Bachelor's Degree Thesis

The F-threshold of radical ideals

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Abstract

F-singularities are singularities of commutative rings of prime characteristic which are related to the Frobenius endomorphism. In order to study them, researchers have introduced numerical invariants such as the Hilbert–Kunz multiplicity and the F-signature, which provide a measure of how singular a ring is from the point of view of the Frobenius map. In this thesis, we introduce some concepts about characteristic p methods and compute the value of one of these invariants, the F-thresholds, for radical ideals in a regular setting.

Keywords: commutative algebra, characteristic p rings, F-thresholds, radical ideals, square-free monomial ideals, edge ideals.

MSC2020: 13A35, 13F55.

Resum

Les F-singularitats són singularitats d'anells commutatius de característica primera relacionats amb l'endomorfisme de Frobenius. Per a poder-les estudiar, s'han introduït invariants numèrics com la multiplicitat de Hilbert–Kunz o la F-signatura, que ofereixen una mesura de com de singular és un anell des del punt de vista de l'aplicació de Frobenius. En aquest treball, introduïm alguns conceptes sobre els mètodes de característica p i calculem el valor d'un d'aquests invariants, els F-llindars, per ideals radicals en un context regular.

Paraules clau: àlgebra commutativa, anells de característica p, F-llindars, ideals radicals, ideals monomials lliures de quadrats, ideals aresta.

MSC2020: 13A35, 13F55.

Resumen

Las F-singularidades son singularidades de anillos commutativos de característica prima relacionados con el endomorfismo de Frobenius. Para poder estudiarlas, se han introducido invariantes numéricos como la multiplicidad de Hilbert–Kunz y la F-signatura, que ofrecen una medida de cómo de singular es un anillo desde el punto de vista de la aplicación de Frobenius. En este trabajo, introducimos algunos conceptos sobre métodos en característica p y calculamos el valor de uno de estos invariantes, los F-umbrales, en el caso de ideales radicales en un contexto regular.

Palabras clave: álgebra conmutativa, anillos de característica p, F-umbrales, ideales radicales, ideales monomiales libres de cuadrados, ideales arista.

MSC2020: 13A35, 13F55.

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"To light a candle is to cast a shadow... "
Ursula K. Le Guin, A Wizard of Earthsea.

Contents

Notation

General conventions

$\mathbb{N}$	The set of natural numbers starting from 0.
$\mathbb{N}^*$	The set of positive natural numbers.
$\mathbb{K}$	A field.
p	A prime number.
$\mathbb{F}_p$	The finite field of p elements.
R	A unitary, commutative, Noetherian ring.
$\mathfrak{m}$	A maximal ideal.

Specific notation

$\operatorname{char} R$	Characteristic of the ring R.
$\dim R$	Krull dimension of the ring R.
$\operatorname{ht}(I)$	Height of an ideal $I \subseteq R$.
$Q(R)$	The fraction field of the integral domain R.
$R[X]$	The ring of polynomials with coefficients in R.
$\mathbb{K}(X)$	The fraction field of $\mathbb{K}[X]$, namely $Q(\mathbb{K}[X])$.
$\overline{\mathbb{K}}$	The algebraic closure of the field $\mathbb{K}$.
$\mu_R(I)$	The minimal number of generators of an ideal $I \subseteq R$. R is often omitted.
$\lfloor x \rfloor$	If $x \in \mathbb{R}$, the largest integer $\leq x$.
I^c	Contraction of an ideal $I \subseteq R$ with respect to a ring homomorphism $f\colon S \to R$.

1 Introduction

1.1 Motivation

The main object of study of this thesis are F-thresholds, a numerical invariant used to study singularities in characteristic p rings. This preliminary definition, however, may raise some elementary questions to a newcomer. We might start by asking **what is a singularity?** Algebraically speaking, a singular ring is a *non-regular* ring.

Definition 1.1.1. A *regular local* ring $(R, \mathfrak{m})$ is a ring where the minimal number of generators of the maximal ideal equals its height (the Krull dimension of R), namely, $\dim(R) = \mu(\mathfrak{m})$. A non local ring R is called *regular* when all of its localisations at prime ideals $R_{\mathfrak{p}}$ are regular.

On the other hand, there is also a geometric notion of singularity.

Definition 1.1.2. Let X be an affine variety such that $X = Z(I) \subseteq \mathbb{A}^n$, with $I = (f_1, \ldots, f_r)$ a prime ideal of $\mathbb{K}[x_1, \ldots, x_n]$, where $\mathbb{K}$ is perfect or has characteristic zero (see Definition 3.1.5). We say that X *is non singular at* $P \in X$ if the Jacobian matrix

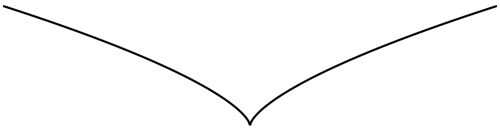
$$\left(\frac{\partial f_i}{\partial x_j}(P)\right)_{i=1,\ldots,r,\ j=1,\ldots,n}$$

has rank $\operatorname{ht}(I)$.

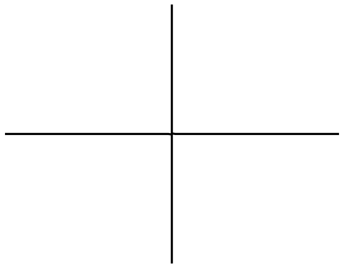
For example, consider the variety given by the zeros of the polynomial $f = xy \in \mathbb{K}[x, y]$. Its Jacobian matrix $(y\ x)$ has rank 0 at $(x, y) = (0, 0)$, so the variety $Z((f))$ is singular at the origin. Likewise, the variety given by the zeros of the polynomial $g = x^2 - y^3 \in \mathbb{K}[x, y]$ is singular at the origin as well. We can actually see these singularities in a graphical representation (Figure 1.1).

These two algebraic and geometric notions of singularity come together by the following theorem, that stems from the work in [Zar47].

Theorem 1.1.3. *Let X be an affine variety such that $X = Z(I) \subseteq \mathbb{A}^n$, with $I = (f_1, \ldots, f_r)$ a prime ideal of $\mathbb{K}[x_1, \ldots, x_n]$, where $\mathbb{K}$ is perfect or has characteristic* *zero. Then, a point $P \in X$ is non-singular if and only if the ring of regular functions in P is regular.*



(A) $0 = x^2 - y^3$



(B) $0 = xy$

FIGURE 1.1: Two singular varieties

While it does not apply to non-perfect fields, this theorem tells us that studying the singularity of a variety is equivalent to studying the singularity of a specific local ring. However, **what does it mean to study a singularity?** Algebraically speaking, there are properties behind and beyond regularity: a singular ring may have other "desirable" properties (for instance, being reduced, a domain, Cohen–Macaulay...), so studying the singularity can mean studying these kind of properties. Geometrically, we can also imagine that there are various kinds of singularities as well. The singularities presented in the previous example are visually different in some way: one is a normal crossing while the other is a cusp. In the second variety, by incrementing the degree of y the cusp becomes more and more narrow. Formalising and analysing these notions is also part of the study of singularities.

A way to study singularities is defining a numerical invariant and deriving properties from its values. For example, let $(R, \mathfrak{m})$ be a local ring, and define the invariant $\mu(\mathfrak{m}) - \operatorname{ht}(\mathfrak{m})$. By Krull's height theorem (2.2.4), $\mu(\mathfrak{m}) - \operatorname{ht}(\mathfrak{m}) \geq 0$. By definition, $\mu(\mathfrak{m}) - \operatorname{ht}(\mathfrak{m}) = 0$ if and only if R is regular. This is a kind of result researchers often seek when studying an invariant.

Another extensively researched approach, called *resolution of singularities*, is the following problem: given a singular variety V, we would like to find whether there exists a non-singular variety W and a map $W \to V$ with particular properties, called birational map. Here, the notion of birational equivalency is introduced, where (very roughly speaking) two varieties are equivalent if one can be bent into the other after removing a few points. Then, the study of singularities through resulution of singularities becomes a problem of classification of varieties up to birational equivalence.

Finally, from our initial definition of F-thresholds, we are left to wonder **what is so particular about characteristic p?** The problem of resolution of singularities is closed for varieties over a field characteristic zero ([Hir64]), but it remains an open problem for characteristic p, except for a few low-dimensional cases. The main difference between positive and zero characteristic is the Frobenius map, a ring homomorphism intrinsic to every characteristic p ring, given by raising its elements to the p-th power. It is precisely the fact that the Frobenius map is a ring morphism that makes it special: this doesn't happen in characteristic zero, where there is no real analogue to the Frobenius map. While characteristic p can be arguably harder than characteristic zero in some contexts, in Chapter 3 we will see that the Frobenius map contains a great deal of information about the ring R, so the study of R inevitably passes through the study of its Frobenius map. As we will see, the Frobenius morphism provides many meaningful results and helpful tools to study characteristic p rings.

1.2 Goals

The goal of this thesis is to introduce characteristic p methods and F-thresholds, and to compute this invariant in some special classes of rings. It will be necessary to introduce many concepts in commutative algebra to understand our motivations and results. The first rings for which we computed an F-threshold were rings defined by edge ideals, a special kind of square-free monomial ideals, both of which carry combinatoric meaning. After some generalisations and reworkings of the proofs, we realised that the same results generalised to square-free monomial ideals, and finally to radical ideals, a much more general hypothesis.

1.3 Structure of the thesis

- Chapter 2 contains some elementary notions on regular sequences and dimension theory needed throughout the thesis.
- Chapter 3 is dedicated to introducing the theory about the Frobenius morphism, including Kunz's theorem. We also motivate and define Frobenius powers and the F-thresholds, our main objects of study.
- In Chapter 4, we explain and prove our main result, and examine its consequences and possible generalisations.
- In Chapter 5, we introduce simplicial complexes and the Stanley–Reisner correspondence, which originally motivated our study of the F-threshold in the square-free monomial case, and explain the link between the two concepts.
- In Chapter 6, we draw conclusions from our work and consider steps to be taken for further research.

2 Prerequisites

In this chapter, we introduce some preliminaries on regular sequences and dimension theory that will be needed throughout the thesis, especially in Chapter 4.

2.1 Regular sequences

Roughly speaking, regular sequences are sequences of elements of a ring R that are "independent" with respect to an R-module M. Regular sequences, in particular Lemma 2.1.7, will be essential to our proofs. Most of the contents of this section are drawn from [BH98], Chapter 1.1 and its exercises.

Definition 2.1.1. Let R be a Noetherian ring, and M an R-module. A sequence $r_1, \dots, r_n \in R$ is an *M-regular sequence* if all of the following conditions hold:

(i) r_1 is a non-zero-divisor in M.

(ii) r_i is a non-zero-divisor in $M/(r_1, \dots, r_{i-1})M$, for all $i = 2, \dots, n$.

(iii) $M/(r_1, \dots, r_n)M \neq 0$.

If (i) and (ii) hold, but not (iii), the sequence is said to be *weakly regular*. We will denote regular sequences using boldface characters, for instance $\mathbf{r} = r_1, \dots, r_n$. Non-zero-divisors are often referred to as *regular elements*, and an R-regular sequence is called a *regular sequence*.

In general, permutations of a regular sequence need not be regular (Example 2.1.2). Nevertheless, in contexts where Nakayama's Lemma is applicable (in particular, local rings), regular sequences can be permuted.

Example 2.1.2. Let $R = \mathbb{K}[x, y, z]$, and $r_1 = x(y-1)$, $r_2 = y$, $r_3 = z(y-1)$. Then, r_1, r_2, r_3 is a regular sequence of R, since:

(i) r_1 is a regular element of R, since R is a domain.

(ii) $r_2 = y$ is a non-zero-divisor in $R/(x(y-1))$, since if $y\overline{f} = \overline{0}$ in $R/(x(y-1))$, then $y \cdot f = g \cdot x(y-1)$ for some $g \in \mathbb{K}[x, y, z]$. Then, y divides g, and $f \in (x(y-1))$, so $\overline{f} = 0$.

We observe that $(x(y-1), y) = (x, y)$, so $R/(r_1, r_2) = R/(x, y) \cong \mathbb{K}[z]$. The class of $r_3 = z(y-1)$ in $R/(x, y)$ is $\overline{r_3} = \overline{z}$, which is a regular element of $R/(x, y) = \mathbb{K}[z]$.

(iii) $R/(r_1, r_2, r_3) \cong \mathbb{K} \neq 0$.

However, r_1, r_3, r_2 is not a regular sequence, since $\overline{x} \cdot \overline{z(y-1)} = \overline{0}$ in $R/(r_1)$, and $\overline{x} \neq \overline{0}$, so r_3 is not a regular element of $R/(r_1)$.

Proposition 2.1.3. *Let R be a Noetherian local ring, M an R-module, and* $\mathbf{r}$ *a finitely generated M-regular sequence. Then, every permutation of* $\mathbf{r}$ *is a regular sequence.*

Proof. Every permutation is a product (composition) of transpositions, so proving following result suffices: let $\mathbf{r} = r_1, \ldots, r_i, r_{i+1}, \ldots, r_n$ be a regular sequence. Then, $\mathbf{r}' = r_1, \ldots, r_{i-1}, r_{i+1}, r_i, \ldots, r_n$ is also a regular sequence. Since the first part of $\mathbf{r}'$, namely $r_1, \ldots, r_{i-1}$, is already a regular sequence, we can focus only on the case $i = 1$. So we will prove that $\mathbf{r}' = r_2, r_1, r_3, \ldots, r_n$ is a regular sequence.

First, we will see that r_2 is a non-zero-divisor of M. This is where the local hypothesis will be needed. Let $K = \ker(r_2 \cdot)$ be the kernel of the multiplication by r_2 on M ($m \mapsto r_2 \cdot m$). We will see that $K = 0$, which is equivalent to r_2 being a non-zero-divisor in M. Let $m \in K$ be such that $r_2 m = 0$. In M/r_1M, $r_2\overline{m} = \overline{0}$, and because r_2 is a non-zero-divisor of M/r_1M, $\overline{m} = \overline{0}$, so there exists an $m' \in M$ such that $m = r_1 m'$. Then, $r_2 r_1 m' = 0$, and because r_1 is a non-zero-divisor of M, $m' \in K$. We have thus seen that for every $m \in K$, $m = r_1 m'$, with $m' \in K$. In other words, $K = r_1 K$. If r_1 was a unit, then $R/(r_1) = 0$ and $R/(\mathbf{r}) = 0$, so $\mathbf{r}$ would not be a regular sequence. Then, by Nakayama's Lemma we conclude that $K = 0$.

Now, we will see that $r_1, r_3, \ldots, r_n$ is an M/r_2M-regular sequence. By hypothesis, $r_3, \ldots, r_n$ is an $M/(r_1, r_2)M$-regular sequence, so we only need to check that r_1 is a regular element of M/r_2M. If we denote the quotient classes in M/r_1M and M/r_2M with one and two bars, respectively, let $\overline{\overline{m}} \in M/r_2M$ be such that $r_1\overline{\overline{m}} = \overline{\overline{0}}$. Lifting back to M, there exists an $m' \in M$ such that $r_1 m = r_2 m'$, and projecting to M/r_1M, we get that $r_2\overline{m'} = \overline{0}$. As r_2 is a regular element of M/r_1M, $\overline{m'} = 0$, so there exists an $m'' \in M$ such that $m' = r_1 m''$. Then, $r_1 m = r_2 r_1 m''$, so $r_1(m - r_2 m'') = 0$, and as r_1 is a regular element of M, $m = r_2 m''$, so in M/r_2M, $\overline{\overline{m}} = 0$, and we conclude that r_1 is a non-zero-divisor in M/r_2M, as we wanted to see.

We have seen that r_2 is a regular element of M, and that $r_1, r_3, \ldots, r_n$ is an M/r_2M-regular sequence, so $r_2, r_1, r_3, \ldots, r_n$ is an M-regular sequence, and we are done. $\square$

Proposition 2.1.4. *Let $0 \to U \to M \to N \to 0$ be a short exact sequence of R-modules, and $\mathbf{r} = r_1, \dots, r_n$ a sequence which is both U-regular and N-regular. Then, $\mathbf{r}$ is M-regular.*

Proof. Let f and g be the maps $U \to M$ and $M \to N$ from the exact sequence, respectively. We will check all three conditions of 2.1.1:

(i) $\underline{r_1 \text{ is a non-zero-divisor in } M}$: Let $m \in M$ be an element such that $r_1 m = 0$. Then, $0 = g(r_1 m) = r_1 g(m)$. Because r_1 is a non-zero-divisor in N, $m \in \ker(g) = \operatorname{Im}(f)$, so there exists a $u \in U$ such that $f(u) = m$. Then, $0 = r_1 f(u) = f(r_1 u)$. As f is injective, $r_1 u = 0$, but r_1 is a non-zero-divisor in U, so $u = 0$ and thus $m = f(u) = 0$.

(ii) $\underline{r_i \text{ is a non-zero-divisor in } M/(r_1, \dots, r_{i-1})M, \text{ for all } i = 2, \dots, n}$: Let I be the ideal generated by the regular sequence, $I = (r_1, \dots, r_{i-1})$. Tensoring the sequence $0 \to U \to M \to N \to 0$ with $\_ \otimes_R R/I$, and as the tensor product is right exact, we get the following exact sequence:

$$U \otimes_R R/I \xrightarrow{f \otimes \iota} M \otimes_R R/I \xrightarrow{g \otimes \iota} N \otimes_R R/I \longrightarrow 0,$$

where ι is the identity map on R/I. Using the isomorphism $R/I \otimes_R C \cong C/IC$ (for any R-module C), we get the following exact sequence

$$U/IU \xrightarrow{f'} M/IM \xrightarrow{g'} N/IN \longrightarrow 0,$$

where f' is given by $\overline{u} \mapsto \widetilde{f(u)}$ and g' is given by $\widetilde{m} \mapsto \overline{\overline{g(m)}}$. If we prove that f' is injective, then we have the following exact sequence:

$$0 \longrightarrow U/IU \xhookrightarrow{f'} M/IM \xrightarrow{g'} N/IN \longrightarrow 0, \qquad (2.1)$$

and then we can use (i) to conclude: because r_i is a non-zero-divisor in U/IU and N/IN, it is a non-zero-divisor in M/IM as well. To check this, we can assume that I is generated by one element, and the general case follows by induction on the length of $\mathbf{r}$. If $I = (r_1)$, let $\overline{u} \in U/r_1U$ be such that $f'(\overline{u}) = \widetilde{0}$. By definition of f', $\widetilde{f(u)} = \widetilde{0}$, so $f(u) \in r_1 M$, and there exists an $m \in M$ such that $f(u) = r_1 m$. In other words, $r_1 m \in \operatorname{Im}(f) = \ker(g)$, so $0 = g(r_1 m) = r_1 g(m)$. As r_1 is a regular element of N, $m \in \ker(g) = \operatorname{Im}(f)$, so there exists a $u' \in U$ such that $f(u) = m$. Then, $f(u) = r_1 m = r_1 f(u') = f(r_1 u')$. Because f is injective, $u = r_1 u'$, and in the quotient U/IU, $\overline{u} = \overline{r_1 u'} = \overline{0}$, so f' is injective.

(iii) $\underline{M/(r_1,\dots,r_n)M \neq 0}$: Let $I = (r_1,\dots,r_n)$. We have seen that there is an exact sequence

$$U/IU \longrightarrow M/IM \twoheadrightarrow N/IN \longrightarrow 0.$$

If $M/IM = 0$, by surjectivity $N/IN = 0$, which is a contradiction by N-regularity of $\mathbf{r}$.

□

Proposition 2.1.5. *Let $r_1,\dots,r_i,\dots,r_n$ and $r_1,\dots,r_i',\dots,r_n$ be M-regular sequences, where M is an R-module. Then, $r_1,\dots,r_ir_i',\dots,r_n$ is an M-regular sequence as well.*

Proof. First, we observe that $r_i,\dots,r_n$ and $r_i',\dots,r_n$ are $M/(r_1,\dots,r_{i-1})M$-regular sequences. Then, the statement is equivalent to showing that $r_ir_i',\dots,r_n$ is a regular sequence in $M/(r_1,\dots,r_{i-1})M$. So it suffices to prove the case $i = 1$. Assume that $r, r_2,\dots,r_n$ and $r', r_2,\dots,r_n$ are M-regular sequences. Then, we want to see that $rr', r_2,\dots,r_n$ is an M-regular sequence as well. It is enough to prove the following:

(i) rr' is a non-zero-divisor in M.

(ii) $r_2,\dots,r_n$ is a regular sequence in $M/(rr')M$.

For (i), let $m \in M$ be such that $rr'm = 0$. Because r is a regular element of M, $r'm = 0$, and since r' is also a regular element of r, $m = 0$. We conclude that rr' is a non-zero-divisor of M.

As for (ii), consider the following short exact sequence:

$$0 \longrightarrow M/rM \xhookrightarrow{f} M/(rr')M \xrightarrow{g} M/r'M \longrightarrow 0, \tag{2.2}$$

where f is the product by r', namely $\overline{m} \mapsto \overline{\overline{r'm}}$, and g is given by $\overline{\overline{m}} \mapsto \widetilde{m}$. Next, we will verify that it is indeed a short exact sequence:

- f is well defined: if $\overline{m} = \overline{m'}$ in M/rM, then $m = m' + rm''$ in M. Applying f, $\overline{\overline{r'm}} = \overline{\overline{r'm'}} + \overline{\overline{r'rm''}} = \overline{\overline{r'm'}}$ in $M/(rr')M$, so we conclude that f is well defined.
- f is injective: let $\overline{m} \in M/rM$ be such that $f(\overline{m}) = \overline{\overline{r'm}} = \overline{\overline{0}}$. Then, $r'm = rr'm'$ in M, so $r'(m - rm') = 0$. As r' is a regular element of M, $m = rm'$, so $m \in rM$ and $\overline{m} = \overline{0}$ in M/rM. Thus, f is injective.

- g is well defined and surjective: consider the diagram

$$\begin{tikzcd} M \arrow[r, two heads, "\pi_1"] \arrow[d, two heads, "\pi_2"'] & M/r'M. \\ M/(rr')M \arrow[ur, dashed, two heads, "g"'] & \end{tikzcd}$$

  Because $(rr')M \subseteq (r')M = \ker(\pi_1)$, by the universal property of the quotient module, there exists a unique homomorphism g that makes the diagram commute, which is given by $\overline{\overline{m}} \mapsto \pi_1(m) = \widetilde{m}$. Furthermore, g is surjective, since π_1 is.

- $\ker(g) = \operatorname{Im}(f)$: We observe that $\ker g = \left\{ \overline{\overline{m}} \in M/(rr')M \;\middle|\; \widetilde{m} = \widetilde{0} \right\}$. Because $\widetilde{m} = \widetilde{0}$ if and only if $m \in r'M$, $\ker(g) = \overline{\overline{r'M}}$. By definition of f, we have $\operatorname{Im}(f) = \overline{\overline{r'M}}$ as well.

Then, since $r_2, \ldots, r_n$ is both M/rM and $M/r'M$-regular, we can apply the previous result (2.1.4) to 2.2 to conclude that $r_2, \ldots, r_n$ is $M/(rr')M$-regular. □

Corollary 2.1.6. Let $r_1, \ldots, r_n$ be M-regular sequences, where M is an R-module. Then, $r_1^{i_1}, \ldots, r_n^{i_n}$ is an M-regular sequence for any $i_1, \ldots, i_n \in \mathbb{N}^*$.

Lemma 2.1.7. *Let $\mathbf{r} = r_1, \ldots, r_n$ be a regular sequence in R. Let $i_1, \ldots, i_n \in \mathbb{N}^*$. Then,*

$$r_1^{i_1-1} \cdot \ldots \cdot r_n^{i_n-1} \notin \left(r_1^{i_1}, \ldots, r_n^{i_n}\right).$$

Proof. Suppose, by contradiction, that

$$r_1^{i_1-1} \cdot \ldots \cdot r_n^{i_n-1} \in \left(r_1^{i_1}, \ldots, r_n^{i_n}\right). \tag{2.3}$$

If R is a local ring, we will prove that for every $j = 1, \ldots, n$, if

$$r_1^{i_1-1} \cdot \ldots \cdot r_j^{i_j-1} \in \left(r_1^{i_1}, \ldots, r_j^{i_j}, r_{j+1}, \ldots, r_n\right),$$

then

$$r_1^{i_1-1} \cdot \ldots \cdot r_{j-1}^{i_{j-1}-1} \in \left(r_1^{i_1}, \ldots, r_{j-1}^{i_{j-1}}, r_j, r_{j+1}, \ldots, r_n\right).$$

Applying this result to 2.3 for $j = n$, and then successively for every $j = n-1, \ldots, 1$, we have $1 \in (r_1, \ldots, r_n)$, which is a contradiction, since $R/(r_1, \ldots, r_n) \neq 0$ as $\mathbf{r}$ is a regular sequence. To prove the result, assume that

$$r_1^{i_1-1} \cdot \ldots \cdot r_j^{i_j-1} \in \left(r_1^{i_1}, \ldots, r_j^{i_j}, r_{j+1}, \ldots, r_n\right).$$

Then, $r_1^{i_1-1}\cdot\ldots\cdot r_j^{i_j-1}$ can be written as a linear combination of $r_1^{i_1},\ldots,r_j^{i_j},r_{j+1},\ldots,r_n$. Let $a_j \in R$ be the coefficient associated to $r_j^{i_j}$. Then, we can write

$$r_1^{i_1-1}\cdot\ldots\cdot r_j^{i_j-1} - a_j r_j^{i_j} \in \left(r_1^{i_1},\ldots,r_{j-1}^{i_{j-1}},r_{j+1},\ldots,r_n\right).$$

Factoring out $r_j^{i_j-1}$, we have

$$\left(r_1^{i_1-1}\cdot\ldots\cdot r_{j-1}^{i_{j-1}-1} - a_j r_j\right) r_j^{i_j-1} \in \left(r_1^{i_1},\ldots,r_{j-1}^{i_{j-1}},r_{j+1},\ldots,r_n\right). \tag{2.4}$$

By 2.1.6, $r_1^{i_1},\ldots,r_{j-1}^{i_{j-1}},r_j^{i_j-1},r_{j+1},\ldots,r_n$ is a regular sequence, and by 2.1.3, any permutation of a regular sequence in a local ring is regular. Thus, we realise that $r_1^{i_1},\ldots,r_{j-1}^{i_{j-1}},r_{j+1},\ldots,r_n,r_j^{i_j-1}$ is a regular sequence, so applying this to 2.4, we have

$$r_1^{i_1-1}\cdot\ldots\cdot r_{j-1}^{i_{j-1}-1} - a_j r_j \in \left(r_1^{i_1},\ldots,r_{j-1}^{i_{j-1}},r_{j+1},\ldots,r_n\right),$$

so we can finally conclude that

$$r_1^{i_1-1}\cdot\ldots\cdot r_{j-1}^{i_{j-1}-1} \in \left(r_1^{i_1},\ldots,r_j,r_{j-1}^{i_{j-1}},r_{j+1},\ldots,r_n\right),$$

just as we wanted to see.

In the general setting, we can reduce to the local case by considering a minimal prime $\mathfrak{p} \in \operatorname{Min}((r_1,\ldots,r_n))$, so $\left(r_1^{i_1},\ldots,r_n^{i_n}\right) \subseteq (r_1,\ldots,r_n) \subseteq \mathfrak{p}$. Localising at $\mathfrak{p}$, we get that $\left(r_1^{i_1},\ldots,r_n^{i_n}\right)_\mathfrak{p} \subseteq (r_1,\ldots,r_n)_\mathfrak{p} \subseteq \mathfrak{p}_\mathfrak{p}$, so $r_1/1,\ldots,r_n/1$ are not units of $R_\mathfrak{p}$. Now, we will check that $r_1/1,\ldots,r_n/1$ is a regular sequence in $R_\mathfrak{p}$:

(i) As localisation is a flat morphism, if r_1 is a non-zero-divisor in R, $\frac{r_1}{1}$ is a non-zero-divisor in $R_\mathfrak{p}$.

(ii) Likewise, as r_i is a non-zero-divisor in $R/(r_1,\ldots,r_{i-1})$, $\frac{\overline{r_i}}{1}$ is a non-zero-divisor in $(R/(r_1,\ldots,r_{i-1}))_\mathfrak{p} \cong R_\mathfrak{p}/(r_1,\ldots,r_{i-1})_\mathfrak{p} = R_\mathfrak{p}/(r_1/1,\ldots,r_{i-1}/1)$, where the isomorphism is given by the exactness of localisation.

(iii) We have seen that $(r_1/1,\ldots,r_n/1) \subseteq \mathfrak{p}_\mathfrak{p} \subsetneq R_\mathfrak{p}$, so $R_\mathfrak{p}/(r_1/1,\ldots,r_{i-1}/1) \neq 0$.

So, if we assume by contradiction that $r_1^{i_1-1}\cdot\ldots\cdot r_n^{i_n-1} \in \left(r_1^{i_1},\ldots,r_n^{i_n}\right)$, in $R_\mathfrak{p}$ we have

$$\frac{r_1^{i_1-1}}{1}\cdot\ldots\cdot\frac{r_n^{i_n-1}}{1} \in \left(\left(\frac{r_1}{1}\right)^{i_1},\ldots,\left(\frac{r_n}{1}\right)^{i_n}\right),$$

which is a contradiction since we saw the result for the local case. $\square$

2.2 Dimension theory

Dimension theory is a fundamental tool in commutative algebra and algebraic geometry, since it provides an algebraic definition of the dimension of an affine variety. As discussed in Chapter 1, the algebraic notion of regularity is tied to a geometrical interpretation of regularity as well. We will use some of the following well known results in dimension theory throughout the thesis, in particular, 2.2.10 will be of utmost importance for the proofs of Chapter 4. Some of the statements in this chapter are offered without proof, one can find such proofs in [AM69], Chapter 11.

Definition 2.2.1. Let $\mathfrak{p}$ be a prime ideal in a commutative ring R. Then, the *height of* $\mathfrak{p}$ is defined as the supremum of the lengths of chains of prime ideals contained in $\mathfrak{p}$. That is,

$$\mathrm{ht}\left(\mathfrak{p}\right) := \sup\left\{t \mid \exists \text{ a chain } \mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_t = \mathfrak{p}\right\}.$$

For a proper ideal $I \subseteq R$, its height is defined as:

$$\mathrm{ht}\left(I\right) := \inf\left\{\mathrm{ht}\left(\mathfrak{p}\right) \mid \mathfrak{p} \supseteq I,\ \mathfrak{p} \text{ prime}\right\}.$$

Definition 2.2.2. The *Krull dimension* of a commutative ring R is defined as the supremum of the lengths of chains of prime ideals in R, namely:

$$\dim R = \sup\left\{\mathrm{ht}\left(\mathfrak{p}\right) \mid \mathfrak{p} \text{ is prime}\right\}.$$

One of the main results of dimension theory is Krull's height theorem, that bounds the height of a prime ideal by its number of generators.

Theorem 2.2.3 (Krull's height theorem)**.** *Let R be a Noetherian ring, $x_1, \ldots, x_n \in R$, and $\mathfrak{p} \in \mathrm{Min}\left((x_1, \ldots, x_n)\right)$. Then, $\mathrm{ht}\left(\mathfrak{p}\right) \leq n$.*

Corollary 2.2.4. Let R be a Noetherian ring, and I an ideal of R. Then, $\mathrm{ht}\left(I\right) \leq \mu(I)$.

The following result offers a sort of converse of Krull's height theorem.

Proposition 2.2.5. *Let R be a Noetherian ring and $\mathfrak{p} \in \mathrm{Spec}(R)$. If $\mathrm{ht}\left(\mathfrak{p}\right) = h$, then there exist $y_1, \ldots, y_h \in R$ such that $\mathfrak{p} \in \mathrm{Min}\left((y_1, \ldots, y_h)\right)$.*

When we apply this proposition to the maximal ideal of a local ring $(R, \mathfrak{m})$, we have the following characterisation of the dimension of a local ring.

Corollary 2.2.6. Let $(R, \mathfrak{m})$ be a Noetherian, local ring. Then,

$$\dim R = \min\left\{c \;\middle|\; \exists x_1, \ldots, x_c \in \mathfrak{m},\ \sqrt{(x_1, \ldots, x_c)} = \mathfrak{m}\right\}. \tag{2.5}$$

Definition 2.2.7. Let $(R, \mathfrak{m})$ be a Noetherian, local ring of dimension $\dim R = d$. A set of elements $\{x_1, \ldots, x_d\} \subseteq \mathfrak{m}$ is called a *system of parameters* if it satisfies 2.5, that is, $\sqrt{(x_1, \ldots, x_d)} = \mathfrak{m}$. An element $x \in R$ is called a *parameter* if it belongs to a system of parameters.

Observation 2.2.8. Let $(R, \mathfrak{m})$ be a Noetherian, local ring, and $x \in \mathfrak{m}$ a non-zero-divisor. Then, x is a parameter.

Lemma 2.2.9. *Let $(R, \mathfrak{m})$ be a Noetherian, local ring. Let $x \in \mathfrak{m}$ be a parameter. Then, $R/(x)$ is local ring of dimension $\dim R - 1$.*

Proof. If x is a parameter, it is part of a system of parameters $\{x, x_2, \ldots, x_d\}$. Let $c = \dim(R/(x))$, so we want to see that $c = d - 1$. We know that $\mathfrak{m} = \sqrt{(x, x_2, \ldots, x_d)}$, so in the quotient $R/(x)$ we have that $\overline{\mathfrak{m}} = \sqrt{(\overline{x_2}, \ldots, \overline{x_d})}$. Because $R/(x)$ is a local ring with maximal ideal $\overline{\mathfrak{m}}$, by 2.2.6 we have $c \leq d - 1$.

On the other hand, if $c = \dim(R/(x))$ and $\overline{\mathfrak{m}}$ is the maximal ideal, again by 2.2.6 there exist $\overline{y_1}, \ldots, \overline{y_c} \in \overline{\mathfrak{m}}$ such that $\overline{\mathfrak{m}} = \sqrt{(\overline{y_1}, \ldots, \overline{y_c})}$. Thus, there exists an $N \in \mathbb{N}^*$ such that $\overline{\mathfrak{m}}^N \subseteq (\overline{y_1}, \ldots, \overline{y_c})$. Lifting back to R, $\mathfrak{m}^N + (x) \subseteq (y_1, \ldots, y_c) + (x)$, so $\mathfrak{m}^N \subseteq (y_1, \ldots, y_c, x) \subseteq \mathfrak{m}$. Taking radicals, we conclude that $\sqrt{(y_1, \ldots, y_c, x)} = \mathfrak{m}$. Therefore, by 2.2.6, $d \leq c + 1 \iff c \geq d - 1$, and we are done. □

Proposition 2.2.10. *Let R be a regular, local, Noetherian ring. Then, R is an integral domain.*

Proof. We will proceed by induction on the dimension of R. If $\dim R = 0$, then $\mu(\mathfrak{m}) = 0$ and $\mathfrak{m} = (0)$, so R is a field and thus a domain. Let $d = \dim(R) > 0$. Then,

$$\mathfrak{m} \not\subseteq \mathfrak{m}^2 \cup \left(\bigcup_{\mathfrak{p} \in \operatorname{Min} R} \mathfrak{p} \right),$$

so there exists an element $a \in \mathfrak{m} \setminus (\mathfrak{m}^2 \cup (\cup_{\mathfrak{p} \in \operatorname{Min} R} \mathfrak{p}))$. As $a \notin \mathfrak{m}^2$, $\overline{a} \neq 0$ in $\mathfrak{m}/\mathfrak{m}^2$, an $R/\mathfrak{m}$-vector space. By Steinitz's exchange lemma, we can extend $\{\overline{a}\}$ to a basis of $\mathfrak{m}/\mathfrak{m}^2$, that is, there exist $\overline{a_1}, \ldots, \overline{a_{d-1}} \in \mathfrak{m}/\mathfrak{m}^2$ such that $\{\overline{a}, \overline{a_1}, \ldots, \overline{a_{d-1}}\}$ is a basis of $\mathfrak{m}/\mathfrak{m}^2$. Then, as a consequence of Nakayama's Lemma, $\{a, a_1, \ldots, a_{d-1}\}$ is a minimal generating set of $\mathfrak{m}$ of cardinality d. By regularity of R, $\{a, a_1, \ldots, a_{d-1}\}$ is a system of parameters of R, so a is a parameter.

Then, by 2.2.9, $R/(a)$ is a local ring of dimension $d - 1$. It is also regular, since $\overline{\mathfrak{m}}$ is generated by $d - 1$ elements. Indeed, $\overline{\mathfrak{m}} = (\overline{a_1}, \ldots, \overline{a_{d-1}})$, and these generators are minimal: suppose without loss of generality that $\overline{a_1}$ can be removed, so $\overline{a_1} = \overline{b_2 f_2} + \ldots + \overline{b_{d-1} a_{d-1}}$ for some $b_2, \ldots, b_{d-1} \in R/(x)$. Lifting back to R, there exists

$c \in (a)$ (so $c = ba$ for some $b \in R$) such that $a_1 = ba + b_2a_2 + \dots b_{d-1}a_{d-1}$, so $\{a, a_1, \dots, a_{d-1}\}$ would not be a minimal generating set of $\mathfrak{m}$.

By induction hypothesis, $R/(a)$ is a domain, so (a) is a prime ideal of R. Because $a \notin \cup_{\mathfrak{p} \in \operatorname{Min} R} \mathfrak{p}$, there exists a $\mathfrak{p} \in \operatorname{Min} R$ such that $\mathfrak{p} \subsetneq (a)$. Now, let $p \in \mathfrak{p}$. As $\mathfrak{p} \subsetneq (a)$, there exists an $x \in R$ such that $p = a \cdot x$. Since $\mathfrak{p}$ is prime and $a \notin \mathfrak{p}$, $x \in \mathfrak{p}$, so $\mathfrak{p} = a\mathfrak{p}$. Then, by Nakayama's Lemma, $\mathfrak{p} = (0)$, so (0) is a prime ideal and thus R is a domain. $\square$

3 The Frobenius map

In this chapter, we introduce some basic concepts about the Frobenius morphism and present Kunz's theorem, a celebrated result that characterizes regular rings in characteristic p. Finally, we introduce the F-thresholds, our main object of study.

3.1 Definitions and basic properties

Definition 3.1.1. Let R be a ring of characteristic p. The *Frobenius endomorphism* is the map:

$$\begin{aligned} F\colon R &\longrightarrow R \\ x &\longmapsto x^p \end{aligned}$$

The e-th iteration of Frobenius ($e \in \mathbb{N}^*$) is denoted as follows:

$$\begin{aligned} F^e\colon R &\longrightarrow R \\ x &\longmapsto x^{p^e} \end{aligned}$$

In the context of a ring of characteristic p, the letter q will be used to represent $q = p^e$.

Proposition 3.1.2. *The Frobenius map is a ring homomorphism.*

Proof. Let $x, y \in R$. Then $F(xy) = (xy)^p = x^p y^p = F(x)F(y)$, since R is commutative. Now, for additivity:

$$F(x+y) = (x+y)^p = \sum_{i=0}^{p} \binom{p}{i} x^i y^{p-i} = \sum_{i=0}^{p} \frac{p!}{i!(p-i)!} x^i y^{p-i} = x^p + y^p = F(x) + F(y).$$

The next-to-last equality is due to the fact that $\binom{p}{i}$ is a multiple of p for all $0 < i < p$, and multiplication by p in R is zero. $\square$

If R has no nilpotent elements, the Frobenius map is injective. In fact, we have:

Proposition 3.1.3. *The Frobenius endomorphism is injective if and only if R is reduced.*

Proof. If the Frobenius morphism F is injective, then so is F^e, for all $e \in \mathbb{N}^*$. Then $\ker F^e = \{x \in R \mid x^{p^e} = 0\} = \{0\}$ for all $e \in \mathbb{N}^*$. Consider a nilpotent element of R: $x \in R$ such that $x^N = 0$ for some $N \in \mathbb{N}^*$. Then let $e' \in \mathbb{N}^*$ such that $N \leq p^{e'}$. Such an e' exists, since N is finite.

Because $N \leq p^{e'}$, then $x^{p^{e'}} = 0$, and $x \in \ker F^{e'} = \{0\}$, so we conclude that $x = 0$. We have seen that the only nilpotent element of R is 0 and, by definition, R is reduced.

For the converse, assume that R is reduced. Then, if there exists an $x \in R$ such that $F(x) = x^p = 0$, then necessarily $x = 0$, and thus F is injective. □

Surjectivity, however, is not characterized by such a result, and requires definitions on its own. Consider the following example:

Example 3.1.4. Let $R = \mathbb{F}_p(X)$. Even though R is a field, the element X is not in the image of F, so F cannot possibly be surjective.

Definition 3.1.5. A ring R of characteristic p[1] is said to be *perfect* when its Frobenius endomorphism is bijective, that is, an automorphism. In particular, when R is reduced and every element of R is a p-th power of an element of R.

Examples 3.1.6.

1. $\mathbb{F}_p$ is perfect.
2. Every algebraically closed field $\mathbb{K}$ is perfect. First, fields are integral domains and thus reduced. Second, for every $a \in \mathbb{K}$, the equation $x^p = a$ has a solution in $\mathbb{K}$, so we have both injectivity and surjectivity of the Frobenius morphism.
3. $\mathbb{F}_p\left[X, X^{1/p}, X^{1/p^2}, \dots\right]$ is a perfect ring, since every p-th root is added.

Noetherian perfect rings are quite special, in fact, one can show the following:

Exercise 3.1.7. A Noetherian perfect ring is a finite, direct product of perfect fields.

Proof. If R is a Noetherian, local ring with maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$, then there exist $y_1, \dots, y_n \in R$ such that $y_i^p = x_i$, for all $i = 1, \dots, n$, so $\mathfrak{m} = (y_1^p, \dots, y_n^p)$. We observe that $(y_1^p, \dots, y_n^p) \subseteq (y_1, \dots, y_n) \subseteq \mathfrak{m}$, so we conclude that $(y_1^p, \dots, y_n^p) = (y_1, \dots, y_n) = \mathfrak{m}$. However, we have the following chain of inclusions:

$$(y_1^p, \dots, y_n^p) \subseteq (y_1, \dots, y_n)^p \subseteq (y_1, \dots, y_n)^2 \subseteq (y_1, \dots, y_n).$$

Both ends of the chain equal $\mathfrak{m}$, so we conclude that $\mathfrak{m}^2 = \mathfrak{m}$, and by Nakayama's Lemma we have that $\mathfrak{m} = (0)$, so R is a field.

[1] It makes sense to define characteristic zero rings as perfect in the context of fields, since a characterisation of perfect fields, and often used as a definition, is that every algebraic field extension of a perfect field is separable ([DF03], Section 13.5). All characteristic zero fields satisfy this.

In the non-local case, let $d = \dim R$. Then, there exists a chain of prime ideals of maximum length d:

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_d,$$

where $\mathfrak{p}_d = \mathfrak{m}$ is a maximal ideal. Then, $R_\mathfrak{m}$ is a local, Noetherian ring, and also perfect. Indeed, $F_\mathfrak{m}$ (the Frobenius map in $R_\mathfrak{m}$) is surjective: let $r/s \in R_\mathfrak{m}$, so $r \in R$ and $s \notin \mathfrak{m}$. Then, because R is perfect, there exists an $r' \in R$ such that $rs^{p-1} = (r')^p$. Thus, $F_\mathfrak{m}(r'/s) = (r')^p/s^p = (rs^{p-1})/s^p = r/s$. A surjective morphism in a Noetherian ring is also injective, so $F_\mathfrak{m}$ is an isomorphism and $R_\mathfrak{m}$ is perfect. As $R_\mathfrak{m}$ is local, we have previously proved that it is a field, so $\mathfrak{m}_\mathfrak{m} = (0)$. Because of the correspondence between prime ideals of $R_\mathfrak{m}$ and prime ideals of R contained in $\mathfrak{m}$, $\dim(R) = 0$, so $\mathfrak{m} \in \operatorname{Min}(R)$, for all $\mathfrak{m} \in \operatorname{Max}(R)$. In particular, there is a finite number of maximal ideals. Additionally, by 3.1.3 we have that R is reduced, so $\sqrt{0} = (0)$, and

$$(0) = \sqrt{0} = \bigcap_{\mathfrak{p} \in \operatorname{Min}(R)} \mathfrak{p} = \bigcap_{\mathfrak{m} \in \operatorname{Max}(R)} \mathfrak{m}.$$

Let $\phi \colon R \to \prod_{\mathfrak{m} \in \operatorname{Max}(R)} R/\mathfrak{m}$. Then, by the Remainder Theorem (as stated in [AM69], Proposition 1.10), ϕ is surjective. Furthermore, because maximal ideals are coprime in pairs, ϕ is also injective. Thus, ϕ is an isomorphism from R to a product of fields, which are perfect since quotient classes and powers commute.

□

The following statements outline how the Frobenius map behaves under localisation.

Observation 3.1.8. If $\mathfrak{p} \subseteq R$ is a prime ideal of R, then $F^{-1}(\mathfrak{p}) = \mathfrak{p}$. This means that the map on $\operatorname{Spec} R$ induced by the Frobenius map, namely, $\mathfrak{p} \longmapsto F^{-1}(\mathfrak{p})$, is the identity.

Proof. On the one hand, if $x \in \mathfrak{p}$, then $F(x) = x^p \in \mathfrak{p}$, so $F(\mathfrak{p}) \subseteq \mathfrak{p}$ and $\mathfrak{p} \subseteq F^{-1}(\mathfrak{p})$. On the other hand, if $x \in F^{-1}(\mathfrak{p})$, then $x^p \in \mathfrak{p}$, and because $\mathfrak{p}$ is prime, $x \in \mathfrak{p}$, so we conclude $F^{-1}(\mathfrak{p}) = \mathfrak{p}$. □

Proposition 3.1.9 (The Frobenius map commutes with localisation)**.** *Let $S \subseteq R$ be a multiplicatively closed set. Then, the following diagram commutes, where the downward arrows are the localisation map:.*

$$\begin{tikzcd} R \arrow[r, "F^e"] \arrow[d] & R \arrow[d] \\ S^{-1}R \arrow[r, "F^e"] & S^{-1}R \end{tikzcd}$$

Proof. It suffices to note that, for any $x \in R$, $\frac{F^e(x)}{1} = \frac{x^q}{1} = \left(\frac{x}{1}\right)^q = F^e\left(\frac{x}{1}\right)$. $\square$

Definition 3.1.10. The *Frobenius power* of an ideal $I \subseteq R$ is the ideal:

$$I^{[p]} := (\{x^p \mid x \in I\})$$

Following the convention of 3.1.1, we write $I^{[q]}$ for $I^{[p^e]}$, and we note that $I^{[p^0]} = I$.

Observation 3.1.11. As $I^{[q]} = (F^e(I))$, if $I = (f_1, \dots, f_n)$, then $I^{[q]} = (f_1^q, \dots, f_n^q)$.

Observation 3.1.12. It holds that $I^{[q]} \subseteq I^q \subseteq I$, and all three ideals have the same radical.

Observation 3.1.13. Let R be a ring of characteristic p. Consider its subset $R^p := \{r^p \mid r \in R\}$. Then, R^p is a subring of R. If R is reduced, there is an isomorfism of rings between $R \cong R^p$ given by the Frobenius map.

Proof. R^p is, by definition, the image of the Frobenius map, so it is a subring. Furthermore, because R is reduced, the Frobenius map is injective, so by Noether's first isomorphism theorem, $R \cong F(R) = R^p$. $\square$

Observation 3.1.14. This isomorphism can be iterated again and again to get $R \cong R^p \cong R^{p^2} \cong R^{p^3} \cong \dots$.

3.2 The Frobenius push-forward

Given a homomorphism of rings $f\colon R \to S$, one can view S as an R-module through the action $r \cdot s := f(r)s$, with $r \in R$ and $s \in S$. This action is called the *restriction of scalars*, and can be used to view S-modules as R-modules. The Frobenius map $F\colon R \to R$ induces an R-module structure on R different from the standard one, given by $r \cdot s = r^p s$, with $r, s \in R$. The $\cdot$ symbol represents the new action given by the restriction of scalars, and no symbol is used for the ordinary product in R. Because it can be confusing to use R as an R-module with two different actions, we often use $F_*(R)$ to denote R as a module over itself via Frobenius. To distinguish the elements of $F_*(R)$ from the scalars we use the notation $F_*(r)$, for $r \in R$. Thus, if $r, s \in R$, we have

$$F_*(r) + F_*(s) = F_*(r+s) \text{ and } r \cdot F_*(s) = F_*(r^p s).$$

This notation can also be applied to the iterated Frobenius F_*^e without any special considerations.

Given a homomorphism of rings $f\colon R \to S$, viewing S as an R-module by restriction of scalars is equivalent to viewing S as an $f(R)$-module by the standard action of S. With this in mind, a natural way of treating $F_*(R)$ is as R viewed as an R^p-module. However, there is another way to interpret $F_*(R)$.

Definition 3.2.1. Let R be a domain, $Q(R)$ its field of fractions and $\overline{Q(R)}$ the algebraic closure of the latter. The *ring of p-th roots of R* is defined as $R^{1/p} := \left\{ x \in \overline{Q(R)} \,\middle|\, x^p \in R \right\}$. An element $x \in R^{1/p}$ such that $r = x^p \in R$ is often denoted as $r^{1/p}$.

Observation 3.2.2. The notation $r^{1/p} \in R^{1/p}$ is well defined thanks to the reducedness for R. Indeed, if there was another $y \in R^{1/p}$ such that $y^p = r = x^p$, then $(x-y)^p = 0$ in R, and reducedness gives us $x = y$.

Clearly, $R \subseteq R^{1/p}$, so we can view $R^{1/p}$ as an R-module with the action given by the extension of scalars on the inclusion. With this structure, $R^{1/p}$ and $F_*(R)$ are isomorphic as R-modules. Before giving a proof, Table 3.1 below illustrates how the two actions operate in the same way on the scalars in the specific case of the scalar multiplication law.

R-Module	$R^{1/p}$	$F_*(R)$
Elements	$s \in R,\ s^p =: t,\ r^{1/p} \in R^{1/p}$	$s \in R,\ r \in F_*(R)$
Law	$s \cdot r^{1/p} = t^{1/p} r^{1/p} = (tr)^{1/p} = (s^p r)^{1/p}$	$s \cdot F_*(r) = F_*(s^p r)$

Table 3.1: Comparison of the scalar multiplication law of $R^{1/p}$ and $F_*(R)$ as R-modules.

Observation 3.2.3. $R^{1/p} \cong F_*(R)$, as R-modules.

Proof. Let x be an element in $R^{1/p}$. Then, consider the map

$$\begin{aligned} \phi\colon R^{1/p} &\longrightarrow F_*(R) \\ x &\longmapsto F_*(x^p). \end{aligned}$$

This map is well-defined since $x^p \in R$, which is equal as a set to $F_*(R)$. It is also a module homomorphism:

- Let $x, y \in R^{1/p}$ be such that $x^p = r$ and $y^p = s$. Then, $\phi(x) + \phi(y) = \phi\left(r^{1/p}\right) + \phi\left(s^{1/p}\right) = F_*(r) + F_*(s) = F_*(r+s) = \phi\left((r+s)^{1/p}\right) = \phi\left(r^{1/p} + s^{1/p}\right) = \phi(x+y)$.
- Let $s \in R$ and $x \in R^{1/p}$ be such that $x^p = r \in R$. Then, $\phi(s \cdot x) = \phi\left(s \cdot r^{1/p}\right) = \phi\left((s^p r)^{1/p}\right) = F_*(s^p r) = s \cdot F_*(r) = s \cdot \phi(r^{1/p}) = s \cdot \phi(x)$.

It is injective: let $x \in R^{1/p}$ and suppose $\phi(x) = 0$. Then, $F_*(x^p) = 0$, so $x^p = 0$ in R. Because R is reduced, we conclude $x = 0$. Regarding surjectivity, if $r \in F_*(R)$, then $r \in R$ (recall that $F_*(R)$ and R are equal as sets). If X is an indeterminate, the equation $X^p = r$ has a solution in $\overline{Q(R)}$. Let x be the solution, which by 3.2.2 is unique and by definition belongs to $R^{1/p}$. Then, $\phi(x) = F_*(x^p) = F_*(r)$, so ϕ is surjective and thus an isomorphism of R-modules. □

Observation 3.2.4. The isomorphism of rings of 3.1.13 works with p-th roots as well, since the image of $R^{1/p}$ under the Frobenius map is R, so we have isomorphisms of rings $R \cong R^{1/p} \cong R^{1/p^2} \cong R^{1/p^3} \cong \dots$.

3.3 Kunz's Theorem

This section is dedicated to a notorious theorem by Ernst Kunz (1969) that characterizes regular rings of prime characteristic.

Theorem 3.3.1. *A ring R is regular if and only if the Frobenius map is flat.*

While the most difficult of the implications is the "backward" (i.e., that if the Frobenius map is flat, the ring is regular), we will only need the direct implication. The proof of the direct implication involves completions, which will not be used in this work, so we present a proof of the direct implication for the case $R = \mathbb{K}[x_1, \dots, x_n]$, which uses the same fundamental ideas as the full proof. A proof of the general case, and of the converse as well, can be found in [Kun69]. Before starting the proof, we need to introduce the following Lemma:

Proposition 3.3.2. *Let A, B be rings such that $A \to B$ is flat. Then, $A[x] \to B[x]$ is flat.*

Proof. Since the tensor product is already right exact, we only have to prove injectivity at the left. Let N, M be $A[x]$-modules, therefore, they are also A-modules. Let $0 \to N \hookrightarrow M$. We want to see that

$$0 \to N \otimes_{A[x]} B[x] \to M \otimes_{A[x]} B[x]$$

is exact. We recall that $B[x] \cong A[x] \otimes_A B$, so we have the isomorphisms:

$$N \otimes_{A[x]} B[x] \cong N \otimes_{A[x]} (A[x] \otimes_A B) \cong \big(N \otimes_{A[x]} A[x]\big) \otimes_A B \cong N \otimes_A B.$$

Likewise, the same holds for M, and thus

$$\begin{array}{ccccc} 0 \longrightarrow & N \otimes_{A[x]} B[x] & \longrightarrow & M \otimes_{A[x]} B[x] \\ & \wr\| & & \wr\| \\ & N \otimes_A B & & M \otimes_A B. \end{array}$$

Because we know that $A \to B$ is flat, $N \otimes_A B \to M \otimes_A B$ is injective, so our desired map is injective as well. $\square$

Proof (of 3.3.1): We will see that R is flat over R^p, that is, that $R^p \hookrightarrow R$ is a flat extension. Firstly, $R^p = \mathbb{K}^p[x_1^p, \ldots, x_n^p]$, and the extension $\mathbb{K}^p[x_1^p, \ldots, x_n^p] \hookrightarrow \mathbb{K}^p[x_1, \ldots, x_n]$ is free, since $\mathbb{K}^p[x_1, \ldots, x_n]$ is a $\mathbb{K}^p[x_1^p, \ldots, x_n^p]$-module with basis $\{x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_i} \mid 0 \leq \leq \alpha_i < p\}$. Because it is free, it is flat as well. Secondly, $\mathbb{K}^p \hookrightarrow \mathbb{K}$ is flat, since it is a field extension (and vector spaces are free). Then, iterating 3.3.2 guarantees us that $\mathbb{K}^p[x_1, \ldots, x_n] \hookrightarrow \mathbb{K}[x_1, \ldots, x_n]$ is a flat extension. In short, we have the following two flat extensions:

$$\mathbb{K}^p[x_1^p, \ldots, x_n^p] \hookrightarrow \mathbb{K}^p[x_1, \ldots, x_n] \hookrightarrow \mathbb{K}[x_1, \ldots, x_n].$$

Because the composition of flat maps is flat, we have that the extension $R^p \hookrightarrow R$ is flat, so R is a flat R^p-module. $\square$

Corollary 3.3.3. As the composition of flat maps is flat, we have that if R is regular, the iterated Frobenius F^e is also flat for all $e \geq 1$.

Example 3.3.4. Let $R = \frac{\mathbb{F}_p[\![x,y]\!]}{(xy)}$. R is not regular, so according to Kunz's Theorem it should not be flat either. First, it is straightforward to justify that R is not regular. (x, y) is the maximal ideal of $\mathbb{F}_p[\![x, y]\!]$, so $(\overline{x}, \overline{y})$ is the maximal ideal of R, and $\mu_R(\overline{x}, \overline{y}) = 2$. On the other hand, as xy is not a unit nor a zero-divisor, by 2.2.9, $\dim(R) = \dim \mathbb{F}_p[\![x, y]\!] - 1 = 1$. Then, R is not regular by definition, since its dimension differs from the minimal number of generators of its maximal ideal.

To see that R is not flat, we will find a map that does not stay injective after tensoring. Let $f\colon R^p \to R^p$ be the map $\overline{a}^p \mapsto \overline{a}^p\overline{y}^p$. We have $\ker f = (\overline{x}^p)$, so

$$f'\colon R^p/(\overline{x}^p) \longrightarrow R^p$$

is injective. Tensoring with R over R^p,

$$\begin{array}{ccc}\frac{R^p}{(\overline{x}^p)} \otimes_{R^p} R & \xrightarrow{f'\otimes_{R^p}\mathrm{id}} & R^p \otimes_{R^p} R \\ \cong & & \cong \\ R/\left(\overline{x}^p\right) & \longrightarrow & R,\end{array}$$

where the lower map is given by $\overline{a} \mapsto \overline{a}^p \overline{y}^p$, which is not injective, as $\overline{x}$ is in its kernel.

3.3.1 Consequences

The flatness of the Frobenius map grants us properties of flat homomorphisms that are key in the development of this work. In particular, the distribution of Frobenius powers over intersections will allow us to work with primary decompositions, and the distribution over colon ideals will be necessary to control containments of Frobenius powers.

Proposition 3.3.5. *Let $f\colon R \to S$ be a flat ring homomorphism. Let I, J be ideals of R, and let $x \in R$ be an element. Then, the following hold:*

(i) $f(I \cap J)S = f(I)S \cap f(J)S$.

(ii) $f(I :_R x)S = f(I)S :_S f(x)$.

Proof. (i) The following is a short exact sequence of R-modules:

$$0 \longrightarrow \frac{R}{I \cap J} \xrightarrow{\alpha} R/I \oplus R/J \xrightarrow{\beta} \frac{R}{I+J} \longrightarrow 0$$

where α and β are given by $\hat{a} \mapsto \left(\bar{a}, \bar{\bar{a}}\right)$ and $\left(\bar{b}, \bar{\bar{c}}\right) \mapsto \widetilde{b-c}$ respectively.

Applying the tensor product $\_ \otimes_R S$ we get the following short exact sequence, since f is flat:

$$0 \longrightarrow \frac{R}{I \cap J}\otimes_R S \xrightarrow{\alpha\otimes_R \mathrm{id}_S} (R/I \otimes_R S)\oplus(R/J \otimes_R S) \xrightarrow{\beta\otimes_R \mathrm{id}_S} \frac{R}{I+J}\otimes_R S \longrightarrow 0$$

Using the isomorphism $R/I \otimes_R M \cong M/IM$ (for an R-module M), and realising that in this case, $I \cdot S = f(I)S$, the sequence becomes:

$$0 \longrightarrow \frac{S}{f(I \cap J)S} \xrightarrow{g} \frac{S}{f(I)S} \oplus \frac{S}{f(J)S} \xrightarrow{h} \frac{S}{f(I+J)S} \longrightarrow 0$$

where now g and h are given by $\hat{s} \mapsto \left(\bar{s}, \overline{\overline{s}}\right)$ and $\left(\bar{t}, \overline{\overline{u}}\right) \mapsto \widetilde{t-u}$.

Because the sequence is exact, g is injective, so

$$\{\hat{0}\} = \ker g = \left\{ \hat{s} \in \frac{S}{f(I \cap J)S} \;\middle|\; \bar{s} = \bar{0},\ \overline{\overline{s}} = \overline{\overline{0}} \right\}.$$

Lifting these two sets, we get $f(I \cap J)S = f(I)S \cap f(J)S$.

(ii) We use the same reasoning, but with the short exact sequence:

$$0 \longrightarrow \frac{R}{I :_R x} \xrightarrow{\alpha} R/I \longrightarrow \frac{R}{(I,x)} \longrightarrow 0,$$

where α is given by $\overline{a} \mapsto \overline{\overline{ax}}$. Tensoring with $\_ \otimes_R S$ and applying the isomorphism seen in (i), we get the short exact sequence:

$$0 \longrightarrow \frac{S}{f\left(I :_R x\right) S} \xrightarrow{g} \frac{S}{f(I)S} \longrightarrow \frac{S}{f\left(I, x\right) S} \longrightarrow 0$$

This time, g is given by $\overline{s} \mapsto \overline{\overline{f(x)s}}$. Just as before, g is injective, and thus

$$\{\overline{0}\} = \ker g = \left\{ \overline{s} \in \frac{S}{f\left(I :_R x\right) S} \;\middle|\; \overline{\overline{f(x)s}} = \overline{\overline{0}} \right\}.$$

Lifting these sets, we get $f\left(I :_R x\right) S = f(I) :_S f(x)$, our desired result.

□

Corollary 3.3.6. Let I, J be ideals of a regular ring R of characteristic p. Then, the previous proposition implies that:

(i) $(I \cap J)^{[q]} = I^{[q]} \cap J^{[q]}$.

(ii) $(I : x)^{[q]} = I^{[q]} : x^q$.

(iii) $(I : J)^{[q]} = I^{[q]} : J^{[q]}$.

Proof. (i) and (ii) are direct from 3.3.5. For (iii), let $J = (f_1, \ldots, f_s) = (f_1) + \ldots + (f_s)$. Then,

$$I : J = I : ((f_1) + \ldots + (f_s)) = (I : f_1) \cap \ldots \cap (I : f_s).$$

Taking Frobenius powers and applying (i) and (ii) consecutively, we get

$$(I{:}J)^{[q]} = ((I : f_1) \cap \ldots \cap (I : f_s))^{[q]} = (I{:}f_1)^{[q]} \cap \ldots \cap (I{:}f_s)^{[q]} = (I^{[q]}{:}f_1^q) \cap \ldots \cap (I^{[q]}{:}f_s^q).$$

As $J^{[q]} = (f_1^q, \ldots, f_s^q) = (f_1^q) + \ldots + (f_s^q)$, we have

$$(I : J)^{[q]} = I^{[q]} : ((f_1^q) + \ldots + (f_s^q)) = I^{[q]} : J^{[q]}.$$

. $\square$

The following is also a handy consequence of Kunz's theorem.

Lemma 3.3.7. *Let I, J be ideals in a regular ring R of characteristic p. Then, $I \subseteq J$ if and only if $I^{[q]} \subseteq J^{[q]}$ for some q.*

Proof. The direct implication is clear and true in the general case. For the converse, suppose by contradiction that $I \nsubseteq J$. Then, the colon $J : I$ is a proper ideal, so there exists $\mathfrak{m} \in \operatorname{Max}(R)$ such that $J : I \subseteq \mathfrak{m}$. Taking Frobenius powers, $(J : I)^{[q]} \subseteq \mathfrak{m}^{[q]}$, and applying 3.3.6 (iii), we get that $J^{[q]} : I^{[q]} = (J : I)^{[q]} \subseteq \mathfrak{m}^{[q]}$. However, our hypothesis is that $I^{[q]} \subseteq J^{[q]}$, so $J^{[q]} : I^{[q]} = R$. Thus, we have $R \subseteq \mathfrak{m}^{[q]} \subseteq \mathfrak{m}$, which is a contradiction with the fact that $\mathfrak{m}$ is a maximal ideal. $\square$

In the following propositions, we will see how the flatness of the Frobenius map influences the behaviour of Frobenius powers regarding associated primes, and how it shines in comparison with ordinary powers.

Proposition 3.3.8. *Let I be a Q-primary ideal in a regular ring R. Then $I^{[q]}$ is also a Q-primary ideal.*

Proof. This is equivalent to showing that $\operatorname{Ass}\left(R/I^{[q]}\right) = \{Q\}$. Since I is Q-primary, $\operatorname{Ass}(R/I) = \operatorname{Min}(I) = \{Q\}$. Let $\mathfrak{p} \in \operatorname{Ass}\left(R/I^{[q]}\right)$, we want to see that $\mathfrak{p} = Q$. Firstly, we know that $I^{[q]} \subseteq \mathfrak{p}$. Taking radicals, we already get that $Q \subseteq \mathfrak{p}$.

We define $\operatorname{MaxAss}\left(R/I^{[q]}\right)$ as the associated primes of $I^{[q]}$ that are maximal with respect to the inclusion. This set is well defined because R is a Noetherian ring. We will show that if $\mathfrak{q} \in \operatorname{MaxAss}\left(R/I^{[q]}\right)$, then $\mathfrak{q} = Q$. This implies that $\mathfrak{p} = Q$, since $\mathfrak{p} \in \operatorname{Ass}\left(R/I^{[q]}\right)$ will be contained in a maximal one, and

$$Q \subseteq \mathfrak{p} \subseteq \mathfrak{q} = Q.$$

So let $\mathfrak{q} \in \operatorname{MaxAss}\left(R/I^{[q]}\right)$. Because $\mathfrak{q}$ is an associated prime, there exists an $\alpha \in R$ such that $\mathfrak{q} = \left(I^{[q]} : \alpha\right)$, so

$$I^{[q]} \subseteq I \subseteq Q \subseteq \mathfrak{q} = \left(I^{[q]} : \alpha\right) \subseteq \left(I^{[q]} : \alpha^q\right).$$

We will now see that $\left(I^{[q]}\colon \alpha\right) = \left(I^{[q]}\colon \alpha^q\right)$. It is known that every maximal element of the family of ideals $F \coloneqq \left\{J\colon \exists \beta \in R \text{ such that } J = \left(I^{[q]}\colon \beta\right)\right\}$ is an associated prime of $I^{[q]}$. Because R is Noetherian, $\left(I^{[q]}\colon \alpha^q\right)$ is contained in one of these maximal elements of F, that is, there exists a $\beta \in R$ such that $\left(I^{[q]}\colon \alpha^q\right) \subseteq \left(I^{[q]}\colon \beta\right)$. However, the latter is maximal and thus an associated prime that contains $\mathfrak{q}$. Because $\mathfrak{q}$ is also a maximal associated prime, we conclude that $\mathfrak{q} = \left(I^{[q]}\colon \alpha\right) = \left(I^{[q]}\colon \alpha^q\right) = \left(I^{[q]}\colon \beta\right)$. Now, by flatness of Frobenius,

$$I^{[q]} \subseteq I \subseteq Q \subseteq \mathfrak{q} = \left(I^{[q]}\colon \alpha\right) = (I\colon \alpha)^{[q]}.$$

Finally, assume by contradiction that there exists an element $\gamma \in \mathfrak{q}$, $\gamma \notin Q$. Then $\gamma \in \mathfrak{q}$ implies that $\gamma\alpha \in I^{[q]} \subseteq I$. Because I is a Q-primary ideal, if $\alpha \notin I$, then there exists an $n \in \mathbb{N}$ such that $\gamma^n \in \sqrt{I} = Q$. However, Q is a prime ideal, so that would imply $\gamma \in Q$, which would be against our assumption. Thus, $\alpha \in I$. Now, $\alpha \in I$ implies that $(I\colon \alpha) = R$, which yields the following contradiction with the fact that $\mathfrak{q}$ is a prime ideal:

$$\mathfrak{q} = \left(I^{[q]}\colon \alpha\right) = (I\colon \alpha)^{[q]} = R^{[q]} = R.$$

This contradiction comes from assuming $Q \subsetneq \mathfrak{q}$, so we conclude $Q = \mathfrak{q}$, as we wanted to see.

□

Corollary 3.3.9. Let I be an ideal of a regular ring R. Then, the Frobenius map preserves associated primes, that is, $\operatorname{Ass}(R/I) = \operatorname{Ass}\left(R/I^{[q]}\right)$.

Proof. Let $I = Q_1 \cap \cdots \cap Q_m$ be an irredundant primary decomposition of I. By flatness of the Frobenius map, $I^{[q]} = Q_1^{[q]} \cap \cdots \cap Q_m^{[q]}$. We will prove that this is an irredundant primary decomposition of $I^{[q]}$.

The previous proposition, 3.3.8, already tells us that $Q_i^{[q]}$ is Q_i-primary, so we only have to see that the given decomposition is irredundant. Firstly, we define $J \coloneqq Q_1 \cap \cdots \cap Q_{m-1}$. Then, we assume by contradiction that the decomposition is redundant. Without loss of generality we can also assume that the redundant factor is $Q_m^{[q]}$, and then $I^{[q]} = Q_1^{[q]} \cap \cdots \cap Q_{m-1}^{[q]} \cap Q_m^{[q]} = Q_1^{[q]} \cap \cdots \cap Q_{m-1}^{[q]}$. By flatness of Frobenius,

$$I^{[q]} = (Q_1 \cap \cdots \cap Q_{m-1} \cap Q_m)^{[q]} = (Q_1 \cap \cdots \cap Q_{m-1})^{[q]} = J^{[q]}.$$

If $I^{[q]} = J^{[q]}$, by 3.3.7 we have that $I = J$. However, that is not the case, since the initial decomposition for I is irredundant, so

$$I = Q_1 \cap \cdots \cap Q_{m-1} \cap Q_m \subsetneq Q_1 \cap \cdots \cap Q_{m-1} = J.$$

Thus, we conclude that the assumption was false, and the decomposition is not redundant. $\square$

Note that both of the previous results are false for ordinary powers. Indeed, if $\mathfrak{p}$ is a prime ideal, $\mathfrak{p}^n$ need not be $\mathfrak{p}$-primary. In contrast, the properties do hold for symbolic powers, a way of defining powers of an ideal that allows us to have more control over its associated primes.

Definition 3.3.10. Let I be an ideal of a Noetherian ring R (any characteristic) without embedded primes[2]. Then, the *n-th symbolic power of I* is defined as:

$$I^{(n)} := \bigcap_{\mathfrak{p} \in \operatorname{Min}(\mathfrak{p})} \left(I_{\mathfrak{p}}^n\right)^c,$$

where $\left(I_{\mathfrak{p}}^n\right)^c$ denotes the contraction with respect to the localisation map of $(I^n)_{\mathfrak{p}} = (I_{\mathfrak{p}})^n$. Then, if $\mathfrak{p}$ is a prime ideal, $\mathfrak{p}^{(n)}$ is the smallest $\mathfrak{p}$-primary ideal that contains $\mathfrak{p}$. Because embedded components can arise in ordinary powers, in general symbolic powers differ from ordinary powers.

Example 3.3.11. Let $I = (xy, xz, yz) \subseteq \mathbb{K}[x, y, z]$. We observe that I is a radical ideal with unique irredundant primary decomposition $I = (x, y) \cap (x, z) \cap (y, z)$.Then, $I^2 = (x^2y^2, x^2z^2, y^2z^2, x^2yz, xy^2z, xyz^2) = (x, y)^2 \cap (x, z)^2 \cap (y, z)^2 \cap (x^2, y^2, z^2)$. In this case, I^2 has the same minimal primes as I, but an (x, y, z)-primary embedded component has arisen. Then, $I^{(2)} = (x, y)^2 \cap (x, z)^2 \cap (y, z)^2 = (x^2y^2, xyz, x^2z^2, y^2z^2)$.

While both symbolic and Frobenius powers offer us specific knowledge about the primary decomposition of an ideal, Frobenius powers also carry information about the order of its generators, just like ordinary powers. In this example, we observe that every generator of I^2 and $I^{[2]} = (x^2y^2, x^2z^2, y^2z^2)$ has order 4, whereas $I^{(2)}$ contains a generator of order 3.

[2]Although symbolic powers can be defined in general, in this context it is more convenient not to do so. For general symbolic powers, 3.3.9 doesn't necessarily hold.

3.4 F-thresholds

If I is an ideal of a characteristic p ring R, we have seen that $I^{[q]} \subseteq I^q$. Most of the times, this bound is tight, that is, if $I^{[q']} \subseteq I^q$, we will have $q' \geq q$. The other containment, however, proves to be much more interesting. We can ask ourselves about the particularities of an N such that, for a fixed q, $I^N \subseteq I^{[q]}$. Or, more generally, about the comparison between Frobenius powers and ordinary powers. In this section, we introduce the F-thresholds, a numerical invariant that explores these kind of relationships. F-thresholds were first defined for regular rings in [MTW04] as a characteristic p analogue to the jumping numbers: a family of invariants in characteristic zero that carry geometric and algebraic information about singularities. Although their definitions are quite different, the cited paper and following ones showed many relations with these characteristic zero invariants as $p >> 0$. Later, F-thresholds were defined in general rings of characteristic p in [HMTW08], and in [DNBP18], F-thresholds were proved to exist in this general setting, since until then their existence had only been proved in some partial cases.

Returning to the our motivation, we observe that because $\sqrt{I} = \sqrt{I^{[q]}}$, there exists an $N_q \in \mathbb{N}$ such that $I^{N_q} \subseteq I^{[q]}$. The following observations allow us to make the dependence of N_q on q more explicit.

Observation 3.4.1. $(I^n)^{[q]} = \left(I^{[q]}\right)^n$.

Proof. If $I = (f_1, \ldots, f_n)$, then

$$(I^n)^{[q]} = (\{(f_1^{\alpha_1})^q \cdot \ldots \cdot (f_n^{\alpha_n})^q \mid \Sigma\alpha_i = n\}) = (\{(f_1^q)^{\alpha_1} \cdot \ldots \cdot (f_n^q)^{\alpha_n} \mid \Sigma\alpha_i = n\}) = \left(I^{[q]}\right)^n.$$

□

Observation 3.4.2. Let J be an ideal of R. Let $\mathfrak{a}$ be an ideal contained in the radical of J, so $\mathfrak{a} \subseteq \sqrt{J}$. Then, there exists an $N \in \mathbb{N}$ such that for all q, $\mathfrak{a}^{N \cdot q} \subseteq J^{[q]}$.

Proof. Since $\mathfrak{a} \subseteq \sqrt{J}$ and R is Noetherian, there exists an $M \in \mathbb{N}$ such that $\mathfrak{a}^M \subseteq J$. Now let $\mathfrak{a} = (f_1, \ldots, f_s)$. If we see that

$$\mathfrak{a}^{M(s(q-1)+1)} \subseteq \left(\mathfrak{a}^{[q]}\right)^M,$$

we will be closer to concluding, since $\left(\mathfrak{a}^{[q]}\right)^M = \left(\mathfrak{a}^M\right)^{[q]} \subseteq J^{[q]}$. By definition,

$$\mathfrak{a}^{M(s(q-1)+1)} = \left(\left\{ f_1^{\alpha_1} \cdot \ldots \cdot f_s^{\alpha_s} \;\middle|\; \sum_{i=1}^{s} \alpha_i = M(s(q-1)+1) \right\}\right)$$

and
$$\left(\mathfrak{a}^{[q]}\right)^M = \left(\left\{ (f_1^q)^{\beta_1} \cdot \ldots \cdot (f_s^q)^{\beta_s} \;\middle|\; \sum_{i=1}^{s} \beta_i = M \right\}\right).$$

Let $g := f_1^{\alpha_1} \cdot \ldots \cdot f_s^{\alpha_s}$ be generator of $\mathfrak{a}^{M(s(q-1)+1)}$, so $\sum_{i=1}^s \alpha_i = M(s(q-1)+1)$. Then, the pigeonhole principle (iterated M times) guarantees us that, in g, there are at least M instances of various f_i having an exponent greater or equal to q (allowing repetitions of i). In other words, g is a multiple of $(f_1^q)^{\gamma_1} \cdot \ldots \cdot (f_s^q)^{\gamma_s}$, with $\sum_{i=1}^s \gamma_i \geq M$, so necessarily $g \in \left(\mathfrak{a}^{[q]}\right)^M$. Because this holds true for any generator, we have the following containment:
$$\mathfrak{a}^{M(s(q-1)+1)} \subseteq \left(\mathfrak{a}^{[q]}\right)^M \subseteq J^{[q]}.$$

Finally, we can take $N := Ms$, so $Nq = Msq \geq M(s(q-1)+1)$ and the desired result holds:
$$\mathfrak{a}^{N \cdot q} \subseteq \mathfrak{a}^{M(s(q-1)+1)} \subseteq J^{[q]}.$$

□

Definition 3.4.3. Let $\mathfrak{a}, J$ be two ideals of R, a characteristic p ring, with $\mathfrak{a} \subseteq \sqrt{J}$. Then, for any $q = p^e$, $e \in \mathbb{N}$, we define

$$\nu_{\mathfrak{a}}^J(q) := \max\left\{t \in \mathbb{N} \;\middle|\; \mathfrak{a}^t \not\subseteq J^{[q]}\right\}.$$

3.4.2 guarantees us the existence of $\nu_{\mathfrak{a}}^J$, since there exists an N such that $\mathfrak{a}^{Nq} \subseteq J^{[q]}$ for all q. Thus, $\nu_{\mathfrak{a}}^J(q) < Nq$.

Proposition 3.4.4. *Let $\mathfrak{a}, J$ be two ideals of R, a characteristic p ring, with $\mathfrak{a} \subseteq \sqrt{J}$. Let e, e' be natural numbers, and let $q = p^e$, $q' = p^{e'}$. Then, the following properties hold:*

(i) *If $\mathfrak{b} \subseteq \mathfrak{a}$, then $\nu_{\mathfrak{b}}^J(q) \leq \nu_{\mathfrak{a}}^J(q)$.*

(ii) *If $J \subseteq I$, then $\nu_{\mathfrak{a}}^I(q) \leq \nu_{\mathfrak{a}}^J(q)$.*

(iii) $\nu_{\mathfrak{a}}^{J^{[q]}}(q') = \nu_{\mathfrak{a}}^J(qq')$.

(iv) *If $n \in \mathbb{N}^*$, $\nu_{\mathfrak{a}^n}^J(q) = \lfloor \frac{\nu_{\mathfrak{a}}^J(q)}{n} \rfloor$.*

(v) *If R is regular, $q'\nu_{\mathfrak{a}}^J(q) \leq \nu_{\mathfrak{a}}^J(q'q)$.*

Proof. (i) Let $v = \nu_{\mathfrak{b}}^J(q)$. By definition, $\mathfrak{b}^v \not\subseteq J^{[q]}$, so surely $\mathfrak{b}^v \subseteq \mathfrak{a}^v \not\subseteq J^{[q]}$. As $\nu_{\mathfrak{a}}^J(q)$ is the maximum t such that $\mathfrak{a}^t \not\subseteq J^{[q]}$, $v \leq \nu_{\mathfrak{a}}^J(q)$.

(ii) Let $v = \nu_{\mathfrak{a}}^J(q)$. By definition, $\mathfrak{a}^{v+1} \subseteq J^{[q]} \subseteq I^{[q]}$. Because $\nu_{\mathfrak{a}}^I(q)$ is the maximum t such that $\mathfrak{a}^t \not\subseteq I^{[q]}$, we necessarily have $\nu_{\mathfrak{a}}^I(q) < v+1$, so $\nu_{\mathfrak{a}}^I(q) \leq v$.

(iii) We observe that $\left(J^{[q]}\right)^{[q']} = J^{[qq']}$. Then, by definition:

$$\nu_{\mathfrak{a}}^{J^{[q]}}(q') = \max\left\{t \in \mathbb{N} \;\middle|\; \mathfrak{a}^t \nsubseteq \left(J^{[q]}\right)^{[q']}\right\} = \max\left\{t \in \mathbb{N} \;\middle|\; \mathfrak{a}^t \nsubseteq J^{[qq']}\right\} = \nu_{\mathfrak{a}}^J(qq').$$

(iv) Let $v = \nu_{\mathfrak{a}^n}^J(q)$. Then, $(\mathfrak{a}^n)^v \nsubseteq J^{[q]}$ and $(\mathfrak{a}^n)^{v+1} \subseteq J^{[q]}$. We notice that, for any $t \in \mathbb{N}$, $(\mathfrak{a}^n)^t \nsubseteq J^{[q]}$ if and only if $nt \leq \nu_{\mathfrak{a}}^J(q)$, so using $t = v$, we have $v \leq \lfloor \frac{\nu_{\mathfrak{a}}^J(q)}{n} \rfloor$. On the other hand, if $t = v+1$ the converse $(\mathfrak{a}^n)^{v+1} \subseteq J^{[q]}$ is true, so $n(v+1) > \nu_{\mathfrak{a}}^J(q)$. Then, $v + 1 > \frac{\nu_{\mathfrak{a}}^J(q)}{n} \geq \lfloor \frac{\nu_{\mathfrak{a}}^J(q)}{n} \rfloor$, and thus $v \geq \lfloor \frac{\nu_{\mathfrak{a}}^J(q)}{n} \rfloor$, which grants us the equality we wanted to prove.

(v) We will prove the result for $q' = p$, namely $p\nu_{\mathfrak{a}}^J(q) \leq \nu_{\mathfrak{a}}^J(pq)$, and the statement follows from a simple induction. First, the regular setting allows us to notice the following fact: in an ideal $I \subseteq R$, if $u^p \in I^{[p]}$, then $u \in I$. Indeed, if $u^p \in I^{[q]}$, then $(u^p) = (u)^{[p]} \subseteq I^{[p]}$, and we can apply 3.3.7 to conclude $(u) \subseteq I$, and thus $u \in I$.

To prove the result, let $v = \nu_{\mathfrak{a}}^J(q)$, so $\mathfrak{a}^v \nsubseteq J^{[q]}$. Then, there exists a $u \in \mathfrak{a}^v$ such that $u \notin J^{[q]}$. By our previous observation, $u \notin J^{[q]}$ implies $u^p \notin \left(J^{[q]}\right)^{[p]} = J^{[pq]}$. As $u \in \mathfrak{a}^v$, $u^p \in (\mathfrak{a}^v)^p = \mathfrak{a}^{pv}$. This proves that $\mathfrak{a}^{pv} \nsubseteq J^{[pq]}$, so $p\nu_{\mathfrak{a}}^J(q) \leq \nu_{\mathfrak{a}}^J(pq)$.

□

Example 3.4.5. The last inequality, (v), does not hold in the non-regular setting. Let $q = p = 3$, $R = \frac{\mathbb{F}_3[x,y]}{(x^4 - y^5)}$ and consider the ideals $\mathfrak{a} = (\overline{x})$, $J = (\overline{y})$. As the ideals are principal, Frobenius and ordinary powers are equal. We have that $\mathfrak{a} \subseteq \sqrt{J}$, since $\overline{x}^4 = \overline{y}^5 \in J$. As $\overline{x}^4 = \overline{y}^5 \in J^{[3]}$, $\mathfrak{a}^4 \subseteq J^{[3]}$. On the other hand, $\mathfrak{a}^3 \nsubseteq J^{[3]}$, because $\overline{x}^3 \notin J^{[3]}$. Therefore, $\nu_{\mathfrak{a}}^J(p) = 3$.

We observe that $\overline{x}^8 = \overline{y}^{10} \in J^{[9]}$, so $\mathfrak{a}^8 \subseteq J^{[9]}$. Thus, $\nu_{\mathfrak{a}}^J(pq) \leq 8$, so $\nu_{\mathfrak{a}}^J(pq) < p \cdot \nu_{\mathfrak{a}}^J(p) = 9$.

Definition 3.4.6. Let $\mathfrak{a}, I$ be two ideals of R, a characteristic p ring, with $\mathfrak{a} \subseteq \sqrt{I}$. Then, we define the *F-threshold of* $\mathfrak{a}$ *with respect to* I as

$$\mathrm{c}^J(\mathfrak{a}) := \lim_{e \to \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}.$$

Observation 3.4.7. The F-threshold is well defined, that is, the limit always exists.

Proof. We will give a short proof of the regular case. The reader can find a complete proof of the general case in [DNBP18], Theorem 3.4.

By 3.4.2, $\nu_{\mathfrak{a}}^{J}(p^e) < Np^e$, so we have a uniform upper bound on $\mathrm{c}^J(\mathfrak{a})$. Thus, if we see that the sequence $\left(\nu_{\mathfrak{a}}^{J}(p^e)\right)_{e\in\mathbb{N}}$ is non-decreasing, we can conclude convergence of $\mathrm{c}^J(\mathfrak{a})$. Let $t_e := \nu_{\mathfrak{a}}^{J}(p^e)$. We will see that $t_e \leq t_{e+1}$.

Assume, by contradiction, that $t_e > t_{e+1}$. As these are natural numbers, we have $t_e \geq t_{e+1} + 1$. By definition of t_{e+1},

$$\mathfrak{a}^{t_{e+1}+1} \subseteq J^{[p^{e+1}]} = \left(J^{[p^e]}\right)^{[p]},$$

and since $t_e \geq t_{e+1} + 1$, we get

$$\left(\mathfrak{a}^{t_e}\right)^{[p]} \subseteq \mathfrak{a}^{t_e} \subseteq \mathfrak{a}^{t_{e+1}+1} \subseteq \left(J^{[p^e]}\right)^{[p]}.$$

Due to the regularity of R, we can use 3.3.7 to deduce $\mathfrak{a}^{t_e} \subseteq J^{[p^e]}$ from $\left(\mathfrak{a}^{t_e}\right)^{[p]} \subseteq \left(J^{[p^e]}\right)^{[p]}$. However, this is a contradiction with the definition of t_e, which is the maximum $t \in \mathbb{N}$ such that $\mathfrak{a}^t \nsubseteq J^{[p^e]}$. □

Proposition 3.4.8. *Let $\mathfrak{a}, J$ be two ideals of R, a characteristic p ring, with $\mathfrak{a} \subseteq \sqrt{J}$. Then, the following properties hold:*

(i) *If $\mathfrak{b} \subseteq \mathfrak{a}$, then $\mathrm{c}^J(\mathfrak{b}) \leq \mathrm{c}^J(\mathfrak{a})$.*

(ii) *If $J \subseteq I$, then $\mathrm{c}^I(\mathfrak{a}) \leq \mathrm{c}^J(\mathfrak{a})$.*

(iii) *If n is a positive integer, $\mathrm{c}^J(\mathfrak{a}^n) = \frac{1}{n} \cdot \mathrm{c}^J(\mathfrak{a})$.*

(iv) $\mathrm{c}^{J^{[q]}}(\mathfrak{a}) = q \cdot \mathrm{c}^J(\mathfrak{a})$.

Proof. The first three are due to taking the limit in 3.4.4 (i), (ii), and (iv) respectively. As for the fourth, with a change of variable and 3.4.4 (iii), we get:

$$\mathrm{c}^J(\mathfrak{a}) = \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^{J}(p^e)}{p^e} = \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^{J}(p\cdot p^e)}{p\cdot p^e} = \frac{1}{p}\cdot \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^{J}(p\cdot p^e)}{p^e} = \frac{1}{p}\cdot \lim_{e\to\infty} \frac{\nu_{\mathfrak{a}}^{J^{[p]}}(p^e)}{p^e} = \frac{1}{p}\cdot \mathrm{c}^{J^{[p]}}(\mathfrak{a}).$$

If $q = p^e$, iterating the result e times we get the desired statement. □

Example 3.4.9. Let $R = \mathbb{K}[x, y, z]$. Let $I = (xy, xz)$. Then, $\nu_I^I(p^e) = 2(p^e - 1)$, so $\mathrm{c}^I(I) = 2$.

Proof. Let $q = p^e$. In order to prove $\nu_I^I(q) = 2(q-1)$, we have to check both $I^{2(q-1)+1} \subseteq I^{[q]}$ and $I^{2(q-1)} \nsubseteq I^{[q]}$.

For the containment, let g be a generator of $I^{2(q-1)+1}$, namely $g = (xy)^\alpha \cdot (xz)^\beta$, with $\alpha + \beta = 2(q-1) + 1$. By the pigeonhole principle, either α or β is greater than or equal to q. We can assume without loss of generality that $\alpha \geq q$. Then, $g \in (xy)^\alpha \subseteq (xy)^q \subseteq ((xy)^q, (xz)^q) = I^{[q]}$. We have seen that all generators of $I^{2(q-1)+1}$ belong to $I^{[q]}$, and so $I^{2(q-1)+1} \subseteq I^{[q]}$.

For the non-containment, consider the element $f := (xy)^{q-1}(xz)^{q-1} \in I^{2(q-1)}$. We will verify $f \notin I^{[q]} = ((xy)^q, (xz)^q)$. If f was in $I^{[q]}$, because f is a monomial and $I^{[q]}$ a monomial ideal, either $(xy)^q$ or $(xz)^q$ divide f. In the first case, y^q would divide f, which is a contradiction since $f = y^{q-1} \cdot (x^{2q-2}z^{q-1})$. Likewise, in the second case z^q would divide f, which is a contradiction as well. $\square$

When trying to find the F-threshold of an ideal, one can use the computer algebra software Macaulay2 to verify a hypothesised value of $\nu_{\mathfrak{a}}^J$ for some small cases.

Example 3.4.10. Let R and I be as in the previous example. Then, the following code can check the containment and non-containment for reasonable values of p and e.

```
p = 7
e = 2
q = p^e

--hypothesised nu
hnu = 2*(q-1)

k = ZZ/p
kk = k[x,y,z]

I = ideal(x*y, x*z)

--calculation of I^[q]
J = ideal(apply(flatten entries generators I, i-> i^q))

--should return false
isSubset(I^(hnu),J)

--should return true
isSubset(I^(hnu + 1),J)

```

Indeed, the last two functions return `false` and `true` respectively.

Furthermore, one can benefit from the FrobeniusThresholds package from [HSTW21] and use the `frobeniusNu` function to compute $\nu_J^{\mathfrak{a}}(q)$ directly, but only in a polynomial ring over a field of characteristic p. This function is implemented as a binary search on a restricted range of t, which then checks containments. By default, ideal containment is checked using the standard `isSubset` function from M2, but there are some optimisations that shine in the polynomial ring setting, such as checking the containment via Frobenius roots (as defined in [BMS08]). There are other optimisations that greatly speed up computations in the case of the threshold of a single polynomial with certain special properties (such as diagonal, binomial, and some others).

The following code illustrates how to use the `frobeniusNu` function and the great speed-up provided by using Frobenius roots instead of the standard ideal containment functions.

```
loadPackage "FrobeniusThresholds"

p = 7
e = 4
q = p^e

--hypothesised nu
hnu = 2*(q-1)

k = ZZ/p
kk = k[x,y,z]

I = ideal(x*y, x*z)

time nu1 = frobeniusNu(e, I, I)
    -- used 79.7303 seconds

time nu2 = frobeniusNu(e, I, I, ContainmentTest =>
FrobeniusRoot)
    -- used 1.05625 seconds

nu2 == hnu
    -- true

```

While no more code is included, as it is not relevant in the presence of proofs, Macaulay2 has been used throughout the development of this work to gain intuitions and find (counter)examples to conjectures.

4 The Frobenius threshold for radical ideals

In this chapter, we will compute the value of the F-threshold for radical ideals in a regular ring. Recall example 3.4.9, which will be key throughout the whole chapter:

Example. Let $R = \mathbb{K}[x, y, z]$. Let $I = (xy, xz)$. Then, $\nu_I^I(q) = 2(q-1)$, so $\mathrm{c}^I(I) = 2$.

Looking at this example and its proof, one might naively guess $\nu_I^I(q) = \mu(I)(q-1)$. Indeed, for radical ideals, the containment $I^{\mu(I)(q-1)+1} \subseteq I^{[q]}$ follows from the proof of 3.4.2, but one quickly realises that this estimate on $\nu_I^I(q)$ may be too high, as in the following example.

Example 4.0.1. Let $R = \mathbb{K}[x, y, z]$. Let $J = (xy, xz, yz)$. Then, $\nu_J^J(q) < 3(q-1)$.

Proof. Consider a generator of $J^{3(q-1)}$, namely $g = (xy)^\alpha (xz)^\beta (yz)^\gamma$, where $\alpha+\beta+\gamma = 3(q-1)$. If α, β or γ are $> q$, then g is a multiple of $(xy)^q$, $(xz)^q$ or $(yz)^q$ respectively, so $g \in J^{[q]}$. However, the only remaining case is $g = (xy)^{q-1}(xz)^{q-1}(yz)^{q-1} = (xy)^{2q-2}z^{2q-2} = (xy)^q (xy)^{q-1} z^{q-2}$, so $g \in J^{[q]}$ as well. Thus, $J^{3(q-1)} \subseteq J^{[q]}$ and by definition, $\nu_J^J < 3(q-1)$. $\square$

Even if, in general, $\nu_I^I(q) \neq \mu(I)(q-1)$, we will see how the regularity and radical hypotheses allow us to employ a generalisation of the techniques used in the proof of 3.4.9, which do rely on the number of generators. This requires us to introduce the following concept.

Definition 4.0.2. Let I be an ideal with no embedded primes. The *big height* of I is the maximum height of the associated (minimal) primes of I, and is sometimes denoted by $\mathrm{bight}(I)$.

Now, we can state the main result of the chapter:

Theorem 4.0.3. *Let R be a regular ring of characteristic p, and let I be a radical ideal in R. Then,*

$$\nu_I^I(q) = H(q-1),$$

where $H = \mathrm{bight}(I)$.

By definition of the F-threshold, this implies that $\mathrm{c}^I(I) = H$. In order to prove the result, we will take inspiration from the ideas of 3.4.9, so for the containment, we will use the pigeonhole principle, and for the non-containment, we will find a particular element of the ordinary power that is not in the Frobenius power. This techniques, however, will only work in certain localisations of I, so we rely on the following key lemma, that ties the proof together by allowing us to check containments at the localisations of associated primes.

Proposition 4.0.4. *Let I and J be ideals in a Noetherian ring R (not necessarily regular). Then, $I \subseteq J \iff I_\mathfrak{p} \subseteq J_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(R/J)$.*

Proof. The direct implication is clear, because localisation is a ring homomorphism. For the converse, assume that $I_\mathfrak{p} \subseteq J_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}(R/J)$. Consider the sets:

$$V := \bigcup_{\mathfrak{p} \in \operatorname{Ass}(R/J)} \mathfrak{p} \text{ and } U := R \setminus V.$$

We will denote the localisation $U^{-1}R$ by R_U. Now, we will prove the containment $I_U \subseteq J_U$, which will aid us in proving $I \subseteq J$, our desired containment. In order to do so, first consider $\operatorname{MaxAss}(R/J)$, the associated primes of J that are maximal with respect to the inclusion. We observe that, because of their maximality,

$$V = \bigcup_{\mathfrak{p} \in \operatorname{Ass}(R/J)} \mathfrak{p} = \bigcup_{\mathfrak{p} \in \operatorname{MaxAss}(R/J)} \mathfrak{p}.$$

So from now on we can consider only the maximal associated primes instead of all associated primes. Now, let $\frac{a}{s} \in I_U$ (we can assume $a \in I$, $s \in U$). We will see that $\frac{a}{s} \in J_U$. If $\mathfrak{p} \in \operatorname{MaxAss}(R/J)$ and $s \in U$, then $s \notin \mathfrak{p}$, so $\frac{a}{s} \in I_\mathfrak{p} \subseteq J_\mathfrak{p}$. This means that, for every $\mathfrak{p} \in \operatorname{MaxAss}(R/J)$, there exists an element $s_\mathfrak{p} \notin \mathfrak{p}$ such that $as_\mathfrak{p} \in J$.

If $\mathfrak{p} \in \operatorname{MaxAss}(R/J)$, then by maximality we have that $\mathfrak{q} \not\subseteq \mathfrak{p}$ for every other $\mathfrak{q} \in \operatorname{MaxAss}(R/J) \setminus \{\mathfrak{p}\}$. This implies that

$$\bigcap_{\mathfrak{q} \in \operatorname{MaxAss}(R/J) \setminus \{\mathfrak{p}\}} \mathfrak{q} \not\subseteq \mathfrak{p},$$

so we can find elements $s'_\mathfrak{p} \in \mathfrak{q}$, $s'_\mathfrak{p} \notin \mathfrak{p}$, for all $\mathfrak{q} \in \operatorname{MaxAss}(R/J) \setminus \{\mathfrak{p}\}$. Now, consider the following element:

$$s' := \sum_{\mathfrak{p} \in \operatorname{MaxAss}(R/J)} s_\mathfrak{p} s'_\mathfrak{p}$$

We observe that this construction guarantees $s' \notin \mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{MaxAss}(R/J)$, so $s' \in U$. Furthermore, because $as_{\mathfrak{p}} \in J$,

$$as' = \sum_{\mathfrak{p} \in \mathrm{MaxAss}(R/J)} as_{\mathfrak{p}} s'_{\mathfrak{p}} \in J.$$

So we conclude that $\frac{a}{1} \in J_U \implies \frac{a}{s} \in J_U$, and we have $I_U \subseteq J_U$.

Finally, we can see $I \subseteq J$. Let $a \in I$. Then, $\frac{a}{1} \in I_U \subseteq J_U \implies \frac{a}{1} = \frac{b}{s}$, with $b \in J$ and $s \in U$, so there exists an $u \in U$ such that $u(sa - b) = 0$. Rearranging the terms, we get $us \cdot a = ub \in J \implies \overline{us \cdot a} = \overline{0}$ in R/J. However, $V = R \setminus U$ is the set of zero divisors of R/J, so $us \in U$ cannot be a zero divisor, and thus we conclude that $\overline{a} = \overline{0} \implies a \in J$. $\square$

4.1 Containment

Let I be a radical ideal of R, a regular ring, and let $H = \mathrm{bight}(I)$. In this section, we will prove the containment $I^{H(q-1)+1} \subseteq I^{[q]}$, for all $q = p^e$, $e \in \mathbb{N}^*$. Because of 4.0.4, we can just check $I_{\mathfrak{p}}^{H(q-1)+1} \subseteq I_{\mathfrak{p}}^{[q]}$ for all $\mathfrak{p} \in \mathrm{Ass}\left(R/I^{[q]}\right)$. However, in 3.3.9 we have seen $\mathrm{Ass}\left(R/I\right) = \mathrm{Ass}\left(R/I^{[q]}\right)$, and it suffices to check for all $\mathfrak{p} \in \mathrm{Ass}\left(R/I\right)$.

Let $I = Q_1 \cap \cdots \cap Q_m$ be the irredundant primary decomposition of I, with Q_1 a primary component with height H (the big height). We recall that, because I is radical, all the primary components are prime and $\mathrm{Min}(I) = \mathrm{Ass}(R/I)$, so if $\mathfrak{p} \in \mathrm{Ass}\left(R/I\right)$, then $\mathfrak{p} = Q_i$ for some $i \in \{1, \ldots, m\}$. Every Q_j $(j \neq i)$ has at least one generator not in Q_i, because the decomposition is irredundant. This generator becomes invertible under localisation by $Q_i = \mathfrak{p}$, so $(Q_j)_{\mathfrak{p}} = R_{\mathfrak{p}}$ for all $j \neq i$. This means that, for all $i \in \{1, \ldots, m\}$,

$$I_{\mathfrak{p}} = (Q_1)_{\mathfrak{p}} \cap \cdots \cap (Q_i)_{\mathfrak{p}} \cap \cdots \cap (Q_m)_{\mathfrak{p}} = (Q_i)_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}.$$

Now, we recall that R is a regular ring, so $R_{\mathfrak{p}}$ is a regular local ring with maximal ideal $\mathfrak{p}_{\mathfrak{p}} = I_{\mathfrak{p}}$. This means that $\mu\left(I_{\mathfrak{p}}\right) = \mathrm{ht}_{R_{\mathfrak{p}}}\left(I_{\mathfrak{p}}\right) = \mathrm{ht}_R\left(\mathfrak{p}\right) = h_i \leq H$, so $I_{\mathfrak{p}}$ is generated by no more than H elements. We name these generators $I_{\mathfrak{p}} = (f_1, \ldots, f_{h_i})$. Next, we will see that

$$I_{\mathfrak{p}}^{h_i(q-1)+1} = (f_1, \ldots, f_{h_i})^{h_i(q-1)+1} \subseteq \left(f_1^q, \ldots, f_{h_i}^q\right) = I_{\mathfrak{p}}^{[q]}.$$

It suffices to show that the generators of $I_{\mathfrak{p}}^{h_i(q-1)+1}$ are inside of $I_{\mathfrak{p}}^{[q]}$. Indeed, consider a generator $f_1^{a_1} \cdot \ldots \cdot f_{h_i}^{a_{h_i}}$, with $a_1 + \ldots + a_{h_i} = h_i(q-1) + 1$. Then, the Pigeonhole Principle guarantees us that $a_k \geq q$ for at least one $k \in \{1, \ldots, h_i\}$, so the element $f_1^{a_1} \cdot \ldots \cdot f_{h_i}^{a_{h_i}}$ is a multiple of f_k^q and we have the desired inclusion.

Because $h_i \leq H$, in general we have $I_{\mathfrak{p}}^{H(q-1)+1} \subseteq I_{\mathfrak{p}}^{[q]}$ for all $\mathfrak{p} \in \mathrm{Ass}\,(R/I)$, as we wanted to see. $\square$

4.2 Non-containment

Let I be a radical ideal of R, a regular ring, and let $H = \mathrm{bight}(I)$. In this section, we will prove the non-containment $I^{H(q-1)} \nsubseteq I^{[q]}$.

Let Q be a primary component of I with maximum height H, so we can write the irredundant primary decomposition as $I = Q \cap Q_1 \cap \cdots \cap Q_m$. This decomposition is unique, since for radical ideals, $\mathrm{Min}(I) = \mathrm{Ass}(R/I)$. Furthermore, all primary components are prime.

We are going to use the same strategy as before. By 4.0.4, it suffices to find a $\mathfrak{p} \in \mathrm{Ass}\left(R/I^{[q]}\right)$ such that $I_{\mathfrak{p}}^{H(q-1)} \nsubseteq I_{\mathfrak{p}}^{[q]}$. That is precisely what happens if we take $\mathfrak{p} = Q \in \mathrm{Ass}(R/I) = \mathrm{Ass}\left(R/I^{[q]}\right)$. Just as with the containment, localizing at a minimal (associated) prime, $I_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}$ is the maximal ideal of the regular local ring $R_{\mathfrak{p}}$. By regularity, $\mu\left(I_{\mathfrak{p}}\right) = \mathrm{ht}_{R_{\mathfrak{p}}}\left(I_{\mathfrak{p}}\right) = \mathrm{ht}_R\left(\mathfrak{p}\right) = H$, so $I_{\mathfrak{p}}$ is generated minimally by H elements: $I_{\mathfrak{p}} = (f_1, \ldots, f_H)$. Since $I_{\mathfrak{p}} = \sqrt{I_{\mathfrak{p}}} = \sqrt{(f_1, \ldots, f_H)}$, $\{f_1, \ldots, f_H\}$ is a system of parameters of $R_{\mathfrak{p}}$. If this system of parameters were a regular sequence, then by 2.1.7, the element $f_1^{q-1} \cdot \ldots \cdot f_H^{q-1} \in I_{\mathfrak{p}}^{H(q-1)}$ would not belong to $I_{\mathfrak{p}}^{[q]} = (f_1^q, \ldots f_H^q)$, so

$$I_{\mathfrak{p}}^{H(q-1)} \nsubseteq I_{\mathfrak{p}}^{[q]}$$

and we would be done. Finally, we will check that this is the case, that is, that $f_1, \ldots, f_H$ is a regular sequence in $R_{\mathfrak{p}}$. Following definition 2.1.1:

(i) By 2.2.10, a regular local ring is a domain, so f_1 is a regular element of $R_{\mathfrak{p}}$.

(ii) We can check inductively: suppose that $f_1, \ldots, f_i$ is a regular sequence. We will check that f_{i+1} is a regular element of $R_{\mathfrak{p}}/(f_1, \ldots, f_i)$. First, $S := R_{\mathfrak{p}}/(f_1, \ldots, f_i)$ is a local ring with maximal ideal $\overline{\mathfrak{p}_{\mathfrak{p}}} = \left(\overline{f_{i+1}}, \ldots, \overline{f_H}\right)$. These $H-i$ generators are minimal. Indeed, suppose without loss of generality that $\overline{f_{i+1}}$ can be removed, so $\overline{f_{i+1}} = \overline{a_{i+2}f_{i+2}} + \ldots + \overline{a_H f_H}$. Lifting to $R_{\mathfrak{p}}$, there exists an element $f \in (f_1, \ldots, f_i)$ such that $f_{i+1} = f + a_{i+2}f_{i+2} + \ldots a_H f_H$, so f_{i+1} would be redundant in $R_{\mathfrak{p}} = (f_1, \ldots, f_H)$, which is a contradiction since we know $\mu_{R_{\mathfrak{p}}}(\mathfrak{p}_{\mathfrak{p}}) = H$. So $\mu_S(\overline{\mathfrak{p}_{\mathfrak{p}}}) = H - i$.

On the other hand, by hypothesis, $\overline{f_i}$ is a non-zero-divisor in $R_{\mathfrak{p}}/(f_1, \ldots, f_{i-1})$, so by 2.2.9

$$\dim(R_{\mathfrak{p}}/(f_1, \ldots, f_i)) = \dim(R_{\mathfrak{p}}/(f_1, \ldots, f_{i-1})) - 1.$$

Iterating, $\dim(R_{\mathfrak{p}}/(f_1, \ldots, f_i)) = \dim(R_{\mathfrak{p}}) - i = H - i$. We have therefore shown that $\dim(S) = \mu_S(\overline{\mathfrak{p}_{\mathfrak{p}}}) = H - i$, so S is a regular local ring, and thus a domain, so $\overline{f_{i+1}}$ is a non-zero-divisor of $\dim(R_{\mathfrak{p}}/(f_1, \ldots, f_i))$.

(iii) $R_{\mathfrak{p}}/(f_1, \ldots, f_H) \neq 0$, since $(f_1, \ldots, f_H) = \mathfrak{p}_{\mathfrak{p}} \subsetneq R_{\mathfrak{p}}$.

□

4.3 Extending the result

The current statement of the result came to be by proving it in a scenario very similar to 3.4.9, namely edge ideals, which are a particular kind of square-free monomial ideals. We then saw that our proof generalised to square-free monomial ideals, and later on to radical ideals. Once we obtained the present result, it was natural to ask ourselves whether the current methods could be further generalised. Given definition 4.0.2 of the big height, it made sense to consider the F-threshold of ideals without embedded primes, a condition slightly more general than radical. The following example, where I has no embedded primes, but is not radical, shows how the result does not generalise in this way.

Example 4.3.1. Let $R = \mathbb{K}[x, y]$, and $I = (x, y)^2 = (x^2, y^2, xy)$. Then, $\mathrm{c}^I(I) = 3/2$.

Proof. By 3.4.8 (iii), $\mathrm{c}^I((x, y)^2) = \frac{1}{2}\mathrm{c}^I((x, y))$. We will now see that $\mathrm{c}^I((x, y)) = 3$ by proving the value of $\nu^I_{(x,y)} = 3(q-1)+1$. As always, this means proving both a containment and a non containment.

(i) $\underline{(x, y)^{3(q-1)+2} \subseteq (x^2, y^2, xy)^{[q]}}$: Let g a generator of $(x, y)^{3(q-1)+2} = (x, y)^{3q-1}$. Then, $g = x^\alpha y^\beta$, with $\alpha + \beta = 3q - 1$. If $\alpha < q$, then $\beta \geq 2q$ and g is a multiple of $y^{2q} \in I^{[q]}$. Likewise, if $\beta < q$, g is a multiple of x^{2q}. Otherwise, if $\alpha, \beta \geq q$, then $g \in (x^q y^q) \subseteq I^{[q]}$, so every generator of $(x, y)^{3(q-1)+2}$ is always in $I^{[q]}$.

(ii) $\underline{(x, y)^{3(q-1)+1} \nsubseteq (x^2, y^2, xy)^{[q]}}$: Consider the element $f := x^{q-1}y^{2q-1} \in (x, y)^{3(q-1)+1}$. Suppose that $x^{q-1}y^{2q-1} \in (x^{2q}, y^{2q}, x^q y^q) = I^{[q]}$. Then, because f is a monomial, and $I^{[q]}$ a monomial ideal, either x^{2q}, y^{2q}, or $x^q y^q$ divide f, which is impossible in every case.

□

The main problem is that the radicality of I was heavily employed in our previous proofs: the core idea being that, for every associated (minimal) prime $\mathfrak{p}$ of I, $R_\mathfrak{p}$ is a regular local ring with maximal ideal $\mathfrak{p}_\mathfrak{p} = I_\mathfrak{p}$, which then allows us to control the number of generators of $I_\mathfrak{p}$. However, in an ideal without embedded primes, we observe that this condition is equivalent to being radical.

Observation 4.3.2. Let I be an ideal without embedded primes. Then, the following are equivalent:

(i) $I_\mathfrak{p} = \mathfrak{p}_\mathfrak{p}$, for every $\mathfrak{p} \in \operatorname{Ass}(R/I) = \operatorname{Min}(I)$.

(ii) I is radical.

Proof. We have seen (ii) $\implies$ (i) in a previous (4.1) proof. To see (i) $\implies$ (ii), let $x \in \sqrt{I} = \cap_{\mathfrak{p}\in \operatorname{Min}(I)}\mathfrak{p}$. Then, $x \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Min}(I)$, so $\frac{x}{1} \in \mathfrak{p}_\mathfrak{p} = I_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Min}(I)$. Thus, $(x)_\mathfrak{p} = (\frac{x}{1}) \subseteq I_\mathfrak{p}$, for all $\mathfrak{p} \in \operatorname{Min}(I) = \operatorname{Ass}(R/I)$. By 4.0.4, $(x) \subseteq I$, so $x \in I$. We have seen that $\sqrt{I} \subseteq I$, and because in general $I \subseteq \sqrt{I}$, we conclude that $\sqrt{I} = I$. $\square$

In light of this observation, we see that relaxing the radical hypothesis in any way will make the core idea of the containment proof fail, so a generalisation of the current results would likely have to employ different techniques.

5 Simplicial complexes and face rings

5.1 Definitions

Simplicial complexes are mathematical objects that, roughly speaking, represent a set of points, edges, triangles and in general, n-simplexes (n-dimensional triangles). While being a field of study in and of themselves, they arise mainly in algebraic topology and combinatorics. Simplicial complexes are mostly studied through algebraic (homological) and combinatoric methods. In this chapter, we introduce some basic concepts about simplicial complexes and relate them to our previous results. The contents of this section are freely drawn from [BH98], Chapter 5.1.

Definition 5.1.1. Let $V = \{v_1, \ldots, v_n\}$ be a finite set. Then, a *simplicial complex* Δ on V is a collection of subsets of V such that for every $F \subseteq G \in \Delta$, then $F \in \Delta$. In this context:

- The elements of V are called *vertices*.
- The elements of Δ, *faces*.
- The maximal faces under inclusion are called *facets*.
- The *dimension* of a face $F \in \Delta$ is defined as $\dim F := |F| - 1$.
- Faces of dimension 1 are called *edges*.
- The *dimension* of Δ is defined as $\dim \Delta := \max \{\dim F \mid F \in \Delta\}$.

Example 5.1.2. Often, when describing a simplicial complex, we slightly abuse the usual notation for sets and omit parentheses and commas between elements of a face. So, if we let $V = \{v_1, v_2, v_3, v_4\}$, then the simplicial complex $\Delta = \{\{v_1, v_2, v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \varnothing\}$ can be written as $\Delta = \{v_1v_2v_3, v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_1, v_2, v_3, v_4, \varnothing\}$. Δ has a total of 10 faces: $\varnothing$ (dimension -1), 4 vertices (dimension 0), 4 edges (dimension 1) and one facet of dimension 2. Thus,

$\dim \Delta = 2$. Simplicial complexes are usually represented graphically, as illustrated by Figure 5.1.

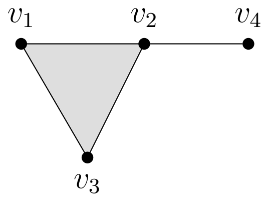


FIGURE 5.1: Example 5.1.2. The shaded area indicates the face $v_1v_2v_3$.

Definition 5.1.3. Let $V = \{v_1, \dots, v_n\}$ be a set of vertices, and $\{F_1, \dots, F_m\} \subseteq \mathcal{P}(V)$ a collection of subsets of V. Then, the (unique) smallest simplicial complex that contains $\{F_1, \dots, F_n\}$ is said to be *generated by* $\{F_1, \dots, F_n\}$, and is denoted by $\langle F_1, \dots, F_m \rangle$. This simplicial complex consists of all subsets $G \subseteq V$ which are contained in some F_i.

Example 5.1.4. Let $V = \{v_1, v_2, v_3, v_4\}$ and $\mathcal{F} := \{v_2v_3v_4, v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_4\} \subseteq \mathcal{P}(V)$. Then, $\langle \mathcal{F} \rangle = \{v_2v_3v_4, v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4, v_1, v_2, v_3, v_4, \varnothing\}$.

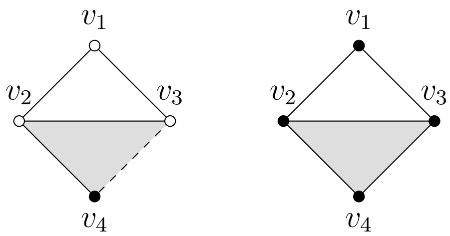


FIGURE 5.2: Example 5.1.3. On the left, $\mathcal{F}$. White points and dashed lines indicate that a vertex or edge (respectively) is not in $\mathcal{F}$. On the right, $\langle \mathcal{F} \rangle$.

5.2 The Stanley–Reisner correspondence

Definition 5.2.1. Let Δ be a simplicial complex on $V = \{v_1, \dots, v_n\}$, and $\mathbb{K}$ a field. Then, the *Stanley–Reisner ring*, or *face ring*, of Δ with respect to $\mathbb{K}$ is defined as

$$\mathbb{K}[\Delta] := \mathbb{K}[x_1, \dots, x_n] / I_\Delta,$$

where I_Δ is the ideal generated by all monomials $x_{i_1}x_{i_1} \cdots x_{i_j}$ such that $v_{i_1}v_{i_1} \cdots v_{i_j} \notin \Delta$.

Example 5.2.2. Consider Example 5.1.2, where $\Delta = \langle v_1v_2v_3, v_2v_4 \rangle$. Then, $I_\Delta = (x_1x_2x_4,\ x_1x_4x_3, x_3x_2x_4, x_1x_4, x_3x_4) = (x_1x_4, x_3x_4)$.

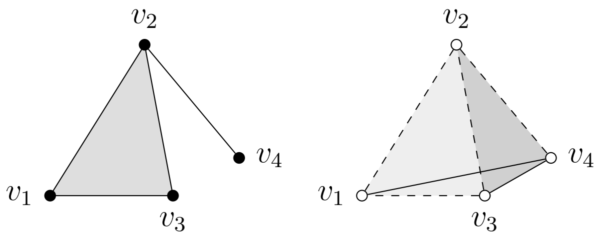


FIGURE 5.3: On the left, Δ. On the right, the non-faces of Δ.

We observe that, by definition, I_Δ is a square-free monomial ideal. The following two propositions describe very particular properties of square-free monomial ideals.

Proposition 5.2.3. *Let I be a square-free monomial ideal. Then, I is a finite intersection of monomial prime ideals (generated by variables).*

Proof. We are going to use the following lemma without proof.

Lemma 5.2.4. *Let I, J be monomial ideals. If $I = (f_1, \dots, f_n)$ and $J = (g_1, \dots, g_m)$, with f_i, g_j monomials for every $1 \leq i \leq n$, $1 \leq j \leq m$, then*

$$I \cap J = (\{\mathrm{lcm}(f_i, g_j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m\})$$

Let $I = (f_1, \dots, f_n)$, where f_i are monomials. We will proceed by induction on n. If $f_1 = x_{j_1} \cdot \ldots \cdot x_{j_r}$, then for $n = 1$ we have

$$I = (x_{j_1} \cdot \ldots \cdot x_{j_r}) = \bigcap_{k=1}^{r} (x_{j_k})\,.$$

In the general case, by the previous Lemma 5.2.4,

$$I = (x_{j_1} \cdot \ldots \cdot x_{j_r}, f_2, \dots, f_n) = \bigcap_{k=1}^{r} (x_{j_k}, f_2, \dots, f_n)\,.$$

By induction hypothesis, $(f_2, \dots, f_n) = \cap_{i=1}^{s} \mathfrak{p}_i$, where $\mathfrak{p}_i$ are prime, monomial ideals (generated by variables). Then,

$$I = \bigcap_{k=1}^{r} \bigcap_{i=1}^{s} (x_{j_k}, \mathfrak{p}_i)\,,$$

and we are done because $(x_{j_k}, \mathfrak{p}_i)$ is a prime, monomial ideal (generated by variables). $\square$

Proposition 5.2.5. *Let I be a monomial ideal. Then I radical if and only if it is square-free.*

Proof. For the direct implication, assume that I is not square free. Then, there is a generator (from a minimal set of monomial generators) of I of the form $x_{i_1}^{\alpha_1} \cdot \ldots \cdot x_{j_r}^{\alpha_r}$, where $\alpha_j > 0$ for all j, and $\alpha_k > 1$ for some k. Let $\alpha = \max\{\alpha_1, \ldots, \alpha_r\}$. Then, $x_{i_1}^{\alpha} \cdot \ldots \cdot x_{j_r}^{\alpha} = (x_{i_1} \cdot \ldots \cdot x_{j_r})^{\alpha}$ is a multiple of $x_{i_1}^{\alpha_1} \cdot \ldots \cdot x_{j_r}^{\alpha_r}$, so $(x_{i_1} \cdot \ldots \cdot x_{j_r})^{\alpha} \in I$. By definition of radical ideal, $x_{i_1} \cdot \ldots \cdot x_{j_r} \in I$, but $x_{i_1} \cdot \ldots \cdot x_{j_r} \notin I$, since by minimality of the generating set no generator divides $x_{i_1} \cdot \ldots \cdot x_{j_r}$. Therefore, I is not radical.

For the converse, if I is square-free, by 5.2.3 it is a finite intersection of prime ideals, which in a Noetherian ring is equivalent to being radical. $\square$

These properties allow us to interpret the primary (prime) decomposition of I_Δ in terms of Δ.

Theorem 5.2.6. *Let Δ be a simplicial complex, and $\mathbb{K}$ a field. Then,*

$$I_\Delta = \bigcap_F \mathfrak{p}_F,$$

where the intersection is taken over all facets F of Δ, and $\mathfrak{p}_F$ denotes the (prime) ideal generated by all x_i such that $v_i \notin F$.

Proof. Because I_Δ is a square-free monomial ideal, by 5.2.3 and 5.2.5 I_Δ is a finite intersection of minimal prime ideals generated by variables. Let $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_r})$ be an ideal generated by variables. We observe that $\mathfrak{p}$ is a minimal prime of I_Δ if and only if $\{v_1, \ldots, v_n\} \setminus \{v_{i_1}, \ldots, v_{i_r}\}$ is a facet of Δ.

Indeed, assume that $\mathfrak{p}$ is a minimal prime of I_Δ and let $F := \{v_{j_1} \ldots v_{j_s}\} = \{v_1, \ldots, v_n\} \setminus \{v_{i_1}, \ldots, v_{i_r}\}$. F is a face of Δ: if it wasn't, then $x_{j_1} \ldots x_{j_s} \in I_\Delta$, so $x_{j_1} \cdot \ldots \cdot x_{j_s} \in \mathfrak{p}$, which is a contradiction because $\mathfrak{p}$ does not contain any x_{j_k}, for any $1 \leq k \leq s$. Furthermore, F is a facet. If it wasn't, then there would be a face of Δ that contains it, so there would exist a $k \in \{i_1, \ldots, i_r\}$ such that $v_{j_1} \ldots v_{j_s} \subsetneq v_{j_1} \ldots v_{j_s} v_k$. If $v_{j_1} \ldots v_{j_s} v_k$ is a face of Δ, then any non-face has to contain at least one of the remaining vertices, so for every monomial generator of I_Δ, there exists an element of $\{x_{i_1}, \ldots, \hat{x_k}, \ldots, x_{i_r}\}$[1] that divides it. Thus, $I_\Delta \subseteq (x_{i_1}, \ldots, \hat{x_k}, \ldots, x_{i_r}) \subsetneq \mathfrak{p} = (x_{i_1}, \ldots, x_{i_r})$, which is a contradiction because $\mathfrak{p}$ is a minimal prime.

Conversely, assume that $F = \{v_{j_1} \ldots v_{j_s}\} = \{v_1, \ldots, v_n\} \setminus \{v_{i_1}, \ldots, v_{i_r}\}$ is a facet. We want to see that $(x_{i_1}, \ldots, x_{i_r}) = \mathfrak{p}$ is a minimal prime of I_Δ. Because F is a face

[1] A hat indicates that the element is missing.

of Δ, the non-faces contain at least one of the vertices not in F, so the monomials associated to non-faces are multiples of $\{x_{i_1}, \dots, x_{i_r}\}$. Thus, $I_\Delta \subseteq \mathfrak{p} = (x_{i_1}, \dots, x_{i_r})$. To see that it is a minimal prime, assume that there exists a prime ideal $\mathfrak{q} \in \mathrm{Min}(I_\Delta)$ such that $I_\Delta \subseteq \mathfrak{q} \subsetneq \mathfrak{p}$. At the beginning of the proof we observed that the minimal primes of I_Δ are generated by variables, so we can assume (rearranging variables) that $\mathfrak{q} = (x_{i_1}, \dots, x_{i_l}) \subsetneq (x_{i_1}, \dots, x_{i_r})$, with $l < r$. Let $F' := \{v_1, \dots, v_n\} \setminus \{v_{i_1}, \dots, v_{i_l}\} = \{v_{j_1} \dots v_{j_s} v_{i_{l+1}} \dots v_{i_r}\}$. Then F' is not a face of Δ, because $F \subsetneq F'$ is a facet, so $x_{j_1} \cdots x_{j_s} x_{i_{l+1}} \cdots x_{i_r} \in I_\Delta \subseteq \mathfrak{q}$. As $\mathfrak{q}$ is a prime ideal, then at least one of the variables $x_{j_1}, \dots, x_{j_s}, x_{i_{l+1}}, \dots x_{i_r}$ is in $\mathfrak{q}$, which is a contradiction since none of them belong to $\mathfrak{q}$. □

Example 5.2.7. Let $\Delta = \langle v_1v_2v_3, v_1v_5, v_3v_4 \rangle$. Then, $I_\Delta = (x_1x_4, x_2x_5, x_2x_4, x_3x_5, x_4x_5)$. The primary decomposition of I_Δ is $I_\Delta = (x_4, x_5) \cap (x_2, x_3, x_4) \cap (x_1, x_2, x_5)$. As stated in 5.2.6, we see that the primary component (x_4, x_5) corresponds to the facet $v_1v_2v_3$, as v_4, v_5 are the vertices that do not belong to $v_1v_2v_3$. Likewise, the components (x_2, x_3, x_4) and (x_1, x_2, x_5) correspond to the facets v_1v_5 and v_3v_4, respectively.

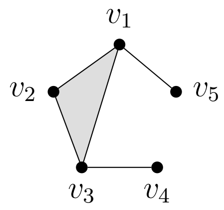


FIGURE 5.4: $\Delta = \langle v_1v_2v_3, v_1v_5, v_3v_4 \rangle$.

We have seen that, given a simplicial complex Δ, one can create an associated square-free monomial ideal I_Δ (the one generated by by the monomials associated to the non-faces of Δ). On the other hand, for every square-free monomial ideal $J \subseteq \mathbb{K}[x_1, \dots, x_n]$, we can find a simplicial complex on $V = \{v_1, \dots, v_n\}$, Δ_J, such that $I_{\Delta_J} = J$. We construct Δ_J in the following way:

$$\Delta_J := \{F \subseteq V \mid F = v_{i_1} \dots v_{i_r},\ x_{i_1} \cdot \dots \cdot x_{i_r} \notin J\}.$$

Δ_J is a simplicial complex because if $F = v_{i_1} \dots v_{i_r} \in \Delta_J$, then $x_{i_1} \cdot \dots \cdot x_{i_r} \notin J$, so any product of a subset of these variables cannot belong to J either. In other words, any subset of F belongs in Δ_J, so Δ_J is indeed a simplicial complex. The procedure described above defines a bijection $\{$*square-free monomial ideals*$\} \leftrightarrow \{$*simplicial complexes*$\}$, better known as the *Stanley–Reisner correspondence*, and illustrated in Table 5.1. This correspondence is inclusion-reversing: if Δ and Δ' are simplicial complexes on the same vertex set, then $\Delta \subseteq \Delta' \iff I_{\Delta'} \subseteq I_\Delta$.

Simplicial complexes		**Square-free monomial ideals**
Δ	$\longmapsto$	I_Δ: Ideal generated by the monomials corresponding to non-faces of Δ.
Δ_J: Simplicial complex generated by the faces that do not correspond to monomials in J.	$\longleftarrow$	J

TABLE 5.1: The Stanley–Reinser correspondence

Example 5.2.8. Let $J = (x_1x_4, x_3x_4) \subseteq \mathbb{K}[x_1, \ldots, x_4]$. Then, $\Delta_J = \{v_1v_2v_3, v_1v_2, v_2v_3, v_1v_3, v_2v_4, v_1, v_2, v_3, v_4, \varnothing\} = \langle v_1v_2v_3, v_2v_4\rangle$. As we saw in 5.2.2,

$$I_{\Delta_J} = (x_1x_4, x_3x_4) = J.$$

5.3 Applications

In Chapter 4, we proved that the F-threshold of a radical ideal I in a regular ring is the big height of the ideal. In general, computing the big height of an ideal requires finding its primary decomposition, which might not be straightforward in some cases. Even with an algorithm, it may be costly computationally speaking. The Stanley–Reisner correspondence gives us another point of view on the primary decomposition of a square-free monomial ideal that allows us to know its big height without computing said decomposition.

Proposition 5.3.1. *Let Δ be a simplicial complex over $V = \{v_1, \ldots, v_n\}$. Then,*

$$\operatorname{bight}(I_\Delta) = n - d - 1,$$

where $d = \min\{\dim F \mid F$ is a facet of $\Delta\}$.

Proof. Because of Theorem 5.2.6 and Proposition 5.2.3, the prime components of I_Δ are generated by variables, so their height is their number of generators. As we have seen, the variables that generate a prime component correspond to vertices that are not in a particular facet of Δ, so an ideal of height H corresponds to a facet with $n-H$ vertices, that is, of dimension $n-H-1$. Thus, the big height will correspond to a facet with minimal number of vertices, and if it has dimension d, then $d = n - \operatorname{bight}(I_\Delta) - 1$, as we wanted to see. $\square$

Because every square-free monomial ideal J is the face ideal of a simplicial complex Δ_J, this proposition allows us to find the big height of a square-free monomial ideal in general, computing just Δ_J, as opposed to using a generalistic method to compute the primary decomposition. As a comparison, the default method for computing a primary decomposition of non-monomial ideals in Macaulay2 is the one proposed by Shimoyama and Yokoyama in [SY96], and it requires computing a Gröbner basis of the ideal, the complexity of which is not fully understood yet [BFS15]. In the monomial case, a more efficient algorithm is used, based on Alexander duality, which is a powerful tool in Stanley–Reisner theory. A detailed description of this algorithm can be found in [EGSS01], pg 77.

6 Conclusions

In summary, we have proved that if I is a radical ideal in a regular ring, then $\mathrm{c}^I(I) = \mathrm{bight}(I)$. Recalling Chapter 1, where we introduced our motivations and presented the F-thresholds as an invariant to measure singularities in characteristic p, one may wonder about the relevance of such a result in a regular setting. Let $H = \mathrm{bight}(I)$. Then, as $\mathrm{ht}\,(I) = \min\{\mathrm{ht}\,(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Min}(I)\}$, $\mathrm{c}^I(I) = \mathrm{ht}\,(I)$ if and only if $\mathrm{ht}\,(\mathfrak{p}) = H$, for all $\mathfrak{p} \in \mathrm{Ass}(R/I)$.

Definition 6.0.1. A ring R is called *equidimensional* if all associate primes have the same height.

Rewording our finding, we see that $\mathrm{c}^I(I) = \mathrm{ht}\,(I)$ if and only if R/I is equidimensional. As mentioned in the introduction, this is the kind of result one would try to derive from an invariant. Because Cohen–Macaulay rings are equidimensional, equidimensionality can be seen as a “first step” towards a ring being Cohen–Macaulay, so it is still a beneficial property for a non-regular ring to have.

As for further work, in Section 4.3 we saw that we have likely reached the limit to generalising the results with our techniques. A new line of work, however, would be to compute the same F-thresholds for other kinds of ideals. A fitting candidate would be the maximal ideals of determinantal rings, which are quotients of a polynomial ring by the ideal generated by minors of generic matrices.

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