THE BERNSTEIN TECHNIQUE
FOR INTEGRO-DIFFERENTIAL EQUATIONS

XAVIER CABRÉ, SERENA DIPIERRO, AND ENRICO VALDINOCI

ABSTRACT. We extend the classical Bernstein technique to the setting of integro-differential operators. As a consequence, we provide first and one-sided second derivative estimates for solutions to fractional equations, including some convex fully nonlinear equations of order smaller than two —for which we prove uniform estimates as their order approaches two. Our method is robust enough to be applied to some Pucci-type extremal equations and to obstacle problems for fractional operators, although several of the results are new even in the linear case. We also raise some intriguing open questions, one of them concerning the “pure” linear fractional Laplacian, another one being the validity of one-sided second derivative estimates for Pucci-type convex equations associated to linear operators with general kernels.

1. INTRODUCTION

The Bernstein technique is a powerful tool to establish derivative estimates, through the use of auxiliary functions and the maximum principle, for solutions of elliptic equations. The goal of this paper is to extend this method to the setting of fractional equations. Up to our knowledge, this is done in the current work for the first time in a systematic way, even for the fractional Laplacian. The technique will allow us to establish first and one-sided second derivative estimates for a large class of integro-differential equations, including some fully nonlinear equations of order smaller than two.

Fractional diffusions arise in classical models, notably in the description of water waves, of atom dislocations in crystals, and in the displacement of an elastic membrane on a thin obstacle. These problems can be efficiently attacked by transforming them into a fractional setting on a lower dimensional (or boundary) object. More recent models include energy transfer in nanotubes, plasma physics, price oscillations in stock markets, and biological dispersals in sparse environments —see e.g. [11] and references therein.

The Bernstein technique —as introduced in the local case by Bernstein himself [8,9]— relies on considering some auxiliary functions which involve the solution, its derivatives, and suitable cutoff functions. In view of certain equations (inequalities, rather) satisfied by the auxiliary functions and thanks to the maximum principle, they allow to estimate higher derivatives of the solution in terms of lower order ones, by paying a price in the size of the reference domain.

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Let us recall this procedure with the simplest example in local equations, the Laplace operator. This will serve as a preparation for the nonlocal framework. Given two functions \( u \) and \( \eta \) (\( u \) must be thought as the solution of an equation, while \( \eta \) will be a cutoff), both smooth enough, consider the auxiliary function

\[
\varphi := \eta^2(\partial_e u)^2 + \sigma u^2, \tag{1.1}
\]

where \( e \in \mathbb{R}^n \) is a unit vector, \(|e| = 1\), and \( \sigma > 0 \) is a constant. We have that

\[
-\Delta \varphi = -2|\nabla \eta|^2(\partial_e u)^2 - 2\eta \Delta \eta (\partial_e u)^2 - 8\eta \nabla \eta \cdot \nabla (\partial_e u) \partial_e u \\
- 2\eta^2 |\nabla \partial_e u|^2 - 2\eta^2 \partial_e u \partial_e \Delta u - 2\sigma |\nabla u|^2 - 2\sigma u \Delta u. \tag{1.2}
\]

Since, by the Cauchy-Schwarz inequality

\[
|8\eta \nabla \eta \cdot \nabla (\partial_e u) \partial_e u| \leq 2\eta^2 |\nabla \partial_e u|^2 + 8|\nabla \eta|^2 (\partial_e u)^2,
\]
equation (1.2) yields

\[
-\Delta \varphi \leq 6|\nabla \eta|^2(\partial_e u)^2 - 2\eta \Delta \eta (\partial_e u)^2 - 2\sigma |\nabla u|^2 + 2\eta^2 \partial_e u (-\Delta) \partial_e u + 2\sigma u (-\Delta u).
\]

In particular, by choosing \( \sigma \geq C_n \|\eta\|^2_{C^2(\mathbb{R}^n)} \) for an appropriate constant \( C_n \) depending on \( n \) (more specifically, on the precise way in which the \( C^2 \)-norm of a function in \( \mathbb{R}^n \) is defined), we obtain that

\[
-\Delta \left( \eta^2(\partial_e u)^2 + \sigma u^2 \right) \leq 2\eta^2 \partial_e u (-\Delta) \partial_e u + 2\sigma u (-\Delta u) \quad \text{if } \sigma \geq C_n \|\eta\|^2_{C^2(\mathbb{R}^n)}. \tag{1.3}
\]

This is a “clean”, key inequality satisfied by any function \( u \) (not necessarily a solution of an equation).

Now, if we assume the function \( u \) to be harmonic, say \( -\Delta u = 0 \) in \( B_1 \subset \mathbb{R}^n \), we deduce from (1.3) that \( -\Delta \varphi \leq 0 \) in \( B_1 \). If, in addition, we take the function \( \eta \in C^\infty_c(B_1) \) to have compact support in \( B_1 \) and to satisfy \( \eta = 1 \) in \( B_{1/2} \), the maximum principle for subharmonic functions ensures that \( \varphi \) attains its maximum along \( \partial B_1 \). As a consequence,

\[
\sup_{B_{1/2}} (\partial_e u)^2 \leq \sup_{B_{1/2}} \varphi \leq \sup_{B_1} \varphi = \sup_{\partial B_1} \varphi = \sigma \sup_{\partial B_1} u^2 \leq \sigma \|u\|^2_{L^\infty(B_1)},
\]

thus yielding an explicit interior gradient estimate for the solution \( u \). As we will see, simple variations of this method, in which higher derivatives are taken into account within the auxiliary function, lead to higher order estimates as well.

In spite of its rather elementary flavor, the Bernstein method is a powerful nonvariational tool that finds applications in several contexts and for a large number of equations. The quadratic auxiliary function above (which is the one that we will consider within the nonlocal setting) finds applications even for second order fully nonlinear uniformly elliptic equations; see the monograph [15, Chapter 9]. More sophisticated auxiliary functions (with other nonlinear dependences on \( u \) and \( \partial_e u \)) lead to gradient estimates for the prescribed mean curvature equation; see e.g. [31] and references therein.

Instead, to the best of our knowledge, the Bernstein method for the quadratic auxiliary function (1.1) has not been yet studied in relation with fractional and integro-differential equations, not even for the fractional Laplacian. In this respect, the closest work that we could find is [10], by Biswas, Jakobsen, and Karlsen, which concerns an integro-differential equation of parabolic type posed in the whole space. Here the Bernstein technique is applied to a quadratic auxiliary function depending on incremental quotients (but not containing the cut-off function \( \eta \)) to
obtain suitable Lipschitz bounds, which are then exploited to deal with the convergence of the discrete scheme under consideration.

In accordance with our proofs and results, we must merge the operators that we treat into two categories. The first one consists of equations built from operators that admit a local extension in one more dimension, as it is the case of the fractional Laplacian. Our second category of equations consists of linear integro-differential operators with general kernels, as well as fully nonlinear operators built from them.

In the next subsections, we describe in detail the framework and results of our work.

1.1. Pucci-type equations in the presence of extensions. We start dealing with the case of Pucci-type equations associated to affine transformations of the fractional Laplacian with elliptic matrices. To built them, given constants $0 < \lambda \leq \Lambda$, we let

$$A = A_{\lambda, \Lambda},$$

where $c_{n,s} > 0$ is a suitable normalizing constant which makes that, when $A = \text{Id}$ is the identity, $L_{\text{Id}}$ becomes a fraction of the classical Laplacian, that is,

$$L_{\text{Id}} = (-\Delta)^s.$$

The above limit is well defined, and finite, whenever $u$ is a $W^{2,\infty} = C^{1,1}$ function (locally) which is bounded in all of $\mathbb{R}^n$. To ease the notation, the principal value P.V. will be omitted from now on.

We can now consider the maximal operator equation

$$M_A u(x) := \sup_{A \in \mathcal{A}} \left( L_A u(x) - g_A(x) \right) = 0 \text{ for all } x \in B_1,$$

where $g_A$ is a given continuous function in the unit ball $B_1 \subset \mathbb{R}^n$, for every $A \in \mathcal{A}$. We will assume continuity of $g_A$ with respect to the parameters $A \in \mathcal{A}$:

if $A_k \to A$ as $k \to +\infty$, then $\lim_{k \to +\infty} g_{A_k}(x) = g_A(x)$ for all $x \in B_1$.

Some existence and regularity results for (1.6) will be described in Appendix D.

By developing a Bernstein technique in this framework, we establish first and one-sided second derivative bounds for solutions of (1.6). Our estimates are uniform as the order of the operators converges to two, providing a unified theory up to the local case, with uniform constants in the bounds. In this respect, note that the operators $L_A$ recover, in the limit $s \nearrow 1$, every second order linear elliptic operator in nondivergence with constant coefficients (see Section 6 in [17] and Remark 5.7 below).

The following is our first result. Here we need the smooth function $u$ to belong to $W^{2,\infty}(\mathbb{R}^n)$, since, within the proofs, the fractional operators will act on derivatives of $u$ up to order two; in this way, second derivatives will be smooth functions bounded in all space.
Theorem 1.1. Given $s \in (0,1)$, $0 < \lambda \leq \Lambda$, and functions $g_A \in W^{1,\infty}(B_1)$ for $A \in \mathcal{A}$, assume (1.7) and let $u \in C^{\infty}(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)$ be a solution of (1.6).

Then,

$$
\sup_{B_{1/2}} |\nabla u| \leq C \left( \|u\|_{L^\infty(\mathbb{R}^n)} + \sup_{A \in \mathcal{A}} \|g_A\|_{W^{1,\infty}(B_1)} \right)
$$

(1.8)

for some constant $C$ depending only on $n$, $\lambda$, and $\Lambda$.

If, in addition, $g_A \in W^{2,\infty}(B_1)$ for all $A \in \mathcal{A}$, then we have

$$
\sup_{B_{1/2}} \partial^2_e u \leq C \left( \|u\|_{L^\infty(\mathbb{R}^n)} + \sup_{A \in \mathcal{A}} \|g_A\|_{W^{2,\infty}(B_1)} \right)
$$

(1.9)

for every $e \in \mathbb{R}^n$ with $|e| = 1$, where $C$ is as before.

Our one-sided second derivative bound (also called semiconcavity bound) is new and somehow surprising. Indeed, since the order of the operator is smaller than two, one should not expect a regularity theory up to the second order. In this respect, some other one-sided second derivative estimates have been previously proved for fractional problems. This has been achieved, in the context of the thin obstacle problem, by Athanasopoulos and Caffarelli. In Corollary 1.10 we will address their result, which uses the Bernstein technique but with a different, less flexible, auxiliary function than in the local theory or in the current work. Their auxiliary function is linear in the second derivatives, while ours is quadratic and thus, as we will see, it has already allowed for applications to more general situations in thin obstacle problems.

Another semiconcavity estimate is that of Mou, which applies to some integro-differential equations under certain (not so simple) hypotheses. The proof in Mou does not rely on the Bernstein technique, but on the Ishii-Lions method.

First derivative bounds have already been proved for large classes of fully nonlinear integro-differential equations, using different methods than ours. Among other papers, we point up the works by Jakobsen and Karlsen, by Caffarelli and Silvestre, and by Barles, Chasseigne, Ciomaga, and Imbert. The seminal work establishes a $C^{1+\alpha}$ bound for a large class of fully nonlinear integro-differential equations that includes Isaacs-type equations made from uniformly elliptic linear operators with general kernels in the class $L_1$. Their proof relies on ABP-type and Harnack inequalities, and thus it is an extension of the Krylov-Safonov local theory. Instead, the work (as did before) relies on the Ishii-Lions method (where an auxiliary function with doubled variables is used) and leads to a Lipschitz bound. It requires Hölder continuous coefficients but allows for weaker ellipticity assumptions. Thus, we are presenting here a third approach that applies to some new equations but not to all of the equations in the papers mentioned above, since we only cover convex equations.

1 As it will be apparent from the proof, estimates (1.8) and (1.9) can be stated in a more precise way in relation with their dependence on the functions $g_A$. This is explained in Remark 5.8.

2 As customary, we say that a function $u$ is semiconcave if there exists $C \geq 0$ such that the function $u(x) - C|x|^2$ is concave.

3 The best regularity theory available for this equation arrives at $C^{1+\varepsilon} \cap C^{2s+\varepsilon}$, with $\max\{1+\varepsilon, 2s+\varepsilon\} < 2$; see Theorem 13.1 and Theorem 1.1, respectively, and Appendix D below.
We point out that the results in both [17] and [4] apply to viscosity solutions. Extending our method to the viscosity framework will require some new ideas that we have not found implemented in the literature, even in the local case. We also recall that solutions of equation (1.6) are not, in general, smooth.\footnote{Recall that in the monograph [15], for instance, the Bernstein technique is carried out only for smooth solutions.}

To prove Theorem 1.1, we first need to extend the Bernstein technique to the simplest linear operator: the fractional Laplacian \((-\Delta)^s\) with \(0 < s < 1\), as defined above. The computations (1.1)-(1.3) for the classical Laplacian will easily carry over the extension operator for the fractional Laplacian, leading to the following analogue of (1.3). Note that the result is uniform as \(s\) tends to 1. To guarantee that the fractional Laplacian is well defined when acting on a smooth function \(u\) and also on the auxiliary functions built from its first derivatives, we will assume that both \(u\) and \(\nabla u\) are bounded in all of \(\mathbb{R}^n\), that is, \(u \in W^{1,\infty}(\mathbb{R}^n)\).

**Proposition 1.2.** Let \(s \in (0, 1)\), \(u \in C^\infty(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)\), \(\eta \in C^\infty(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)\), \(\sigma > 0\), and \(e \in \mathbb{R}^n\) with \(|e| = 1\). Then, we have
\[
(-\Delta)^s \left( \eta^2 (\partial_e u)^2 + \sigma u^2 \right) \leq 2\eta^2 \partial_e u (-\Delta)^s \partial_e u + 2\sigma u (-\Delta)^s u \quad \text{if} \quad \sigma \geq \sigma_0,
\]
for some constant \(\sigma_0\) depending only on \(n\) and \(\|\eta\|_{C^2(\mathbb{R}^n)}\) —and, in particular, independent of \(s\).

The proof of the first derivative estimate in Theorem 1.1 will follow from inequality (1.10) by choosing an appropriate cutoff function \(\eta\), after a change of variables to replace \((-\Delta)^s\) by the operators \(L_A\).

Our method to establish one-sided second derivative bounds will be similar. For this, in (1.10) we first need to replace \(u\) by \(v = \partial_e u\), but since we only expect a one-sided second derivative bound from above, we must consider instead the auxiliary function
\[
\eta^2 (\partial_e v)^2 + \sigma v^2
\]
(1.11)
involving a positive part\footnote{In any case, notice that the smooth setting in the gradient estimate of Theorem 1.1 applies to a large number of equations. Indeed, given any smooth function \(u \in W^{2,\infty}(\mathbb{R}^n)\) we may define \(f := \sup_{A \in A} L_A u\) and \(g_A = f\) for all \(A \in A\). Then, \(u\) solves equation (1.4). In addition, since \(u\) is smooth and has bounded derivatives, one can check that \(g_A = f\) is Lipschitz —which suffices for the validity of the first derivative estimate.} where \(v = \partial_e u\). The analogue of inequality (1.10) for the auxiliary function (1.11) is the content of Proposition 5.6.

For the operators with general kernels treated in the next subsection, we will prove a rather delicate extension of Proposition 1.2. Instead, an analogue inequality for the auxiliary function (1.11) is unknown; see Open problem 1.8.

A similar result to Theorem 1.1 but dealing with linear and with convex fully nonlinear operators of indefinite order will be presented in Subsection 1.3.

### 1.2. Pucci-type equations for general integro-differential operators.
In this paper we also take into account the case of operators with more general kernels, more precisely, kernels which are not pure powers, neither rotationally invariant. This setting takes into account anisotropic environments.
Let \( K : \mathbb{R}^n \to (0, +\infty] \) satisfy
\[
K(z) = K(-z) \quad \text{for all } z \in \mathbb{R}^n \setminus \{0\}
\]
and, for some \( s \in (0, 1) \),
\[
\frac{C_1 s(1 - s)}{|z|^{n+2s}} \leq K(z) \leq \frac{C_2 s(1 - s)}{|z|^{n+2s}} \quad \text{for all } z \in \mathbb{R}^n \setminus \{0\},
\]
where \( 0 < C_1 \leq C_2 \) are given constants. In our main results we will also assume \( K \) to be \( C^2 \) in \( \mathbb{R}^n \setminus \{0\} \) and to satisfy
\[
|z||\nabla K(z)| + |z|^2|D^2 K(z)| \leq C_3 K(z) \quad \text{for all } z \in \mathbb{R}^n \setminus \{0\},
\]
for some constant \( C_3 > 0 \). This is the class \( \mathcal{L}_2 \) of kernels introduced in [17]. We consider the linear operator
\[
\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} (u(x) - u(y)) K(x - y) \, dy
\]
defined, as before, in the principal value sense. The operator is well defined on \( W^{2,\infty} \) functions which are bounded in all of \( \mathbb{R}^n \). When \( K(z) = c_n s |z|^{-n-2s} \), it is the fractional Laplacian.

The assumptions in (1.12)-(1.14) are satisfied by the class of general stable symmetric operators, where the kernels are given by
\[
K(z) := \frac{1}{2} \frac{1}{|z|^{n+2s}} \left( a \left( \frac{z}{|z|} \right) + a \left( -\frac{z}{|z|} \right) \right),
\]
under appropriate hypotheses on the positive function \( a \). See, for instance, (1.3) in [28].

Integro-differential operators with such kernels naturally arise in the Lévy-Khintchine probabilistic formula, to take into account Poisson processes with jumps; see e.g. Section 2.2 in [34]. They possess applications in several fields; see e.g. Sections 1.2 and 1.3 in [34]. In spite of many similarities with the case of the fractional Laplacian, they also present some important differences, in terms of regularity results, with respect to the fractional Laplacian.

We establish first derivative estimates for fully nonlinear equations involving general fractional kernels. To state our result for maximal type operators (other fully nonlinear equations are treated in next subsection), we consider a compact set of indexes \( \mathcal{B} \), as well as kernels \( K_B \) and continuous functions \( g_B \) in \( B_1 \) for \( B \in \mathcal{B} \), satisfying
\[
\text{if } B_j \to B \text{ as } j \to +\infty, \text{ then } \lim_{j \to +\infty} g_{B_j}(x) = g_B(x) \text{ for all } x \in B_1.
\]

**Theorem 1.3.** Let \( \mathcal{B} \) be a compact set and \( \{K_B\}_{B \in \mathcal{B}} \) be kernels satisfying (1.12), (1.13), and (1.14) (all with the same structural constants \( s, C_1, C_2, \) and \( C_3 \)). For \( B \in \mathcal{B} \), let \( g_B \in W^{1,\infty}(B_1) \) and assume that (1.16) is satisfied.

Let \( u \in C^\infty(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n) \) be a solution of
\[
\sup_{B \in \mathcal{B}} \left( \mathcal{L}_{K_B} u(x) - g_B(x) \right) = 0 \quad \text{for all } x \in B_1.
\]

Then,
\[
\sup_{B_{1/2}} |\nabla u| \leq C \left( \|u\|_{L^\infty(\mathbb{R}^n)} + \sup_{B \in \mathcal{B}} \|g_B\|_{W^{1,\infty}(B_1)} \right)
\]
for some constant \( C \) depending only on \( n, s, C_1, C_2, \) and \( C_3 \).
Proposition 1.4. Let \( K \) satisfy (1.12) and (1.13), and let \( \mathcal{L}_K \) be defined by (1.15). Given a function \( u \in C^\infty(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n) \), \( \eta \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), \( e \in \mathbb{R}^n \) with \( |e| = 1 \), and \( \sigma > 0 \), consider

\[
\varphi := \eta^2(\partial_e u)^2 + \sigma u^2.
\]

Then, given \( E \in \mathbb{R} \), the inequality

\[
\mathcal{L}_K \varphi \leq 2\eta^2 \partial_e u \mathcal{L}_K \partial_e u + 2\sigma u \mathcal{L}_K u + E
\]

holds at a point \( x \in \mathbb{R}^n \) if and only if

\[
2 \int_{\mathbb{R}^n} \eta(x) (\eta(x) - \eta(y)) \partial_e u(x) \partial_e u(y) K(x - y) \, dy \leq \int_{\mathbb{R}^n} \left| \eta(x) \partial_e u(x) - \eta(y) \partial_e u(y) \right|^2 K(x - y) \, dy + \sigma \int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) \, dy + E.
\]

It is simple to check that all integrals in (1.19) are well defined in the principal value sense; see the comments following Proposition 2.1. A useful aspect of (1.19) is that all terms in its right-hand side are nonnegative, which is not necessarily the case in (1.18).

Using this result, we will establish the following key inequality for the operator \( \mathcal{L}_K \). It differs from the (still unknown) optimal inequality by a “small error or remainder”. Its proof will contain several quite surprising weighted integral cancellations.

Theorem 1.5. Let \( K \) satisfy (1.12), (1.13), and (1.14), and let \( \mathcal{L}_K \) be defined by (1.15). Let \( u \in C^\infty(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n) \) and \( \eta \in C^\infty(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n) \).

Then, for every \( \varepsilon > 0 \) there exists a constant \( \sigma_\varepsilon > 0 \) depending only on \( \varepsilon \), \( \|\eta\|_{C^2(\mathbb{R}^n)} \), and the structural constants \( n, s, C_1, C_2, \) and \( C_3 \) in (1.13) and (1.14), such that

\[
\mathcal{L}_K \left( \eta^2(\partial_e u)^2 + \sigma_\varepsilon u^2 \right) \leq 2\eta^2 \partial_e u \mathcal{L}_K \partial_e u + 2\sigma_\varepsilon u \mathcal{L}_K u + \varepsilon^2 \|\partial_e u\|_{L^\infty(B_3)}^2 \quad \text{everywhere in } B_2,
\]

for every \( e \in \mathbb{R}^n \) with \( |e| = 1 \).

Note that the error depends on the \( L^\infty \)-norm of the first derivative in a larger ball than the ball where the inequality is claimed.

Theorem 1.5 will serve as the cornerstone to prove the first derivative estimates of Theorem 1.3.

The following are some intriguing open problems on the inequalities satisfied by the auxiliary functions in the Bernstein technique. They concern either the fractional Laplacian or operators with general kernels.
Open problem 1.6. Our proof of (1.10) heavily relies on the extension method. It would be very interesting to prove inequality (1.10) without using the extension. We only know how to do this when the function $u$ is assumed to be $s$-harmonic; see Lemma 2.2 and its proof.\footnote{Note that our proof of Lemma 2.2 (which does not use the extension) is uniform as $s$ tends to 1. This is also the case for the extension proof of (1.10). Instead, Theorem 1.5 (and as a consequence, Theorem 1.3 below) are not uniform. In this respect, it would be very interesting to find a proof of Theorem 1.3 which is uniform as $s \nearrow 1$.}

In view that we know (1.19) to be true with $E = 0$ when $K(z) = |z|^{-n-2s}$, we still find intriguing not to be able to prove it directly in $\mathbb{R}^n$ (for this kernel and with $E = 0$) without using the extension. Finding such a proof could shed light into the following question.

Open problem 1.7. Given a kernel $K$ as above, does Theorem 1.5 hold true without the additional small remainder? That is, does (1.20) hold true with $\varepsilon = 0$ and $\sigma_\varepsilon$ large enough?

Open problem 1.8. We do not know whether one-sided second derivative bounds hold true for general kernels, in the setting of Theorem 1.3. Recall that they do hold, by Theorem 1.1, for the Bellman operator built from affine transformations of the fractional Laplacian.

To establish such a result, one would need to prove an inequality similar to (1.20), but with $u$ and $\partial_\varepsilon u$ replaced by $v$ and $(\partial_\varepsilon v)_+$, respectively, as explained in (1.11) (here the arbitrary function $v$ plays the role of $\partial_\varepsilon u$). Recall that the positive part comes from the fact that we only expect one-sided estimates for second derivatives. For the fractional Laplacian we know that such inequality holds, without an error term, by Proposition 5.6. A corresponding inequality for general kernels, even with an absorbable error term $E$, is unknown. For possible future use, in Proposition 2.1 we state the analogue of criterium (1.19) for the auxiliary function involving the positive part.

1.3. Other fully nonlinear equations and operators of indefinite order. In this subsection we use our methods in the setting of superposition of fractional operators of different orders but having an extension property. Equations of indefinite order describe phenomena in which more than a single diffusion regime takes place.

We consider convex fully nonlinear equations of the following form. Given a positive integer $J$, let $F \in C(\mathbb{R}^J)$. We assume that there exist functions $\alpha_1, \ldots, \alpha_J$ defined in $\mathbb{R}^J$ and constants $\Theta_0 \geq \vartheta_0 > 0$ such that

\begin{equation}
\alpha_j(p) \geq 0 \quad \text{for every } p \in \mathbb{R}^J \text{ and } j \in \{1, \ldots, J\}, \tag{1.21}
\end{equation}

\begin{equation}
\Theta_0 \geq \sum_{j=1}^J \alpha_j(p) \geq \vartheta_0 \quad \text{for every } p \in \mathbb{R}^J, \tag{1.22}
\end{equation}

and

\begin{equation}
F(q) - F(p) \geq \sum_{j=1}^J \alpha_j(p)(q_j - p_j) \quad \text{for every } q \text{ and } p \in \mathbb{R}^J. \tag{1.23}
\end{equation}

By (1.23), $F$ is convex. Note that the hypotheses on $F$ represent, all three together, convexity and a quantification of ellipticity. They are satisfied, for instance, by Bellman-type equations\footnote{Even though the functions $\alpha_j$ are not required to be continuous, they are assumed to be defined everywhere, and not almost everywhere. This is consistent with the general setting of the article, in which the equations are supposed to be satisfied everywhere in a given domain —indeed, our terminology “for every” has to be taken literally, and not in the meaning of “for almost every”. This will be important when proving the maximum principle, since it will require to evaluate the equation at a maximum point. This framework coincides, for instance, with the one of Chapter 3 in the monograph [24].}
built from a finite number of linear operators, which will correspond (see (1.26) below) to $F(p) = \max\{p_1, \ldots, p_J\}$. However, our setting here is more general since we also include operators $F$ of class $C^1$. Indeed, the three assumptions on $F$ are also satisfied if $F$ is $C^1$, convex, nondecreasing in each of its coordinate variables, and satisfies $\Theta_0 \geq \Sigma_{j=1} J \partial_{p_j} F(p) \geq \vartheta_0$ for every $p \in \mathbb{R}^J$—here we take $\alpha_j(p) := \partial_{p_j} F(p)$.

We deal with the superposition of operators of different orders. Given a probability measure $\mu$ on $[0, 1]$, i.e.,

$$\mu \geq 0 \quad \text{and} \quad \mu([0, 1]) = \mu(\mathbb{R}) = 1,$$

we define

$$\mathcal{L}_\mu u(x) := \int_0^1 (-\Delta)^s u(x) \, d\mu(s). \quad (1.25)$$

In case of $\mu$ being a Dirac’s delta at some $s \in [0, 1]$, $\mathcal{L}_\mu$ reduces to the fractional Laplacian $(-\Delta)^s$ (in particular, to the classical Laplacian if $s = 1$). For $s = 0$, $(-\Delta)^0 u = u$. The interest of including $s = 0$ is to allow a unified treatment of fully nonlinear equations and obstacle problems; see Corollary 1.10 below.

The operators $\mathcal{L}_\mu$ have been studied in [13], in relation with Allen-Cahn type equations, through local extension methods.

Given a positive integer $J \in \mathbb{N}$, let $F$ be as above, $\mu_1, \ldots, \mu_J$ be probability measures on $[0, 1]$, and $g_1, \ldots, g_J$ be continuous functions in $B_1$. We consider solutions $u : \mathbb{R}^n \to \mathbb{R}$ of

$$F(\mathcal{L}_{\mu_1} u(x) - g_1(x), \ldots, \mathcal{L}_{\mu_J} u(x) - g_J(x)) = 0 \quad \text{for all } x \in B_1. \quad (1.26)$$

Note that we are dealing with a very general class of equations of indefinite order, which includes the model equation

$$F((-\Delta)^{s_1} u, \ldots, (-\Delta)^{s_J} u) = f(x), \quad (1.27)$$

with $s_j \in [0, 1]$.

A more general framework consists of making affine changes of variables for each index $j$. This establishes a connection between the equations of Subsection 1.1 and those of the current setting; see Remark 5.7 for more details. In addition, in such generality equation (1.26) would include the classical extremal equations built from a finite number of second order linear operators; see [15]. Therefore, since we will establish one-sided second derivative bounds, the convexity assumption on $F$ in the following theorem cannot be dropped, in view of the classical counterexamples to $W^{2, \infty}$ regularity by Nadirashvili and Vlăduţ [27] for nonconvex fully nonlinear equations of second order [11].

Our result establishes first and one-sided second derivative bounds for solutions of (1.26). The estimates are uniform in the number $J$ of operators.

---

9 Indeed, it suffices to define $\alpha_j(p) = 1$ if $j$ is the smallest index for which $p_j = F(p)$ and $\alpha_j(p) = 0$ for all other indexes.

10 Notice that, since here we only want to involve a finite number of linear operators, the associated Bellman-type equations are not, typically, of class $C^1$. This is in contrast with the class of Bellman operators built from infinitely many linear operators (as in Theorems 1.1 and 1.3), which recovers all convex operators (since any convex function can be written as the supremum of linear functions).

11 Two comments are in order here. First, once a one-sided second derivative bound for a second order fully nonlinear uniformly elliptic equation is established, it automatically leads to full second derivative estimates (by using the equation itself; see the Bernstein technique described in Chapter 9 of [15], and in particular inequality (9.5) combined with Lemma 6.4 in [15]). Second, recall that Isaacs equations cover all possible fully nonlinear elliptic equations of second order (see Remark 1.5 in [12]) and that our estimates are independent of the number of operators $J$. 
Theorem 1.9. Given $F$ satisfying (1.21), (1.22), and (1.23), $\mu$ satisfying (1.24), and functions $g_j \in W^{1,\infty}(B_1)$ for $j = 1, \ldots, J$, let $u \in C^\infty(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)$ be a solution of (1.26). Then,
\[
\sup_{B_{1/2}} |\nabla u| \leq C \left( \|u\|_{L^\infty(\mathbb{R}^n)} + |F(0)| + \sup_{j \in \{1, \ldots, J\}} \|g_j\|_{W^{1,\infty}(B_1)} \right)
\]
for some constant $C$ depending only on $n$, $\vartheta_0$, and $\Theta_0$.

If in addition $g_j \in W^{2,\infty}(B_1)$ for $j = 1, \ldots, J$, then we have
\[
\sup_{B_{1/2}} \partial^2_e u \leq C \left( \|u\|_{L^\infty(\mathbb{R}^n)} + |F(0)| + \sup_{j \in \{1, \ldots, J\}} \|g_j\|_{W^{2,\infty}(B_1)} \right)
\]
(1.28)
for every $e \in \mathbb{R}^n$ with $|e| = 1$, where $C$ is as before.

We remark that the estimates of Theorem 1.9 are new, to the best of our knowledge, even in the case when $F$ is linear, even for $\mathcal{L}_{\mu_j} = (-\Delta)^s$, and even when all the functions $g_j$ are taken to be zero.

As in Subsection 1.1 the one-sided second derivative estimate (1.28) is somehow surprising since, for operators which could be of order smaller than two, second derivative estimates are not expected to hold.

Since the operator in Theorem 1.9 is of indefinite order, considering the equation in a ball $B_R$ instead of $B_1$ produces an unusual dependence on the radius $R$ of the right-hand side of the corresponding estimates; see Remark 5.8 for more details.

Theorem 1.9 includes, as a particular case, the obstacle problem for the fractional Laplacian, also called “thin obstacle problem” or “Signorini problem”. For this, we take the measures to be $\mu_1 = \delta_s$ for some $s \in [0, 1]$ and $\mu_2 = \delta_0$, and $F$ to be the max operator.

Corollary 1.10. Let $s \in [0, 1]$, $f$ and $\phi$ be $W^{1,\infty}(B_1)$ functions, and $u \in C^\infty(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)$ be a solution of
\[
\max \left\{ (-\Delta)^s u, \ u - \phi \right\} = f \quad \text{everywhere in } B_1.
\]

Then,
\[
\sup_{B_{1/2}} |\nabla u| \leq C \left( \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{W^{1,\infty}(B_1)} + \|\phi\|_{W^{1,\infty}(B_1)} \right)
\]
for some constant $C$ depending only on $n$.

If in addition $f$ and $\phi$ belong to $W^{2,\infty}(B_1)$, then we have
\[
\sup_{B_{1/2}} \partial^2_e u \leq C \left( \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{W^{2,\infty}(B_1)} + \|\phi\|_{W^{2,\infty}(B_1)} \right)
\]
(1.29)
for every $e \in \mathbb{R}^n$ with $|e| = 1$, where $C$ depends only on $n$.

The gradient estimate of Corollary 1.10 applies to a large number of equations. Namely, as in footnote 5, given any smooth function $u$ in $W^{2,\infty}(\mathbb{R}^n)$, take $\phi = 0$ and define $f := \max\{(-\Delta)^s u, u\}$; note that $f$ is a Lipschitz function.

The bound (1.29) recovers the semiconcavity estimate for the thin obstacle problem, first proved by Athanasopoulos and Caffarelli [1] and later extended by Fernández-Real [22] to the fully nonlinear thin obstacle problem. In these papers the Bernstein technique was already used, but with a less flexible auxiliary function than in the current work: their auxiliary function is linear in the second derivatives, while ours is quadratic. The quadratic structure has already
allowed further applications in obstacle problems. Indeed, in private communication with the authors of [23] (an article that cites ours), Fernández-Real and Jhaveri have used our method in a situation where a polynomial solving the thin obstacle problem is subtracted to the solution and have gotten, in this way, estimates independent of the polynomial. This required the use of the quadratic auxiliary function, as well as the use of incremental quotients.

Organization of the paper. The rest of this paper is organized as follows.

Section 2 is devoted to the arguments needed to treat the operators defined “downstairs” in Subsections 1.2 and 1.3. Subsection 2.1 contains a criterium for the key inequality for auxiliary functions, Proposition 2.1, which will complement Proposition 1.4. In Subsection 2.2 we prove Theorem 1.5, while Subsection 2.3 contains a proof of the key inequality of Proposition 1.2 without using the extension but with the additional assumption that $u$ is $s$-harmonic.

In Section 3 we present the necessary material on linearized operators and maximum principles needed for the proofs of our main results.

In Section 4 we state and prove a general statement (namely, Theorem 4.2) which will be pivotal to obtain the main results of this paper.

Section 5 contains the proofs of those results presented in Subsections 1.1 and 1.3 which deal with operators “with extensions”. More precisely, in Subsection 5.1 we discuss Proposition 1.2 and its variants needed for the proof of the main results, while Subsection 5.2 contains the proofs of Theorem 1.1, Theorem 1.9, and Corollary 1.10.

In Section 6 we deal with operators without an extension property. By suitable scaled estimates, we will be able to “reabsorb” our error or remainder term in Theorem 1.3 and complete the proof of Theorem 1.3.

The three first appendices concern results needed in the paper; Appendix A is of special interest since it establishes a maximum principle for the extension problem which is new, up to our knowledge. On the other hand, Appendix D is of informative nature and discusses existence and regularity issues for the equations of the paper.

2. The key inequalities for general integro-differential operators

2.1. Equivalent formulations of the key inequalities. Here we provide the proof of Proposition 1.4 which will be used in next subsection to establish first derivative estimates. The proof of Proposition 1.4 will also establish the following criterium, a variant of the proposition which involves the positive part of the derivative. If one could prove that inequality (2.2) appearing below holds for an appropriate error $E$, then one-sided second derivative estimates for operators with general kernels would follow; see Open problem 1.8.

Proposition 2.1. Let $\mathcal{L}_K$ be as in (1.15), with $K$ satisfying (1.12) and (1.13). Given $v \in C^\infty(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$, $\eta \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $e \in \mathbb{R}^n$ with $|e| = 1$, $\tau > 0$, and $E \in \mathbb{R}$, consider

$$\psi := \eta^2 (\partial_e v)^2_+ + \tau v^2.$$

Then, the inequality

$$\mathcal{L}_K \psi \leq 2\eta^2 (\partial_e v)_+ + \mathcal{L}_K ((\partial_e v)_+) + 2\tau v \mathcal{L}_K v + E$$

(2.1)
holds at a point $x \in \mathbb{R}^n$ if and only if

$$2 \int_{\mathbb{R}^n} \eta(x) (\eta(x) - \eta(y)) (\partial_\epsilon v)_+(x) (\partial_\epsilon v)_+(y) K(x - y) \, dy$$

$$\leq \int_{\mathbb{R}^n} |\eta(x) (\partial_\epsilon v)_+(x) - \eta(y) (\partial_\epsilon v)_+(y)|^2 K(x - y) \, dy$$

$$+ \tau \int_{\mathbb{R}^n} |v(x) - v(y)|^2 K(x - y) \, dy + E. \tag{2.2}$$

Note that the integrals in (2.2) are finite, since $\eta$ is smooth, locally, and bounded at infinity. At the same time, as in the Introduction, $L_K \psi$ is well defined everywhere in $\mathbb{R}^n$ since $(\partial_\epsilon v)_+^2$ is a locally $W^{2,\infty}$ function which is bounded in all of $\mathbb{R}^n$. However, $(\partial_\epsilon v)_+$ in the right-hand side of (2.1) is only a $C^{1,1}$ function from below (locally). Recall that one says that $\varphi \in C(B_1)$ is \textquotedblleft $C^{1,1}$ from below\textquotedblright\ in $B_1$ if for every $x_0 \in B_1$ there exists $w \in C^{1,1}(B_1) = W^{2,\infty}(B_1)$ such that $w \leq \varphi$ everywhere in $B_1$ and $w(x_0) = \varphi(x_0)$. This setting is sufficient to define the operator pointwise everywhere by having values in $(-\infty) \cup \mathbb{R}$. In addition, we make the convention $0 \cdot (-\infty) = 0$ in the expression $(\partial_\epsilon v)_+ L_K((\partial_\epsilon v)_+)$ in (2.1).

Furthermore, we notice that whenever (2.1) holds true, then also

$$L_K \psi \leq 2\eta^2 (\partial_\epsilon v)_+ L_K \partial_\epsilon v + 2\tau v L_K v + E, \tag{2.3}$$

since this would follow from the above convention when $\partial_\epsilon v(x) \leq 0$ and from the fact that $L_K((\partial_\epsilon v)_+)(x) \leq L_K \partial_\epsilon v(x)$ when $\partial_\epsilon v(x) > 0$.

This observation is relevant since Lemma 3.1 will give control of $L_K \partial_\epsilon v$ from above. This is why, within Section 5 on operators with an extension, the inequality is stated as (2.3) (with $E = 0$), and not as (2.1).

We also point out that the integrals in (1.19) and (2.2) are all well defined, due to the regularity of the functions involved. First, the integral in the left-hand side of (1.19) is well defined in the principal value sense, since, for small $z$,

$$(\eta(x) - \eta(x + z)) \partial_\epsilon u(x + z) K(z) = \left( -\nabla \eta(x) \cdot z + O(|z|^2) \right) (\partial_\epsilon u(x) + O(|z|)) K(z)$$

$$= -\partial_\epsilon u(x) \nabla \eta(x) \cdot z K(z) + O(|z|^2) K(z),$$

and the term $-\nabla \eta(x) \cdot z K(z)$ provides a null contribution to the principal value of the integral over $z \in B_1$, thanks to the symmetry assumption (1.12) (the decay assumption (1.13) will then make the term $O(|z|^2) K(z)$ integrable for $z \in B_1$). A similar argument applies to the first integral in (2.2) where one may assume $(\partial_\epsilon v)_+(x) > 0$.

The second integral in (1.19) is instead a classical Lebesgue integral, since, for small $z$,

$$|\eta(x) \partial_\epsilon u(x) - \eta(x + z) \partial_\epsilon u(x + z)|^2 K(z) \leq \|\eta \partial_\epsilon u\|_{W^{1,\infty}(B_1)}^2 |z|^2 K(z).$$

The same argument applies to the second integral in (2.2), since $(\partial_\epsilon v)_+$ is locally a $W^{1,\infty}$ function. A simpler argument gives that also the last integrals in (1.19) and (2.2) are well defined.

We now establish both Propositions 1.4 and 2.1 in a unified manner.

Proof of Propositions 1.4 and 2.1 Here, we take $G(t)$ to be either $t$ or $t_+$. We also adopt the notation $G^2(t) := (G(t))^2$. We write the proof for $u$, $\sigma$, and $\eta$ to address Proposition 1.4.\footnote{We point out that formally (that is, when all the integrals make sense) the arguments presented here are valid for all nonlinear functions $G$.}
with $G(t) = t$ in this case. By replacing these choices by $v$, $\tau$, and $\eta$, respectively, and with $G(t) = t_+$ now, we will conclude Proposition 2.1.

Given a kernel $K$, we have

$$
\mathcal{L}_K \left( \eta^2 G^2(\partial_v u) + \sigma u^2 \right)(x) - 2\eta^2(x) G(\partial_v u(x)) \mathcal{L}_K G(\partial_v u)(x)
$$

$$
- 2\sigma u(x) \mathcal{L}_K u(x) + \sigma \int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) \, dy
$$

$$
= \int_{\mathbb{R}^n} \left( \eta^2(x) G^2(\partial_v u(x)) - \eta^2(y) G^2(\partial_v u(y)) \right) K(x-y) \, dy
$$

$$
+ \sigma \int_{\mathbb{R}^n} \left( u^2(x) - u^2(y) \right) K(x-y) \, dy
$$

$$
- 2\eta^2(x) G(\partial_v u(x)) \int_{\mathbb{R}^n} \left( G(\partial_v u(x)) - G(\partial_v u(y)) \right) K(x-y) \, dy
$$

$$
- 2\sigma u(x) \int_{\mathbb{R}^n} \left( u(x) - u(y) \right) K(x-y) \, dy + \sigma \int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) \, dy
$$

As a consequence, the inequality

$$
\mathcal{L}_K \left( \eta^2 G^2(\partial_v u) + \sigma u^2 \right) \leq 2\eta^2 G(\partial_v u) \mathcal{L}_K G(\partial_v u) + 2\sigma u \mathcal{L}_K u + E
$$

is pointwise equivalent to

$$
2 \int_{\mathbb{R}^n} \eta(x) \left( \eta(x) - \eta(y) \right) G(\partial_v u(x)) G(\partial_v u(y)) K(x-y) \, dy
$$

$$
\leq \int_{\mathbb{R}^n} \left| \eta(x) G(\partial_v u(x)) - \eta(y) G(\partial_v u(y)) \right|^2 K(x-y) \, dy
$$

$$
+ \sigma \int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) \, dy + E.
$$

From this, as explained above, one deduces Propositions 1.4 and 2.1.

\[ \square \]

2.2. **Proof of the first key inequality with a remainder.** This subsection contains the proof of our main inequality for the auxiliary function in the case of general integro-differential operators.
Proof of Theorem 1.5. By the translation invariance of the problem, we see that, to establish (1.20) in $B_2$, it suffices to prove that

$$2 \int_{\mathbb{R}^n} \eta(0) (\eta(0) - \eta(y)) \partial_\varepsilon u(0) \partial_\varepsilon u(y) K(y) \, dy \leq \int_{\mathbb{R}^n} |\eta(0)\partial_\varepsilon u(0) - \eta(y)\partial_\varepsilon u(y)|^2 K(y) \, dy$$

$$+ \sigma_\varepsilon \int_{\mathbb{R}^n} |u(0) - u(y)|^2 K(y) \, dy + \varepsilon^2 \|\partial_\varepsilon u\|_{L^\infty(B_1)}^2. \quad (2.4)$$

Once this is proved, and using Proposition 1.4 at $x = 0$ with $E := \varepsilon^2 \|\partial_\varepsilon u\|_{L^\infty(B_1)}^2$, the right-hand side of inequality (1.20) in $B_2$ will become $\varepsilon^2 \|\partial_\varepsilon u\|_{L^\infty(B_1)}^2$.

To prove (2.4), we exploit an appropriate cutoff procedure on the gradient of $u$, to suitably remove the singularity of the integrand near the origin in the left-hand side of (2.4), (without spoiling the estimates at infinity). Namely, we consider an odd function $\xi \in C_0^\infty((-2, 2))$ such that $\xi(t) = t$ if $t \in (-1, 1)$, and $|\xi(t)| \leq 2$ for every $t \in \mathbb{R}$. We also consider $\delta \in (0, \frac{1}{2}]$, and set $\xi_\delta(t) := \delta \xi(t/\delta)$. We observe that

$$\xi_\delta(0) = 0, \quad \xi'_\delta(0) = 1, \quad \text{and} \quad \|\xi_\delta\|_{C^2(\mathbb{R})} \leq \frac{C}{\delta}, \quad (2.5)$$

for some universal constant $C$.

We also introduce the map $\mathbb{R}^n \ni y = (y_1, \ldots, y_n) \mapsto Z_\delta(y) := (\xi_\delta(y_1), \ldots, \xi_\delta(y_n))$. Since $\xi_\delta$ is odd, we have that

$$\int_{\mathbb{R}^n} \nabla \eta(0) \cdot Z_\delta(y) K(y) \, dy = 0,$$

in the principal value sense, thanks to the symmetry of the kernel (1.12). Consequently, we have that

$$\int_{\mathbb{R}^n} \eta(0) (\eta(0) - \eta(y)) \partial_\varepsilon u(0) \partial_\varepsilon u(y) K(y) \, dy = \int_{\mathbb{R}^n} \left( \eta(0) (\eta(0) - \eta(y)) \partial_\varepsilon u(0) \partial_\varepsilon u(y) + \eta(0) |\partial_\varepsilon u(0)|^2 \nabla \eta(0) \cdot Z_\delta(y) \right) K(y) \, dy. \quad (2.6)$$

Furthermore, we set

$$I_1(y) := \eta(0) \left( \eta(0) - \eta(y) + \nabla \eta(0) \cdot Z_\delta(y) \right) \partial_\varepsilon u(0) \partial_\varepsilon u(y), \quad (2.7)$$

$$I_2(y) := \partial_\varepsilon u(0) \left( \eta(0) \partial_\varepsilon u(0) - \eta(y) \partial_\varepsilon u(y) \right) \nabla \eta(0) \cdot Z_\delta(y), \quad (2.8)$$

and

$$I_3(y) := (\eta(y) - \eta(0)) \partial_\varepsilon u(0) \partial_\varepsilon u(y) \nabla \eta(0) \cdot Z_\delta(y), \quad (2.9)$$
and we point out that

\[ \begin{align*}
\eta(0) (\eta(0) - \eta(y)) \partial_e u(0) \partial_e u(y) &+ \eta(0) |\partial_e u(0)|^2 \nabla \eta(0) \cdot \mathcal{Z}_\delta(y) \\
&= \eta(0) \left( \eta(0) - \eta(y) + \nabla \eta(0) \cdot \mathcal{Z}_\delta(y) \right) \partial_e u(0) \partial_e u(y) \\
&\quad + \eta(0) \partial_e u(0) \left( \partial_e u(0) - \partial_e u(y) \right) \nabla \eta(0) \cdot \mathcal{Z}_\delta(y) \\
&= I_1(y) + \partial_e u(0) \left( \eta(0) \partial_e u(0) - \eta(y) \partial_e u(y) \right) \nabla \eta(0) \cdot \mathcal{Z}_\delta(y) \\
&\quad + \partial_e u(0) \left( \eta(y) \partial_e u(y) - \eta(0) \partial_e u(0) \right) \nabla \eta(0) \cdot \mathcal{Z}_\delta(y) \\
&= I_1(y) + I_2(y) + I_3(y).
\end{align*} \]

Hence, substituting into (2.6), we obtain

\[ \int_{\mathbb{R}^n} \eta(0) (\eta(0) - \eta(y)) \partial_e u(0) \partial_e u(y) K(y) \, dy = \int_{\mathbb{R}^n} \left( I_1(y) + I_2(y) + I_3(y) \right) K(y) \, dy. \tag{2.10} \]

Now, we set

\[ \varphi_\delta(y) := \eta(0) - \eta(y) + \nabla \eta(0) \cdot \mathcal{Z}_\delta(y), \tag{2.11} \]

and we observe that

\[ \varphi_\delta(0) = 0, \quad \nabla \varphi_\delta(0) = 0, \quad \text{and} \quad \|\varphi_\delta\|_{C^2(\mathbb{R}^n)} \leq C_\delta \|\eta\|_{C^2(\mathbb{R}^n)}, \tag{2.12} \]

for some constant \( C_\delta > 0 \), depending only on \( n \) and \( \delta \), thanks to the properties (2.5) of \( \xi_\delta \).

We also set

\[ J_1 := \int_{\mathbb{R}^n} \partial_e \left( \varphi_\delta(y) K(y) \right) \left( \eta(y) \partial_e u(y) - \eta(0) \partial_e u(0) \right) \left( u(y) - u(0) \right) \, dy \tag{2.13} \]

and

\[ J_2 := \frac{1}{2} \int_{\mathbb{R}^n} \partial_e \left( \partial_e \left( \varphi_\delta(y) K(y) \right) \eta(y) \right) |u(y) - u(0)|^2 \, dy. \tag{2.14} \]

We now perform some integration by parts in \( B_R \). To this end, we use that the integrals involved in the computation are finite and that the boundary terms on \( \partial B_R \) converge to zero as \( R \to +\infty \), thanks to the decay of the kernel and of its derivatives assumed in (1.13) and (1.14).
More precisely, from (2.7) and (2.11), and integrating by parts twice, we find that

\[
\int_{\mathbb{R}^n} I_1(y) K(y) \, dy = \int_{\mathbb{R}^n} \eta(0) \varphi_\delta(y) \partial_e u(0) \partial_e u(y) K(y) \, dy \\
= \int_{\mathbb{R}^n} \varphi_\delta(y) \eta(0) \partial_e u(0) \partial_e (u(y) - u(0)) K(y) \, dy \\
= - \int_{\mathbb{R}^n} \partial_e (\varphi_\delta(y) K(y)) \eta(0) \partial_e u(0) (u(y) - u(0)) \, dy \\
= \int_{\mathbb{R}^n} \partial_e (\varphi_\delta(y) K(y)) (\eta(y) \partial_e u(y) - \eta(0) \partial_e u(0)) (u(y) - u(0)) \, dy \\
- \int_{\mathbb{R}^n} \partial_e (\varphi_\delta(y) K(y)) \eta(y) \partial_e u(y) (u(y) - u(0)) \, dy \\
= J_1 - \frac{1}{2} \int_{\mathbb{R}^n} \partial_e (\varphi_\delta(y) K(y)) \eta(y) \partial_e u(y) - u(0))^2 \, dy \\
= J_1 + \frac{1}{2} \int_{\mathbb{R}^n} \partial_e (\varphi_\delta(y) K(y)) \eta(y) \partial_e u(y) - u(0))^2 \, dy \\
= J_1 + J_2. 
\]

Furthermore, recalling again the bound (1.14) on the first and second derivatives of the kernel and (2.12), we point out that

\[
\left| \partial_e (\varphi_\delta(y) K(y)) \right| \leq \left| \partial_e \varphi_\delta(y) \right| K(y) + \left| \varphi_\delta(y) \right| \left| \partial_e K(y) \right| \\
\leq C_{\delta,\eta} (K(y) + |y| \left| \partial_e K(y) \right|) \leq C_{\delta,\eta} K(y), 
\]

for some constant $C_{\delta,\eta} > 0$, possibly varying from line to line, and depending only on $n$, $\delta$, $\|\eta\|_{C^2(\mathbb{R}^n)}$, and on the structural constant $C_3$ in (1.14) (for convenience, in what follows, we will rename $C_{\delta,\eta}$ allowing dependences also on $s$ and on the constants $C_1$ and $C_2$ in (1.13)).

In addition, using again (1.14) and (2.12),

\[
\left| \partial_e^2 (\varphi_\delta(y) K(y)) \eta(y) \right| \leq C_{\delta,\eta} \left( \left| \partial_e^2 \varphi_\delta(y) K(y) \right| + \left| \partial_e \varphi_\delta(y) \partial_e K(y) \right| + \left| \varphi_\delta(y) \partial_e^2 K(y) \right| \right) \\
\leq C_{\delta,\eta} \left( K(y) + |y| \left| \partial_e K(y) \right| + |y|^2 \left| D^2 K(y) \right| \right) \\
\leq C_{\delta,\eta} K(y).
\]

Hence, from the latter estimate and (2.16),

\[
\left| \partial_e \left( \partial_e (\varphi_\delta(y) K(y)) \eta(y) \right) \right| \leq C_{\delta,\eta} K(y). 
\]

(2.17)
Thus, by the definitions (2.13) and (2.14) of $J_1$ and $J_2$, and the estimates (2.16) and (2.17), using an appropriate Cauchy-Schwarz inequality we get that

$$|J_1| + |J_2|$$

$$\leq C_{\delta, \eta} \left( \int_{\mathbb{R}^n} |\eta(y) \partial_x u(y) - \eta(0) \partial_x u(0)| \, |u(y) - u(0)| \, K(y) \, dy 
+ \int_{\mathbb{R}^n} |u(y) - u(0)|^2 \, K(y) \, dy \right)$$

$$\leq \frac{1}{8} \int_{\mathbb{R}^n} |\eta(y) \partial_x u(y) - \eta(0) \partial_x u(0)|^2 \, K(y) \, dy + C_{\delta, \eta} \int_{\mathbb{R}^n} |u(y) - u(0)|^2 \, K(y) \, dy.$$

This and (2.15) give that

$$\int_{\mathbb{R}^n} I_1(y) \, K(y) \, dy$$

$$\leq \frac{1}{8} \int_{\mathbb{R}^n} |\eta(y) \partial_x u(y) - \eta(0) \partial_x u(0)|^2 \, K(y) \, dy + C_{\delta, \eta} \int_{\mathbb{R}^n} |u(y) - u(0)|^2 \, K(y) \, dy. \tag{2.18}$$

Next, we notice that $|Z_{\delta}(y)| \leq C \, |y|$, for every $y \in \mathbb{R}^n$, for some constant $C$ depending only on $n$. Thus, we have that

$$J_3 := \int_{B_1} |y| \, |Z_{\delta}(y)| \, K(y) \, dy < +\infty. \tag{2.19}$$

In view of the definition (2.8) of $I_2$, noticing that $Z_{\delta}$ is supported in $B_1$, and using a suitable Cauchy-Schwarz inequality, we have that

$$\int_{\mathbb{R}^n} I_2(y) \, K(y) \, dy$$

$$\leq C_\eta \int_{B_1} |\partial_x u(0)||\eta(0)\partial_x u(0) - \eta(y)\partial_x u(y)| \, |Z_{\delta}(y)| \, K(y) \, dy$$

$$\leq \frac{1}{8} \int_{\mathbb{R}^n} |\eta(y)\partial_x u(0) - \eta(0)\partial_x u(0)|^2 \, K(y) \, dy + C_\eta \int_{B_1} |\partial_x u(0)|^2 \, |Z_{\delta}(y)|^2 \, K(y) \, dy$$

$$\leq \frac{1}{8} \int_{\mathbb{R}^n} |\eta(y)\partial_x u(0) - \eta(0)\partial_x u(0)|^2 \, K(y) \, dy + C_\eta \|\partial_x u\|_{L^\infty(B_1)}^2 J_3 \tag{2.20}$$

for some constant $C_\eta > 0$ depending only on $n$ and $\|\eta\|_{C^2(\mathbb{R}^n)}$. Similarly, recalling the definition (2.9) of $I_3$,

$$\int_{\mathbb{R}^n} I_3(y) \, K(y) \, dy \leq C_\eta \int_{B_1} |\eta(y) - \eta(0)| \, |\partial_x u(0)| \, |\partial_y u(y)| \, |Z_{\delta}(y)| \, K(y) \, dy$$

$$\leq C_\eta \int_{B_1} |y| \, |\partial_x u(0)| \, |\partial_y u(y)| \, |Z_{\delta}(y)| \, K(y) \, dy$$

$$\leq C_\eta \|\partial_x u\|_{L^\infty(B_1)}^2 J_3. \tag{2.21}$$

Now we provide a bound on $J_3$. To this aim, we split the integral computation inside $B_{\delta}$, where $Z_{\delta}(y) = y$, and in $B_1 \setminus B_{\delta}$, where $|Z_{\delta}(y)| \leq 2\delta \, n^{1/2}$. In this way, recalling the definition (2.19) of $J_3$, we get that

$$|J_3| \leq C \left( \int_{B_{\delta}} |y|^2 \, K(y) \, dy + \delta \int_{B_1 \setminus B_{\delta}} |y| \, K(y) \, dy \right),$$
for some constant $C$ depending only on $n$. The latter quantity tends to zero as $\delta \searrow 0$, thanks to the bounds \([1.13]\) on the kernel. Given $\varepsilon > 0$, we can therefore take $\delta \in (0,1)$ sufficiently small, in such a way that

$$|J_3| \leq \varepsilon^2.$$ 

After this choice of $\delta$, the constants $C_{\delta,\eta}$ above will be written accordingly, with a slight abuse of notation, as $C_{\varepsilon,\eta}$. As a consequence, collecting the estimates in \([2.18]\), \([2.20]\), and \([2.21]\), we conclude that

$$\int_{\mathbb{R}^n} \left( I_1(y) + I_2(y) + I_3(y) \right) K(y) \, dy$$

$$\leq \frac{1}{4} \int_{\mathbb{R}^n} \left| \eta(y) \partial_{\varepsilon} u(y) - \eta(0) \partial_{\varepsilon} u(0) \right|^2 K(y) \, dy$$

$$+ C_{\varepsilon,\eta} \int_{\mathbb{R}^n} \left| u(y) - u(0) \right|^2 K(y) \, dy + C_{\eta} \varepsilon^2 \left\| \partial_{\varepsilon} u \right\|_{L^\infty(B_1)}^2.$$ 

Therefore, recalling \([2.10]\), we deduce that

$$\int_{\mathbb{R}^n} \eta(0) \left( \eta(0) - \eta(y) \right) \partial_{\varepsilon} u(0) \partial_{\varepsilon} u(y) \, K(y) \, dy$$

$$\leq \frac{1}{4} \int_{\mathbb{R}^n} \left| \eta(y) \partial_{\varepsilon} u(y) - \eta(0) \partial_{\varepsilon} u(0) \right|^2 K(y) \, dy$$

$$+ C_{\varepsilon,\eta} \int_{\mathbb{R}^n} \left| u(y) - u(0) \right|^2 K(y) \, dy + C_{\eta} \varepsilon^2 \left\| \partial_{\varepsilon} u \right\|_{L^\infty(B_1)}^2.$$ 

This establishes \([2.4]\), up to renaming $\varepsilon$ and the constants, by choosing $\sigma_\varepsilon$ large enough. As a consequence, also \([1.20]\) follows. \hfill \square

2.3. "Downstairs" proof of the first key inequality for $s$-harmonic functions. The next result is the particular case of Proposition \([1.2]\) where $u$ is $s$-harmonic. While the proof of Proposition \([1.2]\) will rely on extension methods, we provide here a proof of this particular case without using the extension. Note that, as in Proposition \([1.2]\), $\sigma_0$ is independent of $s$.

**Lemma 2.2.** Let $\eta \in C^\infty(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)$, and assume that $u \in W^{1,\infty}(\mathbb{R}^n)$ is a (weak) solution of

$$(-\Delta)^s u = 0 \quad \text{in } B_1. \quad \tag{2.22}$$

Then, there exists $\sigma_0 > 0$, depending only on $n$ and $\|\eta\|_{C^2(\mathbb{R}^n)}$, such that

$$(-\Delta)^s (\eta^2 \nabla u^2 + \sigma u^2) \leq 0 \quad \text{in } B_1 \quad \text{if } \sigma \geq \sigma_0. \quad \tag{2.23}$$

**Proof (without using the extension problem).** Though the statement of Lemma \([2.2]\) is specific for the fractional Laplacian, we perform the initial part of the proof arguing for a general kernel $K$, to isolate the only point where we will use that $K(z) = c_{n,s}|z|^{-n-2s}$.

The proof relies on several integrations by parts, which carefully take into account oscillations and compensations inside the integrals.

We observe that, by regularity results for \([2.22]\), $u \in C^\infty(B_1)$. Exploiting Proposition \([C.1]\) in order to prove \([2.23]\), it suffices to show, by translation invariance, that

$$2 \int_{\mathbb{R}^n} \eta(0) \left( \eta(0) - \eta(y) \right) \nabla u(0) \cdot \nabla u(y) \, K(y) \, dy$$

$$\leq \int_{\mathbb{R}^n} \left| \eta(0) \nabla u(0) - \eta(y) \nabla u(y) \right|^2 K(y) \, dy + \sigma \int_{\mathbb{R}^n} \left| u(0) - u(y) \right|^2 K(y) \, dy \quad \tag{2.24}$$
To prove this, we call $I_1$ the left-hand side of (2.24). Integrating by parts, we have
\begin{align*}
I_1 &= 2 \int_{\mathbb{R}^n} \eta(0) (\eta(0) - \eta(y)) \nabla u(0) \cdot \nabla (u(y) - u(0)) K(y) \, dy \\
&= -2 \int_{\mathbb{R}^n} \eta(0) \nabla u(0) (u(y) - u(0)) \cdot \nabla \left( (\eta(0) - \eta(y)) K(y) \right) \, dy.
\end{align*}

We remark that, to obtain this integration by parts identity, one must argue in balls $B_R$ and use that the boundary terms on $\partial B_R$ go to zero as $R \to +\infty$ (as well as the integrability in $\mathbb{R}^n$ of the above functions), thanks to the decay of the kernel and of its derivatives assumed in (1.13) and (1.14).

Moreover, from the bound (1.14) on the first derivative of the kernel,
\begin{align*}
\left| \nabla \left( (\eta(0) - \eta(y)) K(y) \right) \right| \leq \left| \nabla \eta(y) \right| K(y) + \left| \eta(0) - \eta(y) \right| \left| \nabla K(y) \right| \leq C K(y),
\end{align*}
where, from now on, $C$ denotes different constants depending only on $n$ and $\|\eta\|_{C^2(\mathbb{R}^n)}$ (in particular, independent of $s$ in our case, that is, when the kernel $K$ is that of the fractional Laplacian). Therefore, we can bound (2.26) as
\begin{align*}
I_1 &= 2 \int_{\mathbb{R}^n} (\eta(y) \nabla u(y) - \eta(0) \nabla u(0)) (u(y) - u(0)) \cdot \nabla \left( (\eta(0) - \eta(y)) K(y) \right) \, dy \\
&= -2 \int_{\mathbb{R}^n} \eta(y) \nabla u(y) (u(y) - u(0)) \cdot \nabla \left( (\eta(0) - \eta(y)) K(y) \right) \, dy \\
&\leq CI_2 - 2 \int_{\mathbb{R}^n} \eta(y) \nabla u(y) (u(y) - u(0)) \cdot \nabla \left( (\eta(0) - \eta(y)) K(y) \right) \, dy,
\end{align*}
with
\begin{align*}
I_2 := \int_{\mathbb{R}^n} \left| \eta(y) \nabla u(y) - \eta(0) \nabla u(0) \right| \left| u(y) - u(0) \right| K(y) \, dy.
\end{align*}

Now, integrating by parts and using again (2.27), we have
\begin{align*}
2 \left| \int_{\mathbb{R}^n} \eta(y) \nabla u(y) (u(y) - u(0)) \cdot \nabla \left( (\eta(0) - \eta(y)) K(y) \right) \, dy \right| \\
&= \left| \int_{\mathbb{R}^n} \eta(y) \nabla \left| u(y) - u(0) \right|^2 \cdot \nabla \left( (\eta(0) - \eta(y)) K(y) \right) \, dy \right| \\
&= \left| \int_{\mathbb{R}^n} \left| u(y) - u(0) \right|^2 \text{div} \left( \eta(y) \nabla \left( (\eta(0) - \eta(y)) K(y) \right) \right) \, dy \right| \\
&\leq C \int_{\mathbb{R}^n} \left| u(y) - u(0) \right|^2 \left| \nabla \eta(y) \right| K(y) \, dy \\
&\quad + \left| \int_{\mathbb{R}^n} \left| u(y) - u(0) \right|^2 \eta(y) \Delta \left( (\eta(0) - \eta(y)) K(y) \right) \, dy \right| \\
&\leq CI_3 + 2|T_1| + |T_2|,
\end{align*}
with
\begin{align*}
I_3 := \int_{\mathbb{R}^n} \left| u(y) - u(0) \right|^2 K(y) \, dy,
\end{align*}
\[ T_1 := \int_{\mathbb{R}^n} |u(y) - u(0)|^2 \eta(y) \nabla \eta(y) \cdot \nabla K(y) \, dy, \]

and

\[ T_2 := \int_{\mathbb{R}^n} |u(y) - u(0)|^2 \eta(y) (\eta(0) - \eta(y)) \Delta K(y) \, dy. \tag{2.28} \]

Clearly, \( I_2 \) and \( I_3 \) are “good terms” which are controlled by the right-hand side of (2.24). Hence, to bound \( I_1 \) it remains to control \(|T_1|\) and \(|T_2|\).

To estimate \( T_1 \), we observe that
\[ |\nabla \eta(y) - \nabla \eta(0)| \leq C |y| \leq C K(y), \]
thanks to the bound (1.14) on the first derivative of the kernel, and therefore, integrating by parts,

\[ |T_1| \leq \left| \int_{\mathbb{R}^n} |u(y) - u(0)|^2 \eta(y) \nabla \eta(0) \cdot \nabla K(y) \, dy \right| + \left| \int_{\mathbb{R}^n} |u(y) - u(0)|^2 \eta(y) (\nabla \eta(y) - \nabla \eta(0)) \cdot \nabla K(y) \, dy \right| \leq \left| \int_{\mathbb{R}^n} \nabla \left( |u(y) - u(0)|^2 \eta(y) \right) \cdot \nabla \eta(0) K(y) \, dy \right| + CI_3 \]

\[ \leq \left| \int_{\mathbb{R}^n} \nabla |u(y) - u(0)|^2 \eta(y) \cdot \nabla \eta(0) K(y) \, dy \right| + CI_3 \]
\[ \leq 2 \left| \int_{\mathbb{R}^n} (u(y) - u(0)) \eta(y) \nabla u(y) \cdot \nabla \eta(0) K(y) \, dy \right| + CI_3 \]
\[ \leq 2 \left| \int_{\mathbb{R}^n} (u(y) - u(0)) \eta(0) \nabla u(y) \cdot \nabla \eta(0) K(y) \, dy \right| + C(I_2 + I_3) \]
\[ = C(I_2 + I_3), \]

where (2.25) has been used in the last line.

Finally, we estimate \( T_2 \). To this end, we take \( \zeta \in C^\infty_c(B_1) \) with \( \zeta = 1 \) in \( B_{1/2} \), and we define

\[ T_3 := \int_{\mathbb{R}^n} |u(y) - u(0)|^2 \eta(y) \nabla \eta(0) \cdot y \zeta(y) \Delta K(y) \, dy. \]

Then, by the second derivative bound (1.14) on the kernel and (2.28), we have

\[ |T_2| = \left| \int_{\mathbb{R}^n} |u(y) - u(0)|^2 \eta(y) (\eta(0) - \eta(y) + \nabla \eta(0) \cdot y \zeta(y)) \Delta K(y) \, dy - T_3 \right| \leq C \int_{\mathbb{R}^n} |u(y) - u(0)|^2 |y|^2 |D^2 K(y)| \, dy + |T_3| \leq CI_3 + |T_3|. \]

Thus, it only remains to bound \( T_3 \). For this, when the kernel \( K \) is that of the fractional Laplacian, we have that

\[ 2(s + 1) \nabla K(y) + y \Delta K(y) \]
\[ = -c_{n,s} (n + 2s) y \left( \frac{2(s + 1)}{|y|^{n+2s+2}} + \sum_{i=1}^{n} \left( \frac{1}{|y|^{n+2s+2}} - \frac{(n+2s+2) y_i^2}{|y|^{n+2s+4}} \right) \right) = 0. \]
Consequently, using once more the first derivative bound (1.14) on the kernel, and integrating again by parts, we find that

\[
|T_3| = 2(s + 1) \left| \int_{\mathbb{R}^n} |u(y) - u(0)|^2 \eta(y) \nabla \eta(0) \cdot \nabla K(y) \, dy \right| \\
\leq 2(s + 1) \left| \int_{\mathbb{R}^n} |u(y) - u(0)|^2 \eta(y) \nabla \eta(0) \cdot \nabla K(y) \, dy \right| \\
+ C \int_{\mathbb{R}^n} |u(y) - u(0)|^2 \eta(y) |\nabla \eta(0)| |y| |\nabla K(y)| \, dy \\
\leq 2(s + 1) \left| \int_{\mathbb{R}^n} \nabla \left( |u(y) - u(0)|^2 \eta(y) \right) \cdot \nabla \eta(0) \zeta(0) K(y) \, dy \right| + CI_3 \\
\leq 2(s + 1) \left| \int_{\mathbb{R}^n} \nabla (u(y) - u(0)) \eta(y) \nabla u(y) \cdot \nabla \eta(0) \zeta(0) K(y) \, dy \right| + CI_3 \\
= 4(s + 1) \left| \int_{\mathbb{R}^n} (u(y) - u(0)) \eta(y) \nabla u(y) \cdot \nabla \eta(0) \zeta(0) K(y) \, dy \right| + CI_3 \\
\leq 4(s + 1) \left| \int_{\mathbb{R}^n} (u(y) - u(0)) \eta(0) \nabla u(0) \cdot \nabla \eta(0) \zeta(0) K(y) \, dy \right| \\
+ 4(s + 1) \left| \int_{\mathbb{R}^n} (u(y) - u(0)) (\eta(y) \nabla u(y) - \eta(0) \nabla u(0)) \cdot \nabla \eta(0) \zeta(0) K(y) \, dy \right| \\
+ CI_3 \\
\leq 0 + C(I_2 + I_3),
\]

where we have exploited (2.25) once more in the last step. \(\square\)

3. Linearized Operator and a Maximum Estimate

3.1. The Linearized Operator. Here we study the linearized equation associated with the nonlinear problem (1.26). To simplify notation, we denote by \(\mathcal{L}_1, \ldots, \mathcal{L}_J\) the linear operators \(\mathcal{L}_{\mu_1}, \ldots, \mathcal{L}_{\mu_J}\). We will always assume that the convexity and ellipticity conditions (1.21), (1.22), and (1.23) are satisfied, for some constants \(\Theta_0 \geq \vartheta_0 > 0\).

Given a function \(u\), we use the short notation

\[
\alpha_j := \alpha_j (\mathcal{L}_1 u(x) - g_1(x), \ldots, \mathcal{L}_J u(x) - g_J(x))
\]

and we consider the operator

\[
Lv := \sum_{j=1}^{J} \alpha_j \mathcal{L}_j v.
\]

It will be important that the functions \(\alpha_j\) are defined at all points of our domain —and not only almost everywhere— since we will need to evaluate them at a maximum point of an auxiliary function. This will always be possible since \(\mathcal{L}_j u\) and \(g_j\) will be finite and well defined at all points, by the regularity assumed on \(u\) and since \(g_j\) are continuous functions.

The relevance of the linearized operator \(L\) is given by the fact that the solution and its derivatives satisfy suitable inequalities with respect to \(L\), as stated in the following result. Here, we remark that since \(g_j\) is not better than semiconcave, its first and second derivatives only exist almost everywhere.
Lemma 3.1. Let \( g_j \) be Lipschitz functions in \( B_1 \) for \( j = 1, \ldots, J \), \( u \in C^\infty(B_1) \cap W^{1,\infty}(\mathbb{R}^n) \) be a solution of (1.26) everywhere in \( B_1 \), and \( e \in \mathbb{R}^n \) satisfy \( |e| = 1 \).

Then,

\[
Lu \geq -F(-g_1,\ldots,-g_J) \quad \text{everywhere in } B_1
\]

and

\[
L \partial_e u = \sum_{j=1}^J \alpha_j \partial_e g_j \quad \text{almost everywhere in } B_1.
\]

In particular,

\[
Lu \geq -|F(0)| - \Theta_0 \sup_{j \in \{1,\ldots,J\}} \|g_j\|_{L^\infty(B_1)} \quad \text{everywhere in } B_1.
\]

If in addition the functions \( g_j \) are semiconcave in \( B_1 \) and \( u \in W^{2,\infty}(\mathbb{R}^n) \), then

\[
L \partial^2_e u \leq \sum_{j=1}^J \alpha_j \partial^2_e g_j \quad \text{almost everywhere in } B_1.
\]

Proof. We prove the first statement by using equation (1.26), in combination with the convexity assumption (1.23), used here with \( q := (-g_1,\ldots,-g_J) \) and \( p := (\mathcal{L}_1 u - g_1,\ldots,\mathcal{L}_J u - g_J) \). In this way, we have that, everywhere in \( B_1 \),

\[
F(-g_1,\ldots,-g_J) - F(\mathcal{L}_1 u - g_1,\ldots,\mathcal{L}_J u - g_J) \geq -\sum_{j=1}^J \alpha_j (\mathcal{L}_1 u - g_1,\ldots,\mathcal{L}_J u - g_J) \mathcal{L}_j u,
\]

which establishes the first statement.

Now, by (1.21), (1.22) and (1.23) (exploited with \( q := 0 \) and \( p := (-g_1,\ldots,-g_J) \)), we see that

\[
|F(0)| - F(-g_1,\ldots,-g_J) \geq F(0) - F(-g_1,\ldots,-g_J) \geq \sum_{j=1}^J \alpha_j (-g_1,\ldots,-g_J) g_j
\]

\[
\geq -\sum_{j=1}^J \alpha_j (-g_1,\ldots,-g_J) \|g_j\|_{L^\infty(B_1)} \geq -\sum_{j=1}^J \alpha_j (-g_1,\ldots,-g_J) \sup_{\ell \in \{1,\ldots,J\}} \|g_\ell\|_{L^\infty(B_1)}
\]

\[
\geq -\Theta_0 \sup_{\ell \in \{1,\ldots,J\}} \|g_\ell\|_{L^\infty(B_1)}.
\]

This observation and the first statement of the lemma lead to the third one.

Now we prove the second statement. To this end, we let \( \varepsilon > 0 \) and exploit the convexity assumption (1.23) at \( x \in B_1 \), with

\[
q := (\mathcal{L}_1 u(x \pm \varepsilon e) - g_1(x \pm \varepsilon e),\ldots,\mathcal{L}_J u(x \pm \varepsilon e) - g_J(x \pm \varepsilon e))
\]

and

\[
p := (\mathcal{L}_1 u(x) - g_1(x),\ldots,\mathcal{L}_J u(x) - g_J(x)).
\]
Using (1.26), this leads to
\[
0 = \frac{1}{\varepsilon} \left\{ F\left( \mathcal{L}_1 u(x \pm \varepsilon e) - g_1(x \pm \varepsilon e), \ldots, \mathcal{L}_j u(x \pm \varepsilon e) - g_j(x \pm \varepsilon e) \right) - F\left( \mathcal{L}_1 u(x) - g_1(x), \ldots, \mathcal{L}_j u(x) - g_j(x) \right) \right\}
\]
\[
\geq \sum_{j=1}^{J} \alpha_j \left( \mathcal{L}_1 u(x) - g_1(x), \ldots, \mathcal{L}_j u(x) - g_j(x) \right) - \mathcal{L}_j u(x \pm \varepsilon e) - g_j(x \pm \varepsilon e) + g_j(x) \right). \]

Taking the limit as \( \varepsilon \searrow 0 \), we thereby find that, for almost every \( x \in B_1 \),
\[
0 \geq \pm \sum_{j=1}^{J} \alpha_j \left( \mathcal{L}_1 u(x) - g_1(x), \ldots, \mathcal{L}_j u(x) - g_j(x) \right) \left( \mathcal{L}_j \partial_e u(x) - \partial_e g_j(x) \right)
\]
\[
= \pm \partial_e u(x) \pm \sum_{j=1}^{J} \alpha_j \partial_e g_j(x).
\]

From this, we deduce the second statement.

Finally, we prove the last statement. For this, exploiting again (1.23) with \( q \) and \( p \) as above, we have that
\[
0 = \frac{1}{\varepsilon^2} \left\{ F\left( \mathcal{L}_1 u(x + \varepsilon e) - g_1(x + \varepsilon e), \ldots, \mathcal{L}_j u(x + \varepsilon e) - g_j(x + \varepsilon e) \right) - F\left( \mathcal{L}_1 u(x) - g_1(x), \ldots, \mathcal{L}_j u(x) - g_j(x) \right) \right\}
\]
\[
+ \left( F\left( \mathcal{L}_1 u(x - \varepsilon e) - g_1(x - \varepsilon e), \ldots, \mathcal{L}_j u(x - \varepsilon e) - g_j(x - \varepsilon e) \right) - F\left( \mathcal{L}_1 u(x) - g_1(x), \ldots, \mathcal{L}_j u(x) - g_j(x) \right) \right)
\]
\[
\geq \sum_{j=1}^{J} \alpha_j \left( \mathcal{L}_1 u(x) - g_1(x), \ldots, \mathcal{L}_j u(x) - g_j(x) \right) - \mathcal{L}_j u(x + \varepsilon e) - \mathcal{L}_j u(x - \varepsilon e) - 2 \mathcal{L}_j u(x)
\]
\[
\geq \sum_{j=1}^{J} \alpha_j \left\{ \mathcal{L}_j u(x + \varepsilon e) + \mathcal{L}_j u(x - \varepsilon e) - 2 \mathcal{L}_j u(x)
\right\}
\]
\[
- \frac{g_j(x + \varepsilon e) + g_j(x - \varepsilon e) - 2g_j(x)}{\varepsilon^2} \right\}.
\]
Sending \( \varepsilon \downarrow 0 \), we conclude that \( 0 \geq \sum_{j=1}^J \alpha_j \left( L_j \partial^2_e u(x) - \partial^2_e g_j(x) \right) \) for almost every \( x \in B_1 \), as desired. \( \square \)

3.2. A maximum estimate. Here we show that the linearized operator \( L \) in (3.2) satisfies the maximum principle, as well as a quantitative maximum estimate in the case of nonzero right-hand sides. This is the content of the following result, which will be proved using a barrier function. As customary, we use the notation \( C_b(\mathbb{R}^n) := C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) to denote the space of bounded and continuous functions over all \( \mathbb{R}^n \).

**Proposition 3.2.** Let \( \gamma_0 \geq 0 \). Let \( \varphi \in C_b(\mathbb{R}^n) \cap W^{2,\infty}(B_1) \) be a nonnegative function in \( \mathbb{R}^n \) such that

\[
L \varphi \leq \gamma_0 \quad \text{everywhere in } B_1,
\]

where \( L \) is given by (3.1)–(3.2) for some \( u \in C^\infty(B_1) \cap W^{1,\infty}(\mathbb{R}^n) \) and continuous functions \( g_1, \ldots, g_J \) in \( B_1 \).

Then,

\[
\sup_{B_1} \varphi \leq \sup_{\mathbb{R}^n \setminus B_1} \varphi + C \gamma_0,
\]

where \( C \) depends only on \( n, \vartheta_0, \) and \( \Theta_0 \). Recall that \( \vartheta_0 \) and \( \Theta_0 \) are the constants in hypothesis (1.22).

**Remark 3.3.** Concerning the statement (3.3), we stress that, in our applications, we will need to use Proposition 3.2 for \( C_b(\mathbb{R}^n) \cap W^{2,\infty}(B_1) \) functions \( \varphi \) which are not \( C^2 \). For instance, in some cases the auxiliary function \( \varphi \) will contain a term involving \( (\partial^2_e u)^2 \) (note that this function is locally \( W^{2,\infty} \) when \( u \) is smooth, even if \( (\partial^2_e u)^2 \) is not \( W^{2,\infty} \) in general). Thus, for a \( C_b(\mathbb{R}^n) \cap W^{2,\infty}(B_1) \) function \( \varphi \), we now define a precise meaning to (3.3) at all points of \( B_1 \), obtaining a finite value for \( L \varphi \).

First, we use the usual principal value definition for all the integro-differential operators considered through the paper, since the regularity of \( \varphi \) is sufficient to compute the integrals (involved in \( L \varphi \)) in the principal value sense, obtaining a finite value.

Secondly, when \( L \varphi \) involves computing the classical Laplacian—as it may be the case for \( \mathcal{L}_\mu \) in (1.25)—, we define \( -\Delta \varphi \) through second incremental quotients as

\[
-\Delta \varphi(x) := \lim \inf_{h \downarrow 0} \sum_{i=1}^n \frac{2 \varphi(x) - \varphi(x + he_i) - \varphi(x - he_i)}{h^2},
\]

which is finite since \( \varphi \in W^{2,\infty} \). As a matter of fact, in our applications, inequality (3.3) will be satisfied even when writing (3.5) with a \( \lim \sup \) instead of \( \lim \inf \).

To establish Proposition 3.2 we start giving an auxiliary barrier function.

**Lemma 3.4.** Let \( s \in (0,1) \) and

\[
\beta(x) := \begin{cases} |x|^2 - 2 & \text{if } x \in B_{10} , \\ 98 & \text{otherwise}. \end{cases}
\]

Then, if \( K \) satisfies (1.12) and (1.13), we have that

\[
\mathcal{L}_K \beta \leq -c \quad \text{everywhere in } B_1,
\]

for some constant \( c > 0 \) depending only on \( n \) and the structural constant \( C_1 \) in (1.13).
Similarly, with $L_\mu$ defined by (1.24)–(1.25), we have that
\[ L_\mu \beta \leq -c \quad \text{everywhere in } B_1, \] (3.7)
for some constant $c > 0$ depending only on $\mu$.

In particular, for every $s \in [0,1]$, 
\[ (-\Delta)^s \beta \leq -c \quad \text{everywhere in } B_1, \] (3.8)
for some constant $c > 0$ depending only on $n$.

**Proof.** Let us start by proving (3.6). Let $x \in B_1$. We observe that if $z \in B_9$ then $|x \pm z| \leq |x| + |z| < 10$, and hence $\beta(x \pm z) = |x \pm z|^2 - 2$. As a consequence, for every $x \in B_1$ and $z \in B_9$,
\[ \beta(x + z) + \beta(x - z) - 2\beta(x) \]
\[ = (|x|^2 + |z|^2 + 2x \cdot z - 2) + (|x|^2 + |z|^2 - 2x \cdot z - 2) - 2(|x|^2 - 2) = 2|z|^2. \]
Accordingly
\[ \int_{B_9} \left( \beta(x + z) + \beta(x - z) - 2\beta(x) \right) K(z) \, dz = 2 \int_{B_9} |z|^2 K(z) \, dz. \] (3.9)

On the other hand, if $z \in \mathbb{R}^n \setminus B_9$, we have that $|x \pm z| \geq |z| - |x| \geq 8$. Therefore, for every $z \in \mathbb{R}^n \setminus B_9$, we have that $\beta(x \pm z) \geq 62$. Since $\beta(x) \leq -1$, we find in this case that
\[ \int_{\mathbb{R}^n \setminus B_9} \left( \beta(x + z) + \beta(x - z) - 2\beta(x) \right) K(z) \, dz \geq 126 \int_{\mathbb{R}^n \setminus B_9} K(z) \, dz. \]
This inequality and (3.9), together with the symmetry (1.12) of the kernel and the lower bound (1.13), lead to
\[ -L_\mu \beta(x) = \frac{1}{2} \int_{\mathbb{R}^n} \left( \beta(x + z) + \beta(x - z) - 2\beta(x) \right) K(z) \, dz \]
\[ \geq \int_{B_9} |z|^2 K(z) \, dz + \int_{\mathbb{R}^n \setminus B_9} K(z) \, dz \]
\[ \geq C_1 \, s \, (1-s) \left( \int_{B_9} \frac{dz}{|z|^{n+2s-2}} + \int_{\mathbb{R}^n \setminus B_9} \frac{dz}{|z|^{n+2s}} \right) \]
\[ \geq \tilde{C} \, C_1, \]
for some constant $\tilde{C}$ only depending on $n$. This gives the desired result (3.6).

Now we prove (3.7). For this, we let $s \in (0,1)$ and apply (3.6) with $K(z) := c_{n,s} \, |z|^{-n-2s}$, obtaining (3.8) with $c > 0$ depending only on $n$ (since it is well known that for this kernel one may take $C_1$ to depend only on $n$). We also notice that $-\Delta \beta = -2n$ and $\beta \leq -1$ in $B_1$, and therefore (3.8) is satisfied also for $s = 0$ and $s = 1$, up to renaming $c$. Consequently, the claim (3.7) follows from the definition (1.25) of $L_\mu$. \hfill \Box

Note that claim (3.8), that we have already proved, can also be regarded as a particular case of (3.7) by taking $\mu$ to be a Dirac’s delta.

With the above barrier at hand, we are in the position of proving the maximum principle with estimate which is suitable for our goals.

**Proof of Proposition 3.2.** We take $\beta$ as in Lemma 3.4 and define
\[ \varphi_*(x) := \varphi(x) + \frac{C \, \gamma_0}{100} \beta(x). \] (3.10)
We claim that
\[
\sup_{\mathbb{R}^n} \varphi_* = \sup_{\mathbb{R}^n \setminus B_1} \varphi_*.
\]
(3.11)

Once this is proved, from the fact that \(-2 \leq \beta \leq 98\), we deduce
\[
\sup_{B_1} \varphi \leq \sup_{B_1} \varphi_* + \frac{2C \gamma_0}{100} \leq \sup_{\mathbb{R}^n \setminus B_1} \varphi_* + \frac{2C \gamma_0}{100} \leq \sup_{\mathbb{R}^n \setminus B_1} \varphi + \frac{98 C \gamma_0}{100} + \frac{2C \gamma_0}{100}
\]
and conclude (3.4).

To prove claim (3.11), it is enough to establish that
\[
\sup_{B_1} \varphi_* = \sup_{\mathbb{R}^n \setminus B_\rho} \varphi_*(x_*)
\]
(3.12)
for every \(\rho \in (0,1)\), since \(\varphi_*\) is continuous in \(\mathbb{R}^n\). For this, we argue by contradiction and assume that there exists \(x_* \in \overline{B}_\rho \subset B_1\) such that \(\varphi_*(x_*) = \sup_{\mathbb{R}^n} \varphi_*\). This leads to \((-\Delta)^s \varphi_*(x_*) \geq 0\) for all \(s \in (0,1)\). We notice that \((-\Delta)^s \varphi_*(x_*) \geq 0\) when \(s = 1\), in view of (3.5), and also when \(s = 0\), since
\[
\varphi_*(x_*) = \sup_{\mathbb{R}^n} \varphi_* \geq \sup_{x \in \mathbb{R}^n \setminus B_{\sqrt{2}}} \left( \varphi(x) + \frac{C \gamma_0}{100} \beta(x) \right) \geq 0.
\]

We therefore obtain that
\[
\mathcal{L}_{\mu_j} \varphi_*(x_*) \geq 0
\]
for all \(j \in \{1,\ldots,J\}\). Finally, recalling that \(\alpha_j \geq 0\) by (1.21) and the definition (3.2) of \(L\), we deduce
\[
L \varphi(x_*) \geq 0.
\]
(3.13)

Now, in light of (3.7), we have that, for all \(j \in \{1,\ldots,J\}\),
\[
\mathcal{L}_{\mu_j} \beta \leq -c \quad \text{everywhere in } B_1,
\]
and consequently, recalling the lower bound in (1.22) on \(\alpha_j\),
\[
L \beta \leq -c \sum_{j=1}^{J} \alpha_j \leq -c \vartheta_0 \quad \text{everywhere in } B_1.
\]

Here \(c\) is the constant in Lemma 3.4. Making use of this inequality, (3.10), and (3.13), and observing that \(x_* \in \overline{B}_\rho \subset B_1\), we conclude
\[
L \varphi(x_*) = L \varphi_*(x_*) - \frac{C \gamma_0}{100} L \beta(x_*) \geq \frac{c C \vartheta_0 \gamma_0}{100}.
\]
This, combined with (3.3), gives that \(c C \vartheta_0 /100 \leq 1\), which provides a contradiction if \(C\) is taken large enough, depending only on \(n\) and \(\vartheta_0\).

This argument establishes claim (3.12) and finishes the proof. \(\Box\)

4. A unified approach towards derivative estimates

Most of our theorems will follow from our next result, Theorem 4.2. It deals with solutions of “linear operators possibly varying from point to point and satisfying a maximum principle with estimate”. To state our result precisely, we consider a family of linear operators \(\{L^{(x)}\}_{x \in B_1}\), of any of the types that we have considered previously in the paper, and we set the following terminology.
Definition 4.1. We say that the family \( \{L(x)\}_{x \in B_1} \) satisfies the maximum principle with estimate in \( B_1 \) if there exists a constant \( C \) such that, for every \( \gamma_0 \geq 0 \) and every nonnegative function \( \varphi \in C_0(\mathbb{R}^n) \cap W^{2, \infty}(B_1) \), the following statement holds true: if

\[
\inf_{y \in B_1} \{L(y)\varphi(x)\} \leq \gamma_0 \quad \text{for all} \ x \in B_1,
\]

then

\[
\sup_{B_1} \varphi \leq \sup_{\mathbb{R}^n \setminus B_1} \varphi + C\gamma_0.
\]

We recall that the computation of \( L(y)\varphi \) with \( \varphi \) only of class \( C_0(\mathbb{R}^n) \cap W^{2, \infty}(B_1) \) is intended in the light of Remark 3.3 (in particular, when \( L(y) \) involves classical second order operators, the incremental quotient setting in (3.5) must be adopted). This comment concerns (4.1) and also the left-hand side of (4.4) below. Note, instead, that the right-hand side of (4.4) is well defined and finite since the functions \( \partial^2_x u, \partial_x u, \) and \( u \) are assumed in the theorem to be smooth and globally bounded.

We now provide a result that will establish first and one-sided second derivative bounds in the case of operators possessing a local extension. It will serve both for the Pucci-type operators as well as for our other fully nonlinear equations defined through the function \( F \). In addition, the same method, properly modified, will be used for equations with no extension property.

We introduce the auxiliary function

\[
\varphi(x) := \overline{\eta}^2(x) \left( \partial^2_x u(x) \right)_+^2 + \tau \eta^2(x) \left( \partial_x u(x) \right)^2 + \sigma \left( u(x) - \sup_{B_1} u \right)^2,
\]

where \( \overline{\eta} \) and \( \eta \) are smooth functions with compact support in \( B_{1/2} \) and \( B_1 \), respectively. In addition, we will need that \( \eta = 1 \) in \( B_{1/2} \) in order to be able to verify the key inequality (4.4) below. As we will see in the proof of the following theorem, the first and second derivative bounds (4.5) and (4.6) will follow from making two different appropriate choices of the pair of cutoff functions \((\overline{\eta}, \eta)\).

Let us explain why we are forced, in this nonlocal theory, to consider the auxiliary function (4.3) involving both \( \partial_x u \) and \( \partial^2_x u \) at the same time, as well as two different cutoff functions. This has not been considered in the local theory. Indeed, for local equations one works first with \( \eta^2(\partial_x u)^2 + \sigma v^2 \) as in (1.1), obtaining first derivative estimates in a ball. One considers next \( \overline{\eta}^2(\partial_v v)^2 + \sigma v^2 \) with \( v = \partial_x u \), as in (1.11). This allows us to control \( \partial^2_x u \) from above in a smaller ball by the previous control of \( \partial_x u \) in the larger ball. However, since in the nonlocal setting the maximum principle involves the whole exterior datum (and not only the boundary datum), this second choice would require to control the first derivatives \( v = \partial_x u \) in all space, which cannot be achieved for an equation posed in a ball. We solve this trouble by introducing the new auxiliary function (4.3). Note anyway that the simpler choice (1.11) would still work if our nonlocal equation were posed in all of \( \mathbb{R}^n \). This is why we have chosen to refer to this simpler test function in the Introduction, especially in order to make the Open problem 1.8 as simple as possible.

Theorem 4.2. Let \( \sigma \geq 1 \) and \( \tau \geq 1 \). Assume that \( \{L(x)\}_{x \in B_1} \) is a family of linear operators satisfying the maximum principle with estimate in \( B_1 \), with constant \( C \), according to Definition 4.1.

Given any \( e \in \mathbb{R}^n \) with \( |e| = 1 \), \( \overline{\eta} \in C^\infty_c(B_{1/2}), \eta \in C^\infty_c(B_1) \) with \( \eta = 1 \) in \( B_{1/2} \), and \( u \in C^\infty(\mathbb{R}^n) \cap W^{2, \infty}(\mathbb{R}^n) \), consider the function \( \varphi \) in (4.3).
Assume that, for every such choice of $\sigma$, $\tau$, $\epsilon$, $\eta$, and $u$, we have
\[ L^{(x)}(x) \leq 2\eta^2(x) \left( \partial^2_e u(x) \right) + L^{(x)}(x) \partial_e u(x) \]
\[ + 2\tau \eta^2(x) \partial_e u(x) L^{(x)}(x) \partial_e u(x) + 2\sigma \left( u(x) - \sup_{B_1} u \right) L^{(x)}(x) \left( u - \sup_{B_1} u \right)(x) \quad (4.4) \]
for every $x \in B_1$.

We then have, for every $u \in C^\infty(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)$,
\[ \sup_{B_1/2} |\partial_x u| \leq C \left( C_1 + (C_0 \|u\|_{L^\infty(B_1)})^{1/2} + \|u\|_{L^\infty(\mathbb{R}^n)} \right) \quad (4.5) \]
and
\[ \sup_{B_1/4} \partial^2_x u \leq C \left( C_2 + C_1 + (C_0 \|u\|_{L^\infty(B_1)})^{1/2} + \|u\|_{L^\infty(\mathbb{R}^n)} \right), \quad (4.6) \]
where
\[ a_0 := \sup_{x \in B_1} \left( L^{(x)}(x) \left( u - \sup_{B_1} u \right)(x) \right), \]
\[ a_1 := \sup_{x \in B_1} |L^{(x)}(x) \partial_e u(x)|, \]
\[ a_2 := \sup_{x \in B_1} \left( L^{(x)}(x) \partial^2_e u(x) \right), \]
and $C$ is a constant depending only on $n$, $\sigma$, and $\tau$.

Note that the functions $\varphi$ in (4.3) belong to $C^\infty_0(\mathbb{R}^n) \cap W^{2,\infty}(B_1)$, as required in the previous section, by basic properties of the positive part and the square power appearing in $(\partial^2_e u)^+$, and since $\partial^2_x u \in C^\infty(\mathbb{R}^n)$.

**Proof of Theorem 4.2** We let
\[ \Phi(x) := a_2 \eta^2(x) (\partial^2_e u(x))^+ + a_1 \eta^2(x) |\partial_e u(x)| + a_0 \|u\|_{L^\infty(B_1)}. \]

We claim that
\[ \sup_{B_1} \varphi \leq C \sup_{B_1} \Phi + \|u\|^2_{L^\infty(\mathbb{R}^n)} \quad (4.7) \]
for some constant $C \geq 1$ depending only on $n$, $\sigma$, and $\tau$. To check this, we use (4.4) to find that, for all $x \in B_1$,
\[ \inf_{y \in B_1} \{ L^{(y)}(x) \} \leq \sup_{B_1} \varphi \leq \sup_{B_1} \Phi + \|u\|^2_{L^\infty(\mathbb{R}^n)} \]
for some constant $C$ depending only on $n$, $\sigma$, and $\tau$. Hence, (4.1) is satisfied with $\gamma_0 := C \sup_{B_1} \Phi$, and thus (4.2) yields that $\sup_{B_1} \varphi \leq \sup_{\mathbb{R}^n \setminus B_1} \varphi + C \gamma_0$. As a consequence, recalling (4.3), we have that $\sup_{B_1} \varphi \leq 4\sigma \|u\|^2_{L^\infty(\mathbb{R}^n)} + C \gamma_0$. This establishes (4.7), as desired.
We stress that the constant $C_2$ in (4.7) does not depend on $\eta$ and $\bar{\eta}$, and hence we can now take appropriate choices for these two functions. Note first that, from (4.7),

$$\sup_{B_1} \left( \eta^2 (\partial^2 \phi)^2_+ + \tau \eta^2 |\partial \phi|^2 \right) \leq \sup_{B_1} \varphi \leq C_2 \left( C \sup_{B_1} \Phi + \|u\|_{L^\infty(\mathbb{R}^n)}^2 \right) \leq C_2 \left( C \left( \sup_{B_1} \left( a_2 \eta^2 (\partial^2 \phi)^2_+ + a_1 \eta^2 |\partial \phi|^2 \right) + a_0 \|u\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}^2 \right) \right) \leq C_2 \left( \frac{1}{2C_2} \sup_{B_1} \left( \eta^2 (\partial^2 \phi)^2_+ + \tau \eta^2 |\partial \phi|^2 \right) \right.

$$

$$\left. + C_2^2 \frac{a_2}{a_1} \sup_{B_1} \eta^2 + C_2^2 \frac{a_1}{a_2} \sup_{B_1} \eta^2 + C a_0 \|u\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}^2 \right),$$

where we have used a Cauchy-Schwarz inequality in the last step, and therefore

$$\frac{1}{2} \sup_{B_1} \left( \eta^2 (\partial^2 \phi)^2_+ + \tau \eta^2 |\partial \phi|^2 \right) \leq C_2^2 \left( C \frac{a_2}{a_1} \sup_{B_1} \eta^2 + C \frac{a_1}{a_2} \sup_{B_1} \eta^2 + C a_0 \|u\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)}^2 \right).$$

We now make our two choices of cutoff functions. First, choosing $\bar{\eta} := 0$ and $\eta$ with $|\eta| \leq 1$ and $\eta = 1$ in $B_{1/2}$, from the last inequality we obtain (4.5). Next, with the same choice of $\eta$, we now choose $\bar{\eta} \in C^\infty_c(B_{1/2})$ satisfying $\bar{\eta} = 1$ in $B_{1/4}$. In this way, we infer (4.6). □

5. OPERATORS HAVING LOCAL EXTENSION

In this section, we prove the main results of Subsections 1.1 and 1.3, namely Theorem 1.9 and Corollary 1.10. To this end, we will exploit the key inequality of Proposition 1.2 in a more general version, as given in the forthcoming Proposition 5.1. To state it, we consider a direction $e \in \mathbb{R}^n$, with $|e| = 1$, and two cutoff functions $\eta$ and $\bar{\eta}$ such that

$$\eta \in C^\infty_c(B_1) \text{ and } \bar{\eta} \in C^\infty_c(B_{1/2}) \text{ with } \eta(x) = 1 \text{ for all } x \in B_{1/2}. \quad (5.1)$$

Note that this setting covers the situation required in Theorem 4.2.

Recall the comments after Definition 4.1 for the precise meaning of the left-hand side of (5.2) below (when $s = 1$), and note that the proposition is uniform in $s$ (in fact, we include the cases $s = 0$ and $s = 1$).

**Proposition 5.1.** Let $\eta$ and $\bar{\eta}$ be as in (5.1), $s \in [0, 1]$, and $\kappa \in \mathbb{R}$. Let $u \in C^\infty(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)$. For every $x \in \mathbb{R}^n$, let

$$\varphi(x) := \eta^2(x) \left( \partial^2 \phi(x) + \tau \eta^2(x) |\partial \phi(x)| \right)^2 + \sigma \left( u(x) - \kappa \right)^2.$$

Then, there exist positive constants $\sigma_0$ and $\tau_0$, depending only on $n$, $\|\eta\|_{C^2(\mathbb{R}^n)}$, and $\|\bar{\eta}\|_{C^2(\mathbb{R}^n)}$, such that

$$(-\Delta)^s \varphi \leq 2\eta^2 (\partial^2 \phi)_+ (-\Delta)^s \partial^2 \phi + 2\tau \eta^2 \partial \phi (-\Delta)^s \partial \phi

+ 2\sigma \left( u - \kappa \right) (-\Delta)^s \left( u - \kappa \right) \quad \text{if } \tau \geq \tau_0 \text{ and } \sigma \geq \sigma_0 \tau. \quad (5.2)$$
everywhere in all of $\mathbb{R}^n$.

The proof of Proposition 5.1 will be given after Corollary 5.4 and will rely on the forthcoming auxiliary calculations.

5.1. **Computations in the extended space.** Throughout this section, we consider $s \in (0, 1)$ and $a := 1 - 2s \in (-1, 1)$. Given $u \in C^\infty(\mathbb{R}^n) \cap W^{2, \infty}(\mathbb{R}^n)$, we take $U$ to be the $s$-harmonic extension of $u$ in $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty) = \mathbb{R}^n \times \mathbb{R}_+$, i.e., the unique bounded solution of the problem

\[
\begin{align*}
\text{div} \left( y^a \nabla U(x, y) \right) & = 0 \quad \text{for all } (x, y) \in \mathbb{R}^{n+1}_+, \\
U(x, 0) & = u(x) \quad \text{for all } x \in \mathbb{R}^n.
\end{align*}
\]

Such solution can be found, for instance, by convolving $u$ against the fractional Poisson kernel; see [16]. As a result, since $u$ is continuous, we have that $U$ is continuous in $\mathbb{R}^{n+1}_+$. It is also clear that $U$ is smooth in $\mathbb{R}^{n+1}_+$. Note that the solution $U$ can also be obtained by direct minimization of an energy functional; see Section 3 in [14].

The uniqueness of the bounded extension $U$ follows from Lemma A.1, which is a more general result than uniqueness that we will need later in this section. See Corollary 3.5 in [14] for an alternative proof of uniqueness.

For notational convenience, we also set $L_a U := -\text{div} \left( y^a \nabla U \right)$.

Of course, no confusion should arise with the linearized operator $L$ introduced in (3.2). For further reference, we point out that

\[ L_a(VW) = (L_aV)W + V(L_aW) - 2y^a \nabla V \cdot \nabla W. \tag{5.3} \]

With a slight abuse of notation, we identify the direction $e \in \mathbb{R}^n$ with the same direction in $\mathbb{R}^{n+1}_+$ when we consider the directional derivatives along $e$. In this setting, we have the following result.

**Lemma 5.2.** Let $\eta$ be as in (5.1) and $\kappa \in \mathbb{R}$. For $(x, y) \in \mathbb{R}^{n+1}_+$, let

\[ \Psi_0(x, y) := (U(x, y) - \kappa)^2 \]

and

\[ \Psi_1(x, y) := \eta^2(x) \left( \partial_e U(x, y) \right)^2. \]

Then,

\[ L_a \Psi_0 = -2y^a |\nabla U|^2 \tag{5.4} \]

and

\[ L_a \Psi_1 \leq -y^a \eta^2 |\nabla \partial_e U|^2 + C \chi_B, y^a \left( \partial_e U \right)^2 \] everywhere in $\mathbb{R}^{n+1}_+$, for some constant $C$ depending only on $n$ and $\|\eta\|_{C^2(\mathbb{R}^n)}$.

The statement and proof of Lemma 5.2 concern a general constant $\kappa \in \mathbb{R}$. Later it will be convenient to choose $\kappa$ to be the supremum of the solution.

**Proof of Lemma 5.2.** By (5.3) we have that $L_a \Psi_0 = 2(L_a U)(U - \kappa) - 2y^a |\nabla U|^2$, whence (5.4) plainly follows.

Now, to prove (5.5), since $L_a U = 0$, and thus $L_a \partial_e U = 0$, we infer that

\[ L_a (\partial_e U)^2 = -2y^a |\nabla \partial_e U|^2. \tag{5.6} \]
Furthermore, by (5.3),
\[ L_a \eta^2 = 2\eta (L_a \eta) - 2y^a |\nabla \eta|^2 = -2y^a \eta \Delta_x \eta - 2y^a |\nabla_x \eta|^2. \] (5.7)
Now, we notice that \( \Psi_1 = \eta^2 (\partial_e U)^2 \). Accordingly, we use (5.3), (5.6), and (5.7) to see that
\[ L_a \Psi_1 = (L_a \eta^2) (\partial_e U)^2 + \eta^2 (L_a (\partial_e U)^2) - 2y^a \nabla_x \eta^2 \cdot \nabla_x (\partial_e U)^2 \]
\[ = -2y^a (\eta \Delta_x \eta + |\nabla_x \eta|^2) (\partial_e U)^2 - 2y^a \eta^2 |\nabla \partial_e U|^2 \]
\[ - 8y^a \eta \partial_e U \nabla_x \eta \cdot \nabla_x \partial_e U. \]
Also, we observe that, by the Cauchy-Schwarz inequality,
\[ 8y^a |\eta \partial_e U \nabla_x \eta \cdot \nabla_x \partial_e U| \leq y^a \eta^2 |\nabla \partial_e U|^2 + 16 y^a |\nabla_x \eta|^2 (\partial_e U)^2. \]
Consequently, we obtain that
\[ L_a \Psi_1 \leq -y^a \eta^2 |\nabla \partial_e U|^2 + 2y^a (|\eta \Delta_x \eta| + 7 |\nabla_x \eta|^2) (\partial_e U)^2. \]
Hence, the desired inequality (5.5) follows, since \( \eta \) is supported in \( B_1 \), \( |\nabla_x \eta| \leq ||\eta||_{C^1(\mathbb{R})} \), and \( |\Delta_x \eta| \leq n ||\eta||_{C^2(\mathbb{R})}. \)

We now start estimating the operator \( L_a \) acting on the second derivatives of \( U \). The following (nonoptimal) inequality is all what we will need subsequently.

**Lemma 5.3.** Let \( \eta \) be as in (5.1). For \( (x,y) \in \mathbb{R}^{n+1}_+ \), let
\[ \Psi_2 (x,y) := \overline{\eta}^2 (x) (\partial_{\text{loc}}^2 U (x,y))^2. \]
Then, there exists a constant \( C \), depending only on \( n \) and \( ||\overline{\eta}||_{C^2(\mathbb{R}^n)} \), such that
\[ L_a \Psi_2 \leq C \chi_{B_1/2} y^a (\partial_{\text{loc}}^2 U)^2 \quad \text{everywhere in } \mathbb{R}^{n+1}_+. \] (5.8)
Concerning claim (5.8), we point out that, since \( U \in C^\infty(\mathbb{R}^{n+1}_+) \), we have that \( \partial^2_{\text{loc}} U \in C^\infty(\mathbb{R}^{n+1}_+) \) and accordingly
\[ (\partial^2_{\text{loc}} U)^2 \in W^{2,\infty}_\text{loc}(\mathbb{R}^{n+1}_+). \] (5.9)
This is sufficient to compute the left-hand side of (5.8) everywhere by writing \( L_a \) in nondivergence form and using the incremental quotient limit definition in (3.5). As an alternative, one could also interpret (5.8) in the weak sense, since (5.9) yields that \( (\partial^2_{\text{loc}} U)^2 \in H^1_{\text{loc}}(\mathbb{R}^{n+1}_+) \).

**Proof of Lemma 5.3.** Recalling (5.3), we have that
\[ L_a \left( \overline{\eta}^2 (\partial_{\text{loc}}^2 U)^2 \right) = L_a \overline{\eta}^2 (\partial_{\text{loc}}^2 U)^2 + \overline{\eta}^2 L_a (\partial_{\text{loc}}^2 U)^2 - 8y^a \overline{\eta} \partial^2_{\text{loc}} U \nabla \overline{\eta} \cdot \nabla \partial_{\text{loc}}^2 U \] (5.10)
and, since \( L_a U = 0 \),
\[ L_a ((\partial^2_{\text{loc}} U)^2) = 2\partial^2_{\text{loc}} U L_a \partial^2_{\text{loc}} U - 2y^a |\nabla \partial^2_{\text{loc}} U|^2 = -2y^a |\nabla \partial^2_{\text{loc}} U|^2. \] (5.11)
Moreover, \( L_a \overline{\eta}^2 = -2y^a \overline{\eta} \Delta_x \overline{\eta} - 2y^a |\nabla_x \overline{\eta}|^2 \). By inserting this and (5.11) into (5.10), we conclude that
\[ L_a \left( \overline{\eta}^2 (\partial_{\text{loc}}^2 U)^2 \right) = -2y^a \overline{\eta} \Delta_x \overline{\eta} (\partial_{\text{loc}}^2 U)^2 - 2y^a |\nabla_x \overline{\eta}|^2 (\partial_{\text{loc}}^2 U)^2 - 2y^a \overline{\eta}^2 |\nabla \partial^2_{\text{loc}} U|^2 \]
\[ - 8y^a \overline{\eta} \partial^2_{\text{loc}} U \nabla_x \overline{\eta} \cdot \nabla_x \partial^2_{\text{loc}} U. \] (5.12)
Now we use the Cauchy-Schwarz inequality, finding that
\[ -8y^a \overline{\eta} \partial^2_{\text{loc}} U \nabla_x \overline{\eta} \cdot \nabla_x \partial^2_{\text{loc}} U \leq y^a \overline{\eta}^2 |\nabla \partial^2_{\text{loc}} U|^2 + 16 y^a |\nabla_x \overline{\eta}|^2 (\partial^2_{\text{loc}} U)^2. \]
Since $\eta$ is supported in $B_{1/2}$, $|\nabla_x \eta| \leq \|\eta\|_{C^1(\mathbb{R})}$, and $|\Delta_x \eta| \leq n \|\eta\|_{C^2(\mathbb{R})}$, the latter estimate and (5.12) give

$$L_a \left( \eta^2 \left( \partial^2_c U \right)^2 \right) \leq C \chi_{B_{1/2}} y^a \left( \partial^2_c U \right)^2 - y^a \eta^2 |\nabla \partial^2_c U|^2 \leq C \chi_{B_{1/2}} y^a \left( \partial^2_c U \right)^2$$

(5.13) everywhere in $\mathbb{R}^{n+1}_+$, for a suitable constant $C$ depending only on $n$ and $\|\eta\|_{C^2(\mathbb{R}^n)}$.

Now, we use this inequality to prove (5.8). For this, let $(x, y) \in \mathbb{R}^{n+1}_+$ and we observe that if $\Psi_2(x, y) > 0$, then $\Psi_2 = \eta^2 \left( \partial^2_c U \right)^2$ in a small neighborhood of $(x, y)$ and thus (5.8) follows in this case directly from (5.13).

If instead $(x, y) \in \mathbb{R}^{n+1}_+$ is such that $\Psi_2(x, y) = 0$, the second incremental quotient definition (3.5) gives, since $\Psi_2 \geq 0$, that $-\Delta \Psi_2(x, y) \leq 0$. Also, since $\Psi_2 \in C^1_{\text{loc}}(\mathbb{R}^{n+1}_+)$ by (5.9), the fact that $\Psi_2 \geq 0 = \Psi_2(x, y)$ leads to $\nabla \Psi_2(x, y) = 0$ and, as a result,

$$L_a \Psi_2(x, y) = -y^a \Delta \Psi_2(x, y) - ay^{a-1} \partial_y \Psi_2(x, y) \leq 0.$$ 

In particular, also in this case $L_a \Psi_2(x, y) \leq C \chi_{B_{1/2}} y^a \left( \partial^2_c U \right)^2$. The proof of (5.8) is thereby complete.

By combining Lemmata 5.2 and 5.3 we obtain:

**Corollary 5.4.** Let $\eta$ and $\pi$ be as in (5.1) and $\sigma$, $\tau$, $\kappa \in \mathbb{R}$. For $(x, y) \in \mathbb{R}^{n+1}_+$, let

$$\Phi(x, y) := \eta^2 (x) \left( \partial^2_c U(x, y) \right)^2 + \tau \eta^2 (x) \left( \partial_d U(x, y) \right)^2 + \sigma \left( U(x, y) - \kappa \right)^2.$$ 

Then, there exist positive constants $\sigma_0$ and $\tau_0$, depending only on $n$, $\|\eta\|_{C^2(\mathbb{R}^n)}$, and $\|\pi\|_{C^2(\mathbb{R}^n)}$, such that $L_a \Phi \leq 0$ everywhere in $\mathbb{R}^{n+1}_+$ if $\tau \geq \tau_0$ and $\sigma \geq \sigma_0 \tau$.

**Proof.** In the notation of Lemmata 5.2 and 5.3 we have that $\Phi = \Psi_2 + \tau \Psi_1 + \sigma \Psi_0$, and consequently

$$L_a \Phi \leq C \chi_{B_{1/2}} y^a \left( \partial^2_c U \right)^2 - \tau y^a \eta^2 |\nabla \partial^2_c U|^2 + C \tau \chi_{B_{1/2}} y^a + \chi_{B_{1/2}} y^a |\nabla U|^2.$$ 

Now, we notice that $|\partial^2_c U| \leq |\nabla \partial_d U|$ and that, by (5.1), $\chi_{B_{1/2}} \leq \eta^2$. Therefore, if $\tau$ is sufficiently large as stated in the corollary, we deduce that

$$L_a \Phi \leq C \tau y^a \left( \partial^2_c U \right)^2 - 2\tau y^a |\nabla U|^2.$$ 

Thus, if $\sigma$ is sufficiently large, we conclude $L_a \Phi \leq 0$, which proves the desired result.

The previous computations allow us to prove the main inequality of this section.

**Proof of Proposition 5.7.** We first consider the case $s = 0$. We have that

$$\varphi - 2\eta^2 \left( \partial^2_c u \right)^2 - 2\tau \eta^2 \left( \partial_d u \right)^2 - 2\sigma (u - \kappa)^2 = -\eta^2 \left( \partial^2_c u \right)^2 - \tau \eta^2 \left( \partial_d u \right)^2 - \sigma (u - \kappa)^2 \leq 0,$$

which establishes the desired result in this case.

Now, we focus on the case $s \in (0, 1)$. From this, the case $s = 1$ can be obtained in the limit\footnote{It is interesting to point out that while we can obtain (5.2) with $s = 1$ by a limit argument as $s \uparrow 1$, when $s = 0$ we needed to perform a direct though simple computation and we could not argue by sending $s \searrow 0$. Indeed, the limit as $s \searrow 0$ of the fractional Laplacian involves a term of the type $s \int_{\mathbb{R}^n \setminus B} u(y) |y|^{-n-2s} \, dy$ which goes to zero if $u$ is compactly supported or decreases sufficiently fast at infinity, but not in general.} (also, one could proceed by direct computations, as in [15]). For $s \in (0, 1)$, we let $U$ be the
s-harmonic extension of \( u \) in \( \mathbb{R}^{n+1}_+ \), and we recall that \( a = 1 - 2s \). By [16], up to a positive multiplicative constant depending only on \( s \)—that we do not write since it plays no role to establish (5.2)— we have

\[
- \lim_{y \searrow 0} y^a \partial_y \partial_x U = (-\Delta)^s \partial_x u \tag{5.14}
\]

and

\[
- \lim_{y \searrow 0} y^a \partial_y \partial_x^2 U = (-\Delta)^s \partial_x^2 u. \tag{5.15}
\]

Moreover, if \( \Phi^* \) is the \( s \)-harmonic extension of \( \varphi \) in \( \mathbb{R}^{n+1}_+ \), we also have that

\[
- \lim_{y \searrow 0} y^a \partial_y \Phi^* = (-\Delta)^s \varphi \tag{5.16}
\]

(recall that \( \varphi \) is locally \( W^{2,\infty} \) and bounded in \( \mathbb{R}^n \)).

The key point is to consider the function

\[
\Phi(x, y) := \overline{\eta}^2(x) \left( \partial_x^2 U(x, y) \right) + \tau \eta^2(x)(\partial_x U(x, y))^2 + \sigma \left( U(x, y) - \kappa \right)^2.
\]

Now, by Corollary 5.4, we know that \( L_a \Phi \leq 0 \) almost everywhere in \( \mathbb{R}^{n+1}_+ \), as long as \( \sigma \) and \( \tau \) are taken as in the statement of Proposition 5.1. Thus, since \( L_a \Phi^* = 0 \),

\[
L_a (\Phi - \Phi^*) \leq 0 \quad \text{everywhere in } \mathbb{R}^{n+1}_+.
\]

Notice also that \( \Phi - \Phi^* \) vanishes in \( \mathbb{R}^n \times \{0\} \), and hence the maximum principle in Lemma A.1 gives that \( \Phi - \Phi^* \leq 0 \) in \( \mathbb{R}^{n+1}_+ \). As a consequence, since this function vanishes in \( \mathbb{R}^n \times \{0\} \), we find that \( \lim_{y \searrow 0} y^a \partial_y (\Phi - \Phi^*) \leq 0 \). Therefore, by (5.16), the definition of \( \Phi \), and recalling again that \( (\partial_x^2 U)^2 \) is a \( W^{2,\infty}_{\text{loc}} \) function,

\[
(-\Delta)^s \varphi \leq - \lim_{y \searrow 0} y^a \partial_y \Phi = - \lim_{y \searrow 0} \left( 2y^a \overline{\eta}^2 \left( \partial_x^2 U \right) \right) + 2y^a \tau \eta^2 \partial_x U \partial_y \partial_x U + 2y^a \sigma \left( U - \kappa \right) \partial_y \partial_x U - y^a \right).
\]

We conclude, since \( \partial_x U(x, 0) = \partial_x u(x) \) and \( \partial_x^2 U(x, 0) = \partial_x^2 u(x) \), that

\[
(-\Delta)^s \varphi \leq - \lim_{y \searrow 0} \left( 2y^a \overline{\eta}^2 \left( \partial_x^2 u \right) \right) + 2y^a \tau \eta^2 \partial_x U \partial_y \partial_x U + 2y^a \sigma \left( u - \kappa \right) \partial_y \partial_x U - 2 \sigma \left( u - \kappa \right) \left( -\Delta \right)^s \left( u - \kappa \right),
\]

where (5.14) and (5.15) were used in the last step.

The following result is a simple, but useful, improvement of Proposition 5.1, in which we obtain that \( \varphi \) is a suitable subsolution with respect to the linearized operator \( (3.2) \) (in our current setting, the linear operators \( \mathcal{L}_1, \ldots, \mathcal{L}_J \) in (3.2) boil down to the linear operators \( \mathcal{L}_{\mu_1}, \ldots, \mathcal{L}_{\mu_J} \)).

In particular, it is not always true that \( \lim_{y \searrow 0} (-\Delta)^s u(x) = u(x) \), since the left-hand side is invariant if one replaces \( u \) by \( u - \kappa \), for \( \kappa \in \mathbb{R} \), while the right-hand side is not; see Proposition 4.4 in [20] for additional details.
Corollary 5.5. Let \( \eta \) and \( \overline{\eta} \) be as in (5.1). Let \( \kappa \in \mathbb{R} \) and \( u \in C^\infty(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n) \). For every \( x \in \mathbb{R}^n \), let

\[
\varphi(x) := \overline{\eta}^2(x) \left( \partial_x^2 u(x) \right)^2 + \tau \, \eta^2 (x) (\partial_x u(x))^2 + \sigma \left( u(x) - \kappa \right)^2.
\]

Then, there exist positive constants \( \sigma_0 \) and \( \tau_0 \), depending only on \( n \), \( \| \eta \|_{C^2(\mathbb{R}^n)} \), and \( \| \overline{\eta} \|_{C^2(\mathbb{R}^n)} \), such that, if \( \tau \geq \tau_0 \) and \( \sigma \geq \sigma_0 \tau \), then in all of \( \mathbb{R}^n \) we have

\[
L \varphi \leq 2\overline{\eta}^2 (\partial_x^2 u)_+ L \partial_x^2 u + 2\tau \, \eta^2 \partial_x u L \partial_x u + 2\sigma (u - \kappa) L(u - \kappa).
\]

Proof. We write (5.2) for every \( s \in [0, 1] \), and we integrate with respect to the measure \( \mu_j \), for every given \( j \in \{1, \ldots, J\} \). We find that

\[
L_{\mu_j} \varphi \leq 2\overline{\eta}^2 (\partial_x^2 u)_+ L_{\mu_j} (\partial_x^2 u) + 2\tau \, \eta^2 \partial_x u L_{\mu_j} \partial_x u + 2\sigma (u - \kappa) L_{\mu_j} (u - \kappa).
\]

Now, we multiply the above expression by \( \alpha_j \), as defined in (3.1), using (1.21), and then we sum the inequality over \( j \in \{1, \ldots, J\} \). In this way, recalling also the definition of \( L \) in (3.2), we conclude the proof. \( \square \)

We remark that Proposition 1.2 is the simplified version of Proposition 5.1 which already leads to first derivative estimates. On the other hand, Proposition 1.2 is stated for a more general cutoff function than the one in Proposition 5.1. The proof of Proposition 1.2 follows the same lines as that of Proposition 5.1 and only requires Lemma 5.2.

Though not explicitly used in this article, we next state the simplest inequality for an auxiliary function which leads to one-sided derivative estimates (at least for global solutions). For more details see the comments before and after (1.11), in Open problem 1.8, and before Theorem 4.2. Its proof also follows the same lines as that of Proposition 5.1.

Proposition 5.6. Let \( s \in (0, 1) \), \( v \in C^\infty(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n) \), \( \eta \in C^\infty(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n) \), \( e \in \mathbb{R}^n \) with \( |e| = 1 \), and \( \sigma > 0 \). Let

\[
\psi := \eta^2 (\partial_x v)_+^2 + \sigma v^2.
\]

Then, in all of \( \mathbb{R}^n \) we have

\[
( -\Delta )^s \psi \leq 2\eta^2 (\partial_x v)_+ ( -\Delta )^s \partial_x v + 2\sigma v ( -\Delta )^s v \quad \text{if} \quad \sigma \geq \sigma_0,
\]

for some constant \( \sigma_0 \) depending only on \( n \) and \( \| \eta \|_{C^2(\mathbb{R}^n)} \) — and, in particular, independent of \( s \).

5.2. Proofs of Theorem 1.1, Theorem 1.9 and Corollary 1.10. We can now prove these results using Theorem 4.2 via an appropriate choice of the linear operators \( \{L(x)\}_{x \in B_1} \).

However, let us first make a remark that connects Theorems 1.1 and 1.9.

Remark 5.7. Theorem 1.9 can be easily extended by changing the definition (1.25) of \( L_{\mu} \) to involve more general operators of the form

\[
\int_0^1 \left( -\sum_{i,j=1}^{n} M_{ij}(s) \partial_{x_i x_j}^2 u \right) s \, d\mu(s),
\]

containing fractions of second order elliptic operators with constant coefficients instead of fractions of the Laplacian. Here, for every \( s \in [0, 1] \) we are given a matrix \( M(s) \in \mathcal{A}_{\Lambda^{-2}, \Lambda^{-2}} \), where the classes \( \mathcal{A} \) were defined in (1.4).

In this way one includes here those equations of Subsection 1.1 built from a finite number \( J \) of linear operators. In fact, since our estimates will be independent of \( J \), it is possible to deduce
Indeed, by the definition (1.5) of $\Delta$, we find that

$$
\Delta (u(x) - u(y)) = u(x) - u(y) - \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|A(x - y)|^{n+2s}} dy,
$$

which coincides with the operator $L_A$ previously considered in (1.3), up to a multiplicative constant. These fractional operators thus possess the same type of extension properties as the fractional Laplacian.

The previous equality can be alternatively checked through Fourier symbols since

$$
\left( \sum_{i,j=1}^{n} M_{ij}(s) \partial_{z_i z_j}^2 \right)^s (x) = c_{n,s} \det A(s) \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|A(s)(x - y)|^{n+2s}} dy,
$$

where $\tilde{\xi} := \frac{\xi}{|A^{1-s}|}$.

**Proof of Theorem 1.1.** Note that, since $A$ is compact and we assume the continuity hypothesis (1.7), observing also that $L_A u(x)$ is continuous with respect to $A$ by the dominated convergence theorem, we have that, given $x \in B_1$, $M_{ij}(x) = L_{A_x} u(x) - g_{A_x}(x)$ for some $A_x \in \mathcal{A}$.

We start verifying the hypotheses of Theorem 4.2 for the family of operators $L(x) = L_{A_x}$. We first check that the maximum principle with estimate, as considered in Definition 4.1, is satisfied in this case. Indeed, assume that (4.1) holds true. By (3.6) applied to the kernel $K(z) = c_{n,s} |A z|^{-n-2s}$, we know that there exists a function $\beta$ with $|\beta| \leq 100$ in $\mathbb{R}^n$ and $L_A \beta \leq -c$ everywhere in $B_1$ for every $A \in \mathcal{A}$. Here $c > 0$ depends only on $n$, $\lambda$, and $\Lambda$.

Define $\tilde{\varphi} := \varphi + \gamma_0 / c$. By (4.1), we have that $\inf_{y \in B_1} \{ L_{A_y} \tilde{\varphi}(x) \} \leq 0$ for every $x \in B_1$. From this (and again the fact that the previous infimum will be equal to $L_{A^{1-s}} \tilde{\varphi}(x)$ for some $A^{1-s}$ perhaps different from $A_x$), one sees that $\sup_{\mathbb{R}^n} \tilde{\varphi}$ cannot be achieved in $B_1$ (except when $\tilde{\varphi}$ is constant), and therefore in any case

$$
\max_{B_1} \tilde{\varphi} \leq \sup_{\mathbb{R}^n \setminus B_1} \varphi + \frac{100 \gamma_0}{c}.
$$

Consequently, we find that

$$
\sup_{B_1} \varphi \leq \sup_{\mathbb{R}^n \setminus B_1} \varphi + \frac{200 \gamma_0}{c},
$$

and hence the maximum principle with estimate, as considered in Definition 4.1, is satisfied.

Now we check that (4.4) holds true in the current situation if we take $\tau$ and $\sigma$ appropriately. Indeed, by the definition (1.5) of $L_A$, we know that

$$
L_A u(x) = \frac{1}{\det A} (-\Delta)^s u_A(A x), \quad u_A(x) := u(A^{-1} x).
$$

Also $\partial_{e_A} u_A(x) = \partial_{e} u(A^{-1} x)$ and $\partial_{e_A}^2 u_A(x) = \partial_{e}^2 u(A^{-1} x)$, with $e_A := A e$. That is, setting $e_A' := e_A / |e_A|$, we have that $\partial_{e_A} u(A^{-1} x) = |e_A| \partial_{e_A} u(A x)$ and $\partial_{e_A}^2 u(A^{-1} x) = |e_A|^2 \partial_{e_A}^2 u(A x)$. Thus, we
have that
\[ (-\Delta)^s \partial_{e_A} u_A(x) = \frac{\det A}{|e_A|} \mathcal{L}_A \partial_{e} u(x) \] and
\[ (-\Delta)^s \partial^2_{e_A} u_A(x) = \frac{\det A}{|e_A|^2} \mathcal{L}_A \partial^2_{e} u(x). \] (5.18)

At the same time, if we let \( \varphi \) be as in (4.3), \( \sigma_A := |e_A|^{-4} \tau \), \( \tau_A := |e_A|^{-2} \tau \), \( \eta_A(x) := \eta(A^{-1}x) \), \( \eta_A(x) := \eta(A^{-1}x) \), and
\[ \varphi_A(x) := \eta_A^2(x) \partial^2_{e_A} u_A(x) + \tau A \eta_A^2(x) \partial_{e_A} u_A(x) \left( \partial_{e_A} u_A(x) \right)^2 + \sigma_A \left( u_A(x) - \sup_{A(B_1)} u_A \right)^2, \]
we find that, if \( x \in B_1 \), and thus \( x = Ax \in A(B_1) \),
\[ \mathcal{L}_A \varphi(x) = \frac{|e_A|^4}{\det A} ( -\Delta )^s \varphi_A(x). \]
As a result, in view of Proposition 5.1, if \( \tau \) and \( \sigma \) are sufficiently large, we conclude that, for every \( x \in B_1 \) (and thus \( x = Ax \in A(B_1) \)),
\[ \mathcal{L}_A \varphi(x) \leq \frac{|e_A|^4}{\det A} \left\{ 2\eta_A^2(x) \left( \partial^2_{e_A} u_A(x) \right) + (-\Delta)^s \partial^2_{e_A} u_A(x) \right\} \]
\[ + 2\tau A \eta_A^2(x) \partial_{e_A} u_A(x) \left( -\Delta \right)^s \partial_{e_A} u_A(x) \]
\[ + 2\sigma A \left( u_A(x) - \sup_{A(B_1)} u_A \right) \left( -\Delta \right)^s \left( u_A - \sup_{A(B_1)} u_A \right)(x) \right\} . \] (5.19)

From this and (5.18), we get
\[ \mathcal{L}_A \varphi(x) \leq \left\{ 2\eta_A^2(x) \left( \partial^2_{e_A} u_A(x) \right) + \mathcal{L}_A \partial^2_{e} u(x) + 2\tau A \eta_A^2(x) \partial_{e} u(x) \mathcal{L}_A \partial_{e} u(x) \right\} \]
\[ + 2\sigma A \left( u(x) - \sup_{B_1} u \right) \mathcal{L}_A \left( u - \sup_{B_1} u \right)(x) \right\} \]
for all \( x \in B_1 \). Hence, inequality (4.4) holds true with \( L(x) = \mathcal{L}_{A_x} \), after choosing, given \( x, A = A_x \).

With this, we are in the position of applying Theorem 4.2. To this end, we estimate the quantities \( a_0, a_1, \) and \( a_2 \) given in its statement. The arguments here are similar to those of Subsection 3.1. First, by equation (1.6),
\[ \mathcal{L}_{A_x} u(x) = g_{A_x}(x) \] (5.20)
for every \( x \in B_1 \). Also, using again equation (1.6), for every \( x, y \in B_1 \) we have that
\[ \mathcal{L}_{A_x} u(y) - g_{A_x}(y) \leq \mathcal{M}_A u(y) = 0 = \mathcal{M}_A u(x) = \mathcal{L}_{A_x} u(x) - g_{A_x}(x). \] (5.21)

Therefore, if \( \varepsilon \in \mathbb{R}^n \) and \( |\varepsilon| = 1 \), given \( h \in (0, 1 - |x|) \), we see that
\[ \mathcal{L}_{A_x} \left( \frac{u(x \pm \varepsilon h) - u(x)}{h} \right) = \mathcal{L}_{A_x} u(x \pm \varepsilon h) - \mathcal{L}_{A_x} u(x) \leq \frac{g_{A_x}(x \pm \varepsilon h) - g_{A_x}(x)}{h}, \]
and, as a consequence, \( \pm \mathcal{L}_{A_e} \partial_e u(x) \leq \pm \partial_e g_{A_e}(x) \) for almost every \( x \in B_1 \). Thanks to the possible sign choice in this inequality, and to the fact that \( g_A \) is a Lipschitz function, and thus pointwise differentiable almost everywhere, we thereby deduce that

\[
\mathcal{L}_{A_e} \partial_e u(x) = \partial_e g_{A_e}(x) \tag{5.22}
\]

for almost every \( x \in B_1 \). In addition, exploiting (5.21) once again,

\[
\mathcal{L}_{A_e} \left( \frac{u(x + he) + u(x - he) - 2u(x)}{h^2} \right) = \mathcal{L}_{A_e} u(x + he) + \mathcal{L}_{A_e} u(x - he) - 2\mathcal{L}_{A_e} u(x) \leq g_{A_e}(x + he) + g_{A_e}(x - he) - 2g_{A_e}(x) \frac{h^2}{h^2}.
\]

Thus, if \( g_{A_e} \) is semiconcave, and hence pointwise twice differentiable almost everywhere, we have that, for almost every \( x \in B_1 \),

\[
\mathcal{L}_{A_e} \partial^2_e u(x) \leq \partial^2_e g_{A_e}(x). \tag{5.23}
\]

In the notation of Theorem 4.2 (with \( L^x \) there corresponding to \( \mathcal{L}_{A_e} \) here), (5.20), (5.22), and (5.23) lead to

\[
a_0 = \sup_{x \in B_1} \left( g_{A_e}(x) \right)_- \leq \sup_{x \in B_1, A \in A} \left( g_A(x) \right)_- = \| (g_A)_- \|_{L^\infty(A \times B_1)},
\]

\[
a_1 = \sup_{x \in B_1} \left| \partial_e g_{A_e}(x) \right| \leq \sup_{x \in B_1, A \in A} \left| \partial_e g_A(x) \right| = \| \partial_e g_A \|_{L^\infty(A \times B_1)},
\]

and

\[
a_2 \leq \sup_{x \in B_1} \left( \partial^2_e g_{A_e}(x) \right)_+ \leq \sup_{x \in B_1, A \in A} \left( \partial^2_e g_A(x) \right)_+ = \| (\partial^2_e g_A)_+ \|_{L^\infty(A \times B_1)}.
\]

These considerations and Theorem 4.2 lead to

\[
\sup_{B_{1/2}} \left| \partial_e u \right| \leq C \left( \| \partial_e g_A \|_{L^\infty(A \times B_1)} + \left( \| (g_A)_- \|_{L^\infty(A \times B_1)} \| u \|_{L^\infty(B_1)} \right)^{1/2} + \| u \|_{L^\infty(\mathbb{R}^n)} \right) \tag{5.24}
\]

and

\[
\sup_{B_{1/4}} \partial^2_e u \leq C \left( (\partial^2_e g_A)_+ \| L^\infty(A \times B_1) + \| \partial_e g_A \|_{L^\infty(A \times B_1)} \right.
\]

\[
\left. + \left( \| (g_A)_- \|_{L^\infty(A \times B_1)} \| u \|_{L^\infty(B_1)} \right)^{1/2} + \| u \|_{L^\infty(\mathbb{R}^n)} \right),
\]

for some constant \( C \) depending only on \( n, \lambda, \) and \( \Lambda \). This gives (1.8), and also (1.9) with \( B_{1/2} \) replaced by \( B_{1/4} \). From this result, it is easy to deduce the second derivative bound as stated in Theorem 1.1 (that is, in \( B_{1/2} \)) by a covering argument. \( \square \)

**Proof of Theorem 1.9.** We use again Theorem 4.2, but now taking, for all \( x \in B_1 \), \( L^x := L \) to be the linearized operator introduced in (3.2). Let us check that the hypotheses of Theorem 4.2 are fulfilled. First of all, by Proposition 3.2 if \( \varphi \in C_b(\mathbb{R}^n) \cap W^{2,\infty}(B_1) \) is nonnegative and satisfies \( L \varphi \leq \gamma_0 \) in \( B_1 \), for some \( \gamma_0 \geq 0 \), then \( \sup_{B_1} \varphi \leq \sup_{\mathbb{R}^n \setminus B_1} \varphi + C \gamma_0 \), with \( C \) depending only on \( n, \vartheta_0, \) and \( \Theta_0 \). This says that \( L \) satisfies the maximum principle in \( B_1 \) with constant \( C \),
according to Definition 4.1. Moreover, choosing $\kappa \ceq \sup_{B_1} u$ in Corollary 5.5, we deduce that (1.4) holds for $\sigma$ and $\tau$ large enough depending only on $n$.

Therefore, from Theorem 4.2 we obtain that

$$\sup_{B_{1/2}} |\partial_c u| \leq C \left( a_1 + \left( a_0 \|u\|_{L^\infty(B_1)} \right)^{1/2} + \|u\|_{L^\infty(\mathbb{R}^n)} \right)$$  \hspace{1cm} (5.25)

and

$$\sup_{B_{1/4}} \partial^2_c u \leq C \left( a_2 + a_1 + \left( a_0 \|u\|_{L^\infty(B_1)} \right)^{1/2} + \|u\|_{L^\infty(\mathbb{R}^n)} \right),$$  \hspace{1cm} (5.26)

where $C$ is a constant depending only on $n$, $\theta_0$, and $\Theta_0$.

The quantities $a_0$, $a_1$, and $a_2$ in the statement of Theorem 4.2 are given by

$$a_0 \ceq \sup_{x \in B_1} \left( L(u - su \mathbb{I}) (x) \right)_-, \quad a_1 \ceq \sup_{x \in B_1} \left| L \partial_c u(x) \right|, \quad \text{and} \quad a_2 \ceq \sup_{x \in B_1} \left( L \partial^2_c u(x) \right)_+.$$

We now use Lemma 3.1. First, taking into account that $L$ could contain the operator $(-\Delta)^0 = \text{Id}$, we have that

$$(L(u - su \mathbb{I}) (x))_- = \max \left\{ -Lu + Lsu \mathbb{I}, 0 \right\}$$

$$\leq |F(0)| + \Theta_0 \sup_{j \in \{1, \ldots, J\}} \|g_j\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}$$

everywhere in $B_1$. Hence, we conclude that

$$a_0 \leq |F(0)| + \Theta_0 \sup_{j \in \{1, \ldots, J\}} \|g_j\|_{L^\infty(B_1)} + \|u\|_{L^\infty(B_1)}.$$

This and the first derivative bound (5.25) lead to (using again Lemma 3.1)

$$\sup_{B_{1/2}} |\partial_c u| \leq C \left( \|u\|_{L^\infty(\mathbb{R}^n)} + |F(0)| + \sup_{j \in \{1, \ldots, J\}} \|g_j\|_{W^{1, \infty}(B_1)} \right).$$

Similarly, in light of the second derivative bound (5.26), we see that

$$\sup_{B_{1/4}} \partial^2_c u \leq C \left( \|u\|_{L^\infty(\mathbb{R}^n)} + |F(0)| + \sup_{j \in \{1, \ldots, J\}} \|g_j\|_{W^{2, \infty}(B_1)} \right).$$

These observations complete the proof of Theorem 4.9 after using a covering argument as at the end of the proof of Theorem 1.1.

Corollary 1.10 follows immediately from Theorem 1.9 as we show next. Alternatively, the corollary could also be established following the lines of the proof of Theorem 1.1—hence, using the max structure of the operator instead of introducing the linearized operator $L$ as in Theorem 1.9.

**Proof of Corollary 1.10**. We are in the setting of Theorem 1.9 with $J = 2$, $\mathcal{L}_{\mu_1} = (-\Delta)^s$, $g_1 = f$, $\mathcal{L}_{\mu_2} = (-\Delta)^0$ (the identity), $g_2 = f + \phi$, and $F$ being the max operator—which satisfies (1.21)-(1.23), with $\theta_0 = \Theta_0 = 1$. Thus, the corollary follows immediately from Theorem 1.9.

\footnote{Here we use that the proof of Theorem 4.2 requires only two choices of pairs $(\eta, \eta)$ for cutoff functions.}
**Remark 5.8.** In view of the previous proofs, the estimates of Theorem 1.1, Theorem 1.9, and Corollary 1.10 can be stated in a more refined manner. Such statements can be found in version 2 of the arXiv preprint of this work [https://arxiv.org/pdf/2010.00376v2.pdf](https://arxiv.org/pdf/2010.00376v2.pdf) where the estimates are more precise in several respects: some quantities on the functions $g_A$ (or $g_j$) involve only their positive or negative parts; the supremums of $g_A$ (or $g_j$) and of its derivatives are considered separately; finally, the equations and estimates are considered in a ball of arbitrary radius $R$, which is particularly interesting in the case of operators of indefinite order (since, for them, the way in which the equation changes after scaling is not as simple as in the definite order case).

**6. General integro-differential operators: proof of Theorem 1.3**

Here, we exploit Theorem 1.5 to establish Theorem 1.3. Because of the presence of an error term in the inequality of Theorem 1.5, the gradient estimates initially obtained will have a remainder or error term. But the fact that the estimates hold at any scale will allow us to reabsorb the error term and get rid of it.

**Proof of Theorem 1.3**. The proof follows closely the lines of that of Theorem 1.1. The main differences are that here we only perform first derivative estimates, whence the choice of the test function must be modified accordingly, and that the gradient estimate initially presents a small error that needs to be reabsorbed via scaled estimates.

The sketch of the proof of Theorem 1.3, emphasizing the technical differences with respect to the proof of Theorem 1.1, goes as follows.

We assume the equation to hold in the ball $B_2$ instead of $B_1$, which we can do up to scaling. The notation $A_x \in A$ in the proof of Theorem 1.1 must be replaced by $B_x \in B$, as well as the notation $L_A$ must be changed here into $L_{KB}$. Also, the continuity of $g_A$ with respect to $A$ given by (1.7) is replaced here by the continuity of $g_B$ with respect to $B$, as given by (1.16). In addition, now $u$ is a solution of equation (1.17) in $B_2$ instead of equation (1.6).

In this way, we find that, given $x \in B_2$, there exists $B_x \in B$ such that

$$0 = \sup_{B \in B} \left( L_{KB} u(x) - g_B(x) \right) = L_{KB_x} u(x) - g_{B_x}(x).$$

Next, proceeding exactly as in the proof of Theorem 1.1, we have that the maximum principle with estimate for the family of operators $L(x) = L_{KB_x}$, as considered in Definition 4.1, is satisfied in this case.

We now proceed as in the proof of Theorem 4.2 (namely, Theorem 4.2 in its generality leads to second derivative estimates, while here we focus on the same setting but restricted to first derivatives). To check the hypotheses of Theorem 4.2 for the family of operators $L^{(x)} = L_{KB_x}$, one needs to establish the validity of a suitable analogue of (4.4), but here with $\tau := 0$ and $\tau := 1$, since we deal only with first derivatives. To this end, instead of using Proposition 5.1 as in the proof of Theorem 1.1, we utilize here Theorem 1.5. But then, in doing so, the right-hand side of the analogue inequality to (4.4) (which holds now in $B_2$ by Theorem 1.5) presents an additional reminder $\varepsilon^2 \|\partial_x u\|_{L^\infty(B_3)}$. Consequently, the first derivative estimate in (5.24) also presents an additional reminder $\varepsilon \|\partial_x u\|_{L^\infty(B_3)}$.

More precisely, the estimate corresponding to (5.24) here reads

$$\sup_{B_1} \|\partial_x u\| \leq C_\varepsilon \left( \|\partial_x g_B\|_{L^\infty(B \times B_2)} + \left( \|g_B\|_{L^\infty(B \times B_2)} \right)^{1/2} + \|u\|_{L^\infty(B_2)} \right)^{1/2} + \varepsilon \|\partial_x u\|_{L^\infty(B_3)}.$$  \hspace{1cm} (6.1)
The form of the right-hand side of this inequality originates from estimating the quantities \(a_0\) and \(a_1\) in the statement of Theorem 4.2. For this, in the present framework the analogue of (5.20) allows us to bound \(a_0\) by \(\|g_B\|_{L^\infty(\mathbb{B} \times B_2)}\) and the analogue of (5.22) provides an estimate on \(a_1\) of the form \(\|\partial_\nu g_B\|_{L^\infty(\mathbb{B} \times B_2)}\), leading to the right-hand side of (6.1).

Now we assume the function \(u\) to be a solution in \(B_3\) and we scale (6.1) in a ball \(B_\rho(x)\), for each \(x \in B_2\) and \(\rho \in (0, 1)\). This can be performed by applying the previous estimate to the function \(u_\rho(y) := u(x + \rho y)\). Using that \(\rho \in (0, 1)\), one finds that

\[
\sup_{B_\rho(x)} \rho |\partial_\nu u| \leq C_\varepsilon \left( \rho^{1+2} \|\partial_\nu g_B\|_{L^\infty(\mathbb{B} \times B_4)} + \left( \rho^{2+} \|g_B\|_{L^\infty(\mathbb{B} \times B_4)} \right) \left( \left\|u\right\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)} \right) \right) + \varepsilon \rho \|\partial_\nu u\|_{L^\infty(B_{3\rho}(x))},
\]

where

\[
\sigma := \|\partial_\nu g_B\|_{L^\infty(\mathbb{B} \times B_4)} + \left( \|g_B\|_{L^\infty(\mathbb{B} \times B_4)} \right) \left( \left\|u\right\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)} \right).
\]

Finally, we combine this scaled estimates with Lemma B.1 (used with \(m := 1\)) to reabsorb the remainder in the left-hand side. In this way we obtain a gradient estimate in \(B_{1/2}\) for solutions in \(B_3\). Now, a standard covering and scaling argument yields the estimate as stated in Theorem 1.3.

\[\square\]

**APPENDIX A. A maximum principle in \(\mathbb{R}^n_{++}\)**

We give here an elementary proof of a maximum principle for the extension operator in the whole halfspace. The result may have appeared somewhere else, but we could not find a reference.

**Lemma A.1.** Let \(V\) be bounded from above in \(\mathbb{R}^n_{++}\) and satisfy \(L_\alpha V \leq 0\) weakly\(^{16}\) in \(\mathbb{R}^n_{++}\). Assume that \(V(x, 0) \leq 0\) for \(x \in \mathbb{R}^n\) (in the trace sense).

Then, \(V \leq 0\) in \(\mathbb{R}^n_{++}\).

**Proof.** Replacing \(V\) by \(V/(1 + \|V^+\|_{L^\infty(\mathbb{R}^n_{++})})\), we can suppose that \(V \leq 1\) in \(\mathbb{R}^n_{++}\). For \(R > 0\), we define \(B_R := \{(x, y) \in \mathbb{R}^n_{++} : |(x, y)| < R\}\) and \(B^+_R := B_R \cap \mathbb{R}^n_{++}\).

Let \(\overline{V} \in C^\infty(\mathbb{R}^{n+1})\) be nonnegative and such that \(\overline{V} = 0\) in \(B_2\) and \(\overline{V} = 1\) in \(\mathbb{R}^{n+1} \setminus B_3\). We define \(\overline{W} : \mathbb{R}^n_{++} \rightarrow \mathbb{R}\) to be the minimizer of the extended Dirichlet energy

\[
\int_{B^+_R} y^\sigma |\nabla U(x, y)|^2 \, dx \, dy
\]

among the functions \(U\) having finite energy and such that \(U = \overline{V}\) along \(\partial B^+_R\). We point out that this is a well posed problem in the sense of Sobolev spaces with Muckenhoupt weights (which indeed possess a suitable notion of trace); see [21] and Theorem 3.2 of [14].

Now, for every \(R > 0\), we let \(W_R(x, y) := W\left(\frac{x}{R}, \frac{y}{R}\right)\). We note that \(W_R(x, y) = \overline{V}\left(\frac{x}{R}, \frac{y}{R}\right) = 1 \geq V(x, y)\) for \((x, y) \in \partial B_{4R} \cap \mathbb{R}^n_{++}\). Furthermore, \(W_R(x, 0) = W\left(\frac{x}{R}, 0\right) = \overline{V}\left(\frac{x}{R}, 0\right) \geq 0 \geq \overline{V}(x, 0)\) for \(x \in \mathbb{R}^n\).
Then, we have that
\[ V(x, y) \leq W_R(x, y) = W \left( \frac{x}{R}, \frac{y}{R} \right) \quad \text{for } (x, y) \in \mathcal{B}_{4R}. \]
(A.1)

We also define \( W_\rho : \mathcal{B}_2 \to \mathbb{R} \) to be the odd reflection of \( W \) in the variable \( y \). Thus, we have that \( -\text{div}(|y|a\nabla W_\rho) = 0 \) weakly in \( \mathcal{B}_2 \). Hence, we can apply the results in [21] (or Theorem 3.3 of [13]) and deduce that \( W_\rho \) is H"older continuous, and thus so is \( W \) up to the boundary \( B_{2R} \times \{0\} \). From this and (A.1) the desired result follows by sending \( R \to +\infty \). \( \square \)

**Appendix B. Scaled inequalities**

We recall a classical result about scaled inequalities which we have used to prove Theorem 1.3.

**Lemma B.1.** Let \( m \geq 0 \) be an integer and \( u \in C^m(B_{2R}) \), with \( B_{3R} \subset \mathbb{R}^n \). Let \( \sigma_0 \geq 0 \), and assume that for every \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon \geq 0 \) such that
\[
\sum_{k=0}^m \rho^k \|D^k u\|_{L^\infty(B_{\rho}(x))} \leq C_\varepsilon \sigma_0 + \varepsilon \sum_{k=0}^m \rho^k \|D^k u\|_{L^\infty(B_{3\rho_0}(x))} \quad (B.1)
\]
for every \( x \in B_2 \) and every \( \rho \in (0, 1) \).

Then, there exists a constant \( C \), depending only\(^1\) on \( m \) and on the value of the constant \( C_\varepsilon \) with \( \varepsilon := (2(m + 1)4^{m+1})^{-1} \), such that
\[
\sum_{k=0}^m \|D^k u\|_{L^\infty(B_{1/2})} \leq C\sigma_0. \quad (B.2)
\]

**Proof.** Let
\[
S(x) := \sum_{k=0}^m (1 - |x|)^{k+1} |D^k u(x)| \quad \text{and} \quad M := \max_{x \in \overline{B_1}} S(x).
\]

Since \( S \) vanishes along \( \partial B_1 \), the maximum of \( S \) in \( \overline{B_1} \) is attained at an interior point: namely, there exists \( x_0 \) with \( |x_0| < 1 \) such that \( S(x_0) = M \).

We set
\[
\rho_0 := \frac{1 - |x_0|}{4} \in (0, 1).
\]

Then, we have that \( B_{3\rho_0}(x_0) \subset B_1 \), and, for all \( x \in B_{3\rho_0}(x_0) \), it holds that \( 1 - |x| \geq 1 - |x_0| - 3\rho_0 = \rho_0 \). As a consequence,
\[
\sum_{k=0}^m \rho_0^k \|D^k u\|_{L^\infty(B_{3\rho_0}(x_0))} \leq \sum_{k=0}^m \rho_0^k \sup_{x \in B_{3\rho_0}(x_0)} \frac{S(x)}{(1 - |x|)^{k+1}} \leq \rho_0^{-1} \sum_{k=0}^m \sup_{x \in B_{3\rho_0}(x_0)} S(x) \leq \rho_0^{-1} (m + 1)M.
\]

\(^1\)It is important to notice that, in the statement of Lemma B.1, the constant \( C \) does not depend only on \( m \) —but on \( m \) and on the value of the function \( C_\varepsilon \) at \( \varepsilon := (2(m + 1)4^{m+1})^{-1} \). When we apply Lemma B.1 to prove Theorem 1.3, this observation is crucial in order to obtain the correct dependencies of the structural constants in Theorem 1.3 (instead of obtaining a universal constant).
From this and (B.1), we infer that

\[ C_\varepsilon \sigma_0 + \varepsilon (m + 1) M \geq \rho_0 C_\varepsilon \sigma_0 + \rho_0 \varepsilon \sum_{k=0}^{m} \rho_0^k \| D^k u \|_{L^\infty(B_{3\rho_0}(x_0))} \geq \rho_0 \sum_{k=0}^{m} \rho_0^k \| D^k u \|_{L^\infty(B_{\rho_0}(x_0))} \]

\[ \geq \sum_{k=0}^{m} \frac{(1 - |x_0|)^{k+1}}{4^{k+1}} |D^k u(x_0)| \geq \sum_{k=0}^{m} \frac{(1 - |x_0|)^{k+1}}{4^{m+1}} |D^k u(x_0)| = \frac{S(x_0)}{4^{m+1}} = \frac{M}{4^{m+1}}. \]

Therefore, taking \( \varepsilon := \frac{1}{2(m+1)4^{m+1}} \), we conclude that \( S \leq M \leq C\sigma_0 \) in \( B_1 \) for some constant \( C \) depending only on \( m \) and on the constant \( C_\varepsilon \) with \( \varepsilon := (2(m+1)4^{m+1})^{-1} \). From this, the bound in (B.2), in half the ball, follows immediately. \( \square \)

**Appendix C. A variant of Proposition 1.4**

We state and prove here a convenient modification of Proposition 1.4 (in which \( \partial_e u \) is replaced by \( \nabla u \)) that is used in the proof of Lemma 2.2.

**Proposition C.1.** Let \( K \) satisfy (1.12) and (1.13), and let \( \mathcal{L}_K \) be defined by (1.15). Given a function \( u \in C^\infty(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n) \), \( \eta \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \), and \( E \in \mathbb{R} \), consider

\[ \varphi := \eta^2 |\nabla u|^2 + \sigma u^2. \]

Then, the inequality

\[ \mathcal{L}_K \varphi \leq 2\eta^2 \nabla u \cdot \mathcal{L}_K \nabla u + 2\sigma u \mathcal{L}_K u + E \]

holds at a point \( x \in \mathbb{R}^n \) if and only if

\[ 2 \int_{\mathbb{R}^n} \eta(x) (\eta(x) - \eta(y)) \nabla u(x) \cdot \nabla u(y) K(x-y) \, dy \leq \int_{\mathbb{R}^n} |\eta(x) \nabla u(x) - \eta(y) \nabla u(y)|^2 K(x-y) \, dy + \sigma \int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) \, dy + E. \]
Proof. We observe that

\[
\mathcal{L}_K \left( \eta^2 |\nabla u|^2 + \sigma u^2 \right) (x) - 2\eta^2(x) \nabla u(x) \cdot \mathcal{L}_K \nabla u(x)
- 2\sigma u(x) \mathcal{L}_K u(x) + \sigma \int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dy
= \int_{\mathbb{R}^n} \left( \eta^2(x) |\nabla u(x)|^2 - \eta^2(y) |\nabla u(y)|^2 \right) K(x - y) dy
+ \sigma \int_{\mathbb{R}^n} (u^2(x) - u^2(y)) K(x - y) dy
- 2\eta^2(x) \nabla u(x) \cdot \int_{\mathbb{R}^n} \left( \nabla u(x) - \nabla u(y) \right) K(x - y) dy
- 2\sigma u(x) \int_{\mathbb{R}^n} (u(x) - u(y)) K(x - y) dy + \sigma \int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dy
= \int_{\mathbb{R}^n} \left( 2\eta^2(x) \nabla u(x) \cdot \nabla u(y) - \eta^2(x) |\nabla u(x)|^2 - \eta^2(y) |\nabla u(y)|^2 \right) K(x - y) dy
= 2 \int_{\mathbb{R}^n} \eta(x) \left( \eta(x) - \eta(y) \right) \nabla u(x) \cdot \nabla u(y) K(x - y) dy
- \int_{\mathbb{R}^n} \left| \eta(x) \nabla u(x) - \eta(y) \nabla u(y) \right|^2 K(x - y) dy.
\]

From this, the proposition follows readily. \hfill \Box

APPENDIX D. EXISTENCE AND REGULARITY OF SOLUTIONS

In order to keep our arguments as simple as possible, in this paper the main computations have been performed assuming that the solution \( u \) is smooth. Note that our auxiliary functions depend on the first or second derivatives of the solution \( u \), and that we need the operator to act on them. Thus, in this article we need the solution to be at least \( C^{1+2s} \) or \( C^{2+2s} \), respectively. Here we discuss known existence and regularity results for the equations that we cover.

Concerning the existence results for concave nonlocal fully nonlinear equations (possibly including also the case of rough kernels), J. Serra proved in Theorem 1.3 of [29] that if \( B \) is a family of indexes and for all \( B \in B \) the kernel \( K_B \) satisfies the evenness and ellipticity conditions (1.12) and (1.13), and \( g_B \) and \( u_0 \) are Hölder continuous, with \( u_0 \) bounded in \( \mathbb{R}^n \setminus B_1 \), then the problem

\[
\begin{cases}
\sup_{B \in B} \left( \mathcal{L}_{K_B} u - g_B \right) = 0 & \text{in } B_1, \\
u = u_0 & \text{in } \mathbb{R}^n \setminus B_1,
\end{cases}
\]

admits a unique viscosity solution which is continuous everywhere and \( C^{2s+\beta}_{\text{loc}}(B_1) \) for some \( \beta \in (0, 1) \). This setting applies to equations (1.6) and (1.17) presented in Theorems 1.1 and 1.3 of this paper.

See also [26] for other existence and regularity results in related (but quite different) fractional settings.

A general approach to regularity is provided by the notion of elliptic operators in nonlinear integro-differential equations arising from Lévy processes, as given by Caffarelli and Silvestre.
in Definition 3.1 of \cite{17}. When our source terms \( g_A, g_B, g_1, \ldots, g_J \) are constants\footnote{The constancy of the source terms is needed to make the operator translation invariant, as requested on page 603 of \cite{17}.} this setting includes our Pucci-type equations \( (1.6) \) and \( (1.17) \), as well as the fully nonlinear framework presented here in \( (1.26) \) (when all the operators have the same order) when \( F : \mathbb{R}^J \to \mathbb{R} \) is differentiable and \( \partial_{p_j} F \in [C^{-1}, C] \) for some \( C > 1 \) and all \( j \in \{1, \ldots, J\} \). Under the assumptions that the source terms \( g_A, g_B, g_1, \ldots, g_J \) are constants, that the operators have all the same order and satisfy \( (1.12), (1.13), \) and a bound for \( \nabla K \) as in \( (1.14) \), in light of Theorem 13.1 in \cite{17} we know that the solutions in Theorems 1.1 and 1.3 are locally \( C^{1+\beta} \) for some \( \beta \in (0,1) \). This also applies to the case dealt with in Theorem 1.9 when all the nonlocal operators \( L_{\mu_j} \) reduce to the fractional Laplacian of some order \( s \) (with the same order \( s \) for all the operators). See also Theorem 27 of \cite{18} for related results with uniform estimates.

Similarly, in the case of vanishing source terms \( f, g_A, \) and \( g_B \), Theorem 1.1 of \cite{19} leads to the local \( C^{2s+\beta} \) regularity for equations \( (1.6) \) and \( (1.17) \) of definite order \( 2s \) —for the latter, assuming that all the kernels satisfy assumptions \( (1.12), (1.13), \) and \( (1.14) \). In particular, since \( 2s + \beta > 2s \), the corresponding equation is satisfied also in the pointwise sense (see e.g. the comment before Theorem 1.1 in \cite{19}, and Proposition 2.1.4 of \cite{30} for a detailed proof of this fact).

In any case, the regularity of type \( C^{1+\beta} \) or \( C^{2s+\beta} \) is not sufficient for the techniques discussed in this paper, and this is the reason for which we are taking additional regularity assumptions in our main results.

General results dealing with higher regularity for nonlocal fully nonlinear equations in bounded domains need to address two difficulties, namely the nonlinearity of the equation and the possible singularity that external data may induce due to the nonlocal structure of the problem. For linear equations satisfied in the whole of \( \mathbb{R}^n \) the regularity theory is well understood, and in this case solutions are \( C^\infty \). Indeed, in this situation one can conclude via a bootstrap argument that solutions are smooth even in case of integro-differential operators with rough kernels, since one can differentiate the equation without introducing errors that come from rough exterior data (e.g., one could proceed as in Section 4 of \cite{33} without having to introduce additional cutoff functions).

To understand the effect that being a global solution has on regularity, one can consider the toy model given by

\[
F((-\Delta)^s u(x), u(x)) = 0 \quad \text{for all } x \in \mathbb{R}^n,
\]

with \( F \) smooth and with first derivatives bounded and bounded away from zero. Assume first that \( s \in (1/2, 1) \). In this case, once \( u \) belongs to \( C^{2s+\beta}(B_1(x_0)) \) for some \( \beta \in (0,1) \) —a regularity which is known from \cite{29} — and for all \( x_0 \in \mathbb{R}^n \), given a direction \( e \), one can differentiate \( (D.1) \) and obtain

\[
\partial_{p_1} F((-\Delta)^s u(x), u(x)) (-\Delta)^s u_e(x) + \partial_{p_2} F((-\Delta)^s u(x), u(x)) u_e(x) = 0 \quad \text{for all } x \in \mathbb{R}^n.
\]

Therefore

\[
(-\Delta)^s u_e(x) = f(x) \quad \text{for all } x \in \mathbb{R}^n,
\]

with

\[
G(p_1, p_2) := \frac{\partial_{p_2} F(p_1, p_2)}{\partial_{p_1} F(p_1, p_2)}
\]

and

\[
f(x) := G((-\Delta)^s u(x), u(x)) u_e(x).
\]
Thus, one can apply the local regularity theory for the fractional Laplacian in $B_{1/2}(x_0)$ (see e.g. Proposition 2.1.11 in [30]) and obtain that
\[
\|u_e\|_{C^{1+\alpha}(B_{1/2}(x_0))} \leq C \left( \|u_e\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1(x_0))} \right) \leq C \|u_e\|_{L^\infty(\mathbb{R}^n)},
\]
for some $\alpha \in (0, 1)$, up to renaming $C$. But, since $s > 1/2$, for every $x_0 \in \mathbb{R}^n$,
\[
|u_e(x_0)| \leq \|u\|_{C^1(B_1(x_0))} \leq \|u\|_{C^{2+\beta}(B_1(x_0))}.
\]
Hence, we have a uniform bound on $\|u\|_{C^{2+\alpha}(B_{1/2}(x_0))}$, for all $x_0 \in \mathbb{R}^n$, and in particular a bound on $\|D^2u\|_{L^\infty(\mathbb{R}^n)}$. Since this is a global bound, one can iterate the procedure and obtain that $u \in C^\infty(\mathbb{R}^n)$. When $s \leq 1/2$, this method needs to be modified, by iterating a Hölder regularity result on the incremental quotients; see e.g. pages 634-635 in [17]. In both cases, we stress that this procedure only works for global solutions, since (D.2) requires $x_0$ to be an arbitrary point in $\mathbb{R}^n$.

The bootstrap method can also be applied to fully nonlinear operators when the equation is satisfied in a bounded domain with good exterior data; see Theorem 1.5 in [32] and Theorem 6 in [7] to implement the bootstrap regularity of Schauder type.

For equations satisfied on bounded domains, when the operator is built, roughly speaking, by the sum of fractional Laplacians of different orders that include the classical Laplacian, then the solutions are typically $C^\infty$, in view of the regularizing effect of the higher order operator. In this setting one can include also some nonlinear terms. A simple example consists of $u \in C^{2s+\beta}(B_1)$ being a solution of the equation
\[
-\Delta u(x) + \tilde{F}(\langle -\Delta \rangle^s u(x)) = 0 \quad \text{for all } x \in B_1,
\]
with $\tilde{F} \in C^\infty(\mathbb{R})$. We then have that the map $x \mapsto f(x) := -\tilde{F}(\langle -\Delta \rangle^s u(x))$ is locally Hölder continuous and hence we can apply the classical Schauder theory to obtain that second derivatives of $u$ are Hölder continuous in $B_{1/2}$. Then, by bootstrapping, we conclude that $u$ has as many derivatives as we wish. We remark that this setting is a particular case of that in [12.7] by choosing $F(p_1, p_2) := p_1 + \tilde{F}(p_2)$ and $s_1 := 1$.

There are other special situations in which $C^\infty$ solutions in a bounded domain can be constructed, as established in Theorem 1.1 of [33]. This paper provides cases of nonlocal fully nonlinear equations (rather concrete ones) whose Dirichlet problems possess a unique and smooth solution. More specifically, as detailed in Definitions 2.3 and 2.10 in [33], one can consider “nice weights” which make the bootstrap regularity compatible with the convex structure of the equation. This setting provides smooth solutions for concave elliptic operators acting on integral expressions of the form
\[
\int_{\mathbb{R}^n} \left( u(x + y) + u(x - y) - 2u(x) \right) \frac{y_i y_j \rho(x, y)}{|y|^{n+2s+2}} \, dy,
\]
where $\rho$ is smooth, bounded, and bounded away from zero, with derivatives of order $j$ in the variable $y$ bounded by $C_j |y|^{-j}$ for some constant $C_j$. In particular, Theorem 1.1 in [33] gives the existence of a $C^\infty$ solution of
\[
\begin{cases}
  \int_{\mathbb{R}^n} (u(x + y) + u(x - y) - 2u(x)) \frac{\rho(x, y)}{|y|^{n+2s}} \, dy = f(x) & \text{for all } x \in B_1, \\
  u = g & \text{in } \mathbb{R}^n \setminus B_1,
\end{cases}
\]
provided that $f$ is $C^\infty$, $g$ is bounded and uniformly continuous, and $\rho$ is a nice weight as above.
In general, however, the nonlocal setting is not expected to always provide $C^\infty$ solutions to general fully nonlinear equations in bounded domains. This is due to the fact that the linearized equation exhibits coefficients which depend on the global data and are in general not better than Hölder continuous (no matter how regular the solution is in the interior of the domain). As a consequence, the Schauder theory cannot be applied to bootstrap regularity. As a matter of fact, the higher regularity theory for fully nonlinear elliptic equations of nonlocal type in bounded domains is, at the moment, a field of research still under investigation. It would be desirable to understand natural assumptions guaranteeing bootstrap regularity of $C^\infty$ type.

References


XAVIER CABRÉ, 1,2
1 ICREA, Pg. Lluís Companys 23, 08010 Barcelona, Spain
2 Universitat Politècnica de Catalunya, IMTech and Departament de Matemàtiques, Diagonal 647, 08028 Barcelona, Spain
E-mail address: xavier.cabre@upc.edu

SERENA DIPIERRO
3 University of Western Australia, Department of Mathematics and Statistics, 35 Stirling Highway, Crawley, Perth, WA 6009, Australia
E-mail address: serena.dipierro@uwa.edu.au

ENRICO VALDINOCI
3 University of Western Australia, Department of Mathematics and Statistics, 35 Stirling Highway, Crawley, Perth WA 6009, Australia
E-mail address: enrico.valdinoci@uwa.edu.au