On approximating shortest paths in weighted triangular tessellations*

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Abstract. We study the quality of weighted shortest paths when a continuous 2-dimensional space is discretized by a weighted triangular tessellation. In order to evaluate how well the tessellation approximates the 2-dimensional space, we study three types of shortest paths: a weighted shortest path $SP_w(s,t)$, which is a shortest path from s to t in the space; a weighted shortest vertex path $SVP_w(s,t)$, which is a shortest path where the vertices of the path are vertices of the tessellation; and a weighted shortest grid path $SGP_w(s,t)$, which is a shortest path whose edges are edges of the tessellation. The ratios $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$, $\frac{\|SVP_w(s,t)\|}{\|SP_w(s,t)\|}$, $\frac{\|SGP_w(s,t)\|}{\|SVP_w(s,t)\|}$ provide estimates on the quality of the approximation.
Given any arbitrary weight assignment to the faces of a triangular tessellation, we prove upper and lower bounds on the estimates that are independent of the weight assignment. Our main result is that $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|} = \frac{2}{\sqrt{3}} \approx 1.15$ in the worst case, and this is tight.

Keywords: Shortest Path, Tessellation, Weighted Region Problem

1 Introduction

Geometric shortest path problems, where the goal is to find an optimal path in a geometric setting, are fundamental problems in computational geometry. In contrast to the classical shortest path problem in graphs, where the space of possible paths is discrete, in geometric settings the space is continuous: the source and target points can be anywhere within a certain geometric domain (e.g., a polygon, the plane, a surface), and the set of possible paths to consider has infinite size. Many variations of geometric shortest path problems exist, depending on the geometric domain, the objective function (e.g., Euclidean metric, link-distance, geodesic distance), or specific domain constraints (e.g., obstacles in the plane, or holes in polygons). Applications of geometric shortest path problems are ubiquitous, appearing in diverse areas such as robotics, video game design, or geographic information science. We refer to Mitchell [?] for a complete survey on geometric shortest path problems.

One of the most general settings for geometric shortest path problems arises when the cost of traversing the domain varies depending on the region. That is, the domain consists of a planar subdivision, that without loss of generality can be assumed to be triangulated. Each region i of the subdivision has a weight w_i, that represents the cost per unit of distance of traveling in that region. Thus, the cost of traversing a region is typically

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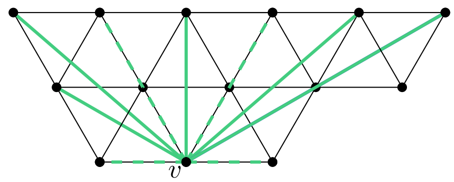
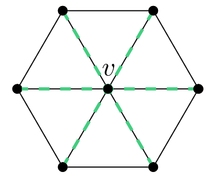
given by the Euclidean distance traversed in the region, multiplied by the corresponding weight. The resulting metric is often called the *weighted region metric*, and the problem of computing a shortest path between two points under this metric is known as the *weighted region problem* (WRP) [12, 13]. The WRP is very general, since it allows to model many well-known variants of geometric shortest path problems. Indeed, having that all weights are equal makes the metric equivalent to the Euclidean metric (up to scaling), while using two different weight values, such as 1 and ∞, allows to model paths amidst obstacles.

Perhaps not surprisingly, the WRP turns out to be a challenging problem. The first algorithm for WRP was a $(1+\varepsilon)$-approximation proposed by Mitchell and Papadimitriou [13], which runs in time $O(n^8 \log\left(\frac{nNW}{w\varepsilon}\right))$, where N is the maximum integer coordinate of any vertex of the subdivision, W and w are, respectively, the maximum finite and the minimum nonzero integer weight assigned to the regions. Substantial research has been devoted to studying faster approximation algorithms and different variants of the problem [1–3] . Approximation schemes for WRP are sophisticated methods that usually are based on variants of continuous Dijkstra, subdividing triangle edges in parts for which crossing shortest paths have the same combinatorial structure (e.g., [13]), or work by computing a discretization of the domain by carefully placing Steiner points (e.g., see [8] for the currently best method of this type). The lack of exact algorithms for WRP is probably justified by algebraic reasons: WRP was recently shown to be impossible to solve in the Algebraic Computation Model over the Rational Numbers [9]. This is a model of computation where one can compute exactly any number that can be obtained from rational numbers by a finite number of basic operations. Efficient algorithms for WRP only exist for a few special cases, e.g., rectilinear subdivisions with L_1 metric [7], or weights restricted to $\{0, 1, \infty\}$ [10].

In applications where the WRP arises, like robotics, gaming or simulation, which usually require efficient and practical algorithms, the problem is simplified in two ways. First, the domain is approximated by using a (weighted) plane subdivision with a simpler structure. Secondly, optimal shortest paths in that simpler subdivision are approximated. The typical way to represent a 2D (or 3D) environment where shortest paths need to be computed is by using *navigation meshes* [16]. These are polygonal subdivisions together with a graph that models the adjacency between the regions. Path planning is then done first on the graph to obtain a sequence of regions to be traversed, and then within each region, for which a shortest geometric path is extracted. Triangles, convex polygons, disks or squares—of different sizes—are among the most frequently used region shapes [16]. Navigational meshes allow efficient path planning in large environments as long as the region weights are limited to $\{1, \infty\}$ (i.e., obstacles only). In case general weights are needed, the complexity of computing the shortest path inside each region requires the use of the simplest possible navigational mesh: *regular grids*. In 2D, the only three types of regular polygons that can be used to tessellate continuous environments are triangles, squares and hexagons. The drawback with a grid is that it imposes a fixed resolution, requiring in general a large number of cells or regions. Still, grids are often used as navigation meshes (even for weights $\{1, \infty\}$), since they are easy to implement, are a natural choice for environments that are grid-based by design (e.g., many game designs), and popular shortest path algorithms such as A^* can be optimized for grids [14].



(a) Some neighbors of a vertex v in the corner-vertex graph.



(b) All neighbors of a vertex v in the 6-corner grid graph.

Fig. 1: Vertex v is connected to its neighbors in a triangular tessellation.

Even when a regular grid is used as a navigation mesh, in practice exact weighted shortest paths are not computed: instead, an approximation is obtained by computing a shortest path on a weighted graph associated to the grid. To this end, two different graphs have been considered in the literature [**?**], *corner-vertex graph* and *k-corner grid graph*, defined next. The baseline to analyze the quality of any approximate path is the weighted shortest path that takes into account the full geometry of each region, as in the WRP. A weighted shortest path will be denoted $SP_w(s,t)$. As already mentioned, exact weighted shortest paths for regions with weights in $\{0, 1, \infty\}$ were studied in [12, 13].

In a *corner-vertex graph* G_{corner}, the vertex set is the set of corners of the tessellation and every pair of vertices is connected by an edge. These graphs can be seen as the complete graphs over the set of vertices. Figure 1a depicts some of the neighbors of a vertex v in the corner-vertex graph. Note that in this graph some edges overlap. A path in this graph is called a *vertex path*; a shortest vertex path between s and t will be denoted $SVP_w(s,t)$, where w makes explicit that this path depends also on a particular weight assignment w.

In a *k-corner grid graph* $G_{k\text{corner}}$, which is a subgraph of a corner-vertex graph, the vertex set is the set of corners of the tessellation, and each vertex is connected by an edge to a predefined set of k neighboring vertices, depending on the tessellation and other design decisions. See Figure 1b for the 6-corner grid graph in a triangular tessellation. (Analogous k-corner grid graphs can be defined for square and hexagonal tessellations.) A path in this graph is called a *grid path*; a shortest grid path between s and t will be denoted $SGP_w(s,t)$.

Shortest vertex paths and shortest grid paths for the case of weights of the cells being 1 or ∞ have been previously studied in [15] and [4], respectively. In all cases, the weight of each graph edge is defined based on the cost of the associated line segment, depending on the weights of the regions that it goes through. More formally, let T_i be a region in a subdivision with weight $\omega_i \in \mathbb{R}_{>0}$. The cost of a segment π_i in the interior of a cell T_i is given by $\omega_i\|\pi_i\|$, where $\|\cdot\|$ is the Euclidean norm. In the case where a segment π lies in the boundary of two cells T_j and T_k, the cost is $\min\{\omega_j, \omega_k\}\|\pi\|$.

Figure 2 shows an example, illustrating the three paths considered: the shortest path $SP_w(s,t)$ (blue), the shortest vertex path $SVP_w(s,t)$ (green), and the shortest grid path $SGP_w(s,t)$ (red) in a 6-corner grid graph. Note that in all figures in this work, cells that are not depicted are considered to have infinite weight.

1.1 Quality bounds for approximation paths

The goal of this work is to understand the relation between $SGP_w(s,t)$, $SVP_w(s,t)$, and the baseline $SP_w(s,t)$. Since $SVP_w(s,t)$ and $SGP_w(s,t)$ are approximations of $SP_w(s,t)$, a fundamental question is: what is the worst-case approximation factor that they can give?

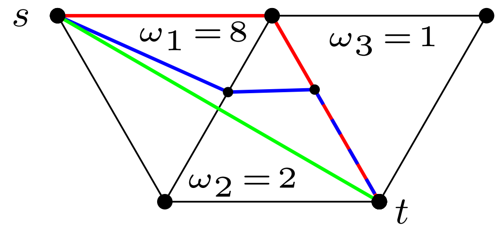


Fig. 2: $SP_w(s,t)$ (blue), $SVP_w(s,t)$ (green), and a $SGP_w(s,t)$ (red) between two corners s and t in $G_{6\text{corner}}$. The cost of each path is 16.75, 17.32 and 18, respectively.

In this paper we focus on weighted tessellations where every face is an equilateral triangle (analog ideas can be used for square and hexagonal grids). In particular, we are interested in upper-bounding the ratios $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$ and $\frac{\|SVP_w(s,t)\|}{\|SP_w(s,t)\|}$, since they indicate the approximation factor of the shortest grid and vertex path, respectively. The ratio $\frac{\|SGP_w(s,t)\|}{\|SVP_w(s,t)\|}$ is also studied, to see which approximation is better.

Almost all previous bounds on the ratio $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$ consider a limited set of weights for the cells. Nash [15] considered only weights in the set $\{1,\infty\}$ and proved that the weight of $SGP_w(s,t)$ in hexagonal $G_{6\text{corner}}$, and $G_{12\text{corner}}$, square $G_{4\text{corner}}$, and $G_{8\text{corner}}$, triangle $G_{6\text{corner}}$, and $G_{3\text{corner}}$ can be, respectively, up to ≈ 1.15, ≈ 1.04, ≈ 1.41, ≈ 1.08, ≈ 1.15, and 2 times the weight of $SP_w(s,t)$. When the weights of the cells are allowed to be in $\mathbb{R}_{>0}$, the only result that we are aware of is for square tessellations and another type of shortest path, with vertices at the center of the cells, for which Jaklin [11] showed that $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|} \leq 2\sqrt{2}$.

The main contribution of this paper is the analysis of the quality of the three types of shortest paths for a triangular grid for $G_{6\text{corner}}$, which is the most natural graph defined on a triangular grid. In contrast to previous work, we allow the weights ω_i to take any value in $\mathbb{R}_{>0}$, so the main challenge here is to obtain tight upper bounds that hold for *any* assignment of region weights. Surprisingly, we show that this is possible: the ratios are upper bounded by constants that are independent of the weights assigned to the regions in the tessellation. Our main result is that $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|} = \frac{2}{\sqrt{3}}$ in the worst case, for any (positive) weight assignment. This implies bounds for the other two ratios considered. Moreover, our upper bound for $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$ is tight, since it matches the lower bound claimed by Nash [15]. Table 1 summarizes our results, together with the previously known lower bounds.

2 $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$ ratio in $G_{6\text{corner}}$ for triangular cells

This section is devoted to obtaining, for two vertices s and t, an upper bound on the ratio $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$ in $G_{6\text{corner}}$ in a triangular tessellation $\mathcal{T}$. We suppose, without loss of generality, that the length of each edge of the triangular cells is 2, in order to have a non-fractional length for the cell height.

Let $(s = u_1, u_2, \ldots, u_n = t)$ be the ordered sequence of consecutive points where $GP_w(s,t)$ and $SP_w(s,t)$ coincide; in case $GP_w(s,t)$ and $SP_w(s,t)$ share one or more seg-

$\frac{\Vert SGP_w(s,t)\Vert}{\Vert SVP_w(s,t)\Vert}$		$\frac{\Vert SGP_w(s,t)\Vert}{\Vert SP_w(s,t)\Vert}$		$\frac{\Vert SVP_w(s,t)\Vert}{\Vert SP_w(s,t)\Vert}$	
Lower bound	Upper bound	Lower bound	Upper bound	Lower bound	Upper bound
$\frac{2}{\sqrt{3}} \approx 1.15$ [15]	$\frac{2}{\sqrt{3}} \approx 1.15$ [5]	$\frac{2}{\sqrt{3}} \approx 1.15$ [15]	$\frac{2}{\sqrt{3}} \approx 1.15$ (Thm. 1)	$\frac{2\sqrt{7\sqrt{3}-12}}{(7-4\sqrt{3})(6\sqrt{2}+\sqrt{7\sqrt{3}-12})} \approx 1.11$ [5]	$\frac{2}{\sqrt{3}} \approx 1.15$ [5]

Table 1: Bounds on the quality of approximations of shortest paths in weighted triangular tessellations for $G_{6\text{corner}}$. The upper bound for the ratio $\frac{\Vert SGP_w(s,t)\Vert}{\Vert SVP_w(s,t)\Vert}$, and the bounds for the ratio $\frac{\Vert SVP_w(s,t)\Vert}{\Vert SP_w(s,t)\Vert}$ are shown in the full version [5].

ments, we define the corresponding points as the endpoints of each of these segments, see Figure 3 for an illustration. Observation 1 below is a special case of the mediant inequality.

Observation 1 *Let $GP_w(s,t)$ and $SP_w(s,t)$ be, respectively, a weighted grid path, and a weighted shortest path, from s to t. Let u_i and u_{i+1} be two consecutive points where $GP_w(s,t)$ and $SP_w(s,t)$ coincide. Then, the ratio $\frac{\Vert GP_w(s,t)\Vert}{\Vert SP_w(s,t)\Vert}$ is at most the maximum of all ratios $\frac{\Vert GP_w(u_i,u_{i+1})\Vert}{\Vert SP_w(u_i,u_{i+1})\Vert}$.*

2.1 Crossing paths and weakly simple polygons

In the weighted version of the problem, conversely to the unweighted version, we need to take into account all the different weights of the regions intersected by $SP_w(s,t)$. In addition, we do not know the shape of the shortest paths $SP_w(s,t)$ and $SGP_w(s,t)$. To solve all these inconveniences, for each $SP_w(s,t)$ we will define a particular grid path called *crossing path* $X(s,t)$, whose behavior will be easier to control. See orange path in Figure 3. Then, the key idea to prove the upper bound on the ratio $\frac{\Vert SGP_w(s,t)\Vert}{\Vert SP_w(s,t)\Vert}$ will be to upper-bound it by the ratio $\frac{\Vert X(s,t)\Vert}{\Vert SP_w(s,t)\Vert}$. To do so, we will analyze the components resulting from the intersection between $SP_w(s,t)$ and $X(s,t)$. Each component will be a weakly simple polygon, which will be the basic unit that we will analyze to obtain our main result. Also, a relation between the weights of some cells intersected by $SP_w(s,t)$ and $X(s,t)$ will be obtained.

Let $(T_1,\ldots,T_n)$ be the ordered sequence of consecutive cells intersected by $SP_w(s,t)$ in the tessellation $\mathcal{T}$. Let v_1^i, v_2^i, v_3^i be the three consecutive corners of the boundary of T_i, $1 \leq i \leq n$. Let $(s = a_1, a_2, \ldots, a_{n+1} = t)$ be the sequence of consecutive points where $SP_w(s,t)$ changes cell in $\mathcal{T}$. In particular, let a_i and a_{i+1} be, respectively, the points where $SP_w(s,t)$ enters and leaves T_i. In a triangular tessellation, the crossing path $X(s,t)$ from a vertex s to a vertex t is defined as follows:

Definition 1. *The crossing path $X(s,t)$ between two vertices s and t in a triangular tessellation $\mathcal{T}$ is defined by the sequence $(X_1,\ldots,X_n)$, where X_i is a sequence of vertices determined by the pair (a_i,a_{i+1}), $1 \leq i \leq n$, as follows. Let $e_1^i \in T_i$ be an edge containing a_i, then:*

- *If $a_{i+1} \in e_1^i$, let $[v,w]$ be the endpoints of e_1^i, where a_i is encountered before a_{i+1} when traversing e_1^i from v to w. Then $X_i = (v,w)$, see Figure 4a.*

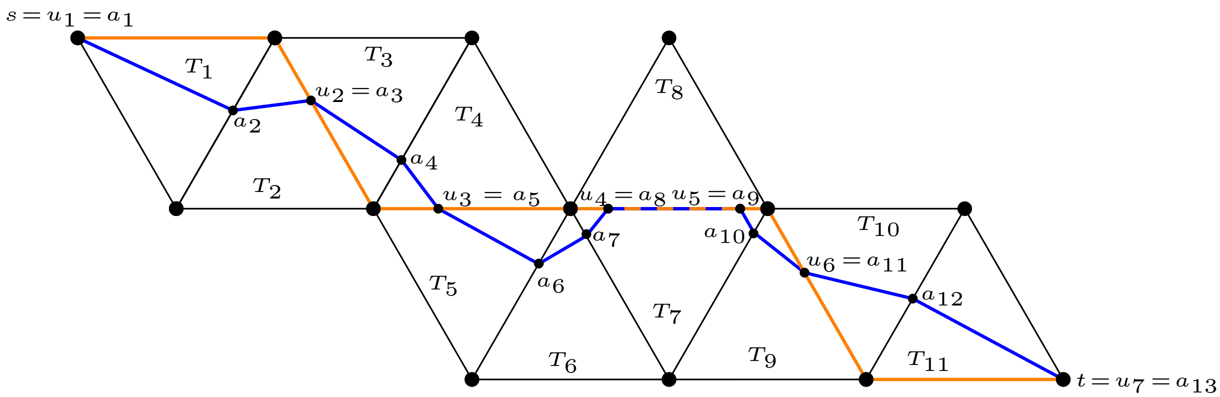


Fig. 3: Weighted shortest path $SP_w(s,t)$ (blue) and the crossing path $X(s,t)$ (orange) from s to t in a triangular tessellation.

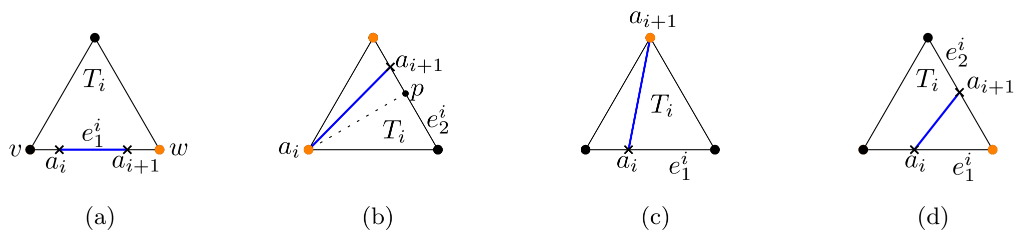


Fig. 4: Some of the positions of the intersection points between $SP_w(s,t)$ (blue) and a cell. The vertices of the crossing path $X(s,t)$ in a triangular cell are depicted in orange.

- *If a_i is an endpoint of e_1^i, let p be the midpoint of the edge $e_2^i \in T_i$ not containing a_i. If $a_{i+1} \in e_2^i$ is to the left of $\overrightarrow{a_ip}$, X_i is a_i and the endpoint of e_2^i to the right of $\overrightarrow{a_ip}$, see Figure 4b. Otherwise, X_i is a_i and the endpoint of e_2^i to the left of $\overrightarrow{a_ip}$.*
- *If a_i is in the interior of e_1^i and a_{i+1} is a corner, $X_i = (a_{i+1})$, see Figure 4c.*
- *If a_i and a_{i+1} belong to the interior of two different edges e_1^i and e_2^i, X_i is the common endpoint of e_1^i and e_2^i, see Figure 4d.*

Let $(s = u_1, u_2, \ldots, u_\ell = t)$ be the sequence of consecutive points where $X(s,t)$ and $SP_w(s,t)$ coincide. The union of $SP_w(s,t)$ and $X(s,t)$ between two consecutive points u_j and u_{j+1}, for $1 \le j < \ell$, induces a weakly simple polygon (see [6] for a formal definition). We distinguish six different types of weakly simple polygons, denoted $P_1, \ldots, P_6$, depending on the number of edges intersected by $SP_w(u_j, u_{j+1})$, see Figure 5. Observe that, by definition of $X(s,t)$, these are the only weakly simple polygons that can arise.

The weakly simple polygons will be an important tool in our proof, since it will be enough to upper bound $\frac{\|X(s,t)\|}{\|SP_w(s,t)\|}$ for each of $P_1, \ldots, P_6$.

Definition 2. *Let u_j and u_{j+1} be two consecutive points in a triangular tessellation, where $X(s,t)$ and $SP_w(s,t)$ coincide. Let p be a common endpoint of the edges of the tessellation that contain u_j and u_{j+1}. A weakly simple polygon induced by u_j and u_{j+1} is of type P_k, for $1 \le k \le 6$, if the subpath $SP_w(u_j, u_{j+1})$ intersects k consecutive edges around p.*

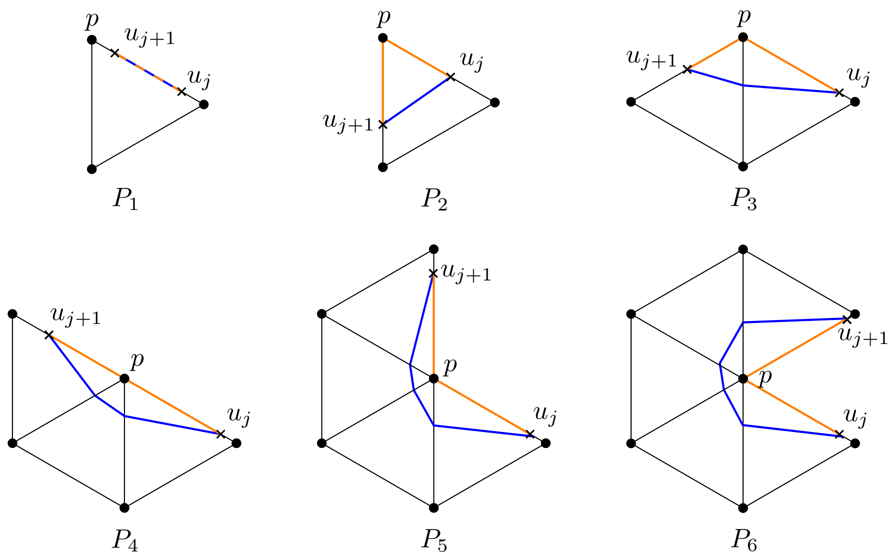
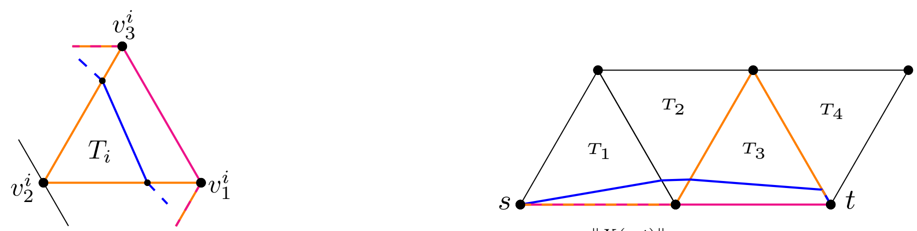


Fig. 5: Some weakly simple polygons P_k, and the subpath of the crossing path $X(s,t)$ (orange) from u_j to u_{j+1} intersecting consecutive triangular cells.



(a) Position of two points p and q in a triangular cell.

(b) The ratio $\frac{\|X(s,t)\|}{\|SP_w(s,t)\|}$ is ≈ 1.4, whereas the ratio $\frac{\|\Pi_i(s,t)\|}{\|SP_w(s,t)\|}$ is almost 1.

Fig. 6: Weighted shortest path $SP_w(s,t)$ (blue), crossing path $X(s,t)$ (orange), and shortcut path $\Pi_i(s,t)$ (purple) intersecting a weakly simple polygon P_2.

2.2 Bounding the ratio for weakly simple polygons

We are now ready to upper bound the ratio $\frac{\|X(u_j,u_{j+1})\|}{\|SP_w(u_j,u_{j+1})\|}$ for each of the six types of weakly simple polygons in $G_{6\text{corner}}$.

First we make a geometric observation that will be needed later. Let p and q be two points that are in the interior of two different edges on the boundary of the same triangular cell. Then, the length of the subpath of the weighted shortest path between p and q is given in Observation 2, which can be proved using the law of cosines.

Observation 2 *Let T_i be a triangular cell, and let (u,v,w) be the three vertices of T_i, in clockwise order. Let $p \in [u,v]$ and $q \in [v,w]$ be two points on the boundary of T_i. Then, $|pq| = \sqrt{|pv|^2 + |vq|^2 - |pv||vq|}$, see Figure 6a.*

We observe that, by definition, for P_1 we have $\frac{\|X(u_j,u_{j+1})\|}{\|SP_w(u_j,u_{j+1})\|} = 1$. Therefore the focus will be on bounding $P_2, \ldots, P_6$. We will begin from the simpler case of $P_3, \ldots, P_6$, and later we will consider P_2, which is substantially more involved. The proof of the lemma below can be found in the full version [5].

Lemma 1. *Let $u_j,\ u_{j+1} \in P_k$, for $3 \le k \le 6$, be two consecutive points where a shortest path $SP_w(s,t)$ and the crossing path $X(s,t)$ coincide in a triangular tessellation $\mathcal{T}$. An upper bound on the ratio $\frac{\|X(u_j,u_{j+1})\|}{\|SP_w(u_j,u_{j+1})\|}$ in $G_{6corner}$ is $\frac{2}{\sqrt{3}}$.*

The grid paths in $G_{6\text{corner}}$ are paths whose edges are edges of the triangular cells. Thus, the ratio $\frac{\|X(u_j,u_{j+1})\|}{\|SP_w(u_j,u_{j+1})\|}$ between two consecutive crossing points u_j and u_{j+1} depends on the weights of these regions. So, the next difficulty that we encountered, related to the crossing path, was that it is possible to find an instance where $SP_w(s,t)$ intersects a weakly simple polygon P_2 such that the ratio $\frac{\|X(s,t)\|}{\|SP_w(s,t)\|}$ is much larger than $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$, see Figure 6b. However, between s and t there are other grid paths shorter than $X(s,t)$ that intersect a P_2. So, in order to obtain an upper bound when $SP_w(s,t)$ intersects a P_2, we will need a finer analysis.

Since the ratio $\frac{\|X(u_j,u_{j+1})\|}{\|SP_w(u_j,u_{j+1})\|}$ is at most $\frac{2}{\sqrt{3}}$ for weakly simple polygons P_k, $k \neq 2$, we will assume from now on, that the ratio is maximized when all weakly simple polygons are of type P_2. Otherwise, we are done.

Definition 3 determines another class of grid paths called *shortcut paths* that gives a tighter upper bound on the ratio $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$ when a weakly simple polygon P_2 is intersected by $SP_w(s,t)$.

Let $\{v_1,\dots,v_n\}$ be a sequence of corners of a triangular tessellation. Then, the grid path $\Pi(s,[v_1,\dots,v_n],t)$ is defined as the path $X(s,v_1) \cup \pi(v_1,\dots,v_n) \cup X(v_n,t)$, where $\pi(v_1,\dots,v_n)$ is the grid path through the vertices $v_1,\dots,v_n$ in that order. We now define shortcut paths.

Definition 3. *Let (u,v,w) be the sequence of vertices of a cell $T_i \in \mathcal{T}$ in clockwise order. If $X(s,t)$ contains the subpath (u,v,w), the shortcut path $\Pi_i(s,t)$ is defined as the grid path $\Pi(s,[u,w],t)$, see purple path in Figure 6a.*

Now, we have all the tools needed to obtain an upper bound on the ratio $\frac{\|X(u_j,u_{j+1})\|}{\|SP_w(u_j,u_{j+1})\|}$ for P_2. By using the shortcut path $\Pi_i(s,t)$, we will be able to obtain a relation between the weights of the cells adjacent to $T_i \in \mathcal{T}$ intersected by the crossing path $X(s,t)$. This relation is given in the next lemma.

Lemma 2. *Let $(T_k,\dots,T_m)$ be the sequence of consecutive cells for which there exists a shortcut path $\Pi_i(s,t)$, $k \le i \le m$, for a given assignment of weights w to the cells of the triangular tessellation. The ratio $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$ is maximized when $\|X(s,t)\| = \|\Pi_i(s,t)\|$ in $G_{6corner}$.*

Proof. Consider an instance for which the ratio $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$ is maximized. Recall this instance just contains weakly simple polygons of type P_2. We will argue that if there is a grid path $GP_w(s,t)$ among $X(s,t)$, $\Pi_i(s,t)$, $k \le i \le m$, that is strictly shorter than the other grid paths, then this instance cannot maximize $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$.

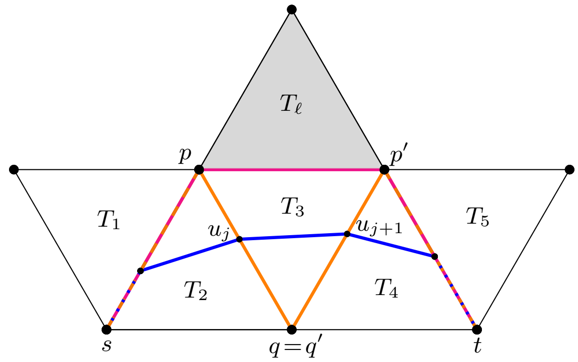


Fig. 7: $SP_w(s,t)$ through a P_2, where $\|\Pi_i(s,t)\| < \|X(s,t)\|$.

Suppose that there is one grid path $GP_w(s,t)$ among $X(s,t)$, $\Pi_i(s,t)$, $k \leq i \leq m$, that is strictly shorter than the other grid paths in the set. Since $GP_w(s,t)$ is a grid path, the ratio $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$ is upper bounded by $\frac{\|GP_w(s,t)\|}{\|SP_w(s,t)\|}$. The objective of the proof is to find another assignment of weights w' for the cells, such that $GP_{w'}(s,t)$ is still a shortest grid path among the grid paths in the set, and $\frac{\|GP_{w'}(s,t)\|}{\|SP_{w'}(s,t)\|} > \frac{\|GP_w(s,t)\|}{\|SP_w(s,t)\|}$.

Let u_j, u_{j+1} be two consecutive points where $GP_w(s,t)$ and $SP_w(s,t)$ coincide. Let T_ℓ be the cell that shares the edge of $\Pi_i(s,t)$ with T_i, see Figure 7. We first set to infinity the weight of all the cells that are not traversed by $SP_w(s,t)$. This way, we ensure that when modifying the weights of some cells, the combinatorial structure of the shortest path is preserved. The weight of the crossing path $X(s,t)$ along the edges of T_i is $2\min\{\omega_{i-1},\omega_i\} + 2\min\{\omega_i,\omega_{i+1}\}$, and the weight of the shortcut path $\Pi_i(s,t)$ along the edges of T_i is $2\min\{\omega_i,\omega_\ell\} = 2\omega_i$ (because $\omega_\ell = \infty$). Let $[p,q]$, and $[p',q']$ be, respectively, the edges containing u_j and u_{j+1}, where $p,p' \in T_\ell$.

– If $GP_w(s,t) = X(s,t)$ then $\|X(s,t)\| < \|\Pi_i(s,t)\|$, and we have that

$$\min\{\omega_{i-1},\omega_i\} + \min\{\omega_i,\omega_{i+1}\} < \omega_i. \tag{1}$$

- Suppose $\omega_i \leq \omega_{i-1}$, then $\omega_i + \min\{\omega_i,\omega_{i+1}\} < \omega_i$, which is not possible since $\min\{\omega_i,\omega_{i+1}\} > 0$. Hence, $\omega_i > \omega_{i-1}$.
- Suppose $\omega_i \leq \omega_{i+1}$, then $\min\{\omega_{i-1},\omega_i\} + \omega_{i+1} < \omega_i$, which is not possible since $\min\{\omega_{i-1},\omega_i\} > 0$. Hence, $\omega_i > \omega_{i+1}$.

These two facts together with Equation 1 imply that $\omega_{i-1} + \omega_{i+1} < \omega_i$. We also have that

$$\frac{\|X(s,t)\|}{\|SP_w(s,t)\|} = \frac{\|X(s,p)\| + 2(\omega_{i-1}+\omega_{i+1}) + \|X(p',t)\|}{\|SP_w(s,u_j)\| + |u_j u_{j+1}|\omega_i + \|SP_w(u_{j+1},t)\|},$$

being $|u_j u_{j+1}| > 0$, so if we decrease the weight ω_i until $\omega_{i-1} + \omega_{i+1} = \omega_i$, the denominator $\|SP_w(s,t)\|$ will decrease, and the numerator $\|X(s,t)\|$ will remain. Hence, the ratio $\frac{\|X(s,t)\|}{\|SP_w(s,t)\|}$ will increase, so we found another weight assignment w' such that $\frac{\|X(s,t)\|}{\|SP_{w'}(s,t)\|} > \frac{\|X(s,t)\|}{\|SP_w(s,t)\|}$ and $\|X(s,t)\| = \|\Pi_i(s,t)\|$.

– Otherwise, if $GP_w(s,t) = \Pi_i(s,t)$ then $\|\Pi_i(s,t)\| < \|X(s,t)\|$, and we have that $\omega_i < \min\{\omega_{i-1}, \omega_i\} + \min\{\omega_i, \omega_{i+1}\}$. We also have that

$$\frac{\|\Pi_i(s,t)\|}{\|SP_w(s,t)\|} = \frac{\|\Pi_i(s,p)\| + 2\omega_i + \|\Pi_i(p',t)\|}{\|SP_w(s,u_j)\| + |u_j u_{j+1}|\omega_i + \|SP_w(u_{j+1},t)\|},$$

given $|u_j u_{j+1}| < 2$. If we increase the weight ω_i until $\omega_i = \omega_{i-1} + \omega_{i+1}$, the numerator $\|\Pi_i(s,t)\|$ will increase faster than the denominator $\|SP_w(s,t)\|$. Hence, the ratio $\frac{\|\Pi_i(s,t)\|}{\|SP_w(s,t)\|}$ will increase, so we found another weight assignment w' such that $\frac{\|\Pi_i(s,t)\|}{\|SP_{w'}(s,t)\|} > \frac{\|\Pi_i(s,t)\|}{\|SP_w(s,t)\|}$ and $\|X(s,t)\| = \|\Pi_i(s,t)\|$.

We are now ready to prove the upper bound on the ratio $\frac{\|X(u_j,u_{j+1})\|}{\|SP_w(u_j,u_{j+1})\|}$ in a P_2. Lemma 3 presents an upper bound on the ratio $\frac{\|X(u_j,u_{j+1})\|}{\|SP_w(u_j,u_{j+1})\|}$, where $u_j, u_{j+1} \in T_i$ are two consecutive points where $X(s,t)$ and $\Pi_i(s,t)$ coincide. Lemma 2 implies that the ratio $\frac{\|X(s,t)\|}{\|SP_w(s,t)\|}$ in $G_{6\text{corner}}$ is maximized when $\|X(s,t)\| = \|\Pi_i(s,t)\|$ for each i such that the shortcut path $\Pi_i(s,t)$ exists. Thus, the ratio is obtained in a weakly simple polygon P_2 when $\|X(s,t)\| = \|\Pi_i(s,t)\|$. Since the exact shape of $SP_w(s,t)$ is unknown, when calculating the ratio in the following Lemma 3, we will maximize the ratio for any position of the points u_j and u_{j+1} where $SP_w(s,t)$ and $X(s,t)$ coincide. The prove of the result is given in the full version [5].

Lemma 3. *Let $u_j,\ u_{j+1} \in P_2$ be two consecutive points in a triangular tessellation $\mathcal{T}$, where a shortest path $SP_w(s,t)$ and the crossing path $X(s,t)$ coincide. Let $u_j,\ u_{j+1} \in T_i$ and $\|X(s,t)\| = \|\Pi_i(s,t)\|$, then an upper bound on the ratio $\frac{\|X(u_j,u_{j+1})\|}{\|SP_w(u_j,u_{j+1})\|}$ in $G_{6corner}$ is $\frac{2}{\sqrt{3}}$.*

Finally, we have all the pieces to prove our main result.

Theorem 1. *In $G_{6corner}$, an upper bound on the ratio $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$ is $\frac{2}{\sqrt{3}}$.*

Figure 8 provides an illustration of the lower bound $\frac{2}{\sqrt{3}}$ on the ratio between the weighted shortest grid path $SGP_w(s,t)$ (red) and the weighted shortest path $SP_w(s,t)$ (blue) claimed by Nash [15]. Hence, the upper bound in Theorem 1 is tight for $G_{6\text{corner}}$.

3 Discussion and future work

We presented upper bounds on the ratio between the lengths of three types of weighted shortest paths in a triangular tessellation. The fact that a compact grid graph such as $G_{6\text{corner}}$ guarantees an error bound of $\approx 15\%$, regardless of weights used, justifies its widespread use in applications in areas such as gaming and simulation, where performance is a priority over accuracy.

Our analysis techniques, presented here for triangular grids, can also be applied to obtain upper bounds for the same ratios in the other two types of regular tessellations, tb

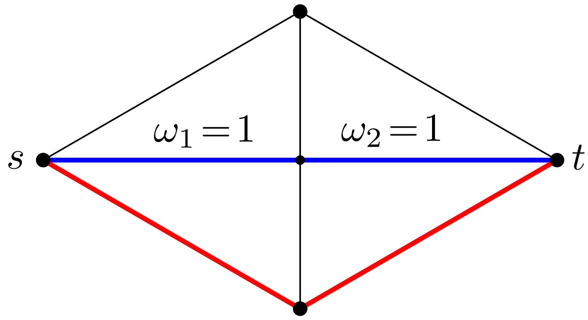


Fig. 8: The ratio $\frac{\|SGP_w(s,t)\|}{\|SP_w(s,t)\|}$ is $\frac{2}{\sqrt{3}}$.

square and hexagonal. The main differences lie in the exact definition of the crossing paths and the weakly simple polygons. Our techniques can also be used to derive upper bounds for another type of grid graphs, where the vertices are cell centers instead of corners (see, e.g., [11, 15]).

For future work, it would be interesting to close the gap for $\frac{\|SVP_w(s,t)\|}{\|SP_w(s,t)\|}$. It is an intriguing question whether the seemingly richer graph $SVP_w(s,t)$ can actually guarantee a better quality factor than $G_{6\text{corner}}$.

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