# A NOTE ON BERNSTEIN-SATO IDEALS

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Abstract. We define the Bernstein-Sato ideal associated to a tuple of ideals and we relate it to the jumping points of the corresponding mixed multiplier ideals.

## 1. Introduction

Let $R$ be either the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ over the complex numbers or the ring $\mathbb{C}\{x_1, \dots, x_n\}$ of convergent power series in the neighbourhood of the origin, or any other point. The *multiplier ideals* of a element $f$ or an ideal $\mathfrak{a}$ in $R$ are a family of nested ideals that play a prominent role in birational geometry (see Lazarsfeld's book [Laz04]). Associated to these ideals we have a set of invariants, the *jumping numbers*, that are intimately related to other invariants of singularities. For instance, Ein, Lazarsfeld, Smith and Varolin [ELSV04] and independently Budur and Saito [BS05], proved that the negatives of the jumping numbers of $f$ in the interval $(0, 1)$ are roots of the *Bernstein-Sato polynomial* of $f$. Budur, Mustaţă and Saito [BMS06] extended the classical theory of Bernstein-Sato polynomials to the case of ideals and also proved that the jumping numbers of an ideal $\mathfrak{a}$ in the interval $(0, 1)$ are roots of the *Bernstein-Sato polynomial* of $\mathfrak{a}$.

There is a natural extension of the theory of multiplier ideals to the context of tuples of germs $F := f_1, \dots, f_\ell$ or tuples of ideals $\boldsymbol{\mathfrak{a}} := \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ in $R$. The main differences that we encounter in this setting is that, whereas the multiplier ideals come with the set of associated jumping numbers, the mixed multiplier ideals come with a set of *jumping walls*. On the other side of the story we have the notion of *Bernstein-Sato ideal* associated to a tuple of germs $F$ given by Sabbah [Sab87]. In the case of a tuple of plane curves, Cassou-Noguès and Libgober [CNL11] related the Bernstein-Sato ideal with the so-called *faces of quasi-adjunction* which is a set of invariants equivalent to the jumping walls.

The aim of this short note is to fill out the theory introducing the notion of Bernstein-Sato ideal associated to a tuple ideals $\boldsymbol{\mathfrak{a}} := \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$. To such purpose we are going to follow the approach given by Mustaţă [Mus19] where he relates the Bernstein-Sato polynomial of a single ideal $\mathfrak{a} = (f_1, \dots, f_r)$ to the reduced Bernstein-Sato polynomial of $g = f_1 y_1 + \cdots + f_r y_r$, where the $y_j$'s are new variables.

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Finally, we show in Theorem 3.11 that the negative of the jumping points of the mixed multiplier ideals of the tuple $\mathbf{\mathfrak{a}}$ that are in the open ball of radius one centered at the origin belong to the zero locus of the Bernstein-Sato ideal of $\mathbf{\mathfrak{a}}$.

The theory of Bernstein-Sato polynomials and its relations with other invariants such as the multiplier ideals is vast and rich. In this note we tried to introduce only the essential concepts that we needed so we recommend those who are not that familiar with these topics to take a look at the surveys of Budur [Bud15], Granger [Gra10] or Jeffries, Núñez-Betancourt and the author [ÀJNB21] for further insight.

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## 2. Bernstein-Sato ideal of a tuple of ideals

Let $R$ be either $\mathbb{C}[x_1, \dots, x_n]$ or $\mathbb{C}\{x_1, \dots, x_n\}$ and denote $\mathfrak{m} = (x_1, \dots, x_n)$ the (homogeneous) maximal ideal. Let $\mathbf{\mathfrak{a}} := \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ be a tuple of ideals in $R$. For each ideal described by a set of generators $\mathfrak{a}_i = \left(f_{i,1}, \dots, f_{i,r_i}\right)$ we consider $g_i = f_{i,1}y_{i,1} + \dots + f_{i,r_i}y_{i,r_i}$ where the $y_{i,j}$'s are new variables. In particular we get a tuple $G := g_1, \dots, g_\ell$ in the ring $A$ that will be either $\mathbb{C}[x_1, \dots, x_n, y_{1,1}, \dots, y_{\ell,r_\ell}]$ or $\mathbb{C}\{x_1, \dots, x_n, y_{1,1}, \dots, y_{\ell,r_\ell}\}$. In the sequel, $d := n + r_1 + \dots + r_\ell$ will denote the number of variables in $A$.

Associated to $R$ or $A$ we have the corresponding ring of differential operators

$$D_R = R\langle \partial_1, \dots, \partial_n \rangle \quad , \quad D_A = A\langle \partial_1, \dots, \partial_n, \partial_{1,1}, \dots, \partial_{\ell,r_\ell} \rangle$$

where $\partial_i$ (resp. $\partial_{i,j}$) is the partial derivative with respect to $x_i$ (resp. $y_{i,j}$). That is, $D_R$ (resp. $D_A$) is the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}}(R)$ (resp. $\mathrm{End}_{\mathbb{C}}(A)$) generated by the ring and the partial derivatives.

**Definition 2.1.** The *Bernstein-Sato ideal* of the tuple $G$ is the ideal $B_G \subseteq \mathbb{C}[s_1, \dots, s_\ell]$ generated by all the polynomials $b(s_1, \dots, s_\ell)$ satisfying the *Bernstein-Sato functional equation*

$$\delta(s_1, \dots, s_\ell) g_1^{s_1+1} \cdots g_\ell^{s_\ell+1} = b(s_1, \dots, s_\ell) g_1^{s_1} \cdots g_\ell^{s_\ell}$$

where $\delta(s_1, \dots, s_\ell) \in D_A[s_1, \dots, s_\ell]$ and $b(s_1, \dots, s_\ell) \in \mathbb{C}[s_1, \dots, s_\ell]$.

Sabbah [Sab87] proved that $B_G \neq 0$ in the convergent power series case. The proof of $B_G \neq 0$ in the polynomial ring case is completely analogous to the classical case of a single element. Indeed, it is enough to consider the local case.

**Remark 2.2.** Briançon and Maisonobe showed in [BM02] that

$$B_G^{\mathbb{C}[x]} = \bigcap_{p \in \mathbb{C}^d} B_G^{\mathbb{C}\{x-p\}}$$

where $B_G^{\mathbb{C}[x]}$ denotes the Bernstein-Sato ideal of a tuple $G$ over the polynomial ring and $B_G^{\mathbb{C}\{x-p\}}$ is the Bernstein-Sato ideal of $G$ in the convergent power series around a point $p \in \mathbb{C}^d$.

Now, since the $g_i$ are pairwise without common factors, we have

$$B_G \subseteq \Big((s_1+1)\cdots(s_\ell+1)\Big).$$

(see [May97, BM99] for details).

**Definition 2.3.** The *reduced Bernstein-Sato ideal* of the tuple $G$ is the ideal $\widetilde{B}_G \subseteq \mathbb{C}[s_1,\dots,s_\ell]$ generated by the polynomials

$$\frac{b(s_1,\dots,s_\ell)}{(s_1+1)\cdots(s_\ell+1)},$$

with $b(s_1,\dots,s_\ell) \in B_G$.

Following the approach given by Mustaţă [Mus19] for the case of a single ideal, we consider the following:

**Definition 2.4.** Let $\mathfrak{a} = \mathfrak{a}_1,\dots,\mathfrak{a}_\ell$ be a tuple of ideals in $\mathcal{O}_{X,O}$ and let $G := g_1,\dots,g_\ell$ be its associated tuple of hypersurfaces. We define the *Bernstein-Sato ideal* of $\mathfrak{a}$ as

$$B_{\mathfrak{a}} := \widetilde{B}_G \subseteq \mathbb{C}[s_1,\dots,s_\ell]$$

Our next result shows that $B_{\mathfrak{a}}$ does not depend on the generators of the ideals $\mathfrak{a}_1,\dots,\mathfrak{a}_\ell$ and thus it is an invariant of the tuple $\mathfrak{a}$.

**Theorem 2.5.** *Let $\mathfrak{a} := \mathfrak{a}_1,\dots,\mathfrak{a}_\ell$ be a tuple of ideals and, for each ideal, consider two different sets of generators $\mathfrak{a}_i = \big(f_{i,1},\dots,f_{i,r_i}\big)$ and $\mathfrak{a}_i = \big(f'_{i,1},\dots,f'_{i,s_i}\big)$. Consider the tuple $G = g_1,\dots,g_\ell$ with $g_i = f_{i,1}y_{i,1} + \cdots + f_{i,r_i}y_{i,r_i}$ and the tuple $G' = g'_1,\dots,g'_\ell$ with $g'_i = f'_{i,1}y'_{i,1} + \cdots + f'_{i,s_i}y'_{i,s_i}$. Then $\widetilde{B}_G = \widetilde{B}_{G'}$.*

*Proof.* Without loss of generality we may assume that, for each ideal $\mathfrak{a}_i$, the set of generators $f'_{i,1},\dots,f'_{i,s_i}$ is just $f_{i,1},\dots,f_{i,r_i},h_i$ for a given $h_i \in \mathfrak{a}_i$. Let $z_1,\dots,z_{r_i}$ such that $h_i = z_1 f_{i,1} + \cdots + z_{r_i} f_{i,r_i}$. Then we have

$$\begin{aligned} g'_i &= f_{i,1}y'_{i,1} + \cdots + f_{i,r_i}y'_{i,r_i} + h_i y'_{i,r_i+1} \\ &= f_{i,1}y'_{i,1} + \cdots + f_{i,r_i}y'_{i,r_i} + (z_1 f_{i,1} + \cdots + z_{r_i} f_{i,r_i})y'_{i,r_i+1} \\ &= f_1(y'_{i,1} + z_1 y'_{i,r_i+1}) + \cdots + f_\ell(y'_{i,r_i} + z_{r_i} y'_{i,r_i+1}). \end{aligned}$$

After a change of variables $y_{i,j} \mapsto y'_{i,j} + z_j y'_{i,r_i+1}$, this polynomial becomes $g_i$. Since Bernstein-Sato ideals do not change by change of variables, we conclude that $B_G = B_{G'}$ and the result follows. ∎

## 3. Mixed multiplier ideals

Let $\pi : X' \longrightarrow X$ be a common log-resolution of a tuple of ideals $\mathbf{\mathfrak{a}} = \mathfrak{a}_1, \ldots, \mathfrak{a}_\ell$ in $R$. Namely, $\pi$ is a birational morphism such that

- $X'$ is smooth,
- $\mathfrak{a}_i \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-F_i)$ for some effective Cartier divisor $F_i$, $i = 1, \ldots, \ell$,
- $\sum_{i=1}^{\ell} F_i + E$ is a divisor with simple normal crossings where $E = Exc(\pi)$ is the exceptional locus.

The divisors $F_i = \sum_j e_{i,j} E_j$ are integral divisors in $X'$ which can be decomposed into their *exceptional* and *affine* part according to the support, i.e. $F_i = F_i^{\mathrm{exc}} + F_i^{\mathrm{aff}}$ where

$$F_i^{\mathrm{exc}} = \sum_{j=1}^{s} e_{i,j} E_j \quad \text{and} \quad F_i^{\mathrm{aff}} = \sum_{j=s+1}^{t} e_{i,j} E_j.$$

Whenever $\mathfrak{a}_i$ is an $\mathfrak{m}$-primary ideal, the divisor $F_i$ is just supported on the exceptional locus. i.e. $F_i = F_i^{\mathrm{exc}}$. We will also consider the *relative canonical divisor*

$$K_\pi = \sum_{i=1}^{s} k_j E_j$$

which is a divisor in $X'$ supported on the exceptional locus $E$ defined by the Jacobian determinant of the morphism $\pi$.

**Definition 3.1.** The *mixed multiplier ideal* associated to a tuple $\mathbf{\mathfrak{a}} = \mathfrak{a}_1, \ldots, \mathfrak{a}_\ell$ of ideals in $R$ and a point $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_\ell) \in \mathbb{R}_{\geqslant 0}^{\ell}$ is defined as

$$\mathcal{J}(\mathbf{\mathfrak{a}}^{\lambda}) := \mathcal{J}(\mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell}) = \pi_* \mathcal{O}_{X'}\left(\lceil K_\pi - \lambda_1 F_1 - \cdots - \lambda_\ell F_\ell \rceil\right)$$

In the classical case of a single ideal we have the notion of *jumping numbers* associated to the sequence of multiplier ideals. The corresponding notion in the context of mixed multiplier ideals is more involved.

**Definition 3.2.** Let $\mathbf{\mathfrak{a}} = \mathfrak{a}_1, \ldots, \mathfrak{a}_\ell$ be a tuple of ideals in $R$. Then, for each $\boldsymbol{\lambda} \in \mathbb{R}_{\geqslant 0}^{\ell}$, we define:

- The *region* of $\boldsymbol{\lambda}$: $\mathcal{R}_{\mathbf{\mathfrak{a}}}(\boldsymbol{\lambda}) = \left\{ \boldsymbol{\lambda'} \in \mathbb{R}_{\geqslant 0}^{\ell} \;\middle|\; \mathcal{J}(\mathbf{\mathfrak{a}}^{\boldsymbol{\lambda'}}) \supseteq \mathcal{J}(\mathbf{\mathfrak{a}}^{\boldsymbol{\lambda}}) \right\}$.
- The *constancy region* of $\boldsymbol{\lambda}$: $\mathcal{C}_{\mathbf{\mathfrak{a}}}(\boldsymbol{\lambda}) = \left\{ \boldsymbol{\lambda'} \in \mathbb{R}_{\geqslant 0}^{\ell} \;\middle|\; \mathcal{J}(\mathbf{\mathfrak{a}}^{\boldsymbol{\lambda'}}) = \mathcal{J}(\mathbf{\mathfrak{a}}^{\boldsymbol{\lambda}}) \right\}$.

The boundaries of these regions is where we have a strict inclusion of ideals. Therefore we may define:

**Definition 3.3.** Let $\mathbf{\mathfrak{a}} = \mathfrak{a}_1, \ldots, \mathfrak{a}_\ell$ be a tuple of ideals in $R$. The *jumping wall* associated to $\boldsymbol{\lambda} \in \mathbb{R}_{\geqslant 0}^{\ell}$ is the boundary of the region $\mathcal{R}_{\mathbf{\mathfrak{a}}}(\boldsymbol{\lambda})$.

In particular, we will be interested in the points of these jumping walls. In the sequel, $B_\varepsilon(\boldsymbol{\lambda})$ stands for the open ball of radius $\varepsilon$ centered at a point $\boldsymbol{\lambda} \in \mathbb{R}^{\ell}$.

**Definition 3.4.** Let $\boldsymbol{\mathfrak{a}} = \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ be a tuple of ideals in $R$. We say that $\boldsymbol{\lambda} \in \mathbb{R}^\ell_{\geqslant 0}$ is a *jumping point* of $\boldsymbol{\mathfrak{a}}$ if $\mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}'}) \supsetneq \mathcal{J}(\boldsymbol{\mathfrak{a}}^{\boldsymbol{\lambda}})$ for all $\boldsymbol{\lambda}' \in \{\boldsymbol{\lambda} - \mathbb{R}^\ell_{\geqslant 0}\} \cap B_\varepsilon(\boldsymbol{\lambda})$ and $\varepsilon > 0$ small enough.

From the definition of mixed multiplier ideals we have that the jumping points $\boldsymbol{\lambda} \in \mathbb{R}^\ell_{\geqslant 0}$ must lie on hyperplanes of the form $H_j : \ e_{1,j}z_1 + \cdots + e_{\ell,j}z_\ell = k_j + \nu_j$ for $j = 1, \dots, s$ and $\nu_j \in \mathbb{Z}_{>0}$.

For $\lambda \in (0,1)$ we have $\mathcal{J}(\mathfrak{a}^\lambda) = \mathcal{J}(g_\alpha^\lambda)$ where $\mathfrak{a} = \big(f_1, \dots, f_r\big)$ is a single ideal in $R$ and $g_\alpha = \alpha_1 f_1 + \cdots + \alpha_r f_r \in R$ with $\alpha_i \in \mathbb{C}$ is a general element (see [Laz04, Prop. 9.2.28]). As a consequence of a more general result of Mustaţă and Popa given in [MP20, Theorem 2.5] we also have a relation between $\mathcal{J}(\mathfrak{a}^\lambda)$ and the multiplier ideal of the associated hypersurface $g = f_1 y_1 + \cdots + f_r y_r$ in $A$.

**Definition 3.5.** Let $\mathcal{J} = \big(Q_1(y), \dots, Q_s(y)\big)$ be an ideal in $A$. Then, $\operatorname{Coeff}(\mathcal{J}) \subseteq R$ is the ideal generated by

$$\{Q_1(\alpha), \dots, Q_s(\alpha) \ | \ \alpha \in \mathbb{C}^r\}$$

The result of Mustaţă and Popa in the form that we need is the following

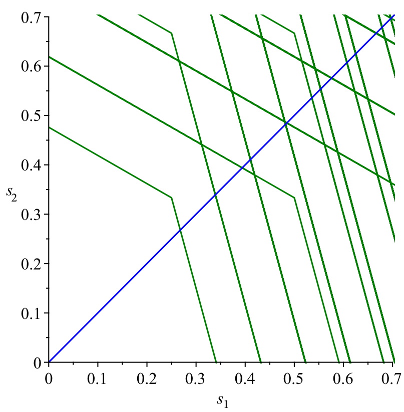
**Proposition 3.6.** *Let $\mathfrak{a} = \big(f_1, \dots, f_r\big)$ be an ideal in $R$ and let $g = f_1 y_1 + \cdots + f_r y_r$ be the associated hypersurface in $A$. Then, for any $\lambda \in \mathbb{Q} \cap (0,1)$ we have*

$$\mathcal{J}(\mathfrak{a}^\lambda) = \operatorname{Coeff}\big(\mathcal{J}(g^\lambda)\big)$$

*In particular, the set of jumping numbers in the interval $(0,1)$ of $\mathfrak{a}$ and $g$ coincide.*

The mixed multiplier ideals version of this result follows immediately from the following observation.

**Remark 3.7.** Consider a ray through the origin $L : (0, \dots, 0) + \mu(\alpha_1, \dots, \alpha_\ell)$ where the $\alpha_i$'s are positive integers.



Then, the jumping points of a tuple $\mathfrak{a} = \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ lying on $L$ are the jumping numbers of the ideal $\mathfrak{a}_1^{\alpha_1} \cdots \mathfrak{a}_\ell^{\alpha_\ell}$

**Corollary 3.8.** *Let $\mathfrak{a} = \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ be a tuple of ideals in $R$, $G := g_1, \dots, g_\ell$ its associated tuple of hypersurfaces and consider $\boldsymbol{\lambda} \in \mathbb{Q}^\ell_{\geqslant 0}$ with Euclidean norm $\| \boldsymbol{\lambda} \| < 1$. Then, $\boldsymbol{\lambda}$ is a jumping point of $\mathfrak{a}$ if and only if it is a jumping point of $G$.*

*Proof.* After Remark 3.7 we may assume that we have a single ideal $\mathfrak{a} = \big(f_1, \dots, f_r\big)$ so its associated hypersurface is $g = f_1 y_1 + \cdots + f_r y_r$. The result then follows from 3.6. □

In order to prove the main result of this section we will need the analytic definition of mixed multiplier ideal associated to a tuple $G = g_1, \dots, g_\ell$.

**Definition 3.9.** Let $G = g_1, \dots, g_\ell$ be a tuple in $A$. Let $\overline{B}_\epsilon(O)$ be a closed ball of radius $\varepsilon$ and center the origin $O \in \mathbb{C}^d$. The mixed multiplier ideal (at the origin $O$) of $G$ associated with $\boldsymbol{\lambda} \in \mathbb{Q}^\ell_{>0}$ is

$$\mathcal{J}(g_1^{\lambda_1} \cdots g_\ell^{\lambda_\ell})_O = \Big\{ h \in A \;\Big|\; \exists \epsilon \ll 1 \text{ such that } \int_{\overline{B}_\epsilon(O)} \frac{|h|^2}{|g_1|^{2\lambda_1} \cdots |g_\ell|^{2\lambda_\ell}} dx dy d\bar{x} d\bar{y} < \infty \Big\}.$$

**Remark 3.10.** As in the case of Bernstein-Sato ideals it is enough to consider this local case since we have

$$\mathcal{J}(g_1^{\lambda_1} \cdots g_\ell^{\lambda_\ell}) = \cap_{p \in \mathbb{C}^d} \mathcal{J}(g_1^{\lambda_1} \cdots g_\ell^{\lambda_\ell})_p.$$

If it is clear from the context we will omit the subscript referring to the point.

**Theorem 3.11.** *Let $\mathfrak{a} = \mathfrak{a}_1, \dots, \mathfrak{a}_\ell$ be a tuple of ideals in $R$. Let $\boldsymbol{\lambda} \in \mathbb{Q}^\ell_{\geqslant 0}$ be a jumping point of $\mathfrak{a}$ with Euclidean norm $\| \boldsymbol{\lambda} \| < 1$. Then $-\boldsymbol{\lambda} \in Z(B_{\mathfrak{a}})$.*

*Proof.* Let $\boldsymbol{\lambda} \in \mathbb{Q}^\ell_{\geqslant 0}$ be a jumping point of the tuple $G = g_1, \dots, g_\ell$ associated to $\mathfrak{a}$ with $\| \boldsymbol{\lambda} \| < 1$ and take $h \in \mathcal{J}(f^{\boldsymbol{\lambda}'}) \smallsetminus \mathcal{J}(f^{\boldsymbol{\lambda}})$ with $\boldsymbol{\lambda}' \in \{\boldsymbol{\lambda} - \mathbb{R}^\ell_{\geqslant 0}\} \cap B_\varepsilon(\boldsymbol{\lambda})$ for $\varepsilon > 0$ small enough. Therefore

$$\frac{|h|^2}{|g_1|^{2\lambda_1'} \cdots |g_\ell|^{2\lambda_\ell'}}$$

is integrable but when we take the limit $\varepsilon \to 0$ we end up with

$$\frac{|h|^2}{|g_1|^{2\lambda_1} \cdots |g_\ell|^{2\lambda_\ell}}$$

that is not integrable. Set $d = n + r_1 + \cdots + r_\ell$ and consider the complex zeta function

$$\int_{\mathbb{C}^d} |g_1|^{2s_1} \cdots |g_\ell|^{2s_\ell} \varphi(x, y, \bar{x}, \bar{y}) dx dy d\bar{x} d\bar{y},$$

where $s_1, \dots, s_\ell$ are indeterminate variables and $\varphi(x, \bar{x}) \in C_c^\infty(\mathbb{C}^d)$ is a *test function*, i.e. an infinitely many times differentiable function with compact support. Moreover $\varphi$ has holomorphic and antiholomorphic part. For any $b(s_1, \dots, s_\ell) \in B_G$ we have a Bernstein-Sato functional equation

$$\delta(s_1, \dots, s_\ell) g_1^{s_1+1} \cdots g_\ell^{s_\ell+1} = b(s_1, \dots, s_\ell) g_1^{s_1} \cdots g_\ell^{s_\ell}$$

Therefore

$$b^2(s_1,\dots,s_\ell)\int_{\mathbb{C}^d}\varphi(x,y,\bar{x},\bar{y})|g_1|^{2s_1}\cdots|g_\ell|^{2s_\ell}dxdyd\bar{x}d\bar{y}=$$
$$=\int_{\mathbb{C}^d}\bar{\delta}^*\delta^*(s_1,\dots,s_\ell)\big(\varphi(x,y,\bar{x},\bar{y})\big)|g_1|^{2(s_1+1)}\cdots|g_\ell|^{2(s_\ell+1)}dxdyd\bar{x}d\bar{y}.$$

where $\bar{\delta}^*$ and $\delta^*$ denote the *conjugate* and the *adjoint* differential operators associated to $\delta$. Notice that $|h|^2\varphi(x,\bar{x})$ is still a test function so

$$b^2(s_1,\dots,s_\ell)\int_{\mathbb{C}^d}|h|^2\varphi(x,y,\bar{x},\bar{y})|g_1|^{2s_1}\cdots|g_\ell|^{2s_\ell}dxdyd\bar{x}d\bar{y}=$$
$$=\int_{\mathbb{C}^d}\bar{\delta}^*\delta^*(s_1,\dots,s_\ell)\big(|h|^2\varphi(x,y,\bar{x},\bar{y})\big)|g_1|^{2(s_1+1)}\cdots|g_\ell|^{2(s_\ell+1)}dxdyd\bar{x}d\bar{y}.$$

Now we take a test function $\varphi$ which is zero outside the ball $\overline{B}_\epsilon(O)$ and identically one on a smaller ball $\overline{B}_{\epsilon'}(O)\subseteq\overline{B}_\epsilon(O)$ and thus we get

$$b^2(s_1,\dots,s_\ell)\int_{\overline{B}_{\epsilon'}(O)}|h|^2|g_1|^{2s_1}\cdots|g_\ell|^{2s_\ell}dxdyd\bar{x}d\bar{y}=$$
$$=\int_{\overline{B}_{\epsilon'}(p)}\bar{\delta}^*\delta^*(s_1,\dots,s_\ell))\big(|h|^2\big)|g_1|^{2(s_1+1)}\cdots|g_\ell|^{2(s_\ell+1)}dxdyd\bar{x}d\bar{y}.$$

Taking $s=-(\lambda_1',\dots,\lambda_\ell')$ we get

$$b^2(-\lambda_1',\dots,-\lambda_\ell')\int_{\overline{B}_{\epsilon'}(O)}\frac{|h|^2}{|g_1|^{2\lambda_1'}\cdots|g_\ell|^{2\lambda_\ell'}}dxdyd\bar{x}d\bar{y}=$$
$$=\int_{\overline{B}_{\epsilon'}(O)}\bar{\delta}^*\delta^*(-\lambda_1',\dots,-\lambda_\ell')\big(|h|^2\big)|g_1|^{2(1-\lambda_1')}\cdots|g_\ell|^{2(1-\lambda_\ell')}dxdyd\bar{x}d\bar{y}$$

but the right-hand side is uniformly bounded for all $\varepsilon>0$. Thus we have

$$b^2(-\lambda_1',\dots,-\lambda_\ell')\int_{\overline{B}_{\epsilon'}(O)}\frac{|h|^2}{|g_1|^{2\lambda_1'}\cdots|g_\ell|^{2\lambda_\ell'}}dxdyd\bar{x}d\bar{y}\le M<\infty$$

for some positive number $M$ that depends on $h$. Then, by the monotone convergence theorem we have to have $b^2(-\lambda_1,\dots,-\lambda_\ell)=0$ and thus $-\boldsymbol{\lambda}\in Z(\tilde{B}_G)=Z(B_{\mathfrak{a}})$. □

## References


[ÀJNB21] Josep Àlvarez Montaner, Jack Jeffries, and Luis Núñez-Betancourt. Bernstein-Sato polynomials in commutative algebra. 2021. Preprint, arXiv:2106.08830. 2

[BM99] Joël Briançon and Hélène Maynadier. Équations fonctionnelles généralisées: transversalité et principalité de l'idéal de Bernstein-Sato. *J. Math. Kyoto Univ.*, 39(2):215–232, 1999. 3

[BM02] Joël Briançon and Philippe Maisonobe. Bernstein-Sato ideals associated to polynomials I, 2002. Unpublished notes. 2

[BMS06] Nero Budur, Mircea Mustaţă, and Morihiko Saito. Bernstein-Sato polynomials of arbitrary varieties. *Compos. Math.*, 142(3):779–797, 2006. 1

[BS05] Nero Budur and Morihiko Saito. Multiplier ideals, $V$-filtration, and spectrum. *J. Algebraic Geom.*, 14(2):269–282, 2005. 1

[Bud15] Nero Budur. Bernstein-Sato polynomials. Lecture notes for the summer school *Multiplier Ideals, Test Ideals, and Bernstein-Sato Polynomials*, at UPC Barcelona, available at https://perswww.kuleuven.be/ u0089821/Barcelona/BarcelonaNotes.pdf, 2015. 2

[CNL11] Pierrette Cassou-Noguès and Anatoly Libgober. Multivariable Hodge theoretical invariants of germs of plane curves. *J. Knot Theory Ramifications*, 20(6):787–805, 2011. 1

[ELSV04] Lawrence Ein, Robert Lazarsfeld, Karen E. Smith, and Dror Varolin. Jumping coefficients of multiplier ideals. *Duke Math. J.*, 123(3):469–506, 2004. 1

[Gra10] Michel Granger. Bernstein-Sato polynomials and functional equations. In *Algebraic approach to differential equations*, pages 225–291. World Sci. Publ., Hackensack, NJ, 2010. 2

[Laz04] R. Lazarsfeld. *Positivity in Algebraic Geometry II*. Springer-Verlag, Berlin, 2004. 1, 5

[May97] Hélène Maynadier. Polynômes de Bernstein-Sato associés à une intersection complète quasi-homogène à singularité isolée. *Bull. Soc. Math. France*, 125(4):547–571, 1997. 3

[MP20] Mirceă Mustaţă and Mihnea Popa. Hodge ideals and minimal exponents of ideals. *Rev. Roumaine Math. Pures Appl.*, 65(3):327–354, 2020. 5

[Mus19] Mircea Mustaţă. Bernstein-Sato polynomials for general ideals vs principal ideals. *Preprint, arXiv:1906.03086*, 2019. 1, 3

[Sab87] Claude Sabbah. Proximité évanescente. II. Équations fonctionnelles pour plusieurs fonctions analytiques. *Compositio Math.*, 64(2):213–241, 1987. 1, 2

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