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Decay of solutions for strain gradient mixtures

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We study antiplane shear deformations for isotropic and homogeneous strain gradient mixtures of the Kelvin-Voigt type in a cylinder. Our aim is to analyze the behaviour of the solutions with respect to the time variable when a dissipative structural mechanism is considered. We study three different cases, each at a time. For each case we prove existence and uniqueness of solutions. We obtain the exponential decay of the solutions in the hyperviscosity and viscosity cases. Exponential decay is also expected when the dissipation is generated by the relative velocity (in the generic case, although a particular combination of the constitutive parameters leads to slow decay). These results are proved with the help of the theory of semigroups.

1 | INTRODUCTION AND BASIC EQUATIONS

A mixture is a material made up of two or more different substances. Mixtures are common in daily life because they are used in a lot of processes as, for example, in steel manufacturing or in the chemical industry. Nowadays, it is quite usual the creation of a composite material combining two materials with different physical and chemical properties with the aim of obtaining a new one with a specific quality (stronger and lighter, for instance).

The study of the continuum theory of mixtures has been an important issue for mathematicians and engineers in the last decades. From a mathematical point of view, the theory of mixtures proposes an interesting set of problems involving partial differential equations (PDE). The origin of the current formulation of the thermomechanical theories of mixtures comes from the works of Truesdell and Toupin [1], Kelly [2], Eringen and Ingram [3, 4], Green and Naghdi [5, 6] and Bowen and Wise [7]. For a deep review of the literature about mixtures with historical references, the reader may consult the papers (or books) of Bowen [8], Atkin and Craine [9, 10], Bedford and Drumheller [11] or Rajagopal and Tao [12]. Some other authors have studied the theory of viscoelastic mixtures. Without trying to be exhaustive, let us highlight the works given in the references [13–18] and [19].

A lot of effort has been done to find qualitative properties of the solutions to the PDE related to the mixture of materials problems. Several results concerning existence, uniqueness, continuous dependence and asymptotic stability can be found in the literature. See, for instance, [20–27] and [28].

In this work we consider a mixture of two interacting materials. Our domain will consist of one cylinder R of constant cross-section, $R = B \times [0, L]$, where B is a bounded two-dimensional region whose boundary, ∂B, is a curve smooth enough to allow the application of the divergence theorem.

First of all, we recall the evolution and constitutive equations which govern the theory we are going to deal with. We follow the guidelines proposed by Ieşan [29]. In this section we present the basic equations that describe the behaviour of a binary mixture of nonsimple elastic solids. Later on, we will introduce several dissipation mechanisms in the system.

The evolution equations we are going to consider are the following:

$$\begin{aligned} \rho_1 \ddot{u}_i &= \tau_{ji,j} - \mu_{rji,rj} - p_i \\ \rho_2 \ddot{w}_i &= \sigma_{ji,j} - \eta_{rji,rj} + p_i \end{aligned} \tag{1}$$

where $\boldsymbol{u}$ and $\boldsymbol{w}$ are the displacements, $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ the partial stress tensors, $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ the partial hyperstresses and $\boldsymbol{p}$ is the internal body force. As usual, ρ_1 and ρ_2 are the mass densities.

The constitutive equations (see [29]) are given by

$$\begin{aligned} \tau_{ji} &= a_{ijrs} u_{r,s} + b_{ijrs} w_{r,s} \\ \sigma_{ji} &= b_{rsij} u_{r,s} + d_{ijrs} w_{r,s} \\ \mu_{ijk} &= g_{isjnmp} u_{n,pm} + h_{isjnpm} w_{n,pm} + f_{misj} d_m \\ \eta_{rsi} &= h_{pmnisr} u_{p,mn} + m_{isrpmn} w_{p,mn} + p_{misr} d_m \\ p_i &= f_{irmn} u_{r,mn} + p_{irmn} w_{r,mn} + a_{ij} d_j \end{aligned} \tag{2}$$

where $d_i = u_i - w_i$ is the relative displacement. Here a_{ijrs}, b_{ijrs}, d_{ijrs}, g_{isjnmp}, h_{isjnpm}, f_{misj}, m_{isrpmn} and p_{misr} are constitutive tensors.

We consider the isotropic and homogeneous case for antiplane shear deformations, that means that we impose in the above equations the following conditions:

$$u_1 = u_2 = w_1 = w_2 = 0, u_3 = u(x_1, x_2), w_3 = w(x_1, x_2). \tag{3}$$

Substituting the constitutive equations into the evolution equations and taking into account the above considerations, we obtain the system of field equations for the unknown components of the displacement vectors (we have simplified the notation a little bit and grouped terms):

$$\begin{cases} \rho_1 \ddot{u} = \mu_1 \Delta u + \mu \Delta w - \gamma_1 \Delta^2 u - \gamma \Delta^2 w - a(u - w), \\ \rho_2 \ddot{w} = \mu \Delta u + \mu_2 \Delta w - \gamma \Delta^2 u - \gamma_2 \Delta^2 w + a(u - w), \end{cases} \tag{4}$$

where μ, $\mu_{\{1,2\}}$, γ, $\gamma_{\{1,2\}}$ and a are constitutive constants and Δ is the two-dimensional Laplacian operator.

As usual in this context, ρ_1 and ρ_2 are assumed to be strictly positive constants. Moreover, in order to guarantee the elastic stability of the materials we will suppose that

$$\mu_1 \mu_2 > \mu^2, \quad \gamma_1 \gamma_2 > \gamma^2, \quad a,\ \mu_1,\ \gamma_1 > 0. \tag{5}$$

These conditions are usual in the analysis of antiplane shear deformations in strain gradient mixtures and are useful to obtain the existence and uniqueness, and they guarantee that the inner energy is positive definite.

We assume also that $\gamma \neq 0$ to ensure the strong coupling between the materials. This condition will be useful to obtain the decay results. Nevertheless, in this case, it is irrelevant if μ is equal to or different from 0.

To have a well-posed problem we need to introduce initial and boundary conditions. As initial conditions we consider:

$$u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = v_0(x), \quad w(x, 0) = w_0(x), \quad \dot{w}(x, 0) = z_0(x). \tag{6}$$

And we set the following boundary conditions:

$$u = \Delta u = w = \Delta w = 0 \text{ on } \partial B. \tag{7}$$

System (4) is conservative. We intend to introduce several dissipation mechanisms in it. The aim of this work is to determine how the solutions to system (4) behave with respect to the time variable depending on the dissipation mechanism we introduce in the constitutive equations.

2 | HYPERVISCOSITY: DISSIPATION GENERATED BY A SECOND-ORDER VELOCITY GRADIENT

We first consider a dissipation generated by a second-order velocity gradient. To be precise we suppose that

$$\mu_{ijk} = g_{isjnmp} u_{n,pm} + h_{isjnpm} w_{n,pm} + f_{misj} d_m + g^*_{isjnmp} \dot{u}_{n,pm}. \tag{8}$$

Therefore the system of field equations for antiplane shear deformations in the case of isotropic and homogeneous mixtures, with our notation, becomes

$$\begin{cases} \rho_1 \ddot{u} = & \mu_1 \Delta u + \mu \Delta w - \gamma_1 \Delta^2 u - \gamma \Delta^2 w - a(u-w) - \gamma^* \Delta^2 \dot{u} \\ \rho_2 \ddot{w} = & \mu \Delta u + \mu_2 \Delta w - \gamma \Delta^2 u - \gamma_2 \Delta^2 w + a(u-w) \end{cases} \tag{9}$$

with $\gamma^* > 0$.

Notice that we obtain a fourth order derivative of $\dot{u}$ with respect to the spatial variables. We call this mechanism *hyperviscosity* because it appears in the definition of the hyperstress.

Remark 1. It is not difficult to see that this system has undamped solutions. Consider $\Phi_n(x)$ the eigenfunctions of the problem

$$\begin{cases} \Delta^2 \Phi - m_n^2 \Phi = 0 & \text{in } B, \\ \Phi = \Delta \Phi = 0 & \text{on } \partial B. \end{cases} \tag{10}$$

Therefore, $\Delta \Phi = -m_n \Phi$ and $\Delta^2 \Phi = m_n^2 \Phi$ with $m_n > 0$ (see [30] for the details). And take, for instance, $u = 0$ and $w = e^{\omega t} \Phi_n(x)$. Replacing them in (9) we get

$$\begin{cases} 0 & = -\mu m_n - \gamma m_n^2 + a, \\ \rho_2 \omega^2 & = -\mu_2 m_n - \gamma_2 m_n^2 - a. \end{cases} \tag{11}$$

Taking appropriate values of ω and a specific combination of a, μ and γ the above expressions can be solutions for (9). Hence, for the rest of this section we will suppose that $\gamma m_n^2 + \mu m_n \neq a$ for all m_n.

2.1 | Existence and uniqueness

We transform our initial-boundary problem in an abstract problem in an appropriate Hilbert space. Let us denote $v = \dot{u}$ and $z = \dot{w}$. We consider the Hilbert space

$$\mathcal{H} = \{(u, v, w, z) \in (H_0^1 \cap H^2) \times L^2 \times (H_0^1 \cap H^2) \times L^2\}. \tag{12}$$

To ease the notation we have written $H_0^1 \cap H^2$ instead of $H_0^1(B) \cap H^2(B)$, and the same for L^2. We will follow the same rule in the rest of the work.

If $U = (u, v, w, z)$ and $U^* = (u^*, v^*, w^*, z^*)$ are two elements of $\mathcal{H}$, we define its inner product as

$$\langle U, U^* \rangle = \frac{1}{2}\int_B \Big(\rho_1 v\overline{v^*} + \rho_2 z\overline{z^*} + \mu_1 \nabla u \overline{\nabla u^*} + \mu_2 \nabla w \overline{\nabla w^*} + \mu(\nabla u \overline{\nabla w^*} + \nabla w \overline{\nabla u^*}) + \gamma_1 \Delta u \overline{\Delta u^*} + \gamma_2 \Delta w \overline{\Delta w^*} + \gamma(\Delta u \overline{\Delta w^*} + \Delta w \overline{\Delta u^*}) + a(u-w)\overline{(u^*-w^*)}\Big)\, dA. \tag{13}$$

As usual, a superposed bar is used to denote the conjugate of a complex number. It is worth noting that this product is equivalent to the usual product in the Hilbert space $\mathcal{H}$ due to the assumptions over the coefficients.

We can rewrite system (9) as follows:

$$\begin{cases} \dot{u} &= v, \\ \dot{v} &= \frac{1}{\rho_1}\big(\mu_1 \Delta u + \mu \Delta w - \gamma_1 \Delta^2 u - \gamma \Delta^2 w - a(u-w) - \gamma^* \Delta^2 v\big), \\ \dot{w} &= z, \\ \dot{z} &= \frac{1}{\rho_2}\big(\mu \Delta u + \mu_2 \Delta w - \gamma \Delta^2 u - \gamma_2 \Delta^2 w + a(u-w)\big). \end{cases} \tag{14}$$

With the above notation, problem (9) with boundary conditions (7) can be stated as

$$\frac{dU}{dt} = \mathcal{A}U, \quad U_0 = (u_0, v_0, w_0, z_0), \tag{15}$$

where (u_0, v_0, w_0, z_0) are the initial conditions (6) and $\mathcal{A}$ is the following 4×4 matrix:

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{\mu_1\Delta - \gamma_1\Delta^2 - aI}{\rho_1} & \frac{-\gamma^*\Delta^2}{\rho_1} & \frac{\mu\Delta - \gamma\Delta^2 + aI}{\rho_1} & 0 \\ 0 & 0 & 0 & I \\ \frac{\mu\Delta - \gamma\Delta^2 + aI}{\rho_2} & 0 & \frac{\mu_2\Delta - \gamma_2\Delta^2 - aI}{\rho_2} & 0 \end{pmatrix}, \tag{16}$$

where I denotes the identity operator.

The domain of the operator $\mathcal{A}$, which will be denoted by $\mathcal{D}(\mathcal{A})$, is

$$\mathcal{D}(\mathcal{A}) = \{(u, v, w, z) \in \mathcal{H} \ : \ v, z \in (H_0^1 \cap H^2), \quad \gamma_1 \Delta^2 u + \gamma \Delta^2 w + \gamma^* \Delta^2 v \in L^2, \gamma \Delta^2 u + \gamma_2 \Delta^2 w \in L^2, \gamma_1 \Delta u + \gamma \Delta w + \gamma^* \Delta v = 0, \gamma \Delta u + \gamma_2 \Delta w = 0 \text{ on } \partial B\}, \tag{17}$$

which is dense in $\mathcal{H}$ since $[C_0^\infty(B)]^4 \subset \mathcal{D}(\mathcal{A})$.

We prove now that system (15) has a unique solution. To do so we show that the operator $\mathcal{A}$ is dissipative and that 0 belongs to the resolvent of $\mathcal{A}$.

Lemma 1. *The operator $\mathcal{A}$ is dissipative. In fact,*

$$\Re\langle \mathcal{A}U, U\rangle = -\frac{\gamma^*}{2}\int_B |\Delta v|^2\, dA. \tag{18}$$

Proof. The proof is straightforward by means of the divergence theorem and making use of the boundary conditions. □

Lemma 2. *Let $\mathcal{A}$ be the above defined matrix. Then, 0 is in the resolvent of $\mathcal{A}$. This is usually written as $0 \in \varrho(\mathcal{A})$ to shorten.*

Proof. For any $\mathcal{F} = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ we will find $U \in \mathcal{H}$ such that $\mathcal{A}U = \mathcal{F}$. Writing this condition term by term we get:

$$\begin{cases} v &= f_1, \\ \mu_1 \Delta u + \mu \Delta w - \gamma_1 \Delta^2 u - \gamma \Delta^2 w - a(u-w) - \gamma^* \Delta^2 v &= \rho_1 f_2, \\ z &= f_3, \\ \mu \Delta u + \mu_2 \Delta w - \gamma \Delta^2 u - \gamma_2 \Delta^2 w + a(u-w) &= \rho_2 f_4. \end{cases} \tag{19}$$

We will solve the above system using the expressions of f_i as series. It is known that each f_i can be written as $f_i = \sum f_n^i \Phi_n(x)$, where $\Phi_n(x)$ are the eigenfunctions of problem (10). From the definition of $\mathcal{H}$, it must be

$$\sum (m_n)^2 (f_n^i)^2 < \infty, \text{ for } i = 1, 3 \text{ and } \sum (f_n^i)^2 < \infty, \text{ for } i = 2, 4. \tag{20}$$

On the other hand, the solutions we are looking for can be written also as series with unknown coefficients:

$$u = \sum u_n \Phi_n(x), \quad v = \sum v_n \Phi_n(x), \quad w = \sum w_n \Phi_n(x), \quad z = \sum z_n \Phi_n(x). \tag{21}$$

If we substitute these expressions in the system, we get straightforward that $v_n = f_n^1$ and $z_n = f_n^3$. Moreover, for each n a new system of equations is obtained:

$$\begin{cases} -a(u_n - w_n) - \gamma_1 m_n^2 u_n - \mu_1 m_n u_n - \gamma m_n^2 w_n - \mu m_n w_n &= \gamma^* f_n^1 m_n^2 + f_n^2 \rho_1, \\ a(u_n - w_n) - \gamma m_n^2 u_n - \mu m_n u_n - \gamma_2 m_n^2 w_n - \mu_2 m_n w_n &= f_n^4 \rho_2. \end{cases} \tag{22}$$

The solution of this system is given by

$$u_n = D_n^{-1} \big(f_n^4 \rho_2 (\gamma m_n^2 + \mu m_n - a) - (a + \gamma_2 m_n^2 + \mu_2 m_n)(\gamma^* f_n^1 m_n^2 + f_n^2 \rho_1) \big) \tag{23}$$

and

$$w_n = D_n^{-1} \big((\gamma m_n^2 + \mu m_n - a)(\gamma^* f_n^1 m_n^2 + f_n^2 \rho_1) - f_n^4 \rho_2 (a + \gamma_1 m_n^2 + \mu_1 m_n) \big) \tag{24}$$

where

$$D_n = (\gamma_1 \gamma_2 - \gamma^2) m_n^4 + (\gamma_2 \mu_1 + \gamma_1 \mu_2 - 2\gamma\mu) m_n^3 + \big(2a\gamma + a(\gamma_1 + \gamma_2) + \mu_1 \mu_2 - \mu^2\big) m_n^2 + a(2\mu + \mu_1 + \mu_2) m_n, \tag{25}$$

which is strictly positive for all m_n. Indeed, the fourth degree coefficient is clearly positive from the conditions over the coefficients (5); the cubic coefficient is positive since

$$0 \le (\sqrt{\gamma_1 \mu_2} - \sqrt{\gamma_2 \mu_1})^2 < \gamma_1 \mu_2 + \gamma_2 \mu_1 - 2|\gamma\mu| \tag{26}$$

and the rest of coefficients are positive since, by the Arithmetic–Geometric Mean Inequality,

$$|\mu| < \sqrt{\mu_1 \mu_2} \le \frac{\mu_1 + \mu_2}{2}. \tag{27}$$

The same holds for the gammas. Hence, it is not difficult to see that $\sum (m_n)^2 u_n^2 < \infty$ and $\sum (m_n)^2 w_n^2 < \infty$.

It is easy to show that the (u, v, w, z) found belongs to the domain of $\mathcal{A}$. Therefore, from (19) we get the desired result.

Finally, taking into account the solutions obtained for u_n, v_n, w_n and z_n, it can be shown that

$$\|U\|_{\mathcal{H}} \le K \|F\|_{\mathcal{H}}, \tag{28}$$

where K is a constant independent of U. □

The fact that the operator $\mathcal{A}$ is dissipative jointly with the fact that $\mathcal{D}(\mathcal{A})$ is closed and dense, the above lemma and the Lumer-Phillips theorem prove the existence and uniqueness of solutions. Hence, we have obtained:

Theorem 1. *The operator $\mathcal{A}$ generates a C_0−semigroup of contractions $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ in $\mathcal{H}$. Therefore, for each $U_0 \in \mathcal{D}(\mathcal{A})$, there exists a unique solution $U(t) \in C^1([0,\infty),\mathcal{H}) \cap C^0([0,\infty),\mathcal{D}(\mathcal{A}))$ to problem (15).*

2.2 | Time decay

Now we will prove that the solutions decay exponentially. To do so, we need to decompose $\mathcal{H}$ as the direct sum of two subspaces: $\mathcal{H} = \mathcal{K}^N \oplus \mathcal{K}$ where $\mathcal{K}^N$ is the finite dimensional subspace generated by the vectors

$$\Omega(h,i,j,k) = (\Phi_h(x), \Phi_i(x), \Phi_j(x), \Phi_k(x)) \quad 1 \leq h,i,j,k \leq N. \tag{29}$$

Notice that $\mathcal{K}^N$ is invariant. That means that the solutions starting at $\mathcal{K}^N$ always belong to $\mathcal{K}^N$.

A solution $U(t)$ of system (9) can also be decomposed as the sum of two elements, $U(t) = U_1(t) + U_2(t)$ where $U_1(t) \in \mathcal{K}^N$ and $U_2(t) \in \mathcal{K}$.

As $U_1(t)$ belongs to a finite dimensional subspace, if all the eigenvalues of $\mathcal{A}$ restricted to $\Omega(h,i,j,k)$ have negative real part, the exponential decay of $U_1(t)$ is guaranteed.

Proposition 1. *All the eigenvalues of $\mathcal{A}$ restricted to $\mathcal{K}^N$ have negative real part.*

Proof. Imposing $u = A_1 e^{\omega t}\Phi_n(x)$ and $w = A_2 e^{\omega t}\Phi_n(x)$, the following homogeneous system on the unknowns A_1 and A_2 is obtained:

$$\begin{pmatrix} a + m_n\mu_1 + m_n^2(\gamma_1 + \omega\gamma^*) + \rho_1\omega^2 & -a + m_n^2\gamma + m_n\mu \\ -a + m_n^2\gamma + m_n\mu & a + m_n^2\gamma_2 + m_n\mu_2 + \rho_2\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{30}$$

Let a_i, for $i = 0,1,2,3,4$, be the coefficients of the fourth degree polynomial, $p(x)$, obtained from the determinant of the coefficients matrix once ω is replaced by x:

$$\begin{aligned} a_0 &= \rho_1\rho_2, \\ a_1 &= \rho_2\gamma^* m_n^2, \\ a_2 &= (\gamma_1\rho_2 + \gamma_2\rho_1)m_n^2 + (\mu_1\rho_2 + \mu_2\rho_1)m_n + a(\rho_1 + \rho_2), \\ a_3 &= \gamma_2\gamma^* m_n^4 + \mu_2\gamma^* m_n^3 + a\gamma^* m_n^2, \\ a_4 &= D_n, \end{aligned} \tag{31}$$

and it is clear that $a_i > 0$ for all $i = 0,1,2,3,4$.

The Routh-Hurwitz theorem (see Dieudonné [31]) will give us the desired result. It says that, if $a_0 > 0$, then all the roots of polynomial

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \tag{32}$$

have negative real part if and only if a_4 and all the leading diagonal minors, M_i, of the matrix

$$\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{pmatrix} \tag{33}$$

are positive.

Straightforward computations give that the leading minors corresponding to polynomial $p(x)$ are:

$$\begin{aligned} M_1 &= a_1, \\ M_2 &= m_n^2(a+\mu_1 m_n+\gamma_1 m_n^2)\rho_2^2\gamma^*, \\ M_3 &= m_n^4(\gamma m_n^2+\mu m_n-a)^2\rho_2^2(\gamma^*)^2, \\ M_4 &= a_4 M_3, \end{aligned} \tag{34}$$

and by hypothesis, $\gamma m_n^2+\mu m_n \neq a$ for all m_n, and by the assumptions over the coefficients (5) we see that they are all positive. □

We now study how $U_2(t)$ decays. To do so we will use the characterization given by Huang [32] or Prüss [33] that we recall here.

Theorem 2. *Let $S(t)=\{e^{\mathcal{A}t}\}_{t\geq 0}$ be a C_0−semigroup of contractions on a Hilbert space $\mathcal{H}$. Then $S(t)$ is exponentially stable if and only if $i\mathbb{R}\subset\varrho(\mathcal{A})$ and $\overline{\lim\limits_{|\lambda|\to\infty}}\,\|(i\lambda\mathcal{I}-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}<\infty$.*

We split the conditions of Theorem 2 in two separate lemmas.

Lemma 3. *Let $\mathcal{A}$ be the matrix operator defined before. Then, $i\mathbb{R}\subset\varrho(\mathcal{A})$.*

Proof. The proof has three steps. The first two refer to the operator $\mathcal{A}$ and are quite standard (for details see, for instance, [34], page 25). We concentrate in the third one, which is specific to each case. Here it reads as follows: suppose that there exist a sequence of real numbers λ_n with $\lambda_n\to\varpi$, $|\lambda_n|<|\varpi|$ and a sequence of vectors $U_n=(u_n,v_n,w_n,z_n)\in\mathcal{D}(\mathcal{A})$ and with unit norm, that is, $\|\Delta u_n\|^2+\|\Delta w_n\|^2+\|v_n\|^2+\|z_n\|^2=1$, such that $\|(i\lambda_n\mathcal{I}-\mathcal{A})U_n\|\to 0$, and we will arrive to a contradiction.

If we write the above expression term by term, we obtain the following conditions:

$$i\lambda_n u_n - v_n \to 0, \text{ in } H^2 \tag{35a}$$

$$i\lambda_n v_n\rho_1-\left(\mu_1\Delta u_n+\mu\Delta w_n-\gamma_1\Delta^2 u_n-\gamma\Delta^2 w_n-a(u_n-w_n)-\gamma^*\Delta^2 v_n\right)\to 0, \text{ in } L^2 \tag{35b}$$

$$i\lambda_n w_n - z_n \to 0, \text{ in } H^2 \tag{35c}$$

$$i\lambda_n z_n\rho_2-\left(\mu\Delta u_n+\mu_2\Delta w_n-\gamma\Delta^2 u_n-\gamma_2\Delta^2 w_n+a(u_n-w_n)\right)\to 0, \text{ in } L^2. \tag{35d}$$

Taking the real part of the product $\langle(i\lambda_n\mathcal{I}-\mathcal{A})U_n,U_n\rangle$ and recalling the result of Lemma 1, it is clear that $\Delta v_n\to 0$ in L^2 and, therefore, $\lambda_n\Delta u_n\to 0$ in L^2.

We now multiply (35b) by w_n to obtain

$$\begin{aligned} i\lambda_n\rho_1\langle v_n,w_n\rangle-\big(\mu_1\langle\Delta u_n,w_n\rangle+\mu\langle\Delta w_n,w_n\rangle-\gamma_1\left\langle\Delta^2 u_n,w_n\right\rangle-\gamma\left\langle\Delta^2 w_n,w_n\right\rangle \\ -a(\langle u_n,w_n\rangle-\langle w_n,w_n\rangle)-\gamma^*\left\langle\Delta^2 v_n,w_n\right\rangle\big)\to 0, \end{aligned} \tag{36}$$

and we notice that since the vector is unitary we have that w_n is bounded and combined with $\lambda_n\Delta u_n\to 0$ in L^2 we get $\langle\Delta u_n,w_n\rangle\to 0$. Also notice that $\langle u_n,w_n\rangle\to 0$, since $\Delta u_n\to 0$ and therefore, by Poincaré inequality $u_n\to 0$. Using these two observations, we can simplify the above expression to get

$$i\lambda_n\rho_1\langle v_n, w_n\rangle - \left(\mu\langle\Delta w_n, w_n\rangle - \gamma_1\left\langle\Delta^2 u_n, w_n\right\rangle - \gamma\left\langle\Delta^2 w_n, w_n\right\rangle + a\langle w_n, w_n\rangle - \gamma^*\left\langle\Delta^2 v_n, w_n\right\rangle\right) \to 0, \quad (37)$$

which applying divergence theorem becomes

$$i\lambda_n\rho_1\langle v_n, w_n\rangle - \left(-\mu|\nabla w_n|^2 - \gamma_1\langle\Delta u_n, \Delta w_n\rangle - \gamma|\Delta w_n|^2 + a|w_n|^2 - \gamma^*\langle\Delta v_n, \Delta w_n\rangle\right) \to 0. \quad (38)$$

Notice that Δw_n is also bounded, and thus, $\langle\Delta u_n, \Delta w_n\rangle \to 0$ and $\langle\Delta v_n, \Delta w_n\rangle \to 0$. z_n is bounded and therefore, by (35c) and Poincaré inequality

$$i\lambda_n\rho_1\langle v_n, w_n\rangle = \rho_1\langle v_n, -i\lambda_n w_n\rangle \approx \rho_1\langle v_n, -z_n\rangle \to 0. \quad (39)$$

With the above observations, equation (37) reduces to

$$\mu|\nabla w_n|^2 + \gamma|\Delta w_n|^2 - a|w_n|^2 \to 0, \quad (40)$$

so, taking a sufficiently small subspace we can find a Poincaré constant such that $\Delta w_n \to 0$ in L^2 (we recall that $\gamma m_n^2 + \mu m_n \neq a$ for all m_n).

Now we multiply (35d) also by w_n and proceeding as before we can reduce it to

$$i\rho_2\lambda_n\langle z_n, w_n\rangle \to 0 \Rightarrow \rho_2\langle z_n, -i\lambda_n w_n\rangle \approx -\rho_2\langle z_n, z_n\rangle \to 0, \quad (41)$$

which finishes the proof because this shows that vector U_n cannot be of unit norm. □

Lemma 4. *Let $\mathcal{A}$ be the operator defined before. Then,* $\overline{\lim}_{|\lambda|\to\infty} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Proof. This result can be proved using the same argument we have followed in the proof of Lemma 3, since throughout the proof we only make use of the fact that $\lim_{n\to\infty}\lambda_n \neq 0$, but it does not depend on λ_n tending to a finite number or to infinity. □

As a consequence of the above lemmas we have the following result.

Theorem 3. *Let (u, w) be a solution to the problem determined by system (9) with boundary conditions (7) and initial conditions (6). Then, (u, w) decays exponentially.*

3 | VISCOSITY: DISSIPATION GENERATED BY A FIRST-ORDER VELOCITY GRADIENT

We study now the dissipation generated by the first-order velocity gradient. This dissipation mechanism is supposedly weaker than the previous one. For this reason, we call this case *viscosity*. To be precise we suppose that

$$\tau_{ji} = a_{ijrs}u_{r,s} + b_{ijrs}w_{r,s} + a^*_{ijrs}\dot{u}_{r,s}. \quad (42)$$

Therefore, with our notation, the system of field equations for antiplane shear deformations in the case of isotropic and homogeneous mixtures becomes

$$\begin{cases} \rho_1\ddot{u} = & \mu_1\Delta u + \mu\Delta w - \gamma_1\Delta^2 u - \gamma\Delta^2 w - a(u-w) + \mu^*\Delta\dot{u} \\ \rho_2\ddot{w} = & \mu\Delta u + \mu_2\Delta w - \gamma\Delta^2 u - \gamma_2\Delta^2 w + a(u-w) \end{cases} \quad (43)$$

with $\mu^* > 0$.

We still assume that $\gamma m_n^2 + \mu m_n \neq a$ for all m_n, because if not, undamped solutions can also be found for this system. The same Hilbert space, $\mathcal{H}$, is considered, with the same inner product.

We will follow the same scheme that we have done in Section 2 and, hence, only the important facts will be stated.

3.1 | Existence and uniqueness

First we rewrite system (43) as follows:

$$\begin{cases} \dot{u} &= v, \\ \dot{v} &= \frac{1}{\rho_1}\left(\mu_1\Delta u + \mu\Delta w - \gamma_1\Delta^2 u - \gamma\Delta^2 w - a(u-w) + \mu^*\Delta v\right), \\ \dot{w} &= z, \\ \dot{z} &= \frac{1}{\rho_2}\left(\mu\Delta u + \mu_2\Delta w - \gamma\Delta^2 u - \gamma_2\Delta^2 w + a(u-w)\right). \end{cases} \tag{44}$$

Therefore, in matrix form the above system is

$$\frac{dU}{dt} = \mathcal{B}U, \quad U_0 = (u_0, v_0, w_0, z_0), \tag{45}$$

where (u_0, v_0, w_0, z_0) are the initial conditions (6) and $\mathcal{B}$ is the following 4×4 matrix:

$$\mathcal{B} = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{\mu_1\Delta - \gamma_1\Delta^2 - aI}{\rho_1} & \frac{\mu^*\Delta}{\rho_1} & \frac{\mu\Delta - \gamma\Delta^2 + aI}{\rho_1} & 0 \\ 0 & 0 & 0 & I \\ \frac{\mu\Delta - \gamma\Delta^2 + aI}{\rho_2} & 0 & \frac{\mu_2\Delta - \gamma_2\Delta^2 - aI}{\rho_2} & 0 \end{pmatrix}. \tag{46}$$

In this case, the domain of the operator $\mathcal{B}$ is

$$\begin{aligned} \mathcal{D}(\mathcal{B}) = \{(u, v, w, z) \in \mathcal{H} \ :\ & v, z \in (H_0^1 \cap H^2), \quad \gamma_1\Delta^2 u + \gamma\Delta^2 w - \mu^*\Delta v \in L^2, \\ & \gamma\Delta^2 u + \gamma_2\Delta^2 w \in L^2, \quad \Delta u = \Delta z = 0 \text{ on } \partial B\}. \end{aligned} \tag{47}$$

Again, it is also dense in the Hilbert space $\mathcal{H}$.

Lemma 5. *The operator $\mathcal{B}$ is dissipative. In fact,*

$$\Re\langle \mathcal{B}U, U\rangle = -\frac{\mu^*}{2}\int_B |\nabla v|^2\, dA. \tag{48}$$

Proof. The proof is straightforward. □

Lemma 6. *Let $\mathcal{B}$ be the above defined matrix. Then, $0 \in \varrho(\mathcal{B})$.*

Proof. We proceed as in the proof of Lemma 2. For any $\mathcal{F} = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ we will find $U \in \mathcal{H}$ such that $\mathcal{B}U = \mathcal{F}$. Writing this condition term by term we get:

$$\begin{cases} v &= f_1, \\ \mu_1\Delta u + \mu\Delta w - \gamma_1\Delta^2 u - \gamma\Delta^2 w - a(u-w) + \mu^*\Delta v &= \rho_1 f_2, \\ z &= f_3, \\ \mu\Delta u + \mu_2\Delta w - \gamma\Delta^2 u - \gamma_2\Delta^2 w + a(u-w) &= \rho_2 f_4. \end{cases} \tag{49}$$

We use again the expressions of f_i as series, with the same notation and with the same conditions we have used in Section 2.

Substituting the expressions of u, v, w and z in the system, we get straightforward that $v_n = f_n^1$ and $z_n = f_n^3$. Moreover, for each n we obtain:

$$\begin{cases} -a(u_n - w_n) - \gamma_1 m_n^2 u_n - \mu_1 m_n u_n - \gamma m_n^2 w_n - \mu m_n w_n &= \rho_1 f_n^2 + \mu^* f_n^1 m_n, \\ a(u_n - w_n) - \gamma m_n^2 u_n - \mu m_n u_n - \gamma_2 m_n^2 w_n - \mu_2 m_n w_n &= f_n^4 \rho_2. \end{cases} \tag{50}$$

The solution of this system is given by

$$u_n = D_n^{-1}\big(f_n^4\rho_2(\gamma m_n^2 + \mu m_n - a) - (a + \gamma_2 m_n^2 + \mu_2 m_n)(f_n^1\mu^* m_n + f_n^2\rho_1)\big) \tag{51}$$

and

$$w_n = D_n^{-1}\big((\gamma m_n^2 + \mu m_n - a)(\mu^* f_n^1 m_n + f_n^2\rho_1) - f_n^4\rho_2(a + \gamma_1 m_n^2 + \mu_1 m_n)\big) \tag{52}$$

where D_n is the same as in the previous section.

Hence, it is not difficult to see that $\sum(m_n)^2 u_n^2 < \infty$ and $\sum(m_n)^2 w_n^2 < \infty$.

The fact that (u, v, w, z) belongs to the domain of $\mathcal{B}$ comes trivially from (49) and the above results. Finally, taking into account the solutions obtained for u_n, v_n, w_n and z_n, it can be shown that

$$\|U\|_{\mathcal{H}} \leq K\|F\|_{\mathcal{H}}, \tag{53}$$

where K is a constant independent of U. □

Theorem 4. *The operator $\mathcal{B}$ generates a C_0−semigroup of contractions $S(t) = \{e^{\mathcal{B}t}\}_{t\geq 0}$ in $\mathcal{H}$. Therefore, for each $U_0 \in \mathcal{D}(\mathcal{B})$, there exists a unique solution $U(t) \in C^1([0,\infty), \mathcal{H}) \cap C^0([0,\infty), \mathcal{D}(\mathcal{B}))$ to problem (45).*

3.2 | Time decay

We prove now that the solutions also decay exponentially. As before, we decompose $\mathcal{H}$ as the direct sum of two subspaces: $\mathcal{H} = \mathcal{K}^N \oplus \mathcal{K}$ (we use the same notation). A solution $U(t)$ of system (43) can be written as $U(t) = U_1(t) + U_2(t)$ where $U_1(t) \in \mathcal{K}^N$ and $U_2(t) \in \mathcal{K}$.

Proposition 2. *All the eigenvalues of $\mathcal{B}$ restricted to $\mathcal{K}^N$ have negative real part.*

Proof. Imposing $u = B_1 e^{\omega t}\Phi_n(x)$ and $w = B_2 e^{\omega t}\Phi_n(x)$, the following homogeneous system on the unknowns B_1 and B_2 is obtained:

$$\begin{pmatrix} a + m_n\mu_1 + m_n^2(\gamma_1 + \omega\gamma^*) + \rho_1\omega^2 & -a + m_n^2\gamma + m_n\mu \\ -a + m_n^2\gamma + m_n\mu & a + m_n^2\gamma_2 + m_n\mu_2 + \rho_2\omega^2 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{54}$$

Let b_i, for $i = 0, 1, 2, 3, 4$ be the coefficients of the fourth degree polynomial, $p(x)$, obtained from the determinant of the coefficients matrix once ω is replaced by x:

$$\begin{aligned} b_0 &= \rho_1\rho_2, \\ b_1 &= \rho_2\mu^* m_n, \\ b_2 &= (\gamma_1\rho_2 + \gamma_2\rho_1)m_n^2 + (\mu_1\rho_2 + \mu_2\rho_1)m_n + a(\rho_1 + \rho_2), \\ b_3 &= \gamma_2\mu^* m_n^3 + \mu_2\mu^* m_n^2 + a\mu^* m_n, \\ b_4 &= D_n, \end{aligned} \tag{55}$$

and it is clear that $b_i > 0$ for all $i = 0, 1, 2, 3, 4$.

The leading minors corresponding to polynomial $p(x)$ are:

$$\begin{aligned} M_1 = &\ b_1, \\ M_2 = &\ m_n(a + \mu_1 m_n + \gamma_1 m_n^2)\rho_2^2\mu^*, \\ M_3 = &\ m_n^2(\gamma m_n^2 + \mu m_n - a)^2\rho_2^2(\mu^*)^2, \\ M_4 = &\ b_4 M_3. \end{aligned} \tag{56}$$

They are all positive and, therefore, Routh-Hurwitz theorem finishes the proof. □

The above proposition shows the exponential decay of $U_1(t)$. We now study $U_2(t)$. To do so, we use again Theorem 2.

Lemma 7. *Let $\mathcal{B}$ be the matrix operator defined above. Then, $i\mathbb{R} \subset \varrho(\mathcal{B})$.*

Proof. We suppose that there exist a sequence of real numbers λ_n with $\lambda_n \to \varpi$, $|\lambda_n| < |\varpi|$ and a sequence of vectors of unit norm $U_n = (u_n, v_n, w_n, z_n) \in \mathcal{D}(\mathcal{B})$ such that $\|(i\lambda_n\mathcal{I} - \mathcal{B})U_n\| \to 0$.

If we write the above expression term by term, we obtain the following conditions:

$$i\lambda_n u_n - v_n \to 0, \text{ in } H^2 \tag{57a}$$

$$i\lambda_n v_n\rho_1 - \left(\mu_1\Delta u_n + \mu\Delta w_n - \gamma_1\Delta^2 u_n - \gamma\Delta^2 w_n - a(u_n - w_n) + \mu^*\Delta v_n\right) \to 0, \text{ in } L^2 \tag{57b}$$

$$i\lambda_n w_n - z_n \to 0, \text{ in } H^2 \tag{57c}$$

$$i\lambda_n z_n\rho_2 - \left(\mu\Delta u_n + \mu_2\Delta w_n - \gamma\Delta^2 u_n - \gamma_2\Delta^2 w_n + a(u_n - w_n)\right) \to 0, \text{ in } L^2. \tag{57d}$$

Taking the real part of the product $\langle(i\lambda_n\mathcal{I} - \mathcal{B})U_n, U_n\rangle$ and recalling (48), it is clear that $\nabla v_n \to 0$ in L^2 and, therefore, $\lambda_n\nabla u_n \to 0$ in L^2.

We now multiply (57b) by u_n and we proceed as in Lemma 3 to obtain

$$\gamma_1|\Delta u_n|^2 + \gamma\langle\Delta w_n, \Delta u_n\rangle \to 0, \tag{58}$$

and we multiply (57d) by u_n as well to get

$$\gamma|\Delta u_n|^2 + \gamma_2\langle\Delta w_n, \Delta u_n\rangle \to 0. \tag{59}$$

From the above expressions and using that $\gamma_1\gamma_2 > \gamma^2$ we get that $|\Delta u_n| \to 0$ and $\langle\Delta w_n, \Delta u_n\rangle \to 0$. Thus, the rest of the proof is as in the previous case. □

Lemma 8. *Let $\mathcal{B}$ be the operator defined above. Then, $\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{B})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.*

Proof. Again, throughout the proof of the previous lemma we only make use of the fact that $\lim_{n\to\infty}\lambda_n \neq 0$, but it does not depend on λ_n tending to a finite number or to infinity. □

Theorem 5. *Let (u, w) be a solution to the problem determined by system (43) with boundary conditions (7) and initial conditions (6). Then, (u, w) decays exponentially.*

4 | DISSIPATION GENERATED BY RELATIVE VELOCITY

Let us study now the dissipation generated by relative velocity. This dissipation mechanism is supposedly even weaker than the others. To be precise we assume that

$$p_i = f_{irmn} u_{r,mn} + p_{irmn} w_{r,mn} + a_{ij} d_j + a^*_{ij} \dot{d}_j. \tag{60}$$

Therefore, following our notation, the system of field equations for antiplane shear deformations in the case of isotropic and homogeneous mixtures becomes

$$\begin{cases} \rho_1 \ddot{u} = & \mu_1 \Delta u + \mu \Delta w - \gamma_1 \Delta^2 u - \gamma \Delta^2 w - a(u-w) - a^*(\dot{u} - \dot{w}) \\ \rho_2 \ddot{w} = & \mu \Delta u + \mu_2 \Delta w - \gamma \Delta^2 u - \gamma_2 \Delta^2 w + a(u-w) + a^*(\dot{u} - \dot{w}) \end{cases} \tag{61}$$

with $a^* > 0$.

Notice that the dissipation has not derivatives with respect to the spatial variables. Notice also that this system is quite different to the previous ones.

Remark 2. This system has undamped solutions. As before, consider $\Phi_n(x)$ the eigenfunctions of (10), but now take, for instance, $u = w = e^{\omega t}\Phi_n(x)$ and substitute them in (61) to get

$$\begin{cases} \rho_1 \omega^2 & = -\mu_1 m_n - \mu m_n - \gamma_1 m_n^2 - \gamma m_n^2, \\ \rho_2 \omega^2 & = -\mu m_n - \mu_2 m_n - \gamma m_n^2 - \gamma_2 m_n^2. \end{cases} \tag{62}$$

This system has a solution for ω whenever

$$[(\mu + \mu_1)\rho_2 - (\mu + \mu_2)\rho_1] m_n + [(\gamma + \gamma_1)\rho_2 - (\gamma + \gamma_2)\rho_1] m_n^2 = 0. \tag{63}$$

Therefore, we assume $[(\mu + \mu_1)\rho_2 - (\mu + \mu_2)\rho_1] + [(\gamma + \gamma_1)\rho_2 - (\gamma + \gamma_2)\rho_1] m_n \neq 0$ for all m_n.

4.1 | Existence and uniqueness

We can rewrite system (61) as follows:

$$\begin{cases} \dot{u} & = & v, \\ \dot{v} & = & \frac{1}{\rho_1}\big(\mu_1 \Delta u + \mu \Delta w - \gamma_1 \Delta^2 u - \gamma \Delta^2 w - a(u-w) - a^*(v-z)\big), \\ \dot{w} & = & z, \\ \dot{z} & = & \frac{1}{\rho_2}\big(\mu \Delta u + \mu_2 \Delta w - \gamma \Delta^2 u - \gamma_2 \Delta^2 w + a(u-w) + a^*(v-z)\big). \end{cases} \tag{64}$$

In matrix form the above system becomes

$$\frac{dU}{dt} = CU, \quad U_0 = (u_0, v_0, w_0, z_0), \tag{65}$$

where (u_0, v_0, w_0, z_0) are the initial conditions (6) and C is the following 4×4 matrix:

$$C = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{\mu_1 \Delta - \gamma_1 \Delta^2 - aI}{\rho_1} & -\frac{a^*}{\rho_1} & \frac{\mu \Delta - \gamma \Delta^2 + aI}{\rho_1} & \frac{a^*}{\rho_1} \\ 0 & 0 & 0 & I \\ \frac{\mu \Delta - \gamma \Delta^2 + aI}{\rho_2} & \frac{a^*}{\rho_2} & \frac{\mu_2 \Delta - \gamma_2 \Delta^2 - aI}{\rho_2} & -\frac{a^*}{\rho_2} \end{pmatrix}. \tag{66}$$

The domain of the operator C is

$$\mathcal{D}(C)=\{(u,v,w,z)\in\mathcal{H}\ :\ v,z\in(H_0^1\cap H^2),\quad \gamma_1\Delta^2u+\gamma\Delta^2w\in L^2,\\ \gamma\Delta^2u+\gamma_2\Delta^2w\in L^2,\quad \Delta u=\Delta z=0 \text{ on } \partial B\}. \tag{67}$$

Again, it is dense in the Hilbert space $\mathcal{H}$.

Lemma 9. *The operator C is dissipative. In fact,*

$$\Re\langle CU,U\rangle=-\frac{a^*}{2}\int_B|v-z|^2\,dA. \tag{68}$$

Proof. The proof is straightforward. □

Lemma 10. *Let C be the above defined matrix. Then, $0\in\varrho(C)$.*

Proof. We proceed as in the proof of Lemma 2. For any $\mathcal{F}=(f_1,f_2,f_3,f_4)\in\mathcal{H}$ we will find $U\in\mathcal{H}$ such that $CU=\mathcal{F}$. Writing this condition term by term we get:

$$\begin{cases} v &= f_1,\\ \mu_1\Delta u+\mu\Delta w-\gamma_1\Delta^2u-\gamma\Delta^2w-a(u-w)-a^*(v-z) &= \rho_1f_2,\\ z &= f_3,\\ \mu\Delta u+\mu_2\Delta w-\gamma\Delta^2u-\gamma_2\Delta^2w+a(u-w)+a^*(v-z) &= \rho_2f_4.\end{cases} \tag{69}$$

As before, we solve the above system using the expressions of f_i,u,v,w,z as series. Doing so, we get straightforward that $v_n=f_n^1$ and $z_n=f_n^3$. Moreover, for each n a new system of equations is obtained:

$$\begin{cases} -a(u_n-w_n)-\gamma_1m_n^2u_n-\mu_1m_nu_n-\gamma m_n^2w_n-\mu m_nw_n &=\rho_1f_n^2+a^*(f_n^1-f_n^3),\\ a(u_n-w_n)-\gamma m_n^2u_n-\mu m_nu_n-\gamma_2m_n^2w_n-\mu_2m_nw_n &=\rho_2f_n^4-a^*(f_n^1-f_n^3).\end{cases} \tag{70}$$

The solution of this system is given by

$$u_n=D_n^{-1}\big(((f_n^1-f_n^3)a^*-f_n^4\rho_2)\big(a-\mu m_n-\gamma m_n^2\big)-\big(a+\mu_2m_n+\gamma_2m_n^2\big)\big((f_n^1-f_n^3)a^*+f_n^2\rho_1\big)\big) \tag{71}$$

and

$$w_n=D_n^{-1}\big((f_n^1-f_n^3)m_na^*((\gamma+\gamma_1)m_n+\mu+\mu_1)+f_n^2\rho_1(-a+\mu m_n+\gamma m_n^2)-f_n^4\rho_2\big(a+\gamma_1m_n^2+\mu_1m_n\big)\big) \tag{72}$$

where D_n is the same as in the previous sections.

Hence, it is not difficult to see that $\sum(m_n)^2u_n^2<\infty$ and $\sum(m_n)^2w_n^2<\infty$.

The fact that (u,v,w,z) belongs to the domain of C comes trivially from (69) and the above results. Finally, taking into account the solutions obtained for u_n, v_n, w_n and z_n, it can be shown that

$$\|U\|_{\mathcal{H}}\le K\|F\|_{\mathcal{H}}, \tag{73}$$

where K is a constant independent of U. □

Theorem 6. *The operator C generates a C_0−semigroup of contractions $S(t)=\{e^{Ct}\}_{t\ge0}$ in $\mathcal{H}$. Therefore, for each $U_0\in\mathcal{D}(C)$, there exists a unique solution $U(t)\in C^1([0,\infty),\mathcal{H})\cap C^0([0,\infty),\mathcal{D}(C))$ to problem (65).*

4.2 Time decay

In this case the decay of the solutions is generically exponential, but there is a specific subcase determined by a concrete combination of the constitutive coefficients that drives the solutions to slow decay.

A solution $U(t)$ to system (61) can also be written as $U(t) = U_1(t) + U_2(t)$ where $U_1(t) \in \mathcal{K}^N$ and $U_2(t) \in \mathcal{K}$.

Proposition 3. *All the eigenvalues of C restricted to $\mathcal{K}^N$ have negative real part.*

Proof. Imposing $u = C_1 e^{\omega t}\Phi_n(x)$ and $w = C_2 e^{\omega t}\Phi_n(x)$, the following homogeneous system on the unknowns C_1 and C_2 is obtained:

$$\begin{pmatrix} a + m_n\mu_1 + m_n^2\gamma_1 + \rho_1\omega^2 + a^*\omega & -a + m_n^2\gamma + m_n\mu - a^*\omega \\ -a + m_n^2\gamma + m_n\mu - a^*\omega & a + m_n^2\gamma_2 + m_n\mu_2 + \rho_2\omega^2 + a^*\omega \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{74}$$

Let c_i, for $i = 0, 1, 2, 3, 4$ be the coefficients of the fourth degree polynomial, $p(x)$, obtained from the determinant of the coefficients matrix once ω is replaced by x:

$$\begin{aligned} c_0 &= \rho_1\rho_2, \\ c_1 &= (\rho_1 + \rho_2)a^*, \\ c_2 &= (\gamma_1\rho_2 + \gamma_2\rho_1)m_n^2 + (\mu_1\rho_2 + \mu_2\rho_1)m_n + a(\rho_1 + \rho_2), \\ c_3 &= (2\gamma + \gamma_1 + \gamma_2)a^* m_n^2 + (2\mu + \mu_1 + \mu_2)a^* m_n, \\ c_4 &= D_n, \end{aligned} \tag{75}$$

and it is clear that $c_i > 0$ for all $i = 0, 1, 2, 3, 4$.

Straightforward computations give that the leading minors corresponding to polynomial $p(x)$ are:

$$\begin{aligned} M_1 &= c_1, \\ M_2 &= a^*(\gamma_1\rho_2^2 + \gamma_2\rho_1^2 - 2\gamma\rho_1\rho_2)m_n^2 + a^*(\mu_1\rho_2^2 + \mu_2\rho_1^2 - 2\mu\rho_1\rho_2)m_n + aa^*(\rho_1 + \rho_2)^2, \\ M_3 &= m_n^2(a^*)^2([(\mu + \mu_1)\rho_2 - (\mu + \mu_2)\rho_1] + [(\gamma + \gamma_1)\rho_2 - (\gamma + \gamma_2)\rho_1]m_n)^2, \\ M_4 &= c_4 M_3. \end{aligned} \tag{76}$$

By the assumptions, they are all positive. Therefore, the Routh-Hurwitz theorem finishes the proof. □

The above proposition shows the exponential decay of $U_1(t)$. We now study $U_2(t)$. In order to do so, we consider two different subcases:

(C1) $[(\gamma + \gamma_1)\rho_2 - (\gamma + \gamma_2)\rho_1] \neq 0$.
(C2) $[(\gamma + \gamma_1)\rho_2 - (\gamma + \gamma_2)\rho_1] = 0$.

Under (C1), we prove now the exponential decay of the solutions. Using again the characterization given by Huang or Prüss.

Lemma 11. *Let C be the matrix operator defined above. Then, if (C1) holds, $i\mathbb{R} \subset \varrho(C)$.*

Proof. We suppose that there exist a sequence of real numbers λ_n with $\lambda_n \to \varpi$, $|\lambda_n| < |\varpi|$ and a sequence of vectors $U_n = (u_n, v_n, w_n, z_n) \in \mathcal{D}(C)$ and with unit norm such that $\|(i\lambda_n\mathcal{I} - C)U_n\| \to 0$.

If we write the above expression term by term, we obtain the following conditions:

$$i\lambda_n u_n - v_n \to 0, \text{ in } H^2 \tag{77a}$$

$$i\lambda_n v_n \rho_1 - \left(\mu_1 \Delta u_n + \mu \Delta w_n - \gamma_1 \Delta^2 u_n - \gamma \Delta^2 w_n - a(u_n - w_n) - a^*(v_n - z_n)\right) \to 0, \text{ in } L^2 \tag{77b}$$

$$i\lambda_n w_n - z_n \to 0, \text{ in } H^2 \tag{77c}$$

$$i\lambda_n z_n \rho_2 - \left(\mu \Delta u_n + \mu_2 \Delta w_n - \gamma \Delta^2 u_n - \gamma_2 \Delta^2 w_n + a(u_n - w_n) + a^*(v_n - z_n)\right) \to 0, \text{ in } L^2. \tag{77d}$$

Taking the real part of the product $\langle (i\lambda_n \mathcal{I} - C)U_n, U_n\rangle$ and recalling (68), it is clear that $|v_n - z_n| \to 0$ in L^2 and, therefore, combining (77a) and (77c) one gets that $i\lambda_n(u_n - w_n) \to 0$ and $(u_n - w_n) \to 0$ in L^2.

We now multiply (77b) and (77d) by $u_n - w_n$ and we proceed as in Lemma 3:

$$\begin{aligned} &\gamma_1 \left\langle \Delta^2 u_n, u_n - w_n \right\rangle + \gamma \left\langle \Delta^2 w_n, u_n - w_n \right\rangle \to 0, \\ &\gamma \left\langle \Delta^2 u_n, u_n - w_n \right\rangle + \gamma_2 \left\langle \Delta^2 w_n, u_n - w_n \right\rangle \to 0, \end{aligned} \tag{78}$$

or equivalently

$$\begin{aligned} &\gamma_1 \langle \Delta u_n, \Delta(u_n - w_n)\rangle + \gamma \langle \Delta w_n, \Delta(u_n - w_n)\rangle \to 0, \\ &\gamma \langle \Delta u_n, \Delta(u_n - w_n)\rangle + \gamma_2 \langle \Delta w_n, \Delta(u_n - w_n)\rangle \to 0. \end{aligned} \tag{79}$$

Therefore, since $\gamma_1\gamma_2 > \gamma^2$ we get that $\langle \Delta u_n, \Delta(u_n - w_n)\rangle \to 0$ and $\langle \Delta w_n, \Delta(u_n - w_n)\rangle \to 0$ as well. Using the distributive property of the scalar product one deduces that $|\Delta u_n|^2 \sim |\Delta w_n|^2$ and $\langle \Delta u_n, \Delta w_n\rangle \sim |\Delta u_n|^2$ as $n \to \infty$.

Let us now write $(77b) \cdot \rho_2 - (77d) \cdot \rho_1$:

$$\begin{aligned} i\lambda_n \rho_1 \rho_2 (v_n - z_n) + (\mu\rho_1 - \mu_1\rho_2)\Delta u_n + (\mu_2\rho_1 - \mu\rho_2)\Delta w_n \\ +(\gamma_1\rho_2 - \gamma\rho_1)\Delta^2 u_n + (\gamma\rho_2 - \gamma_2\rho_1)\Delta^2 w_n \to 0. \end{aligned} \tag{80}$$

Multiplying by u_n, simplifying and using divergence theorem we obtain

$$[(\mu + \mu_1)\rho_2 - (\mu + \mu_2)\rho_1]|\nabla u_n|^2 + [(\gamma + \gamma_1)\rho_2 - (\gamma + \gamma_2)\rho_1]|\Delta u_n|^2 \to 0, \tag{81}$$

therefore, taking a sufficiently small subspace we can find a Poincaré constant such that we have $|\Delta u_n|^2 \to 0$ in L^2.

Finally, we notice that using what we have seen so far (77b) becomes

$$i\lambda_n v_n \rho_1 + \gamma_1 \Delta^2 u_n + \gamma \Delta^2 w_n \to 0, \tag{82}$$

so if we multiply by v_n and divide by λ_n we get

$$i|v_n|^2 \rho_1 + \gamma_1 \left\langle \Delta^2 u_n, \frac{v_n}{\lambda_n} \right\rangle + \gamma \left\langle \Delta^2 w_n, \frac{v_n}{\lambda_n} \right\rangle \to 0, \tag{83}$$

and now using (77a), the fact that $|\Delta u_n|^2 \sim |\Delta w_n|^2$ and integration by parts we arrive to $|v_n|^2 \to 0$. Thus, $|z_n|^2 \to 0$ as well, and we arrive to a contradiction. ☐

Lemma 12. *Let C be the matrix operator defined above. Then, if (C1) holds,* $\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - C)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Proof. Notice that throughout the proof of the previous lemma we only use that $\lim_{n\to\infty}\lambda_n \neq 0$, but it does not depend on λ_n tending to a finite number or to infinity. □

Theorem 7. *Let (u, w) be a solution to the problem determined by system (61) with boundary conditions (7) and initial conditions (6). Then, if (C1) holds, (u, w) decays exponentially.*

We prove now that under (C2) the solutions of system (61) do not decay exponentially. To do so, we use a standard argument. We can prove that there exists a solution of the form $u = C_1 e^{\omega t}\Phi_n(x)$ and $w = C_2 e^{\omega t}\Phi_n(x)$ such that $\Re(\omega) > -\varepsilon$ for all positive ε small enough. This fact implies that we can find a solution ω as near to the imaginary axis as we desire and hence it is impossible to have uniform exponential decay on the solutions to the problem determined by (61) with conditions (6) and (7).

We repeat the argument we did in the proof of Proposition 3. The same polynomial, $p(x)$, is obtained. To prove that there are roots of $p(x)$ as near to the imaginary axis as desired is equivalent to show that for any $\varepsilon > 0$ there are roots of $p(x)$ located to the right hand side of the vertical line $\Re(x) = -\varepsilon$. If we make a translation, this fact is equivalent to show that the polynomial $p(x-\varepsilon)$ has a root with positive real part. The third leading minor of the Routh-Hurwitz matrix corresponding to $p(x-\varepsilon)$ is a fourth degree polynomial on m_n which main coefficient is negative for m_n large enough:

$$M_3 = (-2(\rho_1+\rho_2)a^*\left((\gamma_1\rho_2-\gamma_2\rho_1)^2+4\gamma^2\rho_1\rho_2\right)\varepsilon+\mathcal{O}(\varepsilon^2))m_n^4+p_3(m_n), \tag{84}$$

where $p_3(m_n)$ is a cubic polynomial on m_n.

The above results prove that the solutions for this case do not decay exponentially. Nevertheless, it should be proven that the solutions decay. In order to do so we follow a standard argument.

Lemma 13. *The operator $\mathcal{C}^{-1}$ is compact.*

Proof. We have to see that for any bounded sequence $(\mathcal{F}_n)_{n\in\mathbb{N}} = (f_1^n, f_2^n, f_3^n, f_4^n)_{n\in\mathbb{N}} \in \mathcal{H}$ the sequence $(\mathcal{C}^{-1}\mathcal{F}_n)_{n\in\mathbb{N}} = (u^n, v^n, w^n, z^n)_{n\in\mathbb{N}}$ contains a convergent subsequence.

As in the proof of Lemma 10 the following system of equations is obtained

$$\begin{cases} (\mu_1\Delta-\gamma_1\Delta^2-aI)u^n+(\mu\Delta-\gamma\Delta^2+aI)w^n &= \rho_1 f_2^n + a^*(f_1^n-f_3^n) =: F_1^n, \\ (\mu\Delta-\gamma\Delta^2+aI)u^n+(\mu_2\Delta-\gamma_2\Delta^2-aI)w^n &= \rho_2 f_4^n - a^*(f_1^n-f_3^n) =: F_2^n. \end{cases} \tag{85}$$

On the other hand, we consider the bilinear form $V : (H^2\times H^2)\times(H^2\times H^2)\to\mathbb{R}$ defined as

$$\begin{aligned} V[(u,w),(\tilde{u},\tilde{w})] = \frac{1}{2}\int_B \Big(&\mu_1\nabla u\overline{\nabla\tilde{u}}+\mu_2\nabla w\overline{\nabla\tilde{w}}+\mu(\nabla u\overline{\nabla\tilde{w}}+\nabla w\overline{\nabla\tilde{u}}) \\ &+\gamma_1\Delta u\overline{\Delta\tilde{u}}+\gamma_2\Delta w\overline{\Delta\tilde{w}}+\gamma(\Delta u\overline{\Delta\tilde{w}}+\Delta w\overline{\Delta\tilde{u}})+a(u-w)(\overline{\tilde{u}-\tilde{w}})\Big)\,dA, \end{aligned} \tag{86}$$

which clearly defines an equivalent norm to the $H^2\times H^2$ usual norm. Notice that it corresponds to the inner product (13) taking $v = z = 0$. Notice also that V is coercive:

$$\begin{aligned} V[(u,w),(u,w)] = \frac{1}{2}\int_B (\nabla u,\nabla w)\begin{pmatrix}\mu_1 & \mu\\ \mu & \mu_2\end{pmatrix}\begin{pmatrix}\nabla u\\ \nabla w\end{pmatrix}+(\Delta u,\Delta w)\begin{pmatrix}\gamma_1 & \gamma\\ \gamma & \gamma_2\end{pmatrix}\begin{pmatrix}\Delta u\\ \Delta w\end{pmatrix} \\ +a(u-w)^2 dA \geq K\|(u,w)\|^2_{H^2\times H^2}, \end{aligned} \tag{87}$$

for some constant K, since we are assuming $\mu_1\mu_2 > \mu^2$ and $\gamma_1\gamma_2 > \gamma^2$.

From the definition of V it is clear that

$$V[(u^n, w^n), (u^n, w^n)] = \left\langle(-F_1^n, -F_2^n), (u^n, w^n)\right\rangle_{L^2\times L_2} \\ \leq C\|(u^n, w^n)\|_{L^2\times L^2} \leq C^*\|(u^n, w^n)\|_{H^2\times H^2}, \tag{88}$$

for some constants $C, C^* \in \mathbb{R}$. Therefore, $\|(u^n, w^n)\|_{H^2\times H^2} \leq K$ for some constant $K \in \mathbb{R}$, that is, (u^n, w^n) is bounded in $H^2 \times H^2$. Thus, since $(u^n, w^n) \in H_0^1$ and it is bounded in $H^1 \times H^1$ by Rellich-Kondrachov Theorem (see [35]), there exists a subsequence that we still denote by (u^n, w^n) that converges in $L^2 \times L^2$.

We see now that this subsequence converges in $H^2 \times H^2$. To do so, we see that it is a Cauchy sequence:

$$V[(u^n - u^m, w^n - w^m), (u^n - u^m, w^n - w^m)] \leq C\|(u^n - u^m, w^n - w^m)\|_{L^2\times L^2} \tag{89}$$

and since the sequence is convergent in $L^2 \times L^2$, it is a Cauchy sequence in $H^2 \times H^2$ so, by completeness, it is also convergent in $H^2 \times H^2$.

(v^n, z^n) has a convergent subsequence in $L^2 \times L^2$. This is easy to check since $(v^n, z^n) = (f_1^n, f_3^n) \in (H_0^1 \cap H^2) \times (H_0^1 \cap H^2)$ and therefore (v^n, z^n) is bounded in $H^2 \times H^2$ and, as before, one concludes that it has a convergent subsequence. □

Theorem 8. *Let (u, w) be a solution to the problem determined by system (61) with boundary conditions (7) and initial conditions (6). Then, if (C2) holds, (u, w) decays slowly.*

Proof. Since the operator C^{-1} is compact, $\sigma_p(C) \cap \sigma_{ap}(C) = \sigma_p(C)$, where σ_p and σ_{ap} denote the point and approximate spectrum of the operator, respectively. Thus, using Proposition 3 and the last statement, we have that $\sigma_{ap}(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. Therefore, we can apply Arendt-Batty Theorem (see [36]) which gives us the desired result. □

5 | CONCLUSIONS

We have analyzed the time decay for the solution to the system of partial differential equations that models the behavior of the mixture of two materials of the Kelvin-Voigt type. We have focused on the isotropic and homogeneous case for antiplane shear deformations. We have introduced several dissipation mechanisms in the system, that we analyze separately, and have obtained the following results:

1. Hyperviscosity (the dissipation is given by a fourth order derivative with respect to the spatial variables): exponential decay.
2. Viscosity (the dissipation is given by a second order derivative with respect to the spatial variables): exponential decay.
3. Dissipation generated by the relative velocity (the dissipation has not derivatives with respect to the spatial variables): exponential decay in the generic case and slow decay in a specific situation given by a particular combination of the parameters.

For each system we have also proved the existence and uniqueness of solutions.

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CONFLICT OF INTEREST

The authors have declared no conflict of interest.

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