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A stochastic congested strategy-based transit assignment model with hard capacity constraints

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Abstract

This paper expands on the stochastic version of the strategy-based congested transit assignment problem described in Codina and Rosell (2017b) [A Stochastic Version of the Strategy-Based Congested Transit Assignment Model and a Technique by Smoothing Approximations, Chapter 13 in Advanced Concepts, Methodologies and Technologies for Transportation and Logistics. Springer Series in Advances in Intelligent Systems and Computing 572, J. Zák, Y. Hadas, R. Rossi. (eds.)], specifically by extending it to cases dealing with explicit line capacities. Following this previous work, the stochastic model with explicit line capacities takes into account the stochastic mean waiting times and in-vehicle travel times of passengers at stops during the standard scenario journey of the planning period. This slightly modified model is formulated also as a stochastic variational inequality derived from the deterministic version of the problem with similar properties and characteristics, formulated by the authors in a previous work. For this stochastic capacitated model, two algorithms are presented which may take advantage of simplified simulation models for the bus stops in order to provide realistic mean waiting times for passengers at stops. Both algorithms can be easily adapted to the case of the model with non-strict capacities by Codina and Rosell (2017).

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1. Introduction

Urban public transport is a key component in metropolitan areas and large cities, and it is expected that the demand will continue to increase in the near future. It is for this reason that the planning, modeling and design of transport must take into account congested scenarios. Thus, in recent years, researchers and practitioners in the transportation planning community have developed congested transit assignment models that can take into account the impact of congestion effects on public transportation lines. Not only are these congestion effects that appear in the so-called common lines problem difficult to model, but so are the route choices made by passengers.

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The initial models considered these two factors, but applied them only to uncongested or moderately congested systems. The early paper by Chriqui and Robillard (1975) introduced the concept of attractive lines. In this paper, passenger behaviour is characterized by boarding the first vehicle arriving at a stop of any line that they consider attractive. An algorithm is developed in their paper to calculate the set of attractive lines. As extensions of this seminal paper, there followed the work of Spiess (1984) introducing the notion of strategy on a general multidestination transit network and the work of Nguyen and Pallotino (1988), formalizing the concept of hyperpath, in which passengers were assumed to board the first vehicle arriving at a stop of any line within the attractive set. The works by Cominetti and Correa (2001) and Cepeda et al. (2006) take into account the effects of line capacity limitations, which can be ideally modeled by means of bulk service queues in which servers may serve a maximum number of clients and, consequently, a passenger (client) arriving at a stop may not board the first arriving unit. The approach in these works was referred to as the C3F model (C3F ≡ Cepeda, Correa Cominetti, Florian) in Codina (2013). In this paper, the C3F model was formulated as a variational inequality problem (V.I. problem in the following), and in Codina (2017) a heuristic algorithm is developed in order to solve the strict capacitated C3F model under very highly congested situations. These aspects have been taken into account also in other works, and the reader is directed to the reviews of Fu et al. (2012) and Liu et al.(2010).

Congestion effects on transit assignment models have led to planners considering the aspects of queueing at stops and different headway distributions. The work by Gentile et al. (2016) considers queueing aspects for dealing with delays caused by congestion on transit lines. In the stochastic transit assignment models, routing aspects have been considered with special emphasis, as in Liu et al. (2010) as well as variability in travel times due to daily fluctuations.

Very recently, a stochastic version of the C3F model has been developed by Codina and Rosell (2017b). The model has been formulated as a stochastic variational inequality, which takes into account randomness in travel times and mean waiting time at stops. This model will be referred to as the StochC3F model. The only assumption adopted by the StochC3F model is that mean waiting times at stops adhere to a homoscedastic stochastic model. The model is not based on any route perception cost assumption, such as the random utility maximization found in the works of Lam et al. (1999) or in reliability-based models (Jiang and Szeto (2016)). It also differs from the stochastic model of Cortés et al. (2013), based as well on the deterministic C3F model. Cortés model is based on the stochasticity of passengers boarding decisons. The StochC3F model does not contain explicit limits on passenger flows, which are limited instead by travel time functions and mean waiting time functions at stops.

This paper considers the case of explicit capacity bounds on transit lines and incorporates it into the StochC3F model while also extending the results that were obtained from the deterministic C3F model by very similar means. In addition, two algorithms are proposed for solving the capacitated StochC3F model. The first results from formulating these models as an equivalent deterministic variational inequality developed in Codina and Rosell (2017b), whereas the second one results from the direct statement of the model as a stochastic variational inequality. Both algorithms make use of the Mann-type iterative schemes (Mann (1953)) for solving fixed-point inclusion problems. A relevant feature of these algorithms is that they may incorporate samples coming from simulation queueing models that generate realistic values for travel times and passenger queueing times at stops.

The paper is structured as follows: Section 2 introduces the basic notation and elements intervening in the capacitated StochC3F model. The basic aspects of this model and their properties, developed in Codina and Rosell (2017), are presented in Section 3. In Section 4, the question of strict capacity bounds is presented and introduced for the StochC3F model, parallelling the developments in Codina (2013) for the deterministic C3F model. In Section 5, two solution algorithms are described, whose computational results appear in section 6 and, finally, Section 7 presents a set of concluding remarks.

2. Definitions and notation

The transit network will considered in its expaded form and it will be represented by means of a directed graph $G = (N, A)$, with N being the set of nodes and A the set of links. The sets of emerging and incoming links for a given node $i \in N$ will be denoted by $E(i)$ and $I(i)$. A link will usually be denoted by the index a, or when considered necessary, by (i, j), both in N. The number of trips from i to d will be denoted by g_i^d and constant (inelastic demand). The set of origin-destination (O-D) pairs $\omega = (i, d)$ on the network will be denoted by $W = \{ (i, d) \in N \times N \mid g_i^d > 0 \}$.

D will denote the set of destinations in the network . Also, $N_d = N \setminus \{d\}$ will denote the set of nodes excluding destination $d \in D$. $\hat{N}$ is the subset of nodes representing transit stops, and $\hat{N}_d = \hat{N} \setminus \{d\}$ is derived from it.

The configuration of the expanded transit network is depicted schematically in Figure 1. Transit stops are associated with a node and some of their outgoing links will be boarding links to a transit line. On the other hand, each transit line with vehicles halting at the stop will have a single boarding link a from the stop, a single alighting link and a single dwell link $x(a)$ at the stop. Non-boarding or non-alighting links at transit stops will model movements by walking or connections to other transit stops. The following notation will be used:

- v_a^d : passenger flow at link $a \in A$ with destination $d \in D$.
- $v_i^d = \sum_{a \in E(i)} v_a^d$, i.e., the incoming total flow through node $i \in N$ with destination $d \in D$. $\mathbf{v}_i^d = (..., v_a^d, ...; a \in E(i))$ is a vector of per-destination flows on boarding links at stop i.
- $\mathbf{v}^d = (..., \mathbf{v}_i^d, ...;\ i \in N) \in \mathbb{R}_+^{|A|}$, $d \in D$. The vector of link flows will be denoted by $\mathbf{v} = (..., \mathbf{v}^d, ...;\ d \in D) \in \mathbb{R}_+^{|A||D|}$. However, the vector of total flows on links will not be expressed in bold: $v = \sum_{d \in D} \mathbf{v}^d \in \mathbb{R}_+^{|A|}$. Its components will be $v_a = \sum_{d \in D} v_a^d,\ a \in A$.

The following set V^d of feasible flows for destination $d \in D$ will be defined as:

$$\mathrm{V}^d \stackrel{\Delta}{=} \{\ \mathbf{v}^d \in \mathbb{R}_+^{|A|} \mid \textstyle\sum_{a \in E(i)} v_a^d - \sum_{a \in I(i)} v_a^d = g_i^d, i \in N\ \} \tag{1}$$

The set $\mathcal{V}$ of total link flows will be: $\mathcal{V} = \{\ v \in \mathbb{R}_+^{|A|} \mid \quad v_a = \sum_{d \in D} v_a^d, \quad \mathbf{v}^d \in \mathrm{V}^d\ \}$. Then the feasible set of flows for the transit assignment model will be: $\mathrm{V} = \bigotimes_{d \in D} \mathrm{V}^d$.

The set of boarding links a from stop i will be denoted by $\hat{E}(i)$. To each of these links corresponds a finite effective frequency function $f_a(v)$ (efff in the following), which will depend on the total flows v. Thus, $\hat{E}(i) \stackrel{\Delta}{=} \{\ a \in E(i) \mid f_a(\cdot) < +\infty\ \}$, if $i \in \hat{N}$ or $\hat{E}(i) = \emptyset$, if $i \in N \setminus \hat{N}$. Mean waiting times for boarding at the stops are the reciprocal of the effective frequencies. Thus, $\sigma_a(v) = 1/f_a(v)$. $\sigma_a(\cdot)$ will be referred to as mean travel time functions (mwtf). Travel times on links will be expressed by continuous functions $t_a(v)$, $a \in A$, which are assumed finite on $\mathcal{V}$, i.e., $t_a(v) < +\infty$, $\forall\ v \in \mathcal{V}, a \in A$. An infinite frequency will be assumed for the remaining links of the network.

Fig. 1. Links in the expanded transit network modeling a line serving a transit stop.

3. The stochastic version for the C3F model

A crucial aspect related to congested transit assignment models is the behaviour of link travel time functions and of functions for mean boarding times at stop near line capacities. In general, all models assume that these functions must have increasing values as the flows become larger. One critical aspect, however is how these functions must behave when flows are very close to capacity or are at capacity. Whereas some mathematical formulations require that these functions must remain finite, others admit that these functions may take an infinite value at capacity (and thus, theoretical queueing models for waiting time at stops can be integrated). If capacities are required to hold (i.e., flows beyond capacity for some link cannot be accepted), then imposing them explicitly along with functions that remain always finite usually poses consistency problems in the models solutions. In Codina (2013), the previous two cases are analyzed for the deterministic congested strategy-based transit assignment model or C3F model, whereas in Codina and Rosell (2017a) the algorithmic implications are shown and an algorithm is provided for consistently solving the second type of models (i.e., travel time and mean waiting time at stops that become infinite at line-capacities).

In the stochastic version of the C3F model developed in Codina and Rosell (2017b) (i.e., the StochC3F model) mean waiting times at stops σ are considered as random variables (r.v.). Many factors may contribute to this randomness but the main one is the operation of transit systems during a finite period (several hours). In these conditions the mean waiting time may present strong deviations from the asymptotic values given by traditional queueing models and, further, it can be considered a random variable. It is assumed that in-vehicle travel times t_a are random and follow a model of the form:

$$\tilde{t}_a(v) = \bar{t}_a(v) + \tilde{\epsilon}^t_a(v) \quad (2)$$

where $\bar{t}_a(v)$ is the mean value of the r.v. $\tilde{t}_a(v)$ and the r.v. $\tilde{\epsilon}^t_a(v)$ is assumed to have null mean and distribution parameters that may depend on the total flows v. Notice that $\mathbb{E}[v_a\tilde{t}_a(v)] = v_a\bar{t}_a(v)$. Also, the following homoscedastic model for the waiting time at stops is assumed:

$$\tilde{\sigma}_a(v) = \bar{\sigma}_a(v) + \tilde{\epsilon}^v_a \quad (3)$$

where $\bar{\sigma}_a(v)$ is the mean value of $\tilde{\sigma}_a(v)$ and $\tilde{\epsilon}^v_a$ is a random variable term with null mean, $\mathbb{E}[\tilde{\epsilon}^v_a] = 0$, and with distribution parameters independent of the total flows v. It must be noted that the homoscedasticity assumption in (3) may not be realistic, as shown in the simulations presented in Fonseca et al. (2014) and in Codina and Rosell (2017b), where the dispersion in waiting times at stops is shown to be dependent on the volume of passengers arriving at stops.

One way in which these mathematical problems can be solved is by means of the expected residual minimization (ERM) (see, for instance, [1] Chen et al. (2012)). Basically, for a scenario ω, the corresponding V.I. is $0 \in F(x,\omega) + \mathsf{N}_X(x)$. If $g(x,\omega)$ is a gap function for this V.I., a solution method consists of solving the stochastic optimization problem $Min_{x\in X}\mathbb{E}[g(x,\omega)]$. For the StochC3F model, the gap function to take into account $G_{\mathtt{C3F}}(\mathbf{v},\omega)$ and, considered as an r.v., will be denoted by $\tilde{G}_{\mathtt{C3F}}(\mathbf{v})$.

Let $\tilde{\mathbf{p}}_i = (...,\tilde{p}_a,...;a \in \hat{E}(i))$ be a vector of random variables given by $\tilde{p}_a = p_a + \tilde{\epsilon}_a$ with p_a being a constant and $\tilde{\epsilon}_a$ an r.v., such that $\mathbb{E}[\tilde{\epsilon}_a] = 0$ and that $\mathbb{C}\mathrm{ov}[\tilde{\epsilon}_a,\tilde{\epsilon}_{a'}] = m_{a,a'}$ does not depend on $\mathbf{p}_i = (...,p_a,...;a \in \hat{E}(i))$. Let now the function $\mathcal{H}_i(\cdot)$ be defined as $\mathcal{H}_i(\mathbf{p}_i) = \mathbb{E}[H_i(\tilde{\mathbf{p}}_i)] = \mathbb{E}[\max_{a\in\hat{E}(i)}\{...,\tilde{p}_a,...\}]$. It is known that, in the previous conditions:

$$P_r(\tilde{p}_a \geq \tilde{p}_{a'},\ \forall a' \in \hat{E}(i)) = \partial\mathcal{H}_i/\partial p_a(\mathbf{p}_i) \quad (4)$$

It is shown in Codina and Rosell (2017b) that the StochC3F model can be formulated as the following problem:

$$(\mathtt{stVI})\ \left[\text{Find : } \mathbf{v}^* \in \mathrm{V},\ \text{such that: } \textstyle\sum_{d\in D}\bar{T}^d(\mathbf{v}^*)^\top(\mathbf{v}^d - \mathbf{v}^{d*}) \geq 0,\ \ \forall\mathbf{v} \in V\right] \quad (5)$$

where components $\bar{T}^d_a(\mathbf{v})$ of functional $\bar{T}^d$, $d \in D$, $i \in N_d$ are defined by:

$$\bar{T}^d_a(\mathbf{v}) = \bar{t}_a(v) + \bar{\sigma}_a(v)\left(\Phi^d_i(\mathbf{v})\right)_a,\ if\ a \in \hat{E}(i)\ \ or\ \ \bar{T}^d_a(\mathbf{v}) = \bar{t}_a(v),\ if\ a \in E(i) \setminus \hat{E}(i) \quad (6)$$

$$\text{with}\ \ \Phi^d_i(\mathbf{v}) \stackrel{\Delta}{=} \nabla\mathcal{H}_i(...,v^d_a\bar{\sigma}_a(v),...), i \in \hat{N}_d \quad (7)$$

It is also shown that the primal gap $\bar{G}_P(\mathbf{v})$ for the previous V.I. problem in (5) is $\bar{G}_P(\mathbf{v}) = \max_{\mathbf{u}\in\mathrm{V}}\ (v-u)^\top\bar{t}(v) + \sum_{d\in D}\sum_{d\in D}\ \sum_{i\in\hat{N}_d}(\bar{x}^d_i(\mathbf{v}^d_i) - \bar{\mathrm{x}}^d_i(\mathbf{u}^d_i,v))^\top\Phi^d_i(\mathbf{v})$, where $\bar{\mathrm{x}}^d_i(\mathbf{u}^d_i,v) = \mathbb{E}[\tilde{\mathrm{x}}^d_i(\mathbf{u}^d_i,v)]$ and $\tilde{\mathrm{x}}^d_i(\mathbf{u}^d_i,v) = (...,u^d_a\cdot\tilde{\sigma}_a(v),...;a \in \hat{E}(i))$. Also, $\bar{x}^d_i(\mathbf{v}) = \bar{\mathrm{x}}^d_i(\mathbf{v}^d_i,v)$. Furthermore, it is also shown, by means of Theorem 3.1 below, that the Auslender gap (or primal gap) for V.I. (stVI) in (5) is an upper bound of the ERM gap function $\gamma(\mathbf{v}) = \mathbb{E}[\tilde{G}_{\mathtt{C3F}}(\mathbf{v})]$. This implies that: a) solving V.I. (stVI) is equivalent to solving the ERM problem, or equivalently, the StochC3F model formulated as the variational inequality problem (stVI) in (5), b) if $\mathbf{v}^*$ is a solution of (stVI), then $\gamma(\mathbf{v}^*) = 0$.

Theorem 3.1. *(Codina and Rosell (2017)) If $\bar{G}_P(\mathbf{v})$ is the Auslender's gap for (stVI) in (5) and $\gamma(\mathbf{v}) \stackrel{\Delta}{=} \mathbb{E}[\tilde{G}_{\mathtt{C3F}}(\mathbf{v})]$ is an ERM gap for the StochC3F model formulated as a V.I. in (5), then $\gamma(\mathbf{v}) \leq \bar{G}_P(\mathbf{v})$.*

Remark 1. *It must be noted that, if instead of the model in (3), we were to assume a (less realistic) model of the type $\tilde{x}^d_i(\boldsymbol{v}^d_i,v) = (...,v^d_a\bar{\sigma}_a(v) + \tilde{\epsilon}_a,...;a \in \hat{E}(i))$, with $\tilde{\epsilon}_a$ i.i.d. as Gumbel r.v.'s with zero mean and variance $\pi^2/(6\theta_i)$, $\forall\, a \in \hat{E}(i)$, then $(\Phi^d_i(\boldsymbol{v}))_a = \exp(\theta_i\, v^d_a\bar{\sigma}_a(v))/(\sum_{a'\in\hat{E}(i)}\exp(\theta_i\, v^d_{a'}\bar{\sigma}_{a'}(v)))^{-1}$, where $\theta_i > 0$ is a suitably chosen parameter for scaling the dispersion.*

4. The stochastic capacitated problem

Let $\hat{A}$ be the subset of boarding links in the expanded transit network (see Figure 1) and let A_e be the set of in-vehicle links. If $c_a,\ a \in A_e$, are the line capacities, we will consider the polyhedron X defined as: $X = \{\ \mathbf{v} \in \mathbb{R}^{|D||A|} \mid v_a \leq c_a,\ a \in A_e\ \}$ and its relative interior will be defined by $\mathsf{rint}X = \{\ \mathbf{v} \in \mathbb{R}^{|D||A|} \mid v_a < c_a,\ a \in A_e\ \}$.

The capacitated version will be formulated as a variational inequality similar to (5) and with the inclusion of the simple bounds expressed by the set X:

$$(\texttt{cstVI}) \quad \left[\mathsf{Find}:\ \mathbf{v}^* \in \mathrm{V} \cap X,\ \mathsf{such\ that}: \textstyle\sum_{d \in D} \bar{T}^d(\mathbf{v}^*)^\top (\mathbf{v}^d - \mathbf{v}^{d*}) \geq 0, \quad \forall \mathbf{v} \in V \cap X\right] \tag{8}$$

It can be shown that in the case that (some) capacity bounds are active at the solution, then the strategy-based equilibrium properties are affected.The following assumptions state the conditions under which the solutions of the previous V.I. (cstVI) remain always below capacity (i.e., non-binding capacity bounds) and thus the strategy-based equilibrium properties remain. This is stated formally in Theorem 4.5 below whose proof is omitted for brevity.

Assumption 4.1. *It is assumed that* $\hat{\mathrm{V}} \cap \mathsf{rint}X \neq \emptyset$. *In other words, that acyclic, per-destination, capacity-feasible flows exist, such that the corresponding total link flows are below capacity at in-vehicle links* $a \in A_e$.

Assumption 4.2. *Functions* $\bar{t}_a(\cdot),\ a \in A$ *and* mwtf $\bar{\sigma}_a(\cdot),\ a \in \hat{E}(i),\ i \in \hat{N}$ *are continuous and positive on* $\mathrm{V} \cap \mathsf{rint}X$.

Assumption 4.3. *For any* $\omega \in W$, *there exists at least an "escape" path* γ_ω *on* $G = (N, A)$ *so that any link* $a \in \gamma_\omega$ *verifies that mean travel time functions* $\bar{t}_a(\cdot),\ a \in \gamma_\omega$, *and* mwtf*'s* $\bar{\sigma}_a(\cdot),\ a \in \hat{A} \cap \gamma_\omega$ *are continuous and finite on* $\hat{\mathrm{V}} \cap X$.

Assumption 4.4. *Consider the sets* $A^\infty_\sigma(\mathbf{v}) \triangleq \{\ a \in \hat{A} \mid \bar{\sigma}_a(v) = +\infty\ \}$ *and* $A^\infty_t(\mathbf{v}) \triangleq \{\ a \in A \mid \bar{t}_a(v) = +\infty\ \}$. *For any* $\mathbf{v} \in \{\mathbf{v}' \in \mathrm{V} \cap X \mid \exists a \in A_e,\ v'_a = c_a\}$, $A^\infty_t(\mathbf{v}) \neq \emptyset$ *and/or* $A^\infty_\sigma(\mathbf{v}) \neq \emptyset$. *In other words, if for a capacity-feasible per-destination vector of flows, its corresponding vector of total flows v is at capacity at some in-vehicle link* $a \in A_e$, *then at least a travel time function or a* mwtf σ_a *evaluated at* $\mathbf{v}$ *becomes* ∞.

Theorem 4.5. *Under conditions stated in assumption (4.1), then*

1. *Suppose that travel time functions* $\bar{t}_a(\cdot),\ a \in A$, *and* efff*'s* $\bar{\sigma}_a(\cdot),\ a \in \hat{A}$ *are positive and continuous on* $\mathrm{V} \cap X$. *Then, the C3F VI problem in (5) has at least a solution* $\mathbf{v}^*$ *and any of them is such that* $\mathbf{v}^* \in \hat{\mathrm{V}} \cap X$.
2. *If assumptions 4.2, 4.3 and 4.4 also hold, then the C3F V.I. problem in (5) has at least one solution, and any of its solutions* $\mathbf{v}^*$ *is a) an equilibrium flow for the C3F model, such that b)* $\mathbf{v}^* \in \hat{\mathrm{V}} \cap \mathsf{rint}X$.

5. Two algorithms for the stochastic capacitated C3F model

As a dicrect consequence of the previous Theorem 4.5, the capacitated StochC3F model can then be stated as the following fixed point inclusion problem:

$$\mathbf{v}^* \in \ \mathsf{SolMin}_{\ \mathbf{u} \in \mathrm{V} \cap X} \ \textstyle\sum_{a \in A} \bar{t}_a(\mathbf{v}) u_a + \sum_{d \in D} \sum_{i \in \hat{N}_d} \sum_{a \in \hat{E}(i)} u^d_a\, \bar{\sigma}_a(v) (\Phi_i(\mathbf{v}))_a \tag{9}$$

Notice that, previous inclusion (9) is defined by a linear program whose cost function is parametrized by $\mathbf{v}^*$ and that (9) can be simply stated by $\mathbf{v}^* \in \Psi(\mathbf{v}^*)$, where $\Psi(\mathbf{v})$ is the point-to-set map defined by the solution set of the linear program in (9), which has the structure of a multicommodity network flow problem with capacities for total flows on boarding links

Several alternatives can be considered for solving the previous fixed point inclusion. Typically, the Mann iterative schemes (Mann (1953)) have been widely used. In transportation applications, they are very popular and are known under the denomination of Methods of Successive Averages (MSA). These methods basicaly consist of finding some $\hat{\mathbf{u}} \in \Psi(\mathbf{v}^{(k)})$ and determine the next iterate by $\mathbf{v}^{(k+1} = \mathbf{v}^{(k} + \alpha_k(\hat{\mathbf{u}} - \mathbf{v}^{(k)})$. The most well-known step length form is $\alpha_k = 1/k$ and then the rate of convergence is $\|\mathbf{v}^{(k} - \mathbf{v}^*\|_2 = O(k^{-1/2})$. In this paper, in order to determine the step length, the method of successive weighted averages (MSWA) has been used (Liu et al. (2009)). At each iteration this method k sets the step length as $\alpha_k = k^\nu / \sum_{i=1}^k i^\nu$, with ν being a fixed integer.

To solve this problem the functions $(\Phi_i(\mathbf{v}))_a$ need to be previously evaluated. The analitycal expression for the function $\mathcal{H}$ is only known for a few distributions of probability for the random terms $\tilde{\epsilon}$ in (3). Taking into account their definition in (4) and in (7) an approximated evaluation of $(\Phi_i(\mathbf{v}))_a$ can be made using the following sampling procedure:

- Using Montecarlo methods, initially generate a sample of size n for the random terms $\tilde{\epsilon}^v$ in (3). For each boarding link $a \in \hat{E}(i)$, $i \in \hat{N}$, let $\epsilon_a^{v,(\ell}$, $\ell = 1, 2, ...n$ be this sample.
- If $\mathbf{v}^{(k}$ is the current iterate, then evaluate for each $i \in \hat{N}$ and $d \in D$, the indicators $\delta_a^{d,(\ell}$:

$$\delta_a^{d,(\ell} = 1 \;\text{ if }\; v_a^{d,(k} \cdot (\bar{\sigma}_a(v^{(k)}) + \epsilon_a^{v,(\ell)}) = \max{}_{a' \in \hat{E}(i)} \{v_{a'}^{d,(k} \cdot (\bar{\sigma}_{a'}(v^{(k)}) + \epsilon_{a'}^{v,(\ell)})\} \;\text{ or }\; \delta_a^{d,(\ell} = 0 \;\text{ otherwise} \quad (10)$$

where it is assumed that a tie of the type $\delta_a^{d,(\ell} = \delta_{a'}^{d,(\ell} = ... = \delta_{a''}^{d,(\ell} = 1$ is resolved by choosing at random one of the links $a, a', .., a''$ with its indicator set to 1 and the remaining ones set to 0. Then, an approximation to $(\Phi_i(\mathbf{v}))_a$, $a \in \hat{E}(i)$, $i \in \hat{N}$, $d \in D$, as defined in (7) or (4), will be given by:

$$(\Phi_i(\mathbf{v}))_a \approx n^{-1} \textstyle\sum_{\ell=1}^{n} \delta_a^{d,(\ell} \quad (11)$$

It must be remarked that the case of non-homoscedastic waiting time models at stops, i.e., $\tilde{\sigma}_a(v) = \bar{\sigma}_a(v) + \tilde{\epsilon}_a^v(v)$, could be dealt with very similarly by direct simulation of the bus stop under the conditions imposed by the flows $\mathbf{v}^{(k}$.

The following algorithm assumes an initialization with a capacity-feasible vector $\mathbf{v}^{(0}$

1st Algorithm for the capacitated StochC3F model. `Assuming that` $\bar{G}_P(\mathbf{v}^{(k)}) > \varepsilon$, `at iteration` $k - th$:

`1- Evaluate the functions` $(\Phi_i(\mathbf{v}^{(k)}))_a$ `using the approximation in (11)`
`2- Solve the LP problem (9) for` $\mathbf{v} \equiv \mathbf{v}^{(k}$ `and let` $\hat{\mathbf{u}}$ `be a solution of it;`
`3- MSA step:` $\mathbf{v}^{(k+1} = \mathbf{v}^{(k} + \alpha_k(\hat{\mathbf{u}} - \mathbf{v}^{(k)})$; $k \longleftarrow k + 1$

The second algorithm is based on the formulation of the deterministic capacitated strategy-based transit assignment model formulated in Codina (2013) as a variational inequality and solved in Codina and Rosell (2017). For that V.I. problem the proper gap function can be stated as $\hat{G}_{\texttt{C3F}} : \hat{\mathrm{V}} \cap X \rightarrow \mathbb{R}$, defined as:

$$\hat{G}_{\texttt{C3F}}(\mathbf{v}) = G_{\texttt{C3F}}^{(0}(\mathbf{v}) - \hat{G}_{\texttt{C3F}}^{(1}(v) \quad (12)$$

where $G_{\texttt{C3F}}^{(0}(\mathbf{v}) = \varphi(\mathbf{v}, v)$, $\hat{G}_{\texttt{C3F}}^{(1}(v) = \text{Min}_{\,\mathbf{u} \in \mathrm{V} \cap X}\; \varphi(\mathbf{u}, v)$. A relative gap functon can be stated as $\hat{g}(\mathbf{v}) = \hat{G}_{\texttt{C3F}}(\mathbf{v}) / \hat{G}_{\texttt{C3F}}^{(1}(v)$. The function $\varphi(\cdot,\cdot)$ is defined as: $\varphi(\mathbf{u}, v) = \sum_{d \in D} \left[\sum_{a \in A} t_a(v) u_a^d + \sum_{i \in \hat{N}_d} \max_{a \in \hat{E}(i)} \left\{u_a^d / f_a(v)\right\}\right]$, where $f_a(v) = 1/\sigma_a(v), a \in A_a$ is the reciprocal of the mean waiting time at stops and is the so-called effective frequency function. The second algorithm in this subsection is based on the following corollary 5.1.

Corollary 5.1. (*Codina (2013). Characterization of C3F equilibrium as a fixed point inclusion*). *The conditions stated by assumptions 4.1, 4.2, 4.3 and 4.4 are supposed to hold. For simplicity, let* $\mathrm{w}(\cdot)$ *be defned as* $\mathrm{w}(\mathbf{v}^*) = \sum_{d \in D} \mathbf{v}^{d*}$. *Then,* $\mathbf{v}^*$ *is a solution of the C3F V.I. model iff the following fixed point inclusion is verified:*

$$\mathbf{v}^* \in \mathit{Sol\ Min}_{\,\mathbf{u} \in \mathrm{V} \cap X}\; \varphi(\mathbf{u}, \mathrm{w}(\mathbf{v}^*)) \quad (13)$$

In Codina and Rosell (2017a), the fixed point inclusion problem in (13) is solved also by using a Mann iterative schema. When set explicitly, the optimization problem in the fixed point inclusion in (13) is the following linear problem `[CapPL]`(r, τ), where $r \in \mathbb{R}^{|A_a|}$ and $\tau \in \mathbb{R}^{|A|}$ are parameter vectors set, respectively, to $r_a \equiv 1/\sigma_a(u)$, $a \in A_a$ and $\tau_a \equiv t_a(u)$, $a \in A$.

$$\begin{aligned} \texttt{[CapPL]}(r,\tau) \quad & \text{Min}_{\,\mathbf{v},w} \; \textstyle\sum_{d \in D} \sum_{a \in A} \tau_a v_a^d + \sum_{d \in D} \sum_{i \in \hat{N}} w_i^d \\ & s.t. : v_a^d \le r_a w_i^d, \quad a \in \hat{E}(i),\; i \in N,\; d \in D \\ & \mathbf{v} \in \mathrm{V} \cap X \end{aligned} \quad (14)$$

In Codina and Rosell (2017a) the linear problem `[CapPL]` in (14) is solved with the dual cutting-plane algorithm applied to a partial dualization of the line capacity constraints $v_a \le c_a, a \in A_e$. The resulting dual Lagrangian function is defined by means of a fixed frequency strategy-based transit assignment problem with non-negative costs, which can be solved very efficiently by using the Spiess algorithm in Spiess (1984).

Then, the following algorithm can be devised for solving the stochastic model StochC3F with strict capacities. This algorithm must be initialized by means of a capacity feasible flow $\mathbf{v}^0$ obtained, for instance by solving problem `[CapPL]`$(f^{(0}, t^{(0})$ for some initial frequencies $f^{(0}$ and travel times $t^{(0}$.

2nd Algorithm for the capacitated StochC3F model. `Assuming that` $\hat{g}(\mathbf{v}^{(k)}) > \epsilon$, `at iteration` $k - th$:

`1- Generate a sample` $\hat{\epsilon}_a^t, \hat{\epsilon}_a^v$ `of the perturbation variables` $\tilde{\epsilon}_a^t, \tilde{\epsilon}_a^v$
`2- Calculate new frequencies` $\hat{f}_a^{(k} = (\bar{\sigma}_a(v^{(k)}) + \hat{\epsilon}_a^v)^{-1}$, $a \in \hat{E}(b)$, $b \in \hat{N}_G$ `and travel times` $\hat{t}_a^{(k} = \bar{t}_a(v^{(k)}) + \hat{\epsilon}_a^t$, $a \in A$
`3- Solve [CapPL]`$(\hat{f}^{(k}, \hat{t}^{(k})$ $\rightarrow$ `(`$\hat{\mathbf{v}}$ `is obtained)`
`4- Update:` $\mathbf{v}^{(k+1} = (1 - \alpha_k)\mathbf{v}^{(k} + \alpha_k \hat{\mathbf{v}}$; `(MSA step)`
`5-` $k \leftarrow k + 1$; $\hat{g}(\mathbf{v}^{(k)}) = \hat{G}_{\texttt{C3F}}(\mathbf{v}^{(k)}) / \hat{G}_{\texttt{C3F}}^{(1}(v^{(k)})$

Remark 2. *It must be noted that in step number 1 of the above algorithm, the generation of the perturbation terms can be substituted by a direct simplified model of simulation (as in Fonseca et al. (2015)) of a bus stop so that the stochastic mean waiting time of passengers could adhere to a more realistic model.*

6. Computational results

In order to test the performance of both algorithms, three test networks have been chosen, whose dimensions are shown in table 1 below. Columns **a**, **y**, **x** and **e** display the number of boarding, alighting, dwelling and in-vehicle links of the expanded networks. Transportation lines are assummed to be operated by 50 services each during 5 hours. Correspondingly, passenger waiting times at boarding links have been obtained by simulating, for each boarding link, 50 services on a bulk-service queue of the type $M/D^{[c]}/1$, where servers (buses) had a maximum capacity of 50 passengers. Values for parametrizing the simulations (mean occupancy of buses, boarding and alighting flows) have been drawn at each iteration from $\mathbf{v}^{(k}$. Three congestion levels (Low, Medium, High) have been set for the test networks by assigning suitable multiples of a reference O-D matrix.

The evolution of the corresponding relative gaps in logarithmic scale is shown in figure 2 below for an initially fixed running time. For algorithm 1, the relative gap derivated from the gap $\bar{G}_P$ defined in section 3 is displayed, whereas for algorithm 2, the relative gap derived from the absolute gap (12) is displayed. Both algorithms perform well, excluding the first algorithm for the Zaragoza network in the medium and high congestion cases, where after a fast descent in the gap it enters into an ascent-oscillating phase. It can be observed that algorithm 2 always "reaches its best" much faster than algorithm 1. Also, although taking into acount that both algoritms have been monitored using different gap measures, algorithm 2 seems to perform better than algorithm 1 and that the CPU time per iteration requiered by algorithm 2 is much smaller than the one required for algorithm 1.

Table 1. Dimensions of the test networks.

Network	**a**	**y**	**x**	**e**	**Total**	**Nodes**	**O-D Pairs**	**Dest.**	**Lines**	**Stops**
Winnipeg	950	950	817	950	6,642	2,957	5,533	126	66	327
Zaragoza	3,864	3,864	3,802	3,864	19,585	9,966	4,156	129	31	1,176
Barcelona	142	142	132	142	642	310	88	10	5	18

Fig. 2. Performance of algorithms 1 (left) and 2 (right) on three transit networks.

7. Conclusions

This paper analyzes the stochastic congested strategy-based transit assignment problem, which has been extended to the case in which strict line capacity bounds must be enforced. The main result of this work is that the conditions to verify by the travel time functions and mean waiting time at stops are set, therefore the equilibrium properties of the model are consistent with the concept of strategies. These conditions are natural extensions of those formulated by the authors for the deterministic case in previous papers. In addition, two algorithms are developed for solving the stochastic model. The first one is based on a fixed point reformulation of the variational inequality that is equivalent to the stochastic model and that must be verified by the solutions of the problem. The second algorithm is an adaptation of the algorithm for solving the capacitated deterministic case, and it solves at each iteration an uncongested model by Lagrangian methods. Finally, both algorithms can be adapted in a straightforward manner to the case of the stochastic model with non-strict line capacities developed in Codina and Rosell (2017). It must be remarked also that the algorithms may accomodate simulation queueing models for obtainig samples of the passenger waiting time at boarding queues. Finally, the computational tests carried out show that the second algorithm outperforms the first one which requires larger CPU times to reach their best solutions in terms of the smallest possible gap that the algorithm is able to achieve.

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