Combinatorics – FME
Final Exam – 30th May 2022. 3 hours

- You have to choose 2 problems from problems 1-3 (3.5 points each). Write each problem on a separate paper.
- You have to do Problem 4. (3 points). Problem 4 is on Page 2 of this exam.

(1) Show that, for each non–decreasing function $f : 2^{[n]} \to \{0,1\}$ (namely $X \subset Y$ implies $f(X) \leq f(Y)$), the maximal sets on which f takes the value 0 form an antichain. Deduce that the number of antichains in the Boolean lattice $\mathcal{B}_n$ is the number of such non–decreasing functions. By using the existence of a symmetric chain decomposition in $\mathcal{B}_n$ prove that the number of antichains is at most $3^{\binom{n}{\lfloor n/2 \rfloor}}$.

(2) Let $N = W(2, t^2 + 1)$, where $W(2,t)$ denotes the van der Waerden function for two colors and t–progressions. Show that, for every 2–coloring of $[1, N]$ there is a monochromatic t–progression such that its difference has the same color as the terms of the progression.

(3) Let H be a k–uniform hypergraph such that each point is in at most k edges. Show that, for sufficiently large k, there is a coloring of $V(H)$ with $r = \lfloor k/\log k \rfloor$ colors such that no edge has more than $\lceil 2e \log k \rceil$ points of the same color.

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(4) Let $\mathcal{A}$ and $\mathcal{B}$ be to families of sets over X. We say that $\mathcal{A}$ and $\mathcal{B}$ are *cross-intersecting* if for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $A \cap B \neq \emptyset$. We say that the *rank* of $\mathcal{A}$ is the maximum cardinality of a set in $\mathcal{A}$. Finally, for $x \in X$, we denote by $d_{\mathcal{A}}(x)$ the number of $A \in \mathcal{A}$ such that $x \in A$. Both definitions can be also settled for the family $\mathcal{B}$.

a) Show that if $\mathcal{A}$ has rank a, and $A \in \mathcal{A}$ then there exists $x \in A$ such that $d_{\mathcal{B}}(x) \geq \frac{|\mathcal{B}|}{a}$. State the corresponding result for $d_{\mathcal{A}}(x)$ in terms of the rank of $\mathcal{B}$ (that we denote by b).

Our objective is to show that there is a very popular point for both families. From now on $\mathcal{A}$ and $\mathcal{B}$ will be fixed cross-intersecting families of rank a and b, respectively. Let X_0 be the points in X such that $\frac{d_{\mathcal{A}}(x)}{a} < \frac{1}{2b}$, and $X_1 = X \setminus X_0$.

b) Let $\mathbf{A}$ and $\mathbf{B}$ uniformly at random sets in $\mathcal{A}$ and $\mathcal{B}$, respectively. Show that

$$\sum_{x \in X} p(x \in \mathbf{A} \cap \mathbf{B}) = \sum_{x \in X} p(x \in \mathbf{A}) p(x \in \mathbf{B}) \geq 1.$$

c) Show that there is an $x \in X$ such that $d_{\mathcal{A}}(x) \geq \frac{|\mathcal{A}|}{2b}$ and $d_{\mathcal{B}}(x) \geq \frac{|\mathcal{B}|}{2a}$ (*Hint:* assume the opposite and get a contradiction by studying separately $\sum_{x \in X_1} p(x \in \mathbf{A} \cap \mathbf{B})$ and $\sum_{x \in X_0} p(x \in \mathbf{A} \cap \mathbf{B})$).

d) By an explicit construction, show that the result in *c*) is tight. (*Hint*: take for $X = [b] \times [a]$ and define two cross-intersecting families $\mathcal{A}$ and $\mathcal{B}$ with $|\mathcal{A}| = b$, $|\mathcal{B}| = a$ and all elements in $\mathcal{A}$ have cardinality a (resp. b in $\mathcal{B}$)).

Solution Problem 4:

a) We prove only one of the two inequalities, as the argument is the same. We only need to use a pigeonhole argument: if $d_{\mathcal{B}}(x) \geq \frac{|\mathcal{B}|}{a}$ does not hold for any $x \in A$, then it means that for all $x \in A$ we have that

$$d_{\mathcal{B}}(x) < \frac{|\mathcal{B}|}{a},$$

or, in other words, $a d_{\mathcal{B}}(x) < |\mathcal{B}|$ for each $x \in A$, in particular for the one that makes $d_{\mathcal{B}}(x)$ maximum. But we know that A the cardinality of A is smaller than a, and that A intersects each $B \in \mathcal{B}$, so this expression means that there is some B that does not intersect A, which is a contradiction (one can think that $\sum_{x \in A} d_{\mathcal{B}}(x) \leq a d_{\mathcal{B}}(x_0)$, for a certain $x_0 \in A$, and conclude that this cannot be $< |\mathcal{B}|$ due to the cross-intersecting property).

b) By the cross-intersection property, the event $\exists x (x \in \mathbf{A} \cap \mathbf{B})$ has probability 1. Also, by definition, the events $x \in \mathbf{A}$ and $x \in \mathbf{B}$ are independent. Since the probability of a disjunction of events is at most the sum of the probabilities of the events, we have the statement.

d) We assume the contrary and we will show that $\sum_{x \in X_1} p(x \in \mathbf{A} \cap \mathbf{B}) < 1/2$. The same argument will work to show that $\sum_{x \in X_0} p(x \in \mathbf{A} \cap \mathbf{B}) < 1/2$, which will be a contradiction with b).

So: if we sum over elements in X_1, then x is not in X_0, and consequently, necessarily

$$p(x \in \mathbf{B}) = \frac{d_{\mathcal{B}}(x)}{|\mathcal{B}|} < \frac{1}{2a};$$

if this is not true, then we have at the same time $d_{\mathcal{A}(x)} \geq \frac{|\mathcal{A}|}{2b}$ and $d_{\mathcal{B}}(x) \geq \frac{|\mathcal{B}|}{2a}$, which we are not allowing. So:

$$\sum_{x \in X_1} p(x \in \mathbf{A} \cap \mathbf{B}) = \sum_{x \in X_1} p(x \in \mathbf{A}) p(x \in \mathbf{B}) < \frac{1}{2a} \sum_{x \in X_1} p(x \in \mathbf{A}),$$

Now, expressing the probability in terms of its definition, we have that

$$\frac{1}{2a}\sum_{x\in X_1} p(x\in \mathbf{A}) = \frac{1}{2a}\sum_{x\in X_1}\frac{d_{\mathcal{A}}(x)}{|\mathcal{A}|} = \frac{1}{2a|\mathcal{A}|}\sum_{x\in X_1} d_{\mathcal{A}}(x) = \frac{1}{2a|\mathcal{A}|}\sum_{A\in\mathcal{A}}|A| \leq \frac{a|\mathcal{A}|}{2a|\mathcal{A}|} = \frac{1}{2}.$$

The same argument works summing over X_0, getting that $\sum_{x\in X} p(x\in \mathbf{A}\cap\mathbf{B}) < 1$, which is a contradiction with b).

d) By taking $\mathcal{A} = \{A_i\}_{i=1,\dots,b}$ and $\mathcal{B} = \{B_j\}_{j=1,\dots,a}$ with $A_i = \{(i,1),\dots,(i,a)\}$ and $B_j = \{(1,j),\dots,(b,j)\}$ we get that every $x\in X$ is contained in only one element. So each $x\in X$ satisfies,

$$d_{\mathcal{A}}(x) = d_{\mathcal{B}}(x) = 1 \geq \frac{|\mathcal{A}|}{2b} = \frac{|\mathcal{B}|}{2a}.$$