]$, $f_0 = 0$, $e = 0.05$ and $\mu = 0.2, 0.4, 0.5$, from left to right.

But once we increase the eccentricity, there is a parameter that will be more relevant: the initial value for the true anomaly. For not so small eccentricities, the role of $f_0$ starts being more significant, since the initial position for the primaries will be different (see Figure 14).

![Figure 14: Same configuration, but with $f_0 = 0$ (left) and $f_0 = \pi$ (right).](image)

In Figure 15 we have the distance between the two primaries, using the solution for the Two-Body Problem (22), where we have fixed the semi-major axis, $a = 1$, and we have varied the eccentricity and the initial value for the true anomaly. As we can observe, for small eccentricities there is almost no difference on the distance, no matter what value for $f_0$ we choose. But as we increase the eccentricity, we can observe that the minimum distance is attained for $f_0 = 0$ and $f_0 = 2\pi$, and the maximum distance for $f_0 = \pi$. This makes sense: if we compute these values in (22), we have that $r(0) = (1 - e)$ and $r(\pi) = (1 + e)$. For this argument is not relevant the value of the semi-major axis, since if we increase or decrease $a$, we will have a greater or smaller maximum and minimum, but they still will be the maximum and minimum.
Another consideration regarding the value of $f_0$ is that, if we have in mind the condition for the velocities that we had in (95), it is easy to see that if $f_0 = \pi$ then the velocities, and therefore the modulus, will grown faster if we have $f_0$ near 0 or $2\pi$.

Now that we have justified that it is indeed relevant which value for $f_0$ we are using, and also we can guess that for $f_0$ near $\pi$ we may have some troubles, we have computed, in the same way as before, the angular momentum at the intersection with $\Sigma_1$, for $f_0 = \pi$, and at some point, we have obtained some discontinuities. Then, some questions that naturally arise to us when observing this phenomenon are why these discontinuities appear, and where the angular momentum’s function starts to “break”. In order to find at which moment this phenomenon starts, and therefore study how the orbits look like at these points, we have just done the simulations with a value of the Jacobi constant, and then decreased it by small steps. In Figure 16 (we only show the zoom for $\theta \in [1.5, 1.9]$ because in the rest of the interval these functions behave the same) we have four plots: in the first one the function behaves smoother, then in the second one we can observe that the central part starts to deform until in the third one a discontinuity appears. In the last plot, we can see that this phenomenon is amplified.

Then, we have seen that the effect of the Jacobi constant is pretty strong, since we can have discontinuities if it is small enough; as we said in previous chapters, we can not talk about bounded regions for the motion but indeed it still affects the behaviour of our orbits. Although we are not in the same situation as in the circular case (remember the bounded regions in Figure 4) due to the pulsating coordinates, the Jacobi constant represents (part of the energy of the system, so if we do not impose enough energy to the body, it could be more affected by the influence of the other primary or by the equilibrium points, in concrete by the nearest, $L_1$.

Now let us analyse the orbits that are causing these discontinuities. We have chosen the angle $\theta = 1.7085$, which corresponds to the discontinuity at the right in the third picture at Figure 16, and then we have integrated the corresponding orbit, up to time $\tau_1$ (where $\tau_i$ denotes the necessary time to reach the $i$-th minimum), for each one of the four values of $C$ that we have used in these simulations. In Figure 17 we can see the shape of each orbit, that we have denoted as $o_k$, where $o_k$ is the corresponding orbit for $C_k$, $k = 1, \ldots, 4$. 
Let us interpret these pictures: the first consideration is that the orbits that correspond to the values of $C$ where we still have a continuous function, $C_1 = 5.08$ and $C_2 = 5.0433$, are almost the same. When we decrease the value of $C$, and the discontinuities start, we have that the orbits $o_2$ and $o_3$ are no longer the same, in fact, for both, the orbits barely travel half of the way with respect to the other two (we have marked as a black asterisk the last point of each orbit at the intersection with $\Sigma_1$). In Table 4 we can observe that, on one hand, the time necessary to attain the first minimum, $\tau_1$, is smaller for the orbits that correspond to the $C$ where we have discontinuities, and on the other hand, the distance between the last point of each orbit, and the first primary, $P_1$ is greater for these two orbits that appear when we have discontinuities.

<table>
<thead>
<tr>
<th>Orbit</th>
<th>$C$</th>
<th>$\tau_1$</th>
<th>dist</th>
</tr>
</thead>
<tbody>
<tr>
<td>$o_1$</td>
<td>5.08</td>
<td>1.8394</td>
<td>0.1947</td>
</tr>
<tr>
<td>$o_2$</td>
<td>5.0433</td>
<td>1.8394</td>
<td>0.1947</td>
</tr>
<tr>
<td>$o_3$</td>
<td>5.0432</td>
<td>1.1465</td>
<td>0.4028</td>
</tr>
<tr>
<td>$o_4$</td>
<td>5.0431</td>
<td>1.1277</td>
<td>0.4043</td>
</tr>
</tbody>
</table>

The loops that we can observe at Figure 17 are caused by the proximity of the equilibrium point $L_1$, which has a family of quasi-periodic orbits around it, or what is the same, is a two dimensional torus. The closer we get to that torus, the more this behaviour will be accentuated. It also may happen that an orbit "crosses to the other side", in the sense that this orbit will overcome the effect of $L_1$, and will be attracted to the second primary, $P_2$. We can observe this behavior on Figure 18, where we have in blue the orbits that are
Figure 17: Four orbits $\alpha_k$, with $C = C_k$, $k = 1, \ldots, 4$, $\mu = 0.5$, $e = 0.21$ and $f_0 = \pi$. The red cross is the first primary $P_1$, the red circle is the collinear equilibrium point $L_1$ and the black asterisk is the last point of each orbit in the intersection with $\Sigma_1$.

near $L_1$, then form loops, and finally return close to the first primary, and we also have the orbits plotted in green that are the ones that go to the left of the collinear equilibrium point, and then are more affected by the influence of $P_2$ and go near collision (later on we will talk about this). See [17] for an exhaustive analysis in the circular case about the relation between the Lyapunov periodic orbit around the collinear equilibrium point $L_1$ and the ECOs.

Also, as we can guess by the Table 4, another way of detecting the loops is by looking at the time necessary to reach the $i$-th minimum, $\tau_i$. We have plotted in Figure 19 the function of $\tau_1$ with respect of $\theta \in [0, \pi]$, with a global and a zoom view (the same as we have done for the angular momentum), for $C = 5.0432$, which is the value at which we start to see the discontinuities in Figure 16. If we use this approach, we also obtain that one of the angles at which we have a discontinuity is $\theta = 1.7085$, the same as before.

As a remark, we have seen that the Jacobi constant is a very powerful parameter, in the sense that we can increase it in order to avoid discontinuities (at least for the simulations that we have done, at the moment there is no an analytical result that ensures us that this will always happen, independently of the parameters), so, in order to show this phenomenon, we have at Figure 20 the relation between the value of the eccentricity and the initial condition for $\theta$ that we need for obtaining 1-ECOs. We have three pictures, each one corresponding to a different value for the Jacobi constant: $C = 5$, $C = 7$ and $C = 10$, and we only need to take a look at these pictures to see that for greater values of $C$ we have the four families of
Figure 18: Two types of orbits: in blue the orbits that are attracted by the equilibrium point $L_1$ (red circle) and therefore form loops, and in green the orbits that pass to the left of the equilibrium point, and end up being affected by the second primary, $P_2$ (black cross). Parameters: $\mu = 0.5$, $e = 0.21$, $C = 5$, $f_0 = \pi$, and $\theta \in [1.7, 1.76]$ (green orbits) and $\theta \in [1.76, 1.82]$ (blue orbits).

Figure 19: Time necessary to reach the crossing with $\Sigma_1$, $\tau_1$, for $\mu = 0.5$, $e = 0.21$, $f_0 = \pi$ and $C = 5.0432$. Global (left) and zoom for $\theta \in [1.5, 1.9]$ (right).

initial conditions (even if we have set $f_0 = \pi$, which is the value that mainly causes the phenomena that we have analysed until now: loops and collision with the second primary), while if we are in the situation that $C$ is not big enough, then we no longer have this structure (concretely, we can observe this near the third family of initial conditions, the yellow one), and we also can see that the families are more inclined the more the $C$ grows.
Figure 20: Initial conditions of $\theta$ for the four families of 1-ECOs, for $C = 5$ (left), for $C = 7$ (middle) and $C = 10$ (right), $\mu = 0.5$, $f_0 = \pi$, $e \in [0, 0.25]$.

Up to now, we have seen that the discontinuities are caused by the proximity with the collinear equilibrium point $L_1$, leading to orbits in which loops appear. But, is this the only possibility? If we decrease more the Jacobi constant, for example for $C = 5$, we have in Figure 21 the angular momentum (computed in the same way as before). We have a global view and also a zoom on the interval where we have discontinuities, in order to observe better which shape this function has.

![Angular momentum](image)

Figure 21: Angular momentum up to the crossing with $\Sigma_1$, global (left) and zoom (right), for $\mu = 0.5$, $e = 0.21$, $C = 5$, $f_0 = \pi$.

In this case, we have four angles where we have discontinuities, that we are going to denote as $\theta_i$, $i = 1, \ldots, 4$, where the first one is the one at the left. At a first sight, the shape of the function at the points where we have the discontinuities corresponding to $\theta_1$ and $\theta_4$ seems similar to the situation that we had before,
but in the middle (for $\theta_2$ and $\theta_3$) we can observe that there are two asymptotes. We have computed the corresponding orbits for these four angles (see Figure 22), and, as we have said, we can observe loops in the first and the last orbits, therefore we have these two discontinuities caused by $L_1$. For the other two, we have that these orbits pass very close to the second primary (marked as a black cross), and this is exactly the reason why we have the asymptotic behavior on the angular momentum plot: recall that in order to perform all of these simulations we have had to regularize the system before, because of the singularities, and recall also that the method that we have used is not global, we had only get rid off the singularity regarding $P_1$. Then, for these two orbits we are too close to collision with the other primary, i.e., we are too close to a singularity, and therefore we can not work with this formulation.

Figure 22: The four orbits $o_i$, corresponding to the initial condition for the angle $\theta_i$, where $i = 1, \ldots, 4$, $\mu = 0.5$, $e = 0.21$, $C = 5$, $f_0 = \pi$. The black cross at right denotes the first primary, $P_1$, the black cross at left the second primary, $P_2$, and the red circle is the equilibrium point $L_1$.

To sum up, until now we have seen that we can use values for $\mu$ and $e$ not so small, and we still obtain four families of initial condition for $\theta$, but we need to be careful with which values of $C$ and $f_0$ we use. By all that we have explained until now, from now on we will use suitable values: a high enough Jacobi constant and also $f_0 = 0$. 

Study of $n$-ECOs

Until now, we have focused on the analysis of the 1-ECOs, and we have seen which is the suitable choice for the parameters. Now, we want to dedicate the last part of this chapter to make a small initial exploration to the study of the $n$-ECOs. First, we have computed the initial conditions for $\theta$ for the families of $n$-ECOs, $n = 1, \ldots, 10$, for $f_0 \in [0, 2\pi]$, with $C = 7$ (see Figure 23).

Figure 23: Initial conditions for $\theta$ for the families of $n$-ECOs, $n = 1, \ldots, 10$, from left to right, with $\mu = 0.5$, $C = 7$, $e = 0.1$ and $f_0 \in [0, 2\pi]$.

As we can observe in Figure 23, we obtain four families of initial conditions for the angle $\theta$, at least until a high value for $n$ is reached. In this case, we have that for $n = 9$ and $n = 10$ we no longer have four families, and the phenomenon that we observe is due to the bifurcations: we say that a bifurcation occurs
when we observe that a small change to some parameters of the system causes a sudden qualitative or
topological change in the behavior of the system. In the plots corresponding to the initial condition for the
angles for 9-ECO and 10-ECO, we can see that in the middle region for the $y$ axis, i.e., when $f_0$ is near to
$\pi$, we have some values for $f_0$ where we obtain 6 initial conditions for $\theta$, instead of the 4 values that we had
up to this moment: we can see this in Figure 24, where we have marked the corresponding $\theta_i$, for $i = 1, \ldots, 6$.

![Figure 24: Initial conditions for $\theta$ for the families of 10-ECOs, with $\mu = 0.5$, $C = 7$, $e = 0.1$ and $f_0 \in [0, 2\pi]$. The red line is marked for $f_0 = 2.6770$, and the color dots denote the six initial conditions, $\theta_i$, $i = 1, \ldots, 6$, for which we have a 10-ECOs, for this value of $f_0$.](image)

In Figure 25 we have the orbits $o_i$, that we have computed using as initial condition the angles $\theta_i$, $i = 1, \ldots, 6$, that we had in Figure 24. As we can observe, the orbits that appear due to the bifurcations are pretty simi-
lar, since they come from nearby values for the angle $\theta$ (we can see this for $o_2$ and $o_3$, and also for $o_4$ and $o_5$).

We also have computed the initial values for $\theta$ for which we have $n$-ECOs, but now with a smaller value of
the Jacobi Constant, $C = 5$. As we can observe, we obtain a similar behavior, but it is attained sooner (the
last Figure of 23, regarding 10-ECOs, is pretty similar to the fifth plot in Figure 26, regarding 5-ECOs).
Then, in both cases, $C = 7$ and $C = 5$ we obtain the same phenomenon, bifurcations, but it seems like
when the value of $C$ is smaller, more affected by small changes is the system.

Now, instead of varying $f_0$, we have repeated the simulations, but now for $e \in [0, 0.25]$ (see Figure 27). In
the plots we can see that, from $n = 7$ onward, bifurcations begin to form, starting from very small values
of the eccentricity and then extending this phenomena to higher values of $e$.

To sum up, we have seen that when we increase $n$, we start to observe the emergence of bifurcations,
and also that this phenomenon, in some sense, can be slowed increasing the value of $C$, which raises the
question if it could be achieved a similar analytical result regarding the existence of a value for $C$ for which
we will have only four families of initial condition for $n$-ECOs, as in the circular case.
Figure 25: 10-ECOs, denoted by $o_i$, where each one corresponds to an orbit with $\theta_i$ as initial condition, for $i = 1, ..., 6$ (see Figure 24). Parameters: $\mu = 0.5$, $e = 0.1$, $C = 7$ and $f_0 = 2.6770$. We have marked each petal with a number between 1 and 10, that means which is the order that the orbit follows, and the red asterisk denotes the first primary $P_1$. 
Figure 26: Initial conditions for $\theta$ for the families of $n$-ECOs, $n = 1, \ldots, 10$, from left to right, with $\mu = 0.5$, $C = 5$, $e = 0.1$ and $f_0 \in [0, 2\pi]$. 
Figure 27: Initial conditions for $\theta$ for the families of $n$-ECOs, $n = 1, \ldots, 10$, from left to right, with $\mu = 0.5$, $C = 5$, $f_0 = 0$ and $e \in [0, 0.25]$. 
5. Conclusions

Along this work, we have presented the Restricted Three-Body Problem, first introducing and developing a compact formulation, then regularizing the system in order to get rid of the singularities (only one of them, since we have chosen a local method) and, finally, we have made a thorough numerical study of one particular type of orbits: the ejection-collision orbits. In first chapter we have seen how to transform our initial problem onto an adimensional and synodical system of equations of motion, and we have also observed that we obtain the same shape for both, Circular and Elliptical cases, although there are some differences between them, since in the Elliptical RTBP we have obtained a non-autonomous system, and also we cannot talk about Hill’s Region, due to the effect of the pulsating coordinates. In second chapter we have seen a way to regularize our system, which is easy to understand since it only involves two changes of variables, although the physical meaning of this change is it not obvious, unlike other regularization methods, such as the McGehee method (see [12]), which consists in a change to polar coordinates. Also we have provided the explicit formulation of the regularized equations of motion. Finally, in the chapter dedicated to the numerical computations, we have presented the definition for the $n$-ejection-collision orbits and a suitable algorithm to compute them. We also have analysed the influence of the parameters of the system, and when (and why) we may expect having problems with our method, showing which may be a suitable choice for the values of these parameters. Although we have focused on the analysis of the 1-ECOs, we also have done a first exploration of which is the influence of $n$: the higher it is, the more bifurcations appear, the system becomes more sensitive to small changes, and also the dynamics of the system are richer.

We would like to make a few comments: first of all, we would like to highlight the work that has been done regarding the explicit deduction of the system of equations of motion for the regularized Elliptical RTBP. We also want to recall that we have seen, numerically, that if we use suitable parameters (for $f_0$ near 0 and for $C$ not so small) we can extend the analytical result [18] regarding the existence of four families of initial conditions for 1-ECOs, increasing the parameters $\mu$ and $e$ and still obtaining these four families. And finally, we also want to express our intention to continue working on this topic. Some questions that may be interesting in future work are if we can find an expression for the Jacobi constant for which there exist only four families of initial conditions for the $n$-ECOs, in an analogous way as in the analytical result for the circular case. It could also be interesting to study deeply the family of quasi-periodic orbits associated to the collinear equilibrium point $L_1$, since it has a great influence on the transit orbits between the primaries.
References


