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USING OPTIMAL CONTROL TO OPTIMIZE THE EXTRACTION RATE OF A DURABLE NON-RENEWABLE RESOURCE WITH A MONOPOLISTIC PRIMARY SUPPLIER

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Abstract. The problem dealt with in this paper is that of optimizing the path of the extraction rate (and, consequently, the price) for the monopolistic owner of the primary sources of a totally or partially durable non-renewable resource (such as precious metals or gemstones) in a continuous-time frame, assuming that there is an upper bound on the extraction rate and with an interest rate equal to zero. The durability of the resource implies that, unlike the case of non-durable resources, at any time there is a stock of already-used amounts of the resource that are still potentially reusable, in addition to the resource available in the ground for extraction. The problem is addressed using the Maximum Principle of Pontryagin in the framework of optimal control theory, which allows identifying the patterns that the optimal policies can adopt. In this framework, the Hamiltonian is linear in the control input, which implies a bang-bang control policy governed by a switching surface. There is an underlying geometry to the problem that determines the solutions. It is characterized by the switching surface, its time derivative, the intersection point (if any) and the bang-bang trajectories through this point.

1. **Introduction.** The problem dealt with in this paper involves finding the optimal paths of the extraction rate and the price for the sole owner of the primary sources of a totally or partially durable non-renewable (and therefore, exhaustible) resource, considering continuous time and an upper bound on the extraction rate. The main contribution of our paper is to consider an upper bound on the extraction rate for the first time within the frame of this problem and, using the Maximum Principle of Pontryagin, to identify the patterns of the optimal extraction policies under the adopted assumptions.

Some natural resources are renewable, but others are not. Among the latter, there are non-durable (e.g. fossil fuels and mineral fertilizers) and durable (gemstones, precious metals and other metals like copper) resources. Non-durable resources disappear as such after being burnt or dispersed, whilst durable resources may be reused, perhaps after recycling.

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The distinction between renewable and non-renewable resources has significant implications on the optimal policies of extracting and pricing the resource. In this regard, durable resources have the peculiarity that, at any time, in addition to the resource available in the ground for extraction, there is a stock of already used amounts of the resource that are still potentially reusable.

The problem was modelled in the frameork of Optimal Control where the Maximum Principle of Pontryagin and Dynamic Programming (Hamilton-Jacobi-Bellman equation) are the pillars. Our approach is based on the Pontryagin Principle, as in [8], because although it only provides a necessary condition to be fulfilled by the solution, it is simpler to use as it is formulated in Ordinary Differential equations, while the Hamilton-Jacobi-Bellman equation involves Partial Differential Equations.

The use of control theory for tackling the considered problem is not new, as it is shown in the following paragraphs. However, the potential of this optimization technique has not been fully explored so far. We have assumed a null rate of interest, leaving cases with a different rate of interest for future research. Our approach allows considering cases in which the optimal policy exhausts the resource available in ground, or not.

The present paper considers the case in which there is a sole owner of the primary sources of the resource. Given that if the resource is durable the supply comes from the primary sources and from a stock of the already-used and potentially reusable amounts of the resource, the scenario cannot be strictly called monopolistic. However, as with many authors, the terms monopoly and monopolistic are used in the present paper in this specific sense.

The short literature review shown in this section is, therefore, focused on the monopolistic case. The reader can find a more comprehensive and detailed review in [3], which, using a different technique, deals with the same problem considered in the present paper.

The first two relevant papers to consider the economy of exhaustive resources are [5] and [6], where the Hotelling's rule concerning the price evolution of an exhaustible resource in a competitive market is formulated.

Among the papers and books on the economics of non-renewable resources that have been published after Hotelling's, most focused mainly on the conditions under which the Hotelling rule is valid or not. The majority refer to non-durable resources, whereas those dealing with durable are scarce.

Coase, in [2], analyzes the implications of the durability for a monopolist. Bulow, [1], Suslow, [14], and Malueg and Solow, [9] and [10], study the monopolistic markets of durable, but renewable, goods and compare sales and rental policies. Malueg and Solow [11], [12], are in the same line with respect to durable exhaustible resources.

Other research concerning the economics of durable non-renewable resources focus on the validity of Hotelling's rule for this kind of resources. Stewart, in [13], assumes finite discrete time and, considering the notion of quasi-durability, which is quantified by means of a coefficient corresponding to the fraction of the extracted resource that remains from one period to the next, uses an optimization model and the Lagrange multipliers' technique to compare the strategy of a competitive extractive industry with that of a monopolistic one. It is concluded that Hotelling's rule applies to competitive and monopolistic markets, although in this latter the optimal strategy may lead to falling prices, as opposed to the former. Levhari and Pindyck, [8], instead, use a continuous time infinite horizon formulation and the

Maximum Principle, briefly discuss the case of monopolistic markets and conclude that Hotelling's rule does not hold in them.

Malueg and Solow in [11] analyze in detail the two-periods case under the assumptions of monopoly, static linear demand function, exhaustibility and perfect durability, focusing on the differences that exhaustibility induces in the monopoly equilibrium of durable resources. The same authors, in [12], analyze whether monopoly leads to over-conservation in the case of durable exhaustive resources. They use two models with static linear demand functions and an infinite horizon (a discrete-time model with perfect durability and a continuous-time model in which costs are an increasing function of cumulative production). They obtain similar results, with the general conclusions that monopoly is over-conservative and prices fall monotonically during the production period.

Corominas [3] considers the monopolistic case, with partial or total durability. Using a discrete-time mathematical programming model and Karush, Kuhn and Tucker conditions, analytic solutions and their general properties are obtained in some specific settings.

The discrete-time approach shows itself useful for finding the properties of the optimal policies in specific scenarios and for obtaining numerical solutions in any of a wide range of settings; however, it does not allow the shape of the optimal policies under general assumptions to be described. Instead, considering continuous time, one can expect that optimal control may be useful to this end. As pointed out above, Levhari and Pindyck, in [8], use an optimal control model to deal with the monopolistic case; however, their aim is not to find the optimal path of the extraction rate, but to determine whether Hotelling's rule holds in this case. Thus, the use of optimal control to tackle our problem remains largely unexplored. As Levhari and Pyndick indicate, an optimal control model of the problem may lead, unless some appropriate assumption is introduced, to an unbounded solution. They opt for considering that the cost of extracting the resource is a strictly convex function of the extraction rate. Instead, we assume that the extraction rate is bounded above and the extraction cost is negligible or does not depend on the extraction rate.

The structure of the rest of the paper is as follows. The assumptions adopted and the dynamic of the system are stated in section 2. Section 3 is devoted to the general properties of the optimal solutions, as a starting point to find the diverse patterns of the optimal paths that are deduced in section 4. Some examples are outlined in Section 5. Section 6 closes the paper with the conclusions and proposals of future research.

2. Problem definition.

We consider a non-renewable durable resource with a single primary supplier (the monopolist) possessing x_0 units of the resource. As long as the resource is available, the monopolist is able to decide its extraction rate at any time, $u(t)$, provided that $0 \leq u(t) \leq u_M$. As a result of the extractions the stock in the ground, $x(t)$, available to be put in the market by the monopolist, decreases monotonically. All the extracted resource is sold at a price which is a decreasing function of the stock, $Q(t)$, of the previously sold amounts of the resource that are potentially reusable at time t. Therefore, $P(t) = P(Q(t))$. For any given value of t the price is a decreasing function of $Q(t)$, since the marginal utility of the resource decreases when the amount in the hands of the public increases. Once a unit of the resource is extracted it vanishes at a rate ρ units/(time unit), so is, if $u(t) = 0$, $\mathrm{d}Q/\mathrm{d}t = -\rho Q$. We assume that the extraction cost of a unit of the resource is negligible.

The constrained system dynamics are defined by the following set of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -u, \quad x(t) \geq 0 \tag{1}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = u - \rho Q, \quad Q(t) \geq 0 \tag{2}$$

Given a finite or infinite time horizon, T, and an interest rate, i, constant throughout the time horizon, the monopolist's objective is to maximize the present value of the incomes obtained from the sale of the extracted units of the resource:

$$\int_0^T P(Q(\tau))\, u(\tau)\, e^{-i\tau} \mathrm{d}\tau \tag{3}$$

In the finite horizon case, the objective function could be extended to take into account the value the remaining resource at time T has. This, assuming that this value is proportional to the remaining amount of the resource at T, leads to adding a new term $c\, x(T)$ to the objective function, where c is a positive constant and $x(T)$ is given by

$$x(T) = x_0 - \int_0^T u(\tau)\mathrm{d}\tau \tag{4}$$

Our aim is to analyze the problem in increasing degrees of difficulty that result from combining $i = 0$ or $i \neq 0$, finite or infinite horizon and considering the value of the remaining resource, $x(T)$.

Hence, from now on throughout this paper, $i = 0$ is considered. Then, the objective function reduces to:

$$\int_0^T P(Q(\tau))u(\tau)\mathrm{d}\tau + c\, x(T) \tag{5}$$

Using Equation 4 in the previous equation it is clear that optimizing (5) is equivalent to optimizing

$$\int_0^T \hat{P}(Q(\tau))u(\tau)\mathrm{d}\tau = \int_0^T \left(P(Q(\tau)) - c\right)\, u(\tau)\mathrm{d}\tau \tag{6}$$

In order to simplify the developments and to reduce the number of parameters, the following change of variables is considered,

$$z = \frac{x}{u_M}, \qquad q = \frac{\rho}{u_M}\, Q \quad \text{and} \quad v = \frac{u}{u_M}, \tag{7}$$

then, $q \in [0, 1)$ and $v \in [0, 1]$.

In the new variables, the optimal control problem means maximizing the functional

$$\int_0^T \hat{P}\left(\frac{u_M}{\rho} q(\tau)\right)\, u_M\, v(\tau)\, \mathrm{d}\tau = u_M \int_0^T \hat{P}\left(\frac{u_M}{\rho} q(\tau)\right)\, v(\tau)\, \mathrm{d}\tau,$$

subject to the constraints:

$$0 \leq z(T), \tag{8}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -v, \tag{9}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \rho\,(v - q). \tag{10}$$

Note that the constraint $x(t) \geq 0$ in Equation (1) has been replaced by $z(T) \geq 0$, because $x(t)$ is a decreasing function of time. The constraint on z will result in a transversallity condition on the co-states. Furthermore, since $u_M > 0$, the functional to be maximized can be reduced to

$$\int_0^T p(q(\tau))\, v(\tau)\, \mathrm{d}\tau,$$

where $p(q(\tau)) = \hat{P}\left(\frac{u_M}{\rho}\, q(\tau)\right)$.

In addition, in the new variables only one parameter remains, ρ, which is positive.

3. **General results.**

In order to apply Pontryagin's Maximum Principle (PMP), we state the Hamiltonian:

$$\mathcal{H}(q, \lambda_1, \lambda_2, v) = p(q)\, v - \lambda_1\, v + \lambda_2\, \rho\, (v - q), \tag{11}$$

where λ_1, λ_2 are co-states associated to the state variables z and q respectively. Their dynamics are given by:

$$\frac{\mathrm{d}\,\lambda_1}{\mathrm{d}t} = -\frac{\partial\,\mathcal{H}}{\partial\, z} = 0, \tag{12}$$

$$\frac{\mathrm{d}\,\lambda_2}{\mathrm{d}t} = -\frac{\partial\,\mathcal{H}}{\partial\, q} = -\frac{\mathrm{d}p}{\mathrm{d}q} v + \rho\,\lambda_2 = -\frac{\mathrm{d}p}{\mathrm{d}t}\frac{v}{\rho(v-q)} + \rho\,\lambda_2. \tag{13}$$

and the transversallity conditions result in $\lambda_1(T) = 0$ if the constraint $z(T) \geq 0$ is not considered, which is the case if it is presumed $z(0)$ is large enough as taken in subsection 4.1. The constrained case is considered in subsection 4.2. As for λ_2, since there is no constraint on $Q(T)$, $\lambda_2(T) = 0$.

Note that the Hamiltonian is linear with respect to v. From the PMP, the optimal value of v is found among the arguments that maximize $H(v)$, assuming that all variables different to v are constant. Hence,

$$v = \begin{cases} 0 & \text{if } p(q) - \lambda_1 + \rho\lambda_2 < 0 \\ 1 & \text{if } p(q) - \lambda_1 + \rho\lambda_2 > 0 \end{cases} \tag{14}$$

We are dealing with is a Filippov's[1] system with switching surface $\sigma = 0$, where

$$\sigma(q, \lambda_1, \lambda_2) = p(q) - \lambda_1 + \rho\,\lambda_2. \tag{15}$$

Note that $\frac{\mathrm{d}\sigma}{\mathrm{d}t}$ does not depend on v. Indeed,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\mathrm{d}p}{\mathrm{d}q}\rho\,(v-q) - 0 + \rho\left(-\frac{\mathrm{d}p}{\mathrm{d}q}v + \rho\,\lambda_2\right) = \rho\left(\rho\lambda_2 - q\frac{\mathrm{d}p}{\mathrm{d}q}\right)$$

Hence, $\frac{\mathrm{d}\sigma}{\mathrm{d}t}$ has the same sign at both sides of $\sigma = 0$. As a consequence, $\sigma(t)$, *i.e.* σ evaluated on optimal trajectories, does not present oscillations of infinite frequency around $\sigma = 0$. If an optimal trajectory does eventually cross $\sigma = 0$ it does not immediately recross the switching surface. Actually, as it will be shown, it will not

[1]A Filippov system is a dynamical system with discontinuous right-hand side. A basic reference for these systems is [4].

cross it again. The expected optimal dynamics is a discontinuous dynamics because of the discontinuity in v. Then Equations (9), (10), (12) and (13) should be solved for $v = 0$ and for $v = 1$. This results in:

$$z(t) = z(0), \quad \text{if } v = 0,\ t_0 = 0; \tag{16}$$

$$z(t) = \max\{z(0) - t, 0\}, \quad \text{if } v = 1,\ t_0 = 0; \tag{17}$$

$$q(t) = e^{-\rho\, t}\, q(0), \quad \text{if } v = 0,\ t_0 = 0; \tag{18}$$

$$q(t) = 1 + e^{-\rho\, t}\,(q(0) - 1), \quad \text{if } v = 1,\ t_0 = 0; \tag{19}$$

$$\lambda_1(t) = \lambda_1(0) \quad \forall t \geq 0; \tag{20}$$

$$\lambda_2(t) = e^{\rho\, t}\, \lambda_2(0), \quad \text{if } v = 0 \tag{21}$$

$$\lambda_2(t) = \left(\frac{p(q)}{\rho\,(q(0) - 1)} + C\right) e^{\rho\, t}, \text{if } v = 1,\ t_0 = 0 \tag{22}$$

where $C = \lambda_2(0) - \frac{p(0)}{\rho\,(q(0)-1)}$.

The dynamics in the variables $(z, q, \lambda_1, \lambda_2)$ are decoupled. Indeed, the dynamics for z depends on z and v only and the revenues do not depend on z except for stopping the extraction if z reaches zero. In addition, λ_1 is a constant. As for q and λ_2, their dynamics only depend on q, λ_2 and σ which, in turn, depends on q, λ_2 and λ_1, but λ_1 is a constant. Hence, the optimal control problem may be reduced to the decoupled system given by variables q and λ_2. The dynamics is well defined if σ is strictly positive or negative while on $\sigma = 0$ the input may take any value in the set $[0, 1]$. The curve defined by $\frac{\mathrm{d}\sigma}{\mathrm{d}t} = 0$ is also relevant because it divides the plane (q, λ_2) into regions where σ evaluated on solution trajectories is an increasing or a decreasing function of time.

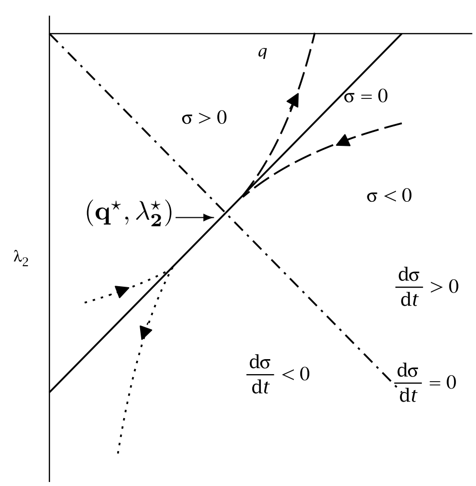

FIGURE 1. The plane (q, λ_2)

Curves $\sigma = 0$, $\frac{\mathrm{d}\sigma}{\mathrm{d}t} = 0$ and two optimal trajectories are outlined in Figure 1. The picture is particularized for a case where $p(q) = a_0 - a_1 q$, $a_0 > 0$, $a_1 > 0$, and $\lambda_1 = 0$ in order to depict an example and to emphasize the important objects. The picture represents the plane (q, λ_2) for $q \geq 0$ and $\lambda_2 \leq 0$ as well as the curve $\sigma = 0$, which in this example is given by the line $\rho\,\lambda_2 = -p(q)$, the curve $\frac{\mathrm{d}\sigma}{\mathrm{d}t} = 0$, which in this example is given by the line $\rho\,\lambda_2 = q\,\frac{\mathrm{d}p}{\mathrm{d}q}$, the regions where $\sigma > 0$ (respectively $\sigma < 0$) as well as where $\frac{\mathrm{d}\sigma}{\mathrm{d}t} > 0$ (respectively, $\frac{\mathrm{d}\sigma}{\mathrm{d}t} < 0$). The picture includes a trajectory in $\frac{\mathrm{d}\sigma}{\mathrm{d}t} < 0$ (dotted line). It starts in $\sigma > 0$ at $(0, \lambda_{20})$, evolves with $v = 1$ until it reaches $\sigma = 0$ then, v switches to $v = 0$ and the trajectory continues in $\{\sigma < 0\} \cap \{\frac{\mathrm{d}\sigma}{\mathrm{d}t} < 0\}$ for all t. For $t \to \infty$, the trajectory evolves to $q = 0$, $\lambda_2 = -\infty$. A trajectory in $\frac{\mathrm{d}\sigma}{\mathrm{d}t} > 0$ (dashed line) is also drawn. It starts in $\sigma < 0$ ($v = 0$) at $(1, \lambda_{21})$ and evolves with ($v = 0$) until reaching $\sigma = 0$, then v switches to $v = 1$ and the trajectory continues in $\{\sigma > 0\} \cap \{\frac{\mathrm{d}\sigma}{\mathrm{d}t} > 0\}$ for all $t > 0$. For $t \to \infty$, the trajectory evolves to $q = 1$ and $\lambda_2 = +\infty$.

It should be noted that

- The domains $\{\sigma < 0\} \cap \{\frac{\mathrm{d}\sigma}{\mathrm{d}t} < 0\}$ and $\{\sigma > 0\} \cap \{\frac{\mathrm{d}\sigma}{\mathrm{d}t} > 0\}$ are invariant for the dynamical system.
- Let (q, λ_2) be a point in $\{\sigma = 0\} \cap \{\frac{\mathrm{d}\sigma}{\mathrm{d}t} < 0\}$, then σ evaluated on the solution trajectory through this point is a decreasing function while, if (q, λ_2) is on $\{\sigma = 0\} \cap \{\frac{\mathrm{d}\sigma}{\mathrm{d}t} > 0\}$, σ evaluated on the solution trajectory through this point is an increasing function.
- If (q, λ_2) belongs to $\{\sigma = 0\} \cap \{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = 0\}$, then v may take any value in the interval $[0, 1]$ and for $v = v^\star = q$, the $q-$component of the intersection point, (q, λ_2) is an equilibrium point of the system denoted, from now on, by $(q^\star, \lambda_2^\star)$. Its coordinates fulfill

$$\rho\,\lambda_2^\star = -p(q) = q^\star\ \left.\frac{\mathrm{d}p}{\mathrm{d}q}\right|_{(q^\star,\lambda_2^\star)}. \tag{23}$$

  And, consequently, for $v = q^\star$ we have

$$\left.\frac{\mathrm{d}q}{\mathrm{d}t}\right|_{(q^\star,\lambda_2^\star)} = 0 \qquad \text{and} \qquad \left.\frac{\mathrm{d}\lambda_2}{\mathrm{d}t}\right|_{(q^\star,\lambda_2^\star)} = 0$$

  $(q^\star, \lambda_2^\star)$ is an equilibrium point while $v = v^\star$. If v changes to $v = 1$ at a given time instant, then the trajectory would leave the equilibrium point.
- Trajectories for $v = 0$ and $v = 1$ at $(q^\star, \lambda_2^\star)$ are tangent to $\sigma = 0$.
- If $q^\star > 1$, the domain $\{\sigma < 0\} \cap \{0 \le q \le 1\}$ is also invariant for the system dynamics. Hence if a trajectory reaches $\sigma = 0$ or starts in $\sigma \le 0$, it goes to $q = 0$ and $\lambda_2 = -\infty$ when $t \to \infty$. The same occurs for $\sigma > 0$. In this case, trajectories go to $q = 1$ and $\lambda_2 = +\infty$ when $t \to \infty$

Note that these remarks are valid for a general $p(q)$ such that $\frac{\mathrm{d}\sigma}{\mathrm{d}t} = 0$ intersects $\sigma = 0$ at a unique point $(q^\star, \lambda_2^\star)$ with $q^\star \ge 0$. In general $\sigma = 0$ and $\frac{\mathrm{d}\sigma}{\mathrm{d}t} = 0$ may intersect or not in the domain $0 \le q \le 1$ and the intersection may contain one point or more. However, in order to make the analysis easier, from now on it will be assumed that σ and $\frac{\mathrm{d}\sigma}{\mathrm{d}t} = 0$ intersect at a unique point, which may lie in the domain $0 \le q \le 1$ or not. Since λ_1 is constant, λ_1 different from zero would lead to a similar diagram except that the line $\sigma = 0$ would be relocated to a parallel line.

The optimal extraction rate will follow a bang-bang pattern that results from the Pontryagin Maximum Principle because the function to be optimized (Equation 11) is linear in the control input v. Hence, maximum values for v are at the beginning or at the end of the interval where the extraction rate belongs: $[0, 1]$. However, to be precise, the optimal extraction rate will not be a pure bang-bang control. As explained in the next section, if the equilibrium point $(q^\star, \lambda_2^\star)$ is in the domain

$0 \leq q \leq 1$, then the normalized optimal extraction rate may be equal to $q^\star$ for a certain time to optimize the cost function.

4. **Optimal control design.** General results stated in the previous section are used here to obtain the optimal control and optimal trajectories. Three trajectories will be relevant in our discussion. Namely, the trajectory with $v = 0$ that reaches $(q^\star, \lambda_2^\star)$, the trajectory with $v = 1$ that reaches $(q^\star, \lambda_2^\star)$ and the trajectory with $v = 1$ and initial conditions $(q^\star, \lambda_2^\star)$.

4.1. **Finite time horizon, T, and $z(0)$ large enough.** In a first optimal control design, suppose that $z(0)$ is large enough so that, with an optimal policy, the product is not exhausted during the interval $[0, T]$. As for the final conditions, $q(T)$ is free as well as $z(T)$ because the hypothesis on $z(0)$. Thus, the transversality conditions of the Pontryagin Maximum Principle, see for instance [7], yield to $\lambda_1(T) = 0$ and $\lambda_2(T) = 0$.

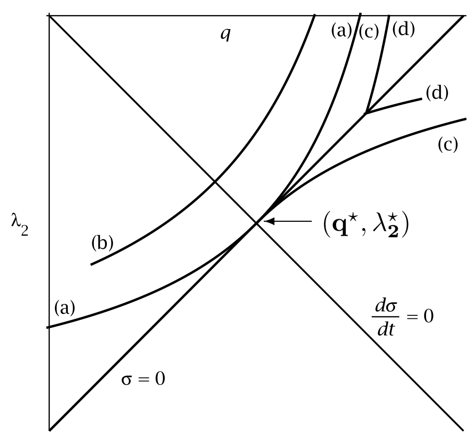

FIGURE 2. Control strategies

The dynamics of the optimum control system are characterized by two curves:

- $\Gamma_1(t)$, the trajectory solution of the control system with $v = 1$, which is tangent to $\sigma = 0$ at $(q^\star, \lambda_2^\star)$. It corresponds to trajectory (a) in Figure 2.
- $\Gamma_0(t)$, the trajectory solution of the control system with $v = 0$, which is tangent to $\sigma = 0$ at $(q^\star, \lambda_2^\star)$. As for this second curve only the arc that ends at the tangent point is relevant because the region defined by $\{\sigma < 0\} \cap \{\frac{d\sigma}{dt} < 0\}$ does not intersect the line $\lambda_2 = 0$. It corresponds to trajectory (c) in Figure 2.

Given initial conditions for z and q, $z(0)$ and $q(0)$, we consider $\lambda_2(0)$ so that the solution of Equations (10), (13) with v defined by Equation(14) fulfills $\lambda_2(T) = 0$. It follows from the geometry just discussed that $v(T) = 1$. In order to find optimal trajectories let us distinguish two cases depending on whether $q^\star > 1$ or $0 < q^\star \leq 1$.

Case $q^\star > 1$. In this case $v(t) = 1$ $\forall t \in [0, T]$ and the trajectory never meets $\sigma = 0$ because if an optimal solution intersected $\sigma = 0$ or started in $\sigma < 0$, it would never reach $\lambda_2 = 0$. Actually, it would go to $(0, -\infty)$ as t goes to ∞, being

$\lambda_2(t)$ a decreasing function. The initial condition for λ_2 may be obtained from Equations (19) and (22).

It should be noted that $p(q(t)) \geq 0\ \forall t \in [0,T]$. Indeed, since $\{\sigma = 0\} \cap \{\frac{\mathrm{d}\sigma}{\mathrm{d}t}\} = \{(q^\star, \lambda_2^\star)\}$ and $q^\star > 1$, Equation (23) is not fulfilled for $q \in [0,1]$. Instead,

$$q\frac{\mathrm{d}p}{\mathrm{d}q} > -p(q), \qquad \forall q \in [0,1].$$

If $\exists q_1 \in [0,1]$ such that $p(q_1) = 0$, then $\left.\frac{\mathrm{d}p}{\mathrm{d}q}\right|_{q=q^\star}$ would be strictly positive which is not possible because $\frac{\mathrm{d}p}{\mathrm{d}q}$ is negative or zero.

Case $0 \leq q^\star \leq 1$. In this case there are several trajectories compatible with $\lambda_2(T) = 0$. Namely,

- $v = 1$ during the whole trajectory.
- $v = 1$ during the whole trajectory, except for a time interval where the trajectory remains at $(q^\star, \lambda_2^\star)$.
- The trajectory starts with $v = 0$ in the domain $\{\sigma < 0\} \cap \{\frac{\mathrm{d}\sigma}{\mathrm{d}t} > 0\}$, it crosses $\sigma = 0$ and continues through $\{\sigma > 0\} \cap \{\frac{\mathrm{d}\sigma}{\mathrm{d}t} > 0\}$ until it reaches $\lambda_2 = 0$ at time $t = T$.
- The trajectory starts with $v = 0$ in the domain $\{\sigma < 0\} \cap \{\frac{\mathrm{d}\sigma}{\mathrm{d}t} > 0\}$, intersects $\sigma = 0$ at the equilibrium point $(q^\star, \lambda_2^\star)$ where it remains for a time interval and continues through the domain $\{\sigma > 0\} \cap \{\frac{\mathrm{d}\sigma}{\mathrm{d}t} > 0\}$ until it reaches $\lambda_2 = 0$ at $t = T$.

Equations from (18) to (22) allow calculating $\lambda_2(0)$ from q_0.

In optimal trajectories $p \geq 0$ while $v = 1$. Indeed, let $(q(t), \lambda_2(t))$ any point of a trajectory with $v = 1$; this means $\sigma\,(q(t), \lambda_2(t)) \geq 0$. Additionally, $\sigma = 0$ intersects $\lambda_2 = 0$ at a point $(q_2, 0)$ where $p(q_2) = 0$ for $q_2 > q(t)$. Thus, since $p(q)$ is a decreasing function of q, $p(q(t)) \geq p(q_2) = 0$.

Optimal control policy Let $q(0)$ be an initial condition for q and, as discussed earlier, let us assume that there is enough of the resource to extract so that $z(t) \geq 0$ for all $t \in [0,T]$. Then, the optimal control policy is defined by the next algorithm:

- If $0 \leq q(0) \leq q^\star$, let t_1 be the time it takes to go on the curve Γ_1 from $q = q(0)$ to $\lambda_2 = 0$. Then:
  - If $t_1 \geq T$, let us take $v = 1$ until reaching $(q^\star, \lambda_2^\star)$, then $v = q^\star$ for $T - t_1$ units of time and then $v = 1$ again until reaching $\lambda_2 = 0$ at $t = T$. The initial condition for λ_2 is given by the λ_2-component of the intersection point $\{q = q(0)\} \cap \Gamma_1$ (trajectory (a) in Figure 2).
  - Otherwise take $v = 1$ for all $t \in [0,T]$. Equation (22) allows computing the appropriate value for $\lambda_2(0)$ (trajectory (b) in Figure 2) .
- If $q^\star \leq q(0) \leq 1$, let t_0 be the time it takes to go on the curve Γ_0 from $q = q(0)$ to $(q^\star, \lambda_2^\star)$ and t_1, the time it takes to go on the curve Γ_1 from $(q^\star, \lambda_2^\star)$ to $\lambda_2 = 0$. Then:
  - If $t_0 + t_1 \geq T$, let us take $v = 0$ until reaching $(q^\star, \lambda_2^\star)$, then $v = q^\star$ for $T - (t_0 + t_1)$ units of time and then $v = 1$ until reaching $\lambda_2 = 0$ at $t = T$. The initial condition for λ_2 is given by the λ_2-component of the intersection point $\{q = q(0)\} \cap \Gamma_0$ (trajectory (c) in Figure 2).
  - Otherwise, let $(q(0), \lambda_{21})$ the intersection point between $q = q(0)$ and $\sigma = 0$ and t_2 the time the trajectory with $v = 1$ takes from $(q(0), \lambda_{21})$ to $\lambda_2 = 0$. Then, if $t_2 \geq T$, take $v = 1$ for all $t \in [0,T]$; the initial condition

for λ_2 may be calculated from Equation (22). If $t_2 < T$, then there exists $t_3 > 0$ and an initial condition for λ_2, $\lambda_2(0)$, so that the trajectory starting at $(q(0), \lambda_2(0))$ with $v = 0$ for $t \in [0, t_3)$ reaches $\sigma = 0$ at $t = t_3$ and, switching the control input to $v = 1$ at this point, it reaches $\lambda_2 = 0$ at time $t = T$. t_3 and the initial condition for λ_2 may be calculated from Equations from (18) to (22) (trajectory (d) in Figure 2).

4.2. **Finite time horizon, T, and $z(0) = z_M$.** Let us consider, in this subsection, the case where only a limited quantity of product can be put on the market. Let z_M be the initial condition for variable z, roughly speaking, the maximum amount of product to be extracted. Our purpose is to determine the optimal control policy fulfilling the constraint $z(t) \geq 0$ for all $t \in [0, T]$. So, let $v_\infty(q(0), T)$ denote the optimum solution obtained when assuming that an unlimited amount of product that can be put on the market and $v_{z_M}(q(0), T)$ the optimum solution in the case $z \in [0, z_M]$. An algorithm for an optimal extraction policy is described below.

If $\int_0^T v_\infty(q_0, T, \tau)\mathrm{d}\tau \leq z_M$, then $v_{z_M}(q_0, T) = v_\infty(q_0, T)$

For the algorithm in the case where $\int_0^T v_\infty(q_0, T, \tau)\mathrm{d}\tau > z_M$, some remarks are in order:

- With the optimal extraction policy, the whole amount of the resource ought to be extracted. This means $z(T) = 0$.
- The problem we are dealing with, in the Pontryagin Principle framework, shows a geometry identical to the one described in previous subsections. In this case, $z(T)$ is not free (actually, $z(T) = 0$). Therefore, λ_1 may not necessarily be 0 but it is still constant. Then, the unique modification in the geometry of this dynamical system with regard to the case where $z(0)$ is large enough is that the curve $\sigma = 0$ has been shifted at a distance λ_1. Consequently, the type of optimal trajectories obtained are the same as in the case studied in the previous section.
- $T > z_M$. Indeed,

$$z_M < \int_0^T v_\infty(q_0, \tau)\mathrm{d}\tau \leq \int_0^T \mathrm{d}\tau = T$$

- Since $z_M < T$, the optimum extraction policy requires extracting over a certain period of time with extraction rate strictly less than 1. This in only possible increasing the time for $v = q^\star$ or for $v = 0$.
- From the geometry of the solutions and the type of optimal solutions, $v_{z_M}(q_0, T, t)$ takes one of these forms:

A

$$v_{z_M}(q_0, T, t) = \begin{cases} 1 & \text{if } t \in [0, t_1) \\ q^\star & \text{if } t \in [t_1, t_2) \\ 1 & \text{if } t \in [t_2, T] \end{cases}$$

B1

$$v_{z_M}(q_0, T, t) = \begin{cases} 0 & \text{if } t \in [0, t_1) \\ q^\star & \text{if } t \in [t_1, t_2) \\ 1 & \text{if } t \in [t_2, T] \end{cases}$$

B2

$$v_{z_M}(q_0, T, t) = \begin{cases} 0 & \text{if } t \in [0, t_1) \\ 1 & \text{if } t \in [t_1, T] \end{cases}$$

B2 appears when spending an extra time with $v = 0$ is enough to get $z(T) = 0$.

- Equations that must hold in case **A**:

$$z_M = (t_2 - t_1)q^\star + T - (t_2 - t_1) \tag{24}$$
$$(q^\star, \lambda_2^\star) = \Gamma_1(q(0), \lambda_2(0), t_1) \tag{25}$$
$$(q_2, 0) = \Gamma_1(q^\star, \lambda_2^\star, T - t_2) \tag{26}$$

Equation (24) states that at time T the resource that can be extracted is depleted, while Equations (25) and (26) describe the solution in variables q and λ_2 in the two intervals where $v = 1$. In the interval $[t_1, t_2)$, for $t_2 > t_1$, the trajectory remains at the equilibrium point $(q^\star, \lambda_2^\star)$ at a constant price.

- Equations that must hold in case **B1**:

$$z_M = (t_2 - t_1)q^\star + T - t_2 \tag{27}$$
$$(q^\star, \lambda_2^\star) = \Gamma_0(q(0), \lambda_2(0), t_1) \tag{28}$$
$$(q_2, 0) = \Gamma_1(q^\star, \lambda_2^\star, T - t_2) \tag{29}$$

Equation (27) states that at time T the extractable resource is depleted. Equations (28) and (29) describe the solution in the variables q and λ_2 in the intervals where $v = 0$ and $v = 1$, respectively. If $t_2 > t_1$, in the interval $[t_1, t_2)$, the trajectory remains on the equilibrium point $(q^\star, \lambda_2^\star)$ at a constant price.

- Equations that must hold in case **B2**:

$$z_M = T - t_1 \tag{30}$$
$$(q(t_1), \lambda_2(t_1)) = \Gamma_0(q(0), \lambda_2(0), t_1) \tag{31}$$
$$(q_2, 0) = \Gamma_1(q(t_1), \lambda_2(t_1, T - t_2) \tag{32}$$

Equation (30) states that at time T the extractable resource is depleted. Equations (31) and (32) describe the solution in the variables q and λ_2 in the intervals where $v = 0$ and $v = 1$, respectively.

- For all the cases (**A**, **B1** and **B2**) the unknowns t_1, t_2 and $\lambda_2(0)$ may be obtained by using numerical methods to solve the equations. The respective dynamics are obtained from Equations (9), (10), (12) and (13) using a bang-bang control except while trajectories must remain in the equilibrium point.
- In **A**, **B1** cases, $\lambda_1 = p(q^\star) + \lambda_2^\star$, while in case **B2**, $\lambda_1 = p(q(t_1)) + \lambda_2(t_1)$. Indeed, from the geometry of the dynamical system $(q^\star, \lambda_2^\star)$ is on the curve $\sigma = 0$. Hence the value of λ_1. Mutatis mutandis for case **B2**.
- A priori, optimal control may take any of the three forms when $v_\infty(q_0, T, 0) = 1$ (the z_∞ optimal control starts with $v = 1$), but only forms **B1** and **B2** are appropriate for $v_\infty(q_0, T, 0) = 0$. In the first case, the relative position between q_0 and of $q^\star$ due to the relocation of $\sigma = 0$ because of $\lambda_1 \neq 0$ determines the appropriate optimal control. However, since in order to do this, the value of λ_1 is needed, so, it may be better to calculate for both cases and take optimum.

5. **Examples.** This section uses some examples to describe several optimal trajectories for u and Q. Note that optimal trajectories in reduced variables have already been described in Figure 2. The simulations presented in this section had been made in the reduced model using variables $(z, q, \lambda_1, \lambda_2)$, while in the figures the original variables u and Q have been depicted. A linear function has been used for $P(Q)$. Actually, $P(Q) = A_0 - A_1\, Q$. Remember that $(q^\star, \lambda_2^\star)$ denotes the equilibrium point, the intersection between $\sigma = 0$ and $\frac{\mathrm{d}\sigma}{\mathrm{d}t} = 0$, which is presumed to be a unique point.

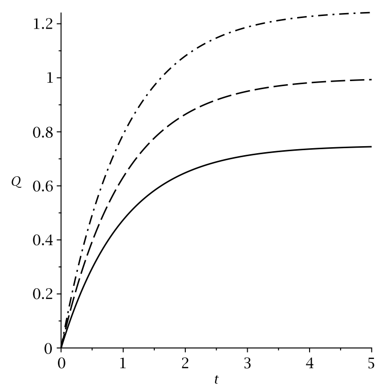

FIGURE 3. Optimal stock trajectories

5.1. $q^\star > 1$. The equilibrium point is not in the range of $q \in [0,1]$. As described previously, the optimal control input is $u = u_M$ for all $t \in [0,T]$ where T stands for the time-horizon of the optimal control problem. The next parameters are common to the simulations: $A_0 = 3$, $A_1 = 1$, $\rho = 1$ and $q_0 = 0$ is a common initial condition. Optimal stock trajectories are depicted in Figure 3. These trajectories were obtained in normalized variables from equations (7), (9), (10) and (8). They correspond to $u_M = \{0.75, 1, 1.25\}$ and the optimal control input $u = u_M$ $\forall t \in [0,T]$. Hence, the normalized optimal input results in $v = 1$, regardless of the values u_M takes. These trajectories are thus obtained from a unique trajectory corresponding to $u_M = 1$ multiplying by 0.75 and 1.25 respectively. As can be seen, trajectories come from multiplying the one corresponding to $u_M = 1$ (dashed line) by $u_M = 0.75$ (continuous line), 1 and 1.25 (dash-dotted line).

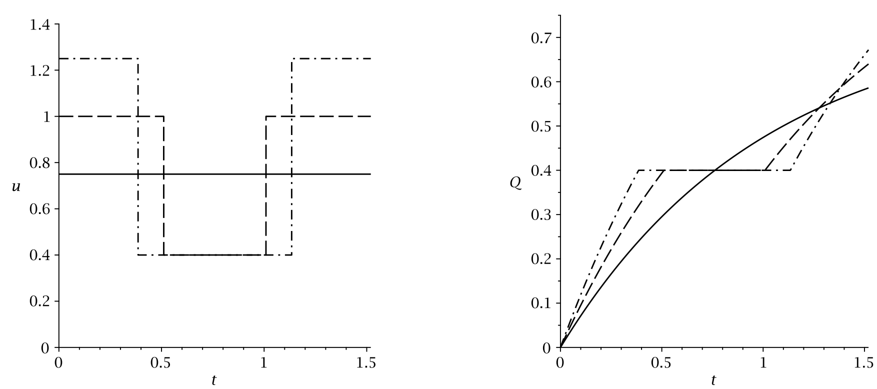

FIGURE 4. Optimal extraction rate (left) and stock trajectories (right)

5.2. $0 < q^\star < 1$. In this case the equilibrium point is in $[0,1]$. Although $q^\star$ depends on the value of u_M, because changes in u_M result in changes in σ, $Q^\star$ is independent from u_M. The next parameters are common to the simulations: $A_0 = 0.8$, $A_1 = 1$, $\rho = 1$ while $q_0 = 0$ is a common initial condition. Three values have been taken

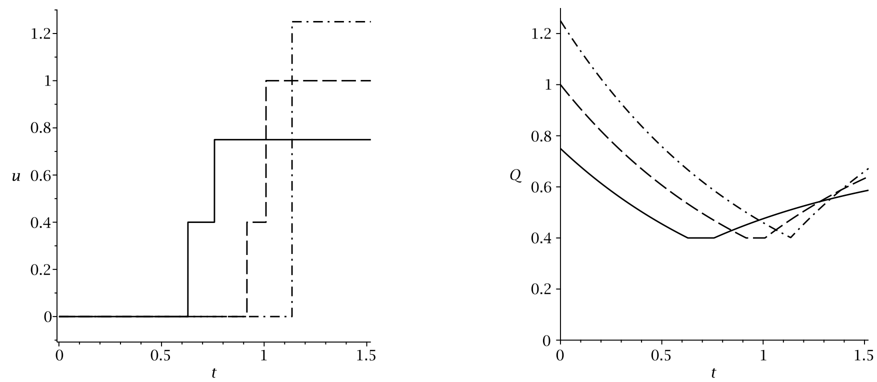

FIGURE 5. Optimal extraction rate (left) and stock trajectories (right)

for u_M. Namely, $u_M = 0.75$ (continuous-trace), $u_M = 1$ (dashed) and $u_M = 1.25$ (dash-dotted).

Let t_1 be the time the q trajectory for $v = 1$ takes to go from $q_0 = 0$ to $q = q^\star$ and t_2 be the time λ_2 takes from $\lambda_2(0) = \lambda_2^\star$ to $\lambda_2 = 0$. Thus, depending on whether $t_1 + t_2 \leq T$, $u = u_M$ for all t in $[0, T]$ ($u_M = 0.75$) or u follows a sequence $u = u_M$, $u = Q^\star$ and $u = u_M$ again ($u_M = 1$ and $u_M = 1.25$).

Optimal control inputs and stock trajectories are depicted in Figure 4. When $u_M = 1$ and $u_M = 1.25$, $t_1 + t_2 < T$. Hence optimal extraction rate remains for $T-(t_1+t_2)$ in $u = q^\star$, the equilibrium value for the input control. When $u_M = 0.75$, $t_1 + t_2 \geq T$, then $u(t) = u_M$ for all t in $[0, T]$.

Figure 5 corresponds to the case of $Q(0) = u_M/\rho$. In this case the specific times to be considered are t_1, the time the trajectory for $v = 0$ takes from $q(0) = 0$ to $q^\star$ and t_2, as defined before, the time the trajectory for $v = 1$ takes from $\lambda_2(0) = \lambda_2^\star$ to $\lambda_2 = 0$. Depending on whether $t_1 + t_2 \leq T$ or not, the optimal control input is given by the sequence $u = 0$, $u = u_M$ ($u_M = 1.25$) or by $u = 0$, $u = Q^\star$ and $u = u_M$ ($u_M = 1$ and $u_M = 0.75$). It should be noted that, according to what was previously explained in the text, the equilibrium value for the input control is the same (0.4) regardless of the value of u_M.

6. **Conclusions and prospects of future research.** The aim of this paper is to determine the extraction policy that maximizes the revenue of the single primary supplier of a totally or partially durable non-renewable resource. The problem is solved within the framework of optimal control theory, assuming that the resource's sale price is a decreasing function of the stock of previously sold amounts of resource that are still reusable, the extraction cost is negligible and there is an upper bound on the extraction rate.

Optimal policy obviously depends on the initial stock, the amount of the resource possessed by the monopolist, the degree of durability of the resource and the planning time horizon. However, the shapes of the optimal policies follow a small number of patterns consisting of a maximum of three phases, in a variable order, in which, respectively, the extraction rate is null, equal to its upper bound or equal to an equilibrium rate that is strictly lower than the upper bound, so that in this phase of extraction the price remains constant. For example, with initial stock equal to 0, the optimal policy can have two phases (initial and final) with the maximum extraction rate, separated by one extraction phase with the equilibrium rate. Depending on the initial conditions and the specific parameters of each particular case, the optimal policy may or may not exhaust the monopolist's resource.

This work leaves two questions open. Namely, the optimization of the present income value with non-zero interest rates, and the treatment of cases with multiple equilibrium points. These are prospects for future research.

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