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Extending classical gauge theories to E-manifolds

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Abstract

Abstract

Gauge theories describe the dynamics of a classical particle with internal degrees of freedom. The natural configuration space is associated to a G-principal bundle $\pi\colon P \longrightarrow M$ and the equations of motion are called Wong's equations. Yang-Mills theories, which are the basis of the standard model of physics, are a gauge theory with structure group $G = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. Weinstein observed that the natural phase space of gauge theories can be obtained by a symplectic reduction, and that Wong's equations are Hamiltonian. For the natural phase space of Yang-Mills theories, Montgomery observed that the natural phase space of Weinstein is a symplectic leaf of a Poisson manifold, and generalized Weinstein's isomorphism with Sternberg's phase space to Poisson manifolds. This result is called the minimal coupling procedure. In this work, we extend this setting to classical gauge theories over E-manifolds, in which the tangent bundle is replace by an integrable subbundle, called the E-tangent bundle. Manifolds with boundary, manifolds with corners, and foliated manifolds are examples of E-manifolds. Using standard results in the literature, we have proved that the formulation of classical gauge and Yang-Mills theories by Weinstein and Montgomery, respectively, hold when the base manifold is taken to be an E-manifold. This result can be applied, for example, to study the classical standard model in a Penrose compactified space-time.

Keywords: E-manifolds, gauge theory, Yang-Mills, minimal coupling, Wong's equations.

MSC2020: 53D05, 70H99.

Resumen

Las teorías gauge describen la dinámica de una partícula clásica con grados internos de libertad. El espacio de configuraciones es un fibrado asociado a un fibrado principal $\pi\colon P \longrightarrow M$ con grupo de estructura G, y las ecuaciones de movimiento reciben el nombre de ecuaciones de Wong. Las teorías Yang-Mills, las cuales son la base del modelo estándar de la física, son una teoría gauge con grupo de estructura $G = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. Weinstein demostró que el espacio de fase natural en una teoría gauge se puede obtener por medio de una reducción simpléctica y que las ecuaciones de Wong son hamiltonianas. Respecto al espacio de fases en una teoría de Yang-Mills, Montgomery observó que el espacio de fases de Weinstein es una hoja simpléctica de un espacio de Poisson y generalizó el isomorfismo de Weinstein con el espacio de fases de Sternberg a variedades de Poisson. Este resultado se conoce como el procedimiento de acoplamiento mínimo. En este trabajo extendemos los resultados conocidos para teorías de gauge clásicas a E-variedades, en las que el fibrado tangente se reemplaza por un subfibrado integrable llamado fibrado E-tangente. Las variedades con borde, las variedades con esquinas y las variedades foliadas son ejemplos de E-variedades. Usando resultados conocidos en la literatura, hemos demostrado que los formalismos de las teorías de gauge y de Yang-Mills clásicas, propuestos por Weinstein y Montgomery, respectivamente, son válidos cuando el espacio base M es una E-variedad. Este resultado puede aplicarse, por ejemplo, para el estudio del modelo estándar clásico en un espacio-tiempo compactificado de Penrose.

Palabras clave: E-variedades, teoría gauge, Yang-Mills, acoplamiento mínimo, ecuaciones de Wong.

MSC2020: 53D05, 70H99.

Resum

Les teories gauge descriuen la dinàmica d'una partícula clàssica amb graus interns de llibertat. L'espai de configuracions és un fibrat associat a un fibrat principal $\pi\colon P \longrightarrow M$ amb grup d'estructura G, i les equacions de moviment reben el nom d'equacions de Wong. Les teories Yang-Mills, les quals són la base del model estàndard de la física, són una teoria gauge amb grup d'estructura $G = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. Weinstein va provar que l'espai de fase natural d'una teoria gauge es pot descriure mitjançant una reducció simplèctica i que les equacions de Wong són hamiltonianes. Respecte de l'espai de fase en una teoria de Yang-Mills, Montgomery va observar que l'espai de fase de Weinstein és una fulla simplèctica d'una varietat de Poisson i va generalitzar l'isomorfisme de Weinstein amb l'espai de fase de Sternberg a varietats de Poisson. Aquest resultat rep el nom d'acoblament mínim. En aquest treball estenem els resultats de Weinstein i Montgomery a E-varietats, on el fibrat tangent se substitueix per un subfibrat anomenat fibrat E-tangent. Les varietats amb vora, les varietats amb cantonades i les varietats foliades són exemples d'E-varietats. Fent servir resultats coneguts, hem provat que els formalismes de les teories clàssiques de gauge i de Yang-Mills, proposades per Weinstein i Montgomery, respectivament, es poden aplicar quan l'espai base M és una E-varietat. Es pot aplicar aquest resultat, per exemple, a l'estudi del model estàndard clàssic en un espai-temps compactificat de Penrose.

Paraules clau: E-varietats, teoria gauge, Yang-Mills, acoblament mínim, equacions de Wong.

MSC2020: 53D05, 70H99.

Preface

This bachelor thesis is the culmination of a five-year journey which started when I joined CFIS. During this period of time I have evolved and grown from many fruitful discussions, from the interaction with many incredible people, and from overcoming difficult moments. This work is not only the result of my own intellectual activity, but also of the many external factors which have influenced me directly or indirectly. It is hard to believe that the outcome of this journey would have been possible without the great amount of help and support I have received. I, unfortunately, cannot give an exhaustive description of everyone who has been a part of my life me during the last five years. I am grateful to all of you, you know who you are.

During the development of this project I have been supported by a collaboration scholarship from the Spanish Ministry of Education. I have also enjoyed the economic support of the Cellex foundation during my studies within the CFIS program. I am grateful to both institutions for their financial support, without whom this work could not have been possible.

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Thanks to the many great partners I have met in Barcelona: the b-lab members, my colleagues from the degrees in mathematics and engineering physics, my freshman students, and the group at the Aleu residence. I would like to specially thank Sandra, Nerea, Asis, Max, Héctor and his family. You all have made Barcelona feel like a home to me. Meeting you has made this experience worthwhile.

Many thanks to my friends from Murcia for the great times together, and for many more to come. Mayca and Jenn, thank you both for being by my side when I needed it. I hope to have been as good as a friend as you have. Lourdes, thank you very much for loving me when I did not have the strength to do so. I will forever be grateful for having you by my side. To Ángel, for being like a brother when I needed you the most. Your friendship has lightened even the darkest moments.

Finally, many thanks to my family. For their support and trust, no matter how hard or dire the situation is. Thank you, dad, for being so cheerful about my achievements and caring about my failures. Mom, thank you for loving me by who I am and not by my results or performance. Thank you for remembering me that life is much broader than what I am sometimes able to see. Andrea, thank you for being such a strong and courageous sister. You are and will continue to be a reference for me.

Contents

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A review on classical physics

1.1 The phase space and Hamilton's equations

In the study of classical mechanics, the objects where physics takes place are smooth manifolds. This formulation is also justified even in the context of Newtonian mechanics, where time and space are separate entities. Taking as base space $Q = \mathbf{R}^3$, there might be constrictions to the movement of certain bodies which hint that the real dynamics take place in a different space. For instance, in a spherical pendulum the distance from a point to the centre of the pendulum remains constant at all times. For this physical system, the possible configurations live within $\mathbf{R}^3$, but are described by the set $\mathbf{S}^2$. Developing a geometric notion of classical mechanics over any differentiable manifold puts in equal footing different physical spaces. Moreover, the coordinate-free language of differential geometry over smooth manifolds highlights the interplay that different geometric concepts play in classical physics.

A theory of classical mechanics over smooth manifolds becomes essential, and not only desirable, when we arrive to the theory of general relativity. The configuration space Q is used to describe both space and time coordinates. In fact, this description of configuration spaces implies that time and space are no longer different concepts; rather, one influences the other and their simultaneous existence is known as *space-time*. The notion of time is now encoded in the metric. In this sense, different choices of coordinates give rise to different ways of measuring time and space. The metric encodes the relationship between both measurements.

In the classical setting (that is, Newtonian and relativistic mechanics) we always assume that the configuration of a particle is described by a smooth manifold. We could even argue that this idealization, whether accurate or not, is an axiom in both theories. Following the discussion on relativistic mechanics, a metric is necessary in order to give a notion of time and space. In fact, the choice of a metric is also decisive in the formulation of Newton's equation's of motion in a geometric setting, even though implicitly. There is no intrinsic notion of acceleration in a smooth manifold; the choice of such an acceleration is known as an affine connection. Any metric determines a special connection known as the Levi-Civita connection. With these observations, we can define what we understand as a physical systems in which classical mechanics can be accurately described. We will additionally impose a technical condition, known as completeness, which ensures that any trajectory remains part of the configuration space for all instants of time. I we do not state it otherwise, manifolds are assumed to be connected.

Definition 1.1 A *physical system* is a complete Riemannian manifold (Q, g). The manifold Q is called the *configuration space*.

We will see, however, that, although Riemannian spaces are the "obvious" objects to describe mechanics, there is an adequate reformulation in a different setting which is more general.

Abstract Newtonian equations of motion

In classical mechanics, the standard paradigm of dynamics is given by Newton's laws. These constitute a complete set of axioms for the evolution of a physical system in any reference frame.

Definition 1.2 — Newton's Law of motion. Let (Q, g) be a Riemannian manifold and consider the Levi-Civita connection ∇. A curve $\gamma\colon I \longrightarrow M$ is a solution of Newton's equation of motion if an only if

$$\nabla_{\dot\gamma}\dot\gamma = 0. \tag{1.1}$$

Observe that this equation does not reflect in any shape or form Newton's third law. This is due to the fact that we will only work with individual particles, and not with systems of multiple particles.

To see why equation (1.1) encompasses Newton's first and second law we have to consider its expression in a local frame of reference. In a geometric setting we understand local frames of reference as local sections of the principal bundle $\mathcal{F}(Q)$, called the *frame bundle* of Q. A local chart (U, φ) defines a section of the frame bundle $\mathcal{F}(U)$ and, accordingly, we will restrict our considerations to this case. Frame bundles are specific cases of principal bundles; a more detailed discussion on them will be presented in section 1.5. In the reference frame φ with coordinates $\boldsymbol{q}$, equation (1.1) is written

$$\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} + \sum_{j,k=1}^{n} \Gamma^i_{jk} \frac{\mathrm{d}q_j}{\mathrm{d}t}\frac{\mathrm{d}q_k}{\mathrm{d}t} = 0, \quad i = 1, \dots, n \tag{1.2}$$

Here, the Christoffel symbols are defined as the smooth functions $\Gamma^i_{jk} \in C^\infty(U)$ satisfying $\nabla_{E_j} E_k = \Gamma^i_{jk} E_i$. This expression generalizes the transformation law for the acceleration of a particle in a classical frame of reference.

A key remark is that, in general, there exists no analogue for Newton's second Law $\ddot{x} = 0$. This is due to the fact that a general connection needs not to be *flat*, that is, a metric may have non-vanishing curvature. To give a physical interpretation of curvature, consider two geodesic flows $\gamma_1(t)$ and $\gamma_2(t)$ and assume that there is a coordinate set (U, φ) for which $\gamma_1(t), \gamma_2(t) \in U$ for every $t \in (-\varepsilon, \varepsilon)$. If ξ describes the coordinates of the difference $\gamma_1 - \gamma_2$, we have

$$\frac{\mathrm{d}^2 \xi_i}{\mathrm{d}t^2} = \sum_{j,k,l=1}^{n} R^i_{jkl} \xi_k \frac{\mathrm{d}\gamma_{1j}}{\mathrm{d}t}\frac{\mathrm{d}\gamma_{2l}}{\mathrm{d}t}, \quad i = 1, \dots, n, \tag{1.3}$$

where R^i_{jkl} is the curvature tensor. Geodesics describe the movement of free particles, upon which there is no exterior force acting. This equation tells us that curvature, in this sense, acts as a kind of "hidden force", modifying the distance between particles in free fall. In the physics literature, these effects are known as *tidal forces*.[1] For the standard Levi-Civita connection in the classical configuration space $Q = \mathbf{R}^3$, the choice of an orthonormal

[1] Roger Penrose. *The Emperor's New Mind*. Oxford landmark science. Oxford University Press, 2016. ISBN 9780198784920

basis extends to a global frame of reference in which equation (1.1) becomes Newton's classical equation $\ddot{x} = 0$.

Having defined the equations of motion of our system, the next step is to study in more depth their structure. From equation (1.2) we can see that the dynamics are of second order in Q, which makes them difficult to handle from a geometric perspective. In standard ordinary differential equations courses, a trick can be used to convert an ordinary differential equation of arbitrary degree to a first order equation, possibly at the cost of increasing the dimension of the phase space. Our goal is to use this trick in a geometric setting to obtain first-order dynamics, which are generated by a vector field, in a different space. This procedure requires an equivalent descriptions of connections in a manifold.

An affine connection on the vector bundle $\tau\colon \mathrm{T}Q \longrightarrow Q$ is equivalent to a splitting of the short exact sequence

$$0 \longrightarrow \ker \tau_* \longrightarrow \mathrm{TT}Q \longrightarrow \mathrm{T}Q \longrightarrow 0 \tag{1.4}$$

It is a well-known fact in differential geometry that affine connections are in correspondence with two objects. The first one is the *connection form*, a linear map $\theta\colon \mathrm{TT}Q \longrightarrow \ker \tau_*$ which satisfies $\theta i = \mathrm{id}_{\ker \tau_*}$. The other object is the *horizontal lift*, a map $X\colon \mathrm{T}Q \longrightarrow \mathrm{TT}Q$ which satisfies $\tau_* X = \mathrm{id}_{\mathrm{T}Q}$. In the language of categories, X is called a section. By definition of vector field, the horizontal lift can be regarded as a vector field $X \in \chi(\mathrm{T}Q)$. Solutions to Newton's equation of motion (1.1) are in correspondence with solutions of the first-order differential equation defined by X. The completeness of the configuration space gives us the following result.

Theorem 1.3 *Let (Q, g) be a riemannian manifold where g is complete. The set of solutions of equation* (1.1)*, which we understand to represent the set of different states of the physical system, is diffeomorphic to* $\mathrm{T}Q$*.* [2]

[2] Daniel S. Freed. Classical field theory and supersymmetry, 2001. URL `https://web.ma.utexas.edu/users/dafr/pcmi.pdf`

This result already characterizes the natural phase space of physical systems as the tangent bundle $\mathrm{T}Q$ instead of the base space Q. However, the structure of the equations of motion, given by the horizontal lift X, has not been studied in detail. For all we know, the solutions to this equations of motion could be extremely complicated. Many of the fundamental results and techniques used in classical mechanics, which include Liouville's theorem, the use of symmetries in the reduction of phase spaces, conserved quantities, and integrability, are better characterized under a Hamiltonian picture of mechanics.

The cotangent bundle and the canonical symplectic structure

Our objective throughout this subsection is to develop the techniques and results needed for the Hamiltonian descripion of mechanics, which is formulated upon the results of symplectic geometry. The construction of the key results in symplectic geometry as presented here is not contextualized at best, and *ad hoc* at worst. We recommend the interested reader to skip to the following subsection for an example of application of the results of this digression to the geodesic flow in $\mathrm{T}Q$. The key observation for Hamiltonian mechanics to be possible is that the cotangent bundle T^*Q has an intrinsic one-form, the Liouville form. Its relevance lies in the associated canonical symplectic form of T^*Q, which governs the Hamiltonian equations of motion.

Definition 1.4 Let Q be a smooth manifold. The one form $\lambda \in \Omega^1(\mathrm{T}^*Q)$, defined by its action on an arbitrary field $X \in \chi(\mathrm{T}^*Q)$ as

$$\langle \lambda, X \rangle = \left\langle \tau_{\mathrm{TT}^*Q}(X), (\tau_{\mathrm{T}^*Q})_*(X) \right\rangle, \tag{1.5}$$

is called the *Liouville form* or the *tautological one form.*

Lemma 1.5 *Let M be a smooth manifold and consider a chart (U, φ) with coordinates $\boldsymbol{q}$. In the natural chart on T^*M with coordinates $\boldsymbol{q}, \boldsymbol{p}$, the Liouville form is expressed as*

$$\lambda = \sum_{i=1}^{n} p_i \, \mathrm{d}q_i. \tag{1.6}$$

Proof. To prove this result take a vector field $X \in \chi(\mathrm{T}^*Q)$ and a local chart (U, φ) with coordinates $\boldsymbol{q}$. We take the induced coordinates $\boldsymbol{q}, \boldsymbol{p}$ in the cotangent bundle T^*Q, which give induced coordinates $\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{s}$ in TT^*Q. In these coordinates, $\tau_{\mathrm{TT}^*Q}(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{s}) = (\boldsymbol{q}, \boldsymbol{p})$, while $(\tau_{\mathrm{T}^*Q})_*(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{s}) = (\boldsymbol{q}, \boldsymbol{r})$. As a consequence, for any vector field $X = r_i(\boldsymbol{q}, \boldsymbol{p})\partial_{q_i} + s_i(\boldsymbol{q}, \boldsymbol{p})\partial_{p_i}$ the expression of the Liouville one-form is

$$\langle \lambda, X \rangle = \left\langle \tau_{\mathrm{TT}^*Q}(X), (\tau_{\mathrm{T}^*Q})_*(X) \right\rangle = \langle \boldsymbol{p}, \boldsymbol{r} \rangle = \sum_{i=1}^{n} p_i r_i(\boldsymbol{p}, \boldsymbol{q}).$$

As a direct consequence of this equation, the Liouville form can be computed locally as

$$\lambda = \sum_{i=1}^{n} p_i \, \mathrm{d}q_i. \qquad \blacksquare$$

Definition 1.6 Let Q be a smooth manifold. The *canonical symplectic form* $\omega \in \Omega^2(\mathrm{T}^*Q)$ is defined as

$$\omega = \mathrm{d}\lambda, \tag{1.7}$$

where λ is the Liouville form (1.5).

Lemma 1.7 *Let Q be a smooth manifold and consider a coordinate chart (U, φ) with coordinates $\boldsymbol{q}$. In natural coordinates $\boldsymbol{q}, \boldsymbol{p}$ of T^*Q the canonical symplectic form is expressed as*

$$\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i. \tag{1.8}$$

Proof. As a direct consequence of the linearity of d and local expression (1.6),

$$\omega = \mathrm{d}\lambda = \sum_{i=1}^{n} \mathrm{d}(p_i \, \mathrm{d}q_i) = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i. \qquad \blacksquare$$

Any smooth map between smooth manifolds $f\colon M \longrightarrow N$ lifts naturally to a map of vector bundles $f_*\colon \mathrm{T}M \longrightarrow \mathrm{T}N$. Moreover, the chain rule $(gf)_* = g_* f_*$ implies that the tangent functor which assigns to each function its tangent map is a covariant functor. In general, there is no such functor assigning to each map $f\colon M \longrightarrow N$ a map of vector bundles $\hat{f}\colon \mathrm{T}^*M \longrightarrow \mathrm{T}^*N$; we can define, however, a contravariant functor $f^*\colon \mathrm{T}^*N \longrightarrow \mathrm{T}^*M$. If f is a diffeomorphism, we can define a covariant functor called the *cotangent lift*.

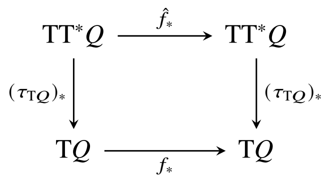
Proposition 1.8 *Let Q be a smooth manifold and consider a diffeomorphism $f \in \mathrm{Diff}(Q)$. There exists a vector bundle morphism $\hat{f} \colon \mathrm{T}^*Q \longrightarrow \mathrm{T}^*Q$ covering f, called the cotangent lift, which additionally satisfies $\widehat{fg} = \hat{f}\hat{g}$.*

Proof. Consider a diffeomorphism $f \in \mathrm{Diff}(Q)$. We define the cotangent lift of a function as $\hat{f} := (f^*)^{-1}$. From the fact that the pullback is a contravariant functor, we see that

$$\widehat{fg} = \left((fg)^*\right)^{-1} = (g^* f^*)^{-1} = (f^*)^{-1}(g^*)^{-1} = \hat{f}\hat{g}. \qquad \blacksquare$$

Cotangent lifts are not really interesting themselves, but rather because they preserve the Lioville one-form. This has important implications in the symplectic properties of lifted maps.

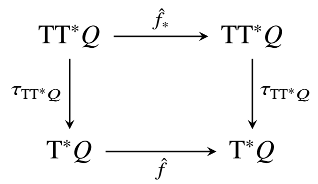
Proposition 1.9 *Let Q be a smooth manifold and consider a diffeomorphism $f \in \mathrm{Diff}(Q)$. The cotangent lift $\hat{f}$ preserves the Liouville form, $\hat{f}^*\lambda = \lambda$. As a consequence, the cotangent lift $\hat{f}$ preserves the canonical symplectic form.*



(1.9)

Proof. The proof is obtained from the commutativity of diagram (1.9), which follows from the fact that $\hat{f}$ is a vector bundle morphism. We will also use the commutativity of diagram (1.10), which is true by definition of tangent map. Taking an arbitrary element $X \in \mathrm{TT}^*Q$, a direct computation using definition (1.5) shows

$$\begin{aligned} \langle \hat{f}^*\lambda, X\rangle &= \langle \lambda, \hat{f}_* X\rangle \\ &= \left\langle \tau_{\mathrm{TT}^*Q}(\hat{f}_* X), (\tau_{\mathrm{T}^*Q})_* \hat{f}_* X \right\rangle \\ &= \left\langle \hat{f}\tau_{\mathrm{TT}^*Q}(X), f_*(\tau_{\mathrm{T}^*Q})_* X \right\rangle \\ &= \left\langle \tau_{\mathrm{TT}^*Q}(X), (\tau_{\mathrm{T}^*Q})_* X \right\rangle \\ &= \langle \lambda, X\rangle. \end{aligned}$$



(1.10)

As a consequence, $\hat{f}^*\lambda = \lambda$. $\blacksquare$

An important example of application of cotangent lifts are Lie group actions on Q. Any Lie group action $\rho \colon G \times Q \longrightarrow Q$ acts on Q by diffeomorphisms and therefore can be lifted to a cotangent map. The lift of all diffeomorphisms of G on Q gives a group action of $\hat{\rho}$ on T^*Q, called the *cotangent lift* of ρ. As a consequence of proposition 1.9, we will see that the action $\hat{\rho}$ is symplectic, a term to be defined.

The Hamiltonian formulation of geodesic flows

Having proved that the cotangent bundle of any smooth manifold Q is Hamiltonian, we are in position to show that the geodesic flow $X \in \chi(\mathrm{T}Q)$ can be recovered from the symplectic form and a smooth function. This method of specifying a vector field is very similar to taking the riemannian gradient of a function.[3] To obtain this result, however, we have to write the equations for the geodesic flow in the cotangent bundle T^*Q. There is no natural isomorphism $\mathrm{T}Q \simeq \mathrm{T}^*Q$; to obtain it we have to consider the musical isomorphisms induced by the metric g.

[3] It is, in some instances, called the *symplectic gradient* of a function. However, it is more commonly refered to as the Hamiltonian vector field generated by a function.

Consider the metric $g \in \mathcal{S}^2Q$. The non-degeneracy condition is equivalent to the map

$$g^\flat\colon \begin{array}{ccc} \chi(Q) & \longrightarrow & \Omega^1(Q) \\ X & \longmapsto & \iota_X g \end{array} \tag{1.11}$$

being a vector bundle isomorphism. Its inverse is denoted by $g^\sharp$ and, together, are called the *musical isomorphisms* of the metric. These maps give an identification $\mathrm{T}Q \simeq \mathrm{T}^*Q$ and, as a consequence, an isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\tau_* & \xrightarrow{\ \iota\ } & \mathrm{TT}Q & \xrightarrow{(\tau_{\mathrm{T}Q})_*} & \mathrm{T}Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g^\flat_*} & & \downarrow{\scriptstyle g^\flat_*} & & \downarrow{\scriptstyle g^\flat} & & \\ 0 & \longrightarrow & \ker\tau_* & \xrightarrow[\ \iota\ ]{} & \mathrm{TT}^*Q & \xrightarrow[\tau_{\mathrm{TT}^*Q}]{} & \mathrm{T}^*Q & \longrightarrow & 0 \end{array}$$

In particular, the geodesic flow X_∇ induces a vector field $g^\flat_* X_\nabla \in \chi(\mathrm{T}^*Q)$. This vector field is also, by definition, a splitting of the induced short exact sequence.

Theorem 1.10 *Let (Q, g) be a complete riemannian manifold and let ∇ be the Levi-Civita connection. If X_∇ is the geodesic flow of ∇, then the pushforward $g^\flat_* X_\nabla := Y_\nabla$ is uniquely defined by the equation*

$$\iota_{Y_\nabla}\omega = -\,\mathrm{d}\langle\lambda, Y_\nabla\rangle. \tag{1.12}$$

Proof. Let φ_t be the flow of the vector field Y_∇. As a consequence of completeness, the flow φ_t is defined for every $t \in \mathbf{R}$. From the fact that $[Y_\nabla, Y_\nabla] = 0$ we have that $(\varphi_t)_* Y_\nabla = Y_\nabla$, we have the following isomorphism of short exact sequences,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\tau_* & \xrightarrow{\ \iota\ } & \mathrm{TT}^*Q & \underset{Y_\nabla}{\overset{\tau_{\mathrm{TT}^*Q}}{\rightleftarrows}} & \mathrm{T}^*Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle (\varphi_t)_*} & & \downarrow{\scriptstyle \varphi_t} & & \\ 0 & \longrightarrow & \ker\tau_* & \xrightarrow[\ \iota\ ]{} & \mathrm{TT}^*Q & \underset{\tau_{\mathrm{TT}^*Q}}{\overset{Y_\nabla}{\leftrightarrows}} & \mathrm{T}^*Q & \longrightarrow & 0 \end{array} \tag{1.13}$$

$$\begin{array}{ccc} \mathrm{TT}^*Q & \xrightarrow{(\varphi_t)_*} & \mathrm{TT}^*Q \\ {\scriptstyle \tau_{\mathrm{TT}^*Q}}\downarrow & & \downarrow{\scriptstyle \tau_{\mathrm{TT}^*Q}} \\ \mathrm{T}^*Q & \xrightarrow[\varphi_t]{} & \mathrm{T}^*Q \end{array} \tag{1.14}$$

We now observe that we can extract the commutative diagram (1.14) from equation (1.13). This diagram is essentially equivalent to diagram (1.10) and, as a consequence, we see that the flow φ_t preserves the Lioville one-form. As a consequence, $\mathcal{L}_{Y_\nabla}\lambda = 0$. This implies, by Cartan's formula, that

$$0 = \mathcal{L}_{Y_\nabla}\lambda = \iota_{Y_\nabla}\,\mathrm{d}\lambda + \mathrm{d}\iota_{Y_\nabla}\lambda.$$

This equation is equivalent to (1.12) by the definition of canonical symplectic form (1.7). ■

$$\begin{array}{ccc} \mathrm{T}Q & \xrightarrow{g^\flat} & \mathrm{T}^*Q \\ {\scriptstyle \tau_{\mathrm{T}Q}}\downarrow & & \downarrow{\scriptstyle \tau_{\mathrm{T}^*Q}} \\ Q & \xrightarrow[\mathrm{id}_M]{} & Q \end{array} \tag{1.15}$$

$$\begin{array}{ccc} \mathrm{TT}Q & \xrightarrow{(g^\flat)_*} & \mathrm{TT}^*Q \\ {\scriptstyle (\tau_{\mathrm{T}Q})_*}\downarrow & & \downarrow{\scriptstyle (\tau_{\mathrm{T}^*Q})_*} \\ \mathrm{T}Q & \xrightarrow[\mathrm{id}_{\mathrm{T}Q}]{} & \mathrm{T}Q \end{array} \tag{1.16}$$

Remark 1.11 The smooth function $\langle\lambda, Y_\nabla\rangle \in C^\infty(\mathrm{T}^*Q)$ can be further simplified. We compute it in a point $\alpha \in \mathrm{T}^*Q$ and denote $X = g^\sharp(\alpha)$, so $\alpha = g^\flat(X)$. From the fact that $g^\flat$ is a vector bundle morphism we have diagram (1.15) which, taking tangent maps, gives diagram (1.16). As a consequence, from

$Y_\nabla = g_*^\flat X_\nabla$ and definition (1.5) we have

$$\begin{aligned}\langle \lambda_\alpha, Y_\nabla(\alpha)\rangle &= \langle \alpha, (\tau_{\mathrm{T}^*M})_* g_*^\flat X_\nabla(X)\rangle \\ &= \langle \alpha, (\tau_{\mathrm{T}M})_* X_\nabla(X)\rangle \\ &= \langle \alpha, X\rangle \\ &= g(X, X).\end{aligned}$$

As a consequence, if we define the kinetic energy $K \in \mathcal{C}^\infty(\mathrm{T}Q)$ at a point $X \in \mathrm{T}Q$ as $K(X) = g(X, X)$, we can conclude that $\langle \lambda, Y_\nabla\rangle = (g^\sharp)^* K$.

1.2 Elements of symplectic geometry

Theorems 1.3 and 1.10 hint that, even though we have followed the definition of configuration space 1.1, the dynamics given by Newton's equation (1.1) are described in the cotangent bundle $M = \mathrm{T}^*Q$. The fundamental object used to describe the equations of motion for geodesics is the canonical symplectic form ω. Motivated by these results, we will review the fundamental results of differential geometry. This approach to mechanics, where the equations of motion are obtained from a function $H \in \mathcal{C}^\infty(M)$ following equation (1.12), is called the *Hamiltonian formulation* of classical mechanics.

The category of symplectic manifolds

To generalize the previous constructions we will impose the necessary conditions to reproduce the proof of theorem 1.10. On one hand, we have used the non-degeneracy condition for ω, which guarantees that X_H is unique in equation (1.12). On the other hand, we have seen that ω is an exact form. It turns out that exactness is too restrictive for our considerations; it is enough to impose that ω is closed.

Definition 1.12 Let M be a smooth manifold. A *symplectic form* is a two-form $\omega \in \Omega^2(M)$ which is closed, $\mathrm{d}\omega = 0$, and non-degenerate. A pair (M, ω), where M is a smooth manifold and ω is a symplectic form, is called a *symplectic manifold*.

Definition 1.13 Let (M, ω) and (N, η) be symplectic manifolds. A smooth map $f\colon M \longrightarrow N$ is said to be a *symplectomorphism* if $f^*\eta = \omega$.

Lemma 1.14 *Let (M, ω), (N, μ) and (L, η) be symplectic manifolds.*

1. *The identity map $\mathrm{id}_M\colon M \longrightarrow M$ is a symplectomorphism.*
2. *If $f\colon M \longrightarrow N$ and $g\colon N \longrightarrow L$ are symplectomorphisms, $gf\colon M \longrightarrow L$ is a symplectomorphism. Moreover, the composition of symplectomorphisms is associative.*

Proof. Firstly, if (M, ω) is a symplectic manifold the identity map $\mathrm{id}_M\colon M \longrightarrow M$ is trivially a symplectomorphism as $\mathrm{id}_M^* = \mathrm{id}_{\Omega(M)}$. Therefore, $\mathrm{id}_M^*\,\omega = \mathrm{id}_{\Omega(M)}\,\omega = \omega$, showing that id_M is a symplectomorphism.

Consider now symplectic manifolds (M, ω), (N, η) and (L, δ) and take symplectomorphisms $f\colon M \longrightarrow N$ and $g\colon N \longrightarrow L$. To see that gf is a symplectomorphism we use the property $(gf)^* = f^*g^*$. As a consequence,

$$(gf)^*\delta = f^*g^*\delta = f^*\eta = \omega, \tag{1.17}$$

showing that gf is a symplectomorphism. Finally, associativity is inherited from the associativity of smooth maps. ■

As a consequence, symplectic manifolds together with symplectomorphisms form the category of symplectic manifolds, denoted by $\mathcal{S}ymp$. In particular, the set of invertible symplectomorphisms of a symplectic manifold (M, ω) is a group, called the symplectomorphic group $\mathrm{Symp}(M, \omega)$.

It is a standard fact in differential geometry that smooth vector fields are infinitesimal generators of local one-parameter subgroups of diffeomorphisms.[4] In a naive sense, vector fields are the infinitesimal counterpart of the diffeomorphism group $\mathrm{Diff}(M)$. A similar construction holds for the group of symplectomorphisms of a symplectic manifold (M, ω).

[4] John Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer New York, NY, 2 edition, 2012. ISBN 978-1-4419-9981-8. DOI: 10.1007/978-1-4419-9982-5

Definition 1.15 Let (M, ω) be a symplectic manifold. A vector field $X \in \chi(M)$ is said to be *symplectic* if $\mathcal{L}_X\omega = 0$.

Observation 1.16 It is easy to check that symplectic vector fields, as well as smooth manifolds, have a Lie algebra structure. The Lie bracket of symplectic vector fields is the natural restriction of the Lie bracket of vector fields. To see that if $X, Y \in \chi_{\mathrm{symp}}(M)$ then $[X, Y] \in \chi_{\mathrm{symp}}(M)$ we use the naturality of the Lie bracket,

$$\mathcal{L}_{[X,Y]}\omega = \mathcal{L}_X\mathcal{L}_Y\omega - \mathcal{L}_Y\mathcal{L}_X\omega = 0.$$

The proof that symplectic vector fields are the infinitesimal generators of symplectomorphisms requires the application of Weinstein's tubular neighbourhood theorem and is beyond the scope of this introductory section. We state the result of without further proof.

Theorem 1.17 — Cannas da Silva. *Let (M, ω) be a symplectic manifold. A small $\mathcal{C}^1$ neighbourhood of* $\mathrm{id} \in \mathrm{Symp}(M, \omega)$ *is homeomorphic to a small neighbourhood of* $0 \in \mathcal{Z}^1(M)$.[5] *As a consequence,*

$$\mathrm{T}_{\mathrm{id}}\,\mathrm{Symp}(M, \omega) \simeq \mathcal{Z}^1(M). \tag{1.18}$$

[5] Ana Cannas da Silva. *Lectures on Symplectic Geometry*. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1 edition, 2008. DOI: 10.1007/978-3-540-45330-7

We will see in proposition 1.23 that the space of closed forms $\mathcal{Z}^1(M)$ is in equivalence with the space of symplectic vector fields $\chi_{\mathrm{symp}}(M)$.

Local structure of symplectic manifolds

The local structure of symplectic manifolds is, rather suprisingly, very trivial. Unlike in riemannian geometry, a symplectic form ω has no local invariants other than the dimension of the manifold. This result can be obtained from a normal expression for symplectic forms, known as Darboux's theorem.

Theorem 1.18 — Darboux. *Let (M, ω) be a symplectic manifold. For every point $p \in M$ there exists a coordinate set (U, φ) with coordinates $(\boldsymbol{q}, \boldsymbol{p})$ such that*

$$\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i. \tag{1.19}$$

Proof. The proof is an application of the skew-symmetric Gram-Schmidt procedure for local sections together with an additional topological argument.

Consider a local open set $U_p \subset M$. To prove this statement we will use construct by induction a family of local vector fields $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ such that $\omega(X_i, X_j) = 0$, $\omega(Y_i, Y_j) = 0$ and $\omega(Y_i, X_j) = \delta_{ij}$. Assume that we already have a set of fields $\{X_1, \ldots, X_k, Y_1, \ldots, Y_k\}$ satisfying this condition and denote $P_k \subset \chi(U)$ their spanned $\mathcal{C}^\infty(U)$ submodule. As $k < n$, we can consider a 1-form $\mu_{k+1} \in P_k^\perp$ and the corresponding vector field Y_{k+1}, defined by the Hamiltonian equation $\iota_{Y_{k+1}}\omega = -\mu_{k+1}$. The following contraction yields

$$\omega(Y_{k+1}, X_i) = \iota_{X_i}\iota_{Y_{k+1}}\omega = \langle \mu_{k+1}, X_i \rangle = 0$$

for any $i \in \{1, \ldots, k\}$ because $\mu_{k+1} \in P_k^\perp$. Similarly we can show that $\omega(Y_{k+1}, Y_i) = 0$ for any $i \in \{1, \ldots, k\}$. There exists a one-form η_{k+1} such that $\langle \eta_{k+1}, Y_{k+1} \rangle = 1$. Additionally, from the previous conditions we may assume that $\eta_{k+1} \in P_k^\perp$. Defining the vector field X_{k+1} as $\iota_{X_{k+1}}\omega = -\,\mathrm{d}\eta_{k+1}$, we can see that

$$\omega(Y_{k+1}, X_{k+1}) = -\iota_{Y_{k+1}}\iota_{X_{k+1}}\omega = \langle \eta_{k+1}, Y_{k+1} \rangle = 1.$$

Similarly, it can be proved that $\omega_(X_{k+1}, X_i) = \omega(X_{k+1}, Y_i) = 0$ for any $i \in \{1, \ldots, k\}$. The claim follows by induction.

We have to construct a set of coordinates using the family of one forms $\{\mu_1, \ldots, \mu_n, \eta_1, \ldots, \eta_n\}$. We start by checking that they are closed. This condition can be seen in the local set of generators $\{X_i, Y_i\}$, and a straightforward computation shows

$$\begin{aligned}
\mathrm{d}\mu_i(X_j, X_k) &= \mathcal{L}_{X_j}\langle \mu_i, X_k \rangle - \mathcal{L}_{X_k}\langle \mu_i, X_j \rangle - \langle \mu_i, [X_j, X_K] \rangle \\
&= \mathcal{L}_{X_j} 0 - \mathcal{L}_{X_k} 0 - \langle \mu_i, 0 \rangle \\
&= 0.
\end{aligned}$$

Similar computations show that $\mathrm{d}\mu_i(Y_j, Y_k) = 0$ and $\mathrm{d}\mu_i(X_j, Y_K) = 0$, so μ_i is closed. An analogous procedure shows that η_i is closed. Now, shrinking the open set U_p if necessary, we may suppose that μ_i, η_i are exact forms, and denote $\mu_i = \mathrm{d}p_i$ and $\eta_i = \mathrm{d}q_i$.

Finally, conditions $\omega(X_i, X_j) = 0$, $\omega(Y_i, Y_j) = 0$ and $\omega(Y_i, X_j) = \delta_{ij}$ and the definition of coordinates q_i, p_i imply that the symplectic form ω can be locally written as

$$\omega = \sum_{i=1}^{n} \mu_i \wedge \eta_i = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

The set of functions q_i, p_i determine a coordinate chart from the non-degeneracy of ω (see equation 1.20). ■

Observe that the expression of ω in these coordinates is the same as the one obtained in (1.8) for the cotangent bundle of a manifold. These coordinates are typically used in the physics literature and are known as *canonical coordinates* or, in a more mathematical language, *Darboux coordinates*.

Even though there are no local invariants in symplectic geometry, the topology of the space imposes restrictions on the symplectic form. This is, once again, in contrast with riemannian geometry: from the paracompactness condition required in the definition of smooth manifold, every manifold admits a metric g. The following result shows that orientability is a obstruction for the existence of symplectic forms.

Theorem 1.19 *Let* (M, ω) *be a symplectic manifold with* $\dim M = 2n$*. The form* $\Omega := \omega^n \in \Omega^{2n}(M)$ *is nowhere vanishing and, as a consequence, is a volume form. It is called the* Liouville volume *of* (M, ω).

Proof. The condition can be checked pointwise and, in particular, locally. Taking canonical coordinates, which always exist by Darboux's theorem 1.18, the expression for the Lioville volume is

$$\omega^n = \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_n \wedge \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_n. \tag{1.20}$$

This form is clearly non-vanishing, which completes the proof. ∎

In the lines of this theorem, we can show that any transformation which preserves the symplectic form will also preserve the Liouville volume form. A direct consequence is that the flow of a physical system modeled over a symplectic manifold (as a result of theorem 1.10) is a volume-preserving transformation.

Theorem 1.20 *Let* (M, ω) *be a symplectic manifold and let* $\Omega \in \Omega^{2n}(M)$ *be its Liouville volume form.*

1. *If* $f \in \mathrm{Symp}(M, \omega)$*, then* $f^*\Omega = \Omega$.
2. *If* $X \in \chi_{\mathrm{symp}}(M)$ *is a symplectic vector field, then* $\mathcal{L}_X\Omega = 0$.

Proof. To prove the first fact, we only have to recall that the pullback of forms commutes with respect to the wedge product. As a consequence, $f^*\Omega = f^*(\omega^n) = (f^*\omega)^n = \omega^n = \Omega$.

The second item can be proved from a more general result following an inductive argument. If $X \in \chi_{\mathrm{symp}}(M)$, then $\mathcal{L}_X\omega^k$ for every $k \in \mathbf{Z}^+$. This result is trivial for $k = 1$ from the definition of symplectic vector field. A direct computation shows

$$\mathcal{L}_X\omega^k = \mathcal{L}_X(\omega \wedge \omega^{k-1}) = (\mathcal{L}_X\omega) \wedge \omega^{k-1} + \omega \wedge (\mathcal{L}_X\omega^{k-1}).$$

The first term vanishes because $\mathcal{L}_X\omega = 0$ by definition, and the second term vanishes because $\mathcal{L}_X\omega^{k-1} = 0$ by the induction hypothesis. Therefore, the claim is proved. As a particular case for $n = k$, $\mathcal{L}_X\Omega = 0$. ∎

Hamiltonian mechanics

In theorem 1.10 we showed that the equations of motion determined by the geodesic flow are determined uniquely by the kinetic energy of the particle through equation (1.12). The vector field Y_∇ is uniquely determined by the non-degeneracy of the symplectic form ω. As any symplectic form is non-degenerate by definition, we can always obtain a vector field X_H from a smooth function $H \in C^\infty(M)$. This construction allows to consider analytical mechanics in any symplectic manifold; we could say that symplectic manifolds are the natural phase spaces in classical mechanics.

Definition 1.21 Let (M, ω) be a symplectic manifold. Given a function $H \in C^\infty(M)$, its *Hamiltonian vector field* or *symplectic gradient* is the unique vector field $X_H \in \chi(M)$ which satisfies

$$\iota_{X_H}\omega = -\,\mathrm{d}H. \tag{1.21}$$

The function H is called the *Hamiltonian*. The set of Hamiltonian vector fields is denoted by $\chi_{\mathrm{Ham}}(M)$.

Remark 1.22 The uniqueness of the Hamiltonian vector field is a consequence of the non-degeneracy of ω. This non-degeneracy can be equivalently stated by saying that the vector bundle map

$$\begin{array}{rccc} \omega^\flat\colon & \chi(M) & \longrightarrow & \Omega^1(M) \\ & X & \longmapsto & \iota_X\omega \end{array} \tag{1.22}$$

is an isomorphism. Its inverse is denoted by $\omega^\sharp$.

In terms of these definition, equation (1.21) is written as $\omega^\flat(X_H) = -\,\mathrm{d}H$ or, equivalently, $X_H = -\omega^\sharp(\mathrm{d}H)$.

The Hamiltonian vector field actually encodes Hamilton's equations of motion from classical mechanics. To see this, we use the expression of the Hamiltonian vector field X_H in a Darboux coordinate chart (U, φ). In the coordinates $(\boldsymbol{q}, \boldsymbol{p})$, we have

$$X_H = \sum_{i=1}^n \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}.$$

The flow of this vector field satisfies the following ODE in Darboux coordinates:

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}, & 1 \leqslant i \leqslant n, \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i}, & 1 \leqslant i \leqslant n. \end{cases}$$

These are exactly Hamilton's equations of motion for a Hamiltonian function $H \in \mathcal{C}^\infty(M)$. Theorem 1.10 shows, as a consequence, that the geodesic flow is Hamiltonian with the induced kinetic energy $(g^\sharp)^*K \in \mathcal{C}^\infty(\mathrm{T}^*Q)$ being the Hamiltonian.

The structure symplectic vector fields and Hamiltonian vector fields is deeply tied to the topological structure of the symplectic manifold M, as the following result shows.

Proposition 1.23 *Let (M, ω) be a symplectic manifold. The morphisms $\omega^\flat, \omega^\sharp$ give isomorphisms of vector spaces $\chi_{\mathrm{symp}}(M) \simeq \mathcal{Z}^1(M)$ and $\chi_{\mathrm{Ham}}(M) \simeq \mathcal{B}^1(M)$.*

Proof. To see that $\chi_{\mathrm{Ham}}(M) \simeq \mathcal{B}^1(M)$ we only have to use the definition of Hamiltonian vector field (1.21) or, equivalently, its characterization $X_H = -\omega^\sharp(\mathrm{d}H)$. We see that, as $\omega^\sharp$ is a $\mathcal{C}^\infty(M)$-module isomorphisms, Hamiltonian vector fields are in one-to-one correspondence with exact one-forms.

To see that $\chi_{\mathrm{symp}}(M) \simeq \mathcal{Z}^1(M)$ we use Cartan's formula for the Lie derivative along a vector field $X \in \chi(M)$ and the closedness of ω,

$$\mathcal{L}_X\omega = \mathrm{d}\iota_X\omega + \iota_X\,\mathrm{d}\omega = \mathrm{d}\iota_X\omega.$$

As a consequence, $X \in \chi_{\mathrm{symp}}(M)$ if and only if $\omega^\flat \in \mathcal{Z}^1(M)$. Given that $\omega^\flat$ is a $\mathcal{C}^\infty(M)$-module isomorphism, we conclude that $\chi_{\mathrm{symp}}(M) \simeq \mathcal{Z}^1(M)$. ■

Corollary 1.24 *In a symplectic manifold* (M, ω) *every Hamiltonian vector field is symplectic.*

Proof. As both isomorphisms in proposition 1.23 are obtained by applications of $\omega^\flat$, $\omega^\sharp$ and $\mathcal{B}^1(M) \subseteq \mathcal{Z}^1(M)$ by the cochain condition of the de Rham complex, we easily conclude that $\chi_{\text{Ham}}(M) \subseteq \chi_{\text{symp}}(M)$. ■

Corollary 1.25 *In a symplectic manifold* (M, ω) *the sets* $\chi_{\text{symp}}(M)$ *and* $\chi_{\text{Ham}}(M)$ *are* **R***-vector spaces.*

Proof. As $\mathcal{B}^1(M)$, $\mathcal{Z}^1(M)$ are **R**-vector spaces and $\omega^\sharp$ is a $C^\infty(M)$-module isomorphism, $\chi_{\text{Ham}}(M)$ and $\chi_{\text{symp}}(M)$ are **R**-vector spaces. ■

1.3 Conserved quantities

Once we have laid out the basic foundations of classical mechanics in the Hamiltonian formalism, our objective is to generalize many well-known results to this setting. In physics one is commonly concerned with *observables*; these are meaningful physical quantities, that is, they can be measured by an observer. Some kind observables will be key in the topological and symplectic properties of physical systems. These are *conserved quantities*, observables preserved along the evolution of the Hamiltonian flow. We will see that conserved quantities are deeply related with constrictions in the dynamics of a Hamiltonian system. In turn, such constrictions can be used to reduce the dimension of the phase space of our system.

Observables in classical physics and Poisson geometry

We define now the idea of observables in classical mechanics. The phase space of a Hamiltonian system is the manifold M; it encodes all the information necessary to describe completely a point-mass particle. We use this motivation as the basis to define observable.

Definition 1.26 Let (M, ω) be a symplectic manifold. An *observable* on M is a smooth function $f \in C^\infty(M)$.

Amongst the many properties of observables to be studied, a natural one is their evolution with the flow of the system. This is, at least, in accordance with the idea that we are generalizing classical mechanics. A straightforward computation with the definition of Hamiltonian vector field shows that the evolution of an observable is given by

$$\mathcal{L}_{X_H} f = \omega(X_H, X_f). \tag{1.23}$$

The right hand side of this equation therefore measures the rate of change of any observable in a physical system. Given its relevance, it receives a distinguished place in symplectic geometry.

Definition 1.27 — Poisson bracket. Let (M, ω) be a symplectic manifold. We define the *Poisson bracket* of two observables $f, g \in C^\infty(M)$ as

$$\{f, g\} \coloneqq \omega(X_f, X_g). \tag{1.24}$$

Under this notation, equation (1.23) is nothing but the classical Poisson equation for the evolution of an observable.[6]. We present now some properties that this Poisson bracket satisfies. Items 1 to 3 are trivial from the properties of the symplectic form and the Hamiltonian vector field. Item 4, on the other hand, requires more work and is a consequence of the closedness of ω. For this purpose, we will introduce an auxiliary result which is interesting on its own.

[6] As a reminder, the Poisson equation in classical mechanics states that for any observable $f \in \mathcal{C}^\infty(M)$ we have

$$\dot{f} = \{H, f\}. \tag{1.25}$$

Proposition 1.28 *Let (M, ω) be a symplectic manifold and consider the Poisson bracket* (1.24)*. The assignation of Hamiltonian vector fields is a Lie algebra anti-homomorphism, that is, for any $f, g \in \mathcal{C}^\infty(M)$ we have*

$$X_{\{f,g\}} = [X_f, X_g]. \tag{1.26}$$

Proof. The proof uses the identity $\iota_{[X,Y]} = \mathcal{L}_X \iota_Y - \iota_Y \mathcal{L}_X$ together with Cartan's formula, from which

$$\begin{aligned}
\iota_{[X_f,X_g]}\omega &= \mathcal{L}_{X_f}\iota_{X_g}\omega - \iota_{X_g}\mathcal{L}_{X_f}\omega \\
&= \mathrm{d}\iota_{X_f}\iota_{X_g}\omega + \iota_{X_f}\,\mathrm{d}\iota_{X_g}\omega - \iota_{X_g}\,\mathrm{d}\iota_{X_f}\omega - \iota_{X_g}\iota_{X_f}\,\mathrm{d}\omega \\
&= \mathrm{d}(\omega(X_g, X_f)) - \iota_{X_f}\,\mathrm{d}^2 g + \iota_{X_g}\,\mathrm{d}^2 f \\
&= -\,\mathrm{d}\{f, g\}.
\end{aligned}$$

For this chain of equalities, we have used the definition of Hamiltonian vector field (1.21), the closedness of ω, and the cochain condition $\mathrm{d}^2 = 0$. ■

Proposition 1.29 *Let (M, ω) be a symplectic manifold. The Poisson bracket satisfies the following properties:*

1. *It is* linear *in each variable,* $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$.
2. *It is* skew-symmetric*,* $\{f, g\} = -\{g, f\}$.
3. *It satisfies* Leibniz's rule*,* $\{fg, h\} = f\{g, h\} + \{f, h\}g$.
4. *It satisfies* Jacobi's identity*,* $\{f, \{g, h\}\} + \{f, \{g, h\}\} + \{f, \{g, h\}\} = 0$.

Proof. The first three items are straightforward consequences of the definition of Hamiltonian vector field and the properties of the symplectic form ω.

To see Jacobi's identity, we use proposition 1.28 together with Poisson's equation (1.23). Applying this result to arbitrary functions $f, g, h \in \mathcal{C}^\infty(M)$,

$$\begin{aligned}
\{\{f, g\}, h\} &= \mathcal{L}_{X_{\{f,g\}}} \\
&= \mathcal{L}_{[X_f,X_g]} h \\
&= \mathcal{L}_{X_f}\mathcal{L}_g h - \mathcal{L}_{X_g}\mathcal{L}_{X_f} h \\
&= \{f, \{g, h\}\} - \{g, \{f, h\}\}.
\end{aligned}$$

This equation, using the skew-symmetry of the bracket $\{\cdot, \cdot\}$, implies Jacobi's identity. ■

These properties were obtained from the bracket (1.24) defined in terms of the symplectic form ω. We can use these properties to define a Poisson bracket in a more general sense, generalizing the construction (1.24).

Definition 1.30 — Poisson bracket. Let M be a smooth manifold. A *Poisson bracket* is a map $\{\cdot,\cdot\}\colon \mathcal{C}^\infty(M)\times\mathcal{C}^\infty(M)\longrightarrow\mathcal{C}^\infty(M)$ which satisfies properties 1 to 4 in proposition 1.29. A pair $(M,\{\cdot,\cdot\})$, where M is a smooth manifold and $\{\cdot,\cdot\}$ is a Poisson bracket, is called a *Poisson manifold*.

We can give a different characterization of Poisson structures. Leibniz's rule is a derivation rule for the Poisson bracket. From this condition we can prove, using standard methods with partitions of unity, that the Poisson bracket is local in both variables; that is, if $f, f' \in \mathcal{C}^\infty(M)$ and $g, g' \in \mathcal{C}^\infty(M)$ agree in some neighbourhood $p\subset U$, then $\{f,g\}(p)=\{f',g'\}(p)$. This result, combined with linearity, implies that the Poisson bracket can be recovered from a tensor field $\varPi\in\varGamma(\mathcal{T}^2M)$ as

$$\{f,g\}=\langle \mathrm{d}f\otimes\mathrm{d}g,\varPi\rangle. \tag{1.27}$$

From the skew-symmetry condition, the tensor field actually factors to a bivector field $\varPi\in\chi^2(M)$ and equation (1.27) is transformed into

$$\{f,g\}=\langle \mathrm{d}f\wedge\mathrm{d}g,\varPi\rangle. \tag{1.28}$$

The bivector $\varPi$ encodes properties 1, 2 and 3 in proposition 1.29. Jacobi's identity has to be imposed as an additional integrability condition $[\varPi,\varPi]=0$.[7] In fact, the considerations made here are actually equivalent to the properties in 1.29 defining the Poisson structure; that is, any bivector field $\varPi\in\chi^2(M)$ satisfying $[\varPi,\varPi]=0$ defines a Poisson structure by means of formula (1.28). This motivates a more geometric definition of Poisson structures and Poisson manifold.

[7] The bracket $[\cdot,\cdot]$ is called the *Schouten-Nijenhuis bracket* and endows the vector space of multivector fields $\chi^\bullet(M)$ with a graded algebra structure. The existence of this bracket is, in fact, equivalent to the existence of the exterior differential d and the Lie bracket of vector fields $[\cdot,\cdot]$.

Definition 1.31 — Poisson tensor. Let M be a smooth manifold. A *Poisson tensor* is a bivector field $\varPi\in\chi^2(M)$ that satisfies the integrability condition $[\varPi,\varPi]=0$.

The previous results show that there exists a one-to-one correspondence between Poisson brackets and Poisson tensors. We will use the characterization in terms of the tensor $\varPi$ for the rest of the thesis.

Given Poisson manifolds, we can consider a subclass of smooth maps which preserve the structure. This gives rise to the notion of *Poisson morphism*. As expected, Poisson morphisms will satisfy the properties of morphisms in a category.

Definition 1.32 — Poisson morphisms. Let $(M,\varPi)$ and $(N,\varLambda)$ be Poisson manifolds. A smooth map $f\colon M\longrightarrow N$ is said to be *Poisson* or a *Poisson morphism* if

$$f_*\varPi=\varLambda. \tag{1.29}$$

Lemma 1.33 *Let $(M,\varPi)$, $(N,\varLambda)$ and $(L,\varDelta)$ be Poisson manifolds.*

1. *The identity map $\mathrm{id}_M\colon M\longrightarrow M$ is a Poisson morphism.*
2. *If $f\colon M\longrightarrow N$ and $g\colon N\longrightarrow L$ are Poisson morphisms, $gf\colon M\longrightarrow L$ is a Poisson morphism. Moreover, the composition of Poisson morphisms is associative.*

Proof. Firstly, the identity map $\mathrm{id}_M\colon M\longrightarrow M$ is a Poisson morphism. This follows from the fact that $(\mathrm{id}_M)_* = \mathrm{id}_{\mathcal{X}(M)}$. As a consequence, $(\mathrm{id}_M)_*\Pi = \mathrm{id}_{\mathcal{X}(M)}\,\Pi = \Pi$.

Secondly, given Poisson manifolds (M,Π), (N,Λ) and (L,Δ) and Poisson maps $f\colon M \longrightarrow N$, $g\colon N \longrightarrow L$, the chain rule implies $(gf)_* = g_*f_*$. Therefore, we see that

$$(gf)_*\Pi = g_*f_*\Pi = g_*\Lambda = \Delta,$$

showing that gf is a Poisson morphism. The composition of maps is associative, as it is inherited from the associativity for smooth maps. ■

The class of Poisson manifolds together with Poisson maps give rise to the category of *Poisson manifolds*, denoted by $\mathcal{P}ois$.

Local structure

Observe that a Poisson tensor Π is defined by the condition $[\Pi,\Pi] = 0$, which is local. Additionally, the Poisson bracket (1.28) is also local. As a consequence, the restriction of Π to an open set $U \subset M$ is a Poisson tensor. Throughout this section we will explore the local expression of Poisson tensors.

Proposition 1.34 *Let (M,Π) be a Poisson manifold and consider a chart (U,φ) with coordinates $\boldsymbol{x}$. If Π_{ij} are the local coefficients of Π in coordinates $\boldsymbol{x}$, then for any functions $f,g \in \mathcal{C}^\infty(M)$ we have*

$$\{f,g\} = \sum_{i,j=1}^{n} \Pi_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}. \tag{1.30}$$

Proof. From the definition of the Poisson bracket from the Poisson tensor (1.28), a direct computation yields

$$\begin{aligned}\{f,g\} &= \frac{1}{2}\sum_{i,j=1}^{n} \Pi_{ij}\left(\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}\right)(\mathrm{d}f,\mathrm{d}g)\\ &= \frac{1}{2}\sum_{ij=1}^{n} \Pi_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \Pi_{ij}\frac{\partial g}{\partial x_i}\frac{\partial f}{\partial x_j}\\ &= \sum_{i,j=1}^{n} \Pi_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}.\end{aligned}$$ ■

Corollary 1.35 *Let (M,Π) be a Poisson manifold and consider a chart (U,φ) with coordinates $\boldsymbol{x}$. The structure functions are given by $\Pi_{ij} = \{x_i,x_j\}$.*

Proof. A direct application of equation (1.30) to the smooth functions $x_i, x_j \in \mathcal{C}^\infty(U)$ shows that

$$\{x_i,x_j\} = \sum_{k,l=1}^{n} \Pi_{ij}\frac{\partial x_i}{\partial x_k}\frac{\partial x_j}{\partial x_l} = \sum_{k,l=1}^{n} \Pi_{ij}\delta_{ik}\delta_{jl} = \Pi_{ij}.$$ ■

Observation 1.36 In a symplectic manifold (M,ω) with the Poisson structure Π_ω induced from the symplectic form, the structure functions in a Darboux chart with coordinates $\boldsymbol{q}, \boldsymbol{p}$ are $\{q_i, q_j\} = 0$, $\{p_i, p_j\} = 0$ and $\{q_i, p_j\} = \delta_{ij}$. As a consequence, the Poisson tensor is

$$\Pi_\omega = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}. \tag{1.31}$$

This gives the standard expression for the Poisson bracket of two observables in classical mechanics,

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}.$$

Jacobi's identity has not been taken into account in the local description of Poisson tensors. It acts as a constraint in the structure functions of any chart. To prove this result, we begin with a preliminary statement regarding the local expression of the Jacobi identity.

Proposition 1.37 *Let (M, Π) be a Poisson manifold and consider a local chart (U, φ) for which the structure constants of Π are denoted by $\Pi_{ij} \in \mathcal{C}^\infty(U)$. If $f, g, h \in \mathcal{C}^\infty(M)$, then*[8]

$$\begin{aligned}&\{f,\{g,h\}\} + \{f,\{g,h\}\} + \{f,\{g,h\}\}\\ &= \sum_{i,j,k=1}^{n} \left(\sum_{l=1}^{n} \Pi_{il}\frac{\partial \Pi_{jk}}{\partial x_l} + \Pi_{jl}\frac{\partial \Pi_{ki}}{\partial x_l} + \Pi_{kl}\frac{\partial \Pi_{ij}}{\partial x_l} \right) \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial x_k}.\end{aligned} \tag{1.32}$$

[8] In some references, the following expression is defined as the *Jacobiator* J, which acts on smooth functions $J(f, g, h)$. Equation (1.32) shows that the Jacobiator, similarly to the Poisson tensor, can be recovered from a trivector field $\mathcal{J} \in \chi^3(M)$, and that its local expression is

$$\mathcal{J} = \frac{1}{6}\sum_{i,j,k=1}^{n}\left(\sum_{l=1}^{n}\sum_{\text{cyc.}}\Pi_{il}\frac{\partial \Pi_{jk}}{\partial x_l}\right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k}.$$

Proof. For the computation we use the expression in local coordinates 1.30. We first carry the computation[9] for a single term, obtaining

[9] We temporarily drop the summation symbols and use instead Einstein's summation convention to ease the notation as much as possible.

$$\begin{aligned}\{f,\{g,h\}\} &= \Pi_{ij}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}\left(\Pi_{kl}\frac{\partial g}{\partial x_k}\frac{\partial h}{\partial x_l}\right)\\ &= \Pi_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_k}\frac{\partial h}{\partial x_l}\frac{\partial \Pi_{kl}}{\partial x_j} + \Pi_{ij}\Pi_{kl}\frac{\partial f}{\partial x_i}\frac{\partial^2 g}{\partial x_j \partial x_k}\frac{\partial h}{\partial x_l}\\ &+ \Pi_{ij}\Pi_{kl}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_k}\frac{\partial^2 h}{\partial x_j \partial x_l}.\end{aligned}$$

Using this expression for each term and regrouping the obtained expressions we have

$$\begin{aligned}&\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}\\ &= \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial x_k}\left(\Pi_{il}\frac{\partial \Pi_{jk}}{\partial x_l} + \Pi_{jl}\frac{\partial \Pi_{ki}}{\partial x_l} + \Pi_{kl}\frac{\partial \Pi_{ij}}{\partial x_l}\right)\\ &+ \frac{\partial g}{\partial x_i}\frac{\partial h}{\partial x_j}\left(\Pi_{ij}\Pi_{jl}\frac{\partial^2 f}{\partial x_l \partial x_k} - \Pi_{il}\Pi_{jk}\frac{\partial^2 f}{\partial x_l \partial x_k}\right)\\ &+ \frac{\partial h}{\partial x_i}\frac{\partial f}{\partial x_j}\left(\Pi_{ij}\Pi_{jl}\frac{\partial^2 g}{\partial x_l \partial x_k} - \Pi_{il}\Pi_{jk}\frac{\partial^2 g}{\partial x_l \partial x_k}\right)\\ &+ \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\left(\Pi_{ij}\Pi_{jl}\frac{\partial^2 h}{\partial x_l \partial x_k} - \Pi_{il}\Pi_{jk}\frac{\partial^2 h}{\partial x_l \partial x_k}\right)\end{aligned}$$

We can now eliminate the last three sums. Considering now fixed indexes i, j, the terms between parentheses can be simplified because f is a smooth function and Schwarz's theorem can be applied. Swapping indexes k and l

we can see that

$$\Pi_{ik}\Pi_{jl}\frac{\partial^2 f}{\partial x_l \partial x_k} - \Pi_{il}\Pi_{jk}\frac{\partial^2 f}{\partial x_l \partial x_k} = \Pi_{ik}\Pi_{jl}\frac{\partial^2 f}{\partial x_l \partial x_k} - \Pi_{il}\Pi_{jk}\frac{\partial^2 f}{\partial x_k \partial x_l} \\ = 0.$$

As this happens for every $1 \leqslant i, j, k, l \leqslant n$, all the sums vanish. As a consequence, we have that

$$\{f, \{g, h\}\} + \{f, \{g, h\}\} + \{f, \{g, h\}\} \\ = \sum_{i,j,k=1}^{n} \left(\sum_{l=1}^{n} \Pi_{il}\frac{\partial \Pi_{jk}}{\partial x_l} + \Pi_{jl}\frac{\partial \Pi_{ki}}{\partial x_l} + \Pi_{kl}\frac{\partial \Pi_{ij}}{\partial x_l} \right) \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial x_k}.$$

as we wanted to prove. ■

In contrast to symplectic geometry, there is no local normal form for Poisson structures. The closest analogue is known as Weinstein's splitting theorem, which we state without proof.

Theorem 1.38 — Weinstein. *Let (M, Π) be a Poisson manifold. For every point $p \in M$ there exists a coordinate chart (U, φ) with coordinates $\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{r}$ such that*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} + \sum_{i,j=1}^{l} \Pi_{ij}(\boldsymbol{r}) \frac{\partial}{\partial r_i} \wedge \frac{\partial}{\partial z_j}. \tag{1.33}$$

and $\Pi_{ij}(\mathbf{0}) = 0$. The former term is denoted by Π_{S}, while the latter is denoted by Π_{T}; these structures are called symplectic *and* transverse *Poisson structures, respectively.*

Hamiltonian dynamics and Poisson cohomology

The fundamental motivation to study symplectic and Poisson geometry, at least from a physicist's point of view, is that such geometric structures encode the dynamical information of a classical physical system. The fundamental construction in this regard is the definition of Hamiltonian vector field (1.21). There exists an analogue construction for Hamiltonian vector fields in the realm of Poisson geometry. Here, the role of the musical isomorphism $\omega^\sharp$ is played by the *anchor map $\Pi^\sharp$*. In this case, the map is not necessarily a vector bundle isomorphism given that Π might not have maximal rank. Hamiltonian vector fields in the Poisson sense are defined by means of Poisson's equation (1.25) for the time evolution of observables.

Definition 1.39 Let (M, Π) be a Poisson manifold. To any smooth function $H \in \mathcal{C}^\infty(M)$ we associate its *Hamiltonian vector field*, defined as the unique vector field $X_H \in \chi(M)$ such that

$$\mathcal{L}_{X_H} f = \{H, f\}. \tag{1.34}$$

Remark 1.40 We can rewrite this equation in terms of the Poisson tensor, in a similar notation to the definition of Hamiltonian vector field in the symplectic setting (1.21). By definition of the action of Π on smooth functions, we have that $\{f, g\} = \langle \mathrm{d}f \wedge \mathrm{d}g, \Pi \rangle$. This equation allows us to define the Hamiltonian vector field directly as $X_H = \iota_{\mathrm{d}H}\Pi$.

The last equation is reminiscent to the definition of Hamiltonian vector fields in symplectic geometry in terms of the musical isomorphism $\omega^\sharp$. We introduce, inspired by this observation, the anchor map. Hamiltonian vector fields will be consequently characterized as the image of exact one-forms under the anchor map.

Definition 1.41 Let (M, Π) be a Poisson manifold. The anchor map is defined as the vector bundle morphism

$$\begin{array}{rccc} \Pi^\sharp\colon & \mathrm{T}^*M & \longrightarrow & \mathrm{T}M \\ & \alpha & \longmapsto & \iota_\alpha \Pi. \end{array} \tag{1.35}$$

The definition of Hamiltonian vector field by Poisson's equation (1.34) together with Jacobi's identity imply that the assignation of Hamiltonian vector fields is a Lie algebra morphism.

Proposition 1.42 *Let (M, Π) be a Poisson manifold. If $H, K \in C^\infty(M)$ are smooth functions, then*

$$[X_H, X_K] = X_{\{H,K\}}. \tag{1.36}$$

Proof. To show this result, we will use the characterization of smooth vector fields as derivations. Take an arbitrary function $f \in C^\infty(M)$. Using the naturality of the Lie bracket,

$$\begin{aligned} \mathcal{L}_{[X_H, X_K]} f &= \mathcal{L}_{X_H} \mathcal{L}_{X_K} f - \mathcal{L}_{X_K} \mathcal{L}_{X_H} f \\ &= \{H, \{K, f\}\} - \{K, \{H, f\}\} \\ &= -\{f, \{H, K\}\} \\ &= \mathcal{L}_{X_{\{H,K\}}} f. \end{aligned}$$

From the fact that f is an arbitrary function, we conclude equation (1.36). ■

In proposition 1.23 we saw that there exists a relationship between the topology of the manifold M and the structure of the space of Hamiltonian vector fields. It is natural to wonder if similar considerations hold for Poisson manifolds. The structure of Hamiltonian vector fields and Poisson vector fields can be recovered from a cohomology, different in this case from de Rham's cohomology.

Proposition 1.43 *Let (M, Π) be a Poisson manifold. The family of maps*

$$\begin{array}{rccc} \mathrm{d}_\Pi^l\colon & \chi^l(M) & \longrightarrow & \chi^{l+1}(M) \\ & X & \longmapsto & [\Pi, X] \end{array} \tag{1.37}$$

satisfies $\mathrm{d}_\Pi^{l+1} \, \mathrm{d}_\Pi^l = 0$.

Proof. To prove this statement we consider an arbitrary multivector field $X \in \chi^l(M)$ and compute $\mathrm{d}_\Pi^{l+1} \, \mathrm{d}_\Pi^l(X)$ which is, by definition, equivalent to $[\Pi, [\Pi, X]]$. By the graded Jacobi identity we have

$$\begin{aligned} [\Pi, [\Pi, X]] &= [[\Pi, \Pi], X] + (-1)^{(2-1)(2-1)} [\Pi, [\Pi, X]] \\ &= -[\Pi, [\Pi, X]]. \end{aligned}$$

Here we have used the fact that $[\Pi, \Pi] = 0$ by definition of Poisson tensor. As a consequence, $[\Pi, [\Pi, X]] = 0$, which is what we wanted to prove. ■

This result implies that we can define a map d_{Π} in the graded algebra $\chi^{\bullet}(M)$ similar to the exterior differential in smooth geometry such that $\mathrm{d}_{\Pi}^2 = 0$. This induces a cochain complex structure in $\chi^{\bullet}(M)$, called the *Lichnerowicz complex*. The resulting cohomology is called *Poisson cohomology*. For this cohomology the set of chains is denoted by $\mathcal{Z}^l_{\Pi}(M)$, while the set of boundaries is denoted by $\mathcal{B}^l_{\Pi}(M)$. The cohomology groups are represented by $H^l_{\Pi}(M)$.

The following proposition gives a direct analogue to proposition 1.23 in the Poisson case.

Proposition 1.44 *Let (M, Π) be a Poisson manifold.*

1. *A vector field $X \in \chi^1(M)$ is Poisson if and only if $X \in \mathcal{Z}^1_{\Pi}(M)$.*
2. *A vector field $X \in \chi^1(M)$ is Hamiltonian if and only if $X \in \mathcal{B}^1_{\Pi}(M)$.*

Proof. Regarding item 1, from the properties of the Schouten-Nijenhuis bracket[10] we can see that $\mathcal{L}_X\Pi = [X, \Pi] = [\Pi, X]$. As a consequence, $\mathcal{L}_X\Pi = 0$ if and only if $[\Pi, X] = 0$. Item 1 follows by definition from this result.

[10] Ivan Kolář, Slovák Jan, and Peter W. Milchor. *Natural Operations in Differential Geometry*. Springer Berlin, Heidelberg, 1 edition, 1993. ISBN 978-3-540-56235-1. DOI: 10.1007/978-3-662-02950-3

Regarding item 2, we use the characterization of the Hamiltonian vector field X_H as image under the anchor map, $X_H = \Pi^{\sharp}(\mathrm{d}H) = \iota_{\mathrm{d}H}\Pi$. Using a property of the Schouten-Nijenhuis bracket and the graded skew-symmetry, $\iota_{\mathrm{d}H}\Pi = -[H, \Pi] = -[\Pi, H]$. Therefore, $X_H = -\mathrm{d}_{\Pi}H$ and, as a consequence, $X_H \in \mathcal{B}^1_{\Pi}(M)$. This equation also shows that every vector field $X \in \mathcal{B}^1_{\Pi}(M)$ is Hamiltonian. ■

1.4 Reduction of phase spaces by Lie group actions

The reduction of a physical system by a group action has been one of the most successful techniques in classical mechanics. It was used, for example, by Euler to show that the two-body problem is integrable and to find explicit integration for the equations of motion. It also plays a important role in the description of the phase spaces of gauge theories that we will encounter in section 1.5. For this purpose, we will present the basic results on the reduction of phase spaces.

Marsden-Weinstein reduced spaces

We understand that a Lie group G acts on a physical system (M, ω), described as a symplectic manifold and a Hamiltonian function $H \in C^{\infty}(M)$, if the group action preserves all the data of the system. Therefore, we shall impose that the symplectic form is preserved by the group action, arriving to the notion of symplectic group action.

Definition 1.45 Let (M, ω) be a symplectic manifold. A group action $\rho\colon G \times M \longrightarrow M$ is said to be *symplectic* if the induced group morphism $\rho_{\bullet}\colon G \longrightarrow \mathrm{Diff}(M)$ restricts to the set of symplectomorphisms.

In the line of the previous discussion, we impose that the Hamiltonian defining the equations of motion is invariant by the action of G. Under these assumptions, the Hamiltonian flow of the system can be descibed in the quotient space M/G.

Proposition 1.46 *Let (M, ω) be a symplectic manifold and consider a Hamiltonian $H \in C^\infty(M)$. Let $\rho \colon G \times M \longrightarrow M$ be a proper, free, symplectic group action such that H is G-invariant. Then,*

1. *The quotient space M/G is a smooth manifold and the induced structure $\Lambda := \pi_* \Pi$ is well defined and Poisson, where $\pi \colon M \longrightarrow M/G$ is the natural projection and Π is the natural Poisson structure induced from ω.*
2. *The Hamiltonian vector field X_H is π-projectable, and $\pi_* X_H = X_{[H]}$. Here $[H] \in C^\infty(M/G)$ is the Hamiltonian function H taken over classes of equivalence.*
3. *The flow φ_t of X_H is π-projectable and $\pi_*\varphi_t = \psi_t$, where ψ_t is the flow of $X_{[H]}$.*

Proof. For item 1, the quotient space M/G is a smooth manifold by freeness and properness of ρ. From the fact that ρ is a symplectic action, we can see that $(\rho_\bullet)_*\Pi = \Pi$. As a consequence, Π is π-projectable, obtaining a bivector field $\Lambda := \pi_*\Pi$ and the projection π is automatically a Poisson map. From the naturality of the Schouten-Nijenhuis bracket,

$$[\Lambda, \Lambda] = [\pi_*\Pi, \pi_*\Pi] = \pi_*[\Pi, \Pi] = 0, \tag{1.38}$$

and Λ is a Poisson structure.

Regarding item 2, we may regard the function $[H]$ as the projection of H under π, which exists by G-invariance of H. Therefore, we can say that $[H]$ is π-related with H. Once again, using the properties of the Schouten-Nijenhuis bracket we see

$$\pi_* X_H = -\pi_*[\Pi, H] = -[\pi_*\Pi, \pi_* H] = -[\Lambda, [H]] = X_{[H]}.$$

This proves the desired result.

Item 3 is a straightforward consequence of item 2. ■

As a consequence of this result, we can see that the Hamiltonian motion generated by H in M can be studied in the Poisson manifold M/G, which has a reduced number of degrees of freedom or, in mathematical terms, a lower dimension. It turns our that the Hamiltonian dynamics in M/G can be reduced to a smaller space. This is not only true for the reduction of a symplectic manifold by a symplectic group action, but for every Hamiltonian vector field in a Poisson manifold. The result is a corollary of Weinstein's splitting theorem 1.38.

Corollary 1.47 — Symplectic foliation of Poisson manifolds. *Let (M, Π) be a Poisson manifold. There exists a foliation $\mathcal{F}_\Pi$ of Π, such that every leaf $\mathcal{F}_\Pi(p)$ has a symplectic form ω induced by the symplectic Poisson structure [11], $\Pi_\omega = \Pi_S$. Moreover, every Hamiltonian vector field is tangent to $\mathcal{F}_\Pi$.*

[11] Hence the name "symplectic".

Proof. Weinstein's splitting theorem gives a chart which can be used to describe the foliation. In coordinates $\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{r}$ the symplectic leaf at $p = \mathbf{0}$ is described by $\boldsymbol{r} = 0$. We notice that, from the condition $\Pi_{ij}(\mathbf{0}) = 0$, Hamiltonian vector fields at $\boldsymbol{r} = \mathbf{0}$ are automatically tangent to $\mathcal{F}_\Pi(p)$. We notice that the Poisson structure Π_S restricts to $\mathcal{F}_\Pi(p)$ and $i_*\Pi_S = \Pi$. Because Π_S has maximal rank, it defines a symplectic form in $\mathcal{F}_\Pi(p)$. ■

With these result we are ready to present the main result concerning the reduction of a Hamiltonian system by a symplectic group action.

Theorem 1.48 — Marsden, Weinstein. *Let (M, ω) be a symplectic manifold and consider a Hamiltonian function $H \in \mathcal{C}^\infty(M)$. Let $\rho\colon G \times M \longrightarrow M$ be a proper, free and symplectic group action such that H is G-invariant. For a point $p \in M$, the Hamiltonian flow φ_t generated by H is π-projectable, is tangent to $\mathcal{F}_\Pi(\pi(p))$ and the restriction is Hamiltonian with Hamiltonian function $\pi_* H$.*

Noether's theorem and the moment map

In classical physics, conserved quantities have been used to study the dynamics of physical systems and, more importantly, to reduce them. This follows the discussion of the symplectic group actions and the reduction of the phase space of a physical system. The relationship between these two concepts is known as Noether's theorem. We introduce conserved quantities and prove Noether's theorem which, in the notation of modern differential geometry, is surprisingly easy.

Definition 1.49 Let (M, Π) be a Poisson manifold and consider a Hamiltonian function $H \in \mathcal{C}^\infty(M)$. An observable $f \in \mathcal{C}^\infty(M)$ is called a *conserved quantity* if[12]

$$\mathcal{L}_{X_H} f = 0. \tag{1.39}$$

[12] In the spirit of Poisson's equation (1.25), conserved quantities are also said to *Poisson commute* with H.

Theorem 1.50 — Noether. *Let (M, Π) be a Poisson manifold and consider a Hamiltonian $H \in \mathcal{C}^\infty(M)$. A function $f \in \mathcal{C}^\infty(M)$ is a conserved quantity if and only if the Hamiltonian H is invariant by the Hamiltonian flow of f.*

Proof. The proof simply relies on noticing that, by Poisson's equation and the skew-symmetry of Poisson's bracket,

$$\mathcal{L}_{X_H} f = \{H, f\} = -\{f, H\} = -\mathcal{L}_{X_f} H.$$

As a consequence, f is a conserved quantity if and only if $\mathcal{L}_{X_f} H = 0$. This is equivalent to saying that $\varphi_t^* H = H$ whenever φ_t is defined, where φ_t is the flow of X_f. ■

This result justifies the importance of conserved quantities in classical mechanics. A method for finding conserved quantities was found by Poisson using his eponymous bracket. In fact, although Poisson geometry has become an important field of study on its own, the main motivation for the introduction of the Poisson bracket was to find conserved quantities in physical systems.

Theorem 1.51 — Poisson. *Let (M, Π) be a Poisson manifold and consider a Hamiltonian $H \in \mathcal{C}^\infty(M)$. If $f, g \in \mathcal{C}^\infty(M)$ are conserved quantities, then $\{f, g\} \in \mathcal{C}^\infty(M)$ is also a conserved quantity.*

Proof. This theorem is a direct application of equation (1.36), as

$$\mathcal{L}_{X_{\{f,g\}}} h = \mathcal{L}_{[X_f, X_g]} h = \mathcal{L}_{X_f} \mathcal{L}_{X_g} h - \mathcal{L}_{X_g} \mathcal{L}_{X_f} h = 0.$$ ■

This result can be equivalently read in algebraic terms: the set of conserved quantities in a Hamiltonian system is a Lie subalgebra of $(\mathcal{C}^\infty(M), \{\cdot, \cdot\})$.

Noether's theorem states that one-parameter subgroups can be recovered from conserved quantities using the Hamiltonian flows. We have studied the Marsden-Weinstein reduction of a symplectic manifold under a symplectic group action. Motivated by this result, we can wonder about the circumstances in which there exists such a correspondence for Lie group actions. In order to use a similar argument to that in Noether's theorem, we consider group actions $\rho\colon G \times M \longrightarrow M$ which act in Hamiltonian fashion; in other words, we ask for the fundamental action to restrict to Hamiltonian vector fields, $\hat{\rho}\colon \mathfrak{g} \longrightarrow \chi_{\mathrm{Ham}}(M)$. We call these actions *weakly Hamiltonian*.

We study now the structure of weakly-Hamiltonian actions. Given that the fundamental action $\hat{\rho}$ is linear, $\mathfrak{g}$ is finite-dimensional, and the assignation of Hamiltonian vector fields $X_\bullet$ is surjective, we can always define a lift $\mu^\bullet\colon \mathfrak{g} \longrightarrow C^\infty(M)$ which makes diagram (1.40) commute. In this setting, the map μ^* is called the *comoment map* of the action, and it plays the role of the generators of motion in the proof of theorem 1.50.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^\infty_{\mathrm{Cas}}(M) & \xrightarrow{\iota} & C^\infty(M) & \xrightarrow{X_\bullet} & \chi_{\mathrm{Ham}}(M) & \longrightarrow & 0 \\ & & & & & \nwarrow^{\mu^\bullet} & \uparrow{\scriptstyle\hat{\rho}} & & \\ & & & & & & \mathfrak{g} & & \end{array} \tag{1.40}$$

This diagram encodes the Hamiltonian equations of motion for any fundamental vector field of ρ, $\hat{\rho}(X) = X_{\mu^X}$ or, in other terms,

$$\iota_{\hat{\rho}(X)}\omega = -\,\mathrm{d}\mu^X. \tag{1.41}$$

We can define now the *moment map* of the action, $\mu\colon M \longrightarrow \mathfrak{g}^*$, by the duality induced from the inner pairing $\langle\cdot,\cdot\rangle\colon \mathfrak{g}^* \times \mathfrak{g} \longrightarrow \mathbf{R}$ as

$$\langle \mu(p), X\rangle = \mu^X(p), \tag{1.42}$$

for every $p \in M$ and $X \in \mathfrak{g}$. The existence of X is actually equivalent to the linearity of μ^*, which is in turn equivalent to the condition $\hat{\rho}\colon \mathfrak{g} \longrightarrow \chi_{\mathrm{Ham}}(M)$. The moment map of the action will play the role of conserved quantities in the proof of Noether's theorem.

With the previous construction we have one part of Noether's theorem: the moment map of a weakly Hamiltonian group action is a conserved quantity. We cannot prove, however, that the choice of a moment map is also equivalent to a global group action by Hamiltonian vector fields. The issue is that the correspondence between local and global actions is governed by Lie's third theorem. The information for the group structure in G is given by the Lie algebra structure of $\mathfrak{g}$. Without further assumptions, we cannot ensure that the assignation $\hat{\rho}\colon \mathfrak{g} \longrightarrow \chi_{\mathrm{Ham}}(M)$ is a Lie algebra anti-homomorphism, and Lie's third theorem cannot be used.

Let us assume that we have a comoment map $\mu^\bullet\colon \mathfrak{g} \longrightarrow C^\infty(M)$. The commutativity of diagram (1.40) completely defines the assignation $\hat{\rho}$. Given that $X_\bullet$ is a Lie algebra homomorphism by proposition 1.42, we impose that $\mu^\bullet$ is a Lie algebra anti-homomorphism. In terms of the moment map, this condition is equivalent to requiring μ to be Ad^*-invariant.[13]

[13] Zuoqin Wang. The moment map. URL `http://staff.ustc.edu.cn/~wangzuoq/Courses/15S-Symp/Notes/Lec08.pdf`

Definition 1.52 Let (M, ω) be a symplectic manifold and consider a connected Lie group action $\rho\colon G \times M \longrightarrow M$. We say that ρ is a *Hamiltonian group action* if there exists a function $\mu\colon M \longrightarrow \mathfrak{g}^*$, called the *moment map*, that satisfies the following properties:

1. If we define the function $\mu^X(p) = \langle \mu(p), X \rangle$, the fundamental action satisfies Hamilton's equations of motion
$$\iota_{\hat\rho(X)}\omega = -\,\mathrm{d}\mu^X. \tag{1.43}$$

2. The moment map intertwines the group action and the coadjoint representation,
$$\mathrm{Ad}^*_\bullet\, \mu = \mu\rho_\bullet. \tag{1.44}$$

Theorem 1.53 — Noether. *Let (M, ω) be a symplectic manifold with a Hamiltonian function $H \in \mathcal{C}^\infty(M)$.*

1. *If $\rho\colon G \times M \longrightarrow M$ is a Hamiltonian group action and H is G-invariant, the function μ^X is a conserved quantity for each $X \in \mathfrak{g}$.*

2. *If we are given an* Ad**-equivariant map μ such that X_{μ^X} is a complete vector field and $\{H, \mu^X\} = 0$ for each $X \in \mathfrak{g}$, then we have a Lie group action ρ for which H is G-invariant.*

Proof. Regarding the first item, from equation (1.41) and applying the same argument as in theorem 1.50 from Poisson's equation,
$$\mathcal{L}_{\hat\rho(X)}H = \{\mu^X, H\} = -\{H, \mu^X\} = -\mathcal{L}_{X_H}\mu^X.$$
Given the fact that H is G-invariant, we have that $\mathcal{L}_{\hat\rho(X)}H = 0$. Therefore, μ^X is a conserved quantity.

To prove the converse, from the fact that μ is Ad*-equivariant we know that the assignation $\mu^\bullet$ is a Lie algebra morphism. As a consequence, $\hat\rho \coloneqq X_{\mu^\bullet}$ is a Lie algebra anti-homomorphism, given that
$$X_{\mu^{[X,Y]}} = X_{-\{\mu^X, \mu^Y\}} = -[X_{\mu^X}, X_{\mu^Y}].$$
As the vector fields $\hat\rho(X)$ are complete by definition, we can apply Lie's third theorem to integrate the infinitesimal action $\hat\rho$ to a Lie group action $\rho\colon G \times M \longrightarrow M$. Given that $\mathcal{L}_{\hat\rho(X)}H = \{H, \mu^X\} = 0$ by assumption, we conclude that H is G-invariant. ■

1.5 Gauge theories in classical mechanics

Gauge theories describe point-mass particles, modelled over a manifold M, with internal symmetries. Given a point $p \in M$, the possible configurations of symmetry are understood to be described in terms of a smooth manifold Q. This pointwise picture is naturally extended over the whole manifold M using the notion of fibre bundle.[14]

[14] Many definitions and results in this chapter have been extracted from the book of Kolář, Jan, and Milchor.

Definition 1.54 Let M and Q be smooth manifolds. A *smooth fibre bundle B with fibre Q* is a surjective map $\pi\colon B \longrightarrow M$ such that for every point $p \in M$ there exists a open set U such that $\psi\colon \pi^{-1}(U) \longrightarrow U \times Q$ is a diffeomorphism, making diagram (1.45) commute. The pair (U, ψ) is a *fibre chart*. Here, B is said to be the *total space* and M is the *base space*.

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \psi\ } & U \times Q \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & U & \end{array} \tag{1.45}$$

Definition 1.55 Let $\tau_M : B_M \longrightarrow M$ and $\tau_N : B_N \longrightarrow N$ be fibre bundles. A *morphism of fibre bundles* is a pair (F, f), where $F : B_M \longrightarrow B_N$ and $f : M \longrightarrow N$ are smooth maps which make diagram (1.46) commutative. It is said that F *covers* f.

$$\begin{array}{ccc} B_M & \xrightarrow{F} & B_N \\ {\scriptstyle \tau_M}\downarrow & & \downarrow{\scriptstyle \tau_N} \\ M & \xrightarrow[f]{} & N \end{array} \tag{1.46}$$

Remark 1.56 Observe that the commutativity of diagram (1.46) implies that the image of the fibre $F(\pi_M^{-1}(p)) \subseteq \tau_N^{-1}(f(p))$. Indeed, take any point $q \in \pi_M^{-1}(p) \subset B_M$. As a consequence of the definition of fibre bundle morphism, $f(p) = f\tau_M(q) = \tau_N f(q)$, and therefore $f(q) \in \pi_N^{-1}(f(p))$, as we wanted to prove.

This remark shows that the map f is completely determined by F. We have $f(p) = \tau_N F(q)$, where $p \in M$ and $q \in \tau_M^{-1}(p)$. This construction does not depend on the choice of q.

Proposition 1.57 *Let $\tau_M : B_M \longrightarrow M$, $\tau_N : B_N \longrightarrow N$ and $\tau_L : B_L \longrightarrow L$ be fibre bundles.*

1. *The identity morphism* $\mathrm{id}_{B_M} : B_M \longrightarrow B_M$ *is a fibre bundle morphism*
2. *If $F : B_M \longrightarrow B_N$ and $G : B_N \longrightarrow B_L$ are fibre bundles covering f and g, respectively, $GF : B_M \longrightarrow B_L$ is a fibre bundle morphism covering gf. Moreover, composition of fibre bundle morphisms is associative.*

Proof. The identity map id_{B_M} is obviously smooth and it covers id_M, making the pair $(\mathrm{id}_{B_M}, \mathrm{id}_M)$ a fibre bundle morphism.

$$\begin{array}{ccccc} B_M & \xrightarrow{F} & B_N & \xrightarrow{G} & B_L \\ \downarrow{\scriptstyle \tau_M} & & \downarrow{\scriptstyle \tau_N} & & \downarrow{\scriptstyle \tau_L} \\ M & \xrightarrow[f]{} & N & \xrightarrow[g]{} & L \end{array} \tag{1.47}$$

Regarding the composition of fibre bundle morphisms, the composition GF is a fibre bundle morphism given the commutativity of diagram (1.47). Associativity follows from the associativity of smooth maps. ■

It is straightforward to check that fibre bundle morphisms satisfy, indeed, the properties of the morphisms of a category. Fibre bundles together with their morphisms give rise to a category $\mathcal{FBun}$, the category of *fibre bundles*. Additionally, the second statement in proposition 1.57 shows that we have a *base functor* $\eta : \mathcal{FBun} \longrightarrow \mathcal{Man}$ taking each fibre bundle to its base, $\eta(B_M) = M$, and each fibre bundle morphism to its base map, $\eta(F, f) = f$.

The smooth structure in smooth manifolds is determined by a maximal atlas of coordinate charts. This definition encodes the idea that the smooth structure of a manifold can be defined locally with additional properties, which allows for patching local charts in a smooth sense. The same idea is somewhat present in the definition of fibre bundle in terms of fibre charts. We say that an open cover by trivializing charts (U_α, ψ_α) is a *fibre bundle atlas*, and denote it $\mathcal{A}(B, \pi)$. If $U_\alpha \cap U_\beta := U_{\alpha\beta}$, the composition $\psi_\alpha \psi_\beta^{-1} : \pi^{-1}(U_{\alpha\beta}) \longrightarrow U_{\alpha\beta} \times Q$ satisfies $\psi_\alpha \psi_\beta^{-1}(x, q) = (x, \psi_{\alpha\beta}(q))$, given the commutativity of diagram (1.45). The maps $\psi_{\alpha\beta} : U_{\alpha\beta} \longrightarrow \mathrm{Diff}(Q)$ are called the *transition functions*. It is straightforward from their definition that these functions satisfy the cocycle condition $\psi_{\alpha\beta}\psi_{\beta\gamma} = \psi_{\alpha\gamma}$ in $U_{\alpha\beta\gamma}$ and $\psi_{\alpha\alpha} = \mathrm{id}_{U_\alpha}$.

The converse statement also holds. That is, given a fibre bundle atlas $\mathcal{A}$, there exists a fibre bundle $\pi : B \longrightarrow M$ with fibre Q such that $\mathcal{A} = \mathcal{A}(B, \pi)$.

Principal bundles, associated bundles and Ehresmann connections

Let us now consider a fixed fibre bundle $\pi\colon B \longrightarrow M$ with fibre Q, understood once again to represent the configuration space of a particle in a manifold M. We consider the group of automorphisms $\mathrm{Aut}(B) \subseteq \mathrm{Hom}_{\mathcal{FBun}}(B, B)$, which is the group of invertible fibre bundle morphisms $f\colon B \longrightarrow B$. The base functor defines group morphism $\eta\colon \mathrm{Aut}(B) \longrightarrow \mathrm{Diff}(M)$. In this sense, any automorphism $f \in \mathrm{Aut}(B)$ gives a transformation in the base space M and a change of internal symmetry. The kernel $\ker\eta$ plays a fundamental role in the theory; elements of $\ker\eta$ are, according to the previous interpretation of automorphisms in B, transformations of B which only change the internal symmetry of the particle.

Definition 1.58 Let $\pi\colon B \longrightarrow M$ be a fibre bundle with fibre Q. The kernel of $\eta\colon \mathrm{Aut}(B) \longrightarrow \mathrm{Diff}(M)$ is called the *gauge group of* B, $\mathrm{Gau}(B) \coloneqq \ker\eta$.

We have encoded internal degrees of freedom of point-mass particles in fibre bundles, enlarging the configuration space of our physical system. We will also consider the action of a Lie group G in our bundle Q by gauge transformations, $\rho_\bullet \in \mathrm{Gau}(M)$. The diffeomorphisms induced by the action transform internal degrees of freedom of particles in M; in this sense, the action reflects symmetries in the physical theory. These objects are the start for gauge theories.

Definition 1.59 Let $\tau\colon B \longrightarrow M$ be a fibre bundle with typical fibre Q. A *G-structure* in B consists of the following objects.

1. A left action $l\colon G \times Q \longrightarrow Q$ on the fibre.
2. A fibre bundle atlas (U_α, ψ_α) for which the transition functions are compatible with the action l. In other words, there exists a family $\varphi_{\alpha\beta} \longrightarrow G$ satisfying the cocycle condition $\varphi_{\alpha\beta}\varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$ in $U_{\alpha\beta\gamma}$ and $\varphi_{\alpha\alpha} = \mathrm{id}_{U_\alpha}$ such that $\psi_{\alpha\beta} = l_{\varphi_{\alpha\beta}}$. In this case, (U_α, ψ_α) is a *G-atlas* and $\varphi_{\alpha\beta}$ are the *transition functions*.

As before, any G-atlas completely determines the fibre bundle $\pi\colon B \longrightarrow M$ and the G-structure.

Definition 1.60 Let M be a smooth manifold and let G be a Lie group. A *principal G-bundle* is a fibre bundle $\pi\colon P \longrightarrow M$ with a right group action $r\colon P \times G \longrightarrow P$ such that:

1. It is π-invariant, $\pi\rho_g = \pi$ por every $g \in G$.
2. It is free and transitive on each fibre $\pi^{-1}(p)$, where $p \in M$.

Principal bundles are special classes of G-structures in fibre bundles.[15] The freeness and transitiveness of the action r in each fibre implies that there exists a diffeomorphism $\pi^{-1}(p) \simeq G$. Principal bundles represent gauge theories for which the internal degrees of freedom agree with the group structure of the theory. These objects are fundamental in the theory because they are universal models for fibre bundles with G-structures. Any G-fibre bundle gives rise to a G-principal bundle; moreover, any G-fibre bundle can be obtained from a principal G-bundle as an associated bundle. The following

[15] Ivan Kolář, Slovák Jan, and Peter W. Milchor. *Natural Operations in Differential Geometry*. Springer Berlin, Heidelberg, 1 edition, 1993. ISBN 978-3-540-56235-1. DOI: 10.1007/978-3-662-02950-3

theorem, which contains this discussion, shows that principal bundles are the relevant objects in gauge theories.

Theorem 1.61 *Every fibre bundle $\tau\colon B \longrightarrow M$ with a G-structure determines a G-principal bundle $\pi\colon P \longrightarrow M$.*

Conversely, for a G-principal bundle $\pi\colon P \longrightarrow M$ and a smooth manifold Q with a G-action $l\colon G\times Q \longrightarrow Q$, the quotient $P\times_G Q$ is a fibre bundle with fibre Q and a natural G-structure which fits in diagram (1.48). *Moreover, if $(U_\alpha, \varphi_\alpha)$ is a principle bundle atlas with cocycle transition functions $\varphi_{\alpha\beta}$, this cocycle determines a G-structure for $P\times_G Q$ with the action l.*

$$\begin{array}{ccc} P\times Q & \xrightarrow{\;q\;} & P\times_G Q \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ P & \xrightarrow[\pi]{} & M \end{array} \tag{1.48}$$

So far, we have proved that any gauge theory with structure group G can be recovered from a certain G-principal bundle by the procedure of association. This result is important in the analysis of configuration spaces for particles in gauge theories and it gives the motivation for a definition of gauge theories from principal bundles. However, we have not been concerned with the equations of motion in gauge theories. For Newton's equation (1.1), the generalization of acceleration to an arbitrary manifold had to specify an affine connection in $\mathrm{T}M$. Following this line of thought, we will use a more general concept of connection introduced by Ehresmann. If the principal bundle $\tau\colon B \longrightarrow M$ has a G-structure, we can additionally require an Ehresmann connection to be invariant by the G-action. This definition aims to represent that gauge transformations do not change the physics of gauge theories.

Definition 1.62 Let $\pi\colon B \longrightarrow M$ be a fibre bundle. An *Ehresmann connection* is a splitting of the following short exact sequence:

$$0 \longrightarrow \ker\tau_* \longrightarrow \mathrm{T}B \longrightarrow \pi^*\mathrm{T}M \longrightarrow 0 \tag{1.49}$$

The vector bundle $\ker\tau_*$ is called the *vertical bundle*.

Definition 1.63 Let $\pi\colon B \longrightarrow M$ be a fibre bundle with a G-structure. A *G-invariant Ehresmann connection* is a splitting of the following short exact sequence:

$$0 \longrightarrow \ker\tau_*/G \longrightarrow \mathrm{T}B/G \longrightarrow \mathrm{T}M \longrightarrow 0 \tag{1.50}$$

Remark 1.64 It can be checked that any G-invariant Ehresmann connection 1.63 gives rise to an Ehresmann connection 1.62.

Similarly, principal connections can be defined over principal bundles requiring an invariance condition by the G-action. Given that principal G-bundles are examples of fibre bundles with a G-structure, remark 1.64 is still valid. We will use the following result, which trivializes the vertical bundle of a principal bundle.

Lemma 1.65 *Let $\pi\colon P \longrightarrow M$ be a G-principal bundle. The vector bundle morphism*

$$\begin{array}{cccc} \varphi\colon & P\times\mathfrak{g} & \longrightarrow & \ker\pi_* \\ & (p, X) & \longmapsto & (\psi_p)_*(X) \end{array} \tag{1.51}$$

is an isomorphism. Here, $\psi_p\colon G \longrightarrow M$ is the orbit map of the action r.

Definition 1.66 Let $\pi\colon P \longrightarrow M$ be a principal G-bundle. A *principal connection* on P is a splitting of the following short exact sequence,

$$0 \longrightarrow P\times\mathfrak{g} \longrightarrow \mathrm{T}P/G \longrightarrow \mathrm{T}M \longrightarrow 0, \tag{1.52}$$

called the *Atiyah sequence*.

Theorem 1.61 states that the configuration space of any gauge theory can be recovered from a principal G-bundle. The dynamics of the system, encoded in the Ehresmann connection, satisfy the same property.

Theorem 1.67 *Let $\pi\colon P \longrightarrow M$ be a principal G-bundle and let Φ be a principal connection. For any smooth manifold Q and action $l\colon G\times Q \longrightarrow Q$, there exists an induced connection $\overline{\Phi}$ which fits the commutative diagram* (1.53).

$$\begin{array}{ccccc} \mathrm{T}(P\times Q) & \longleftrightarrow & \mathrm{T}P\times \mathrm{T}Q & \xrightarrow{\Phi\times\mathrm{id}} & \mathfrak{g}\times\mathcal{H}_P\times \mathrm{T}Q \\ \Big\downarrow{\scriptstyle \mathrm{T}q} & & \Big\downarrow{\scriptstyle \mathrm{T}q} & & \Big\downarrow{\scriptstyle \mathrm{T}q} \\ \mathrm{T}(P\times_G Q) & \longleftrightarrow & \mathrm{T}P\times_G \mathrm{T}Q & \xrightarrow[\overline{\Phi}]{} & (\mathfrak{g}\times\mathcal{H}_P)\times_G \mathrm{T}Q \end{array} \tag{1.53}$$

The phase space of gauge theories, Wong's equations and Weinstein's spaces

So far, we have developed the language of principal bundles and principal connections, which are fundamental for the formulation of gauge theories in physics. We, however, have not explicitly stated the equations of motion induced by a G-invariant Ehresmann connection (or a gauge field in the physics literature). In the Lagrangian approach to mechanics, the curvature of the connection together with the Hodge duality induced from the metric in the base space M can be used to define a Lagrangian function. We will not take this approach, however. Following the geometric ideas of Weinstein, we will understand connections not as fields, but as projections of symplectic manifolds to the standard phase space of our particle T^*M. This surjection allows us to pull back a Hamiltonian function $H \in C^\infty(M)$

To arrive to this geometric interpretation of gauge theories we have to use equivalent characterizations of a principal connection as defined in (1.52). The following result is borrowed from homological algebra and is known as the splitting lemma. It is true over any abelian category. The reader should therefore observe that this result has to be proved *ad-hoc*, given that the category of vector bundles is not abelian.

Proposition 1.68 — Splitting lemma. *Let us consider a principal G-bundle $\pi\colon P \longrightarrow M$. Then, the following are equivalent.*

1. *A splitting of the Atiyah sequence* (1.52).
2. *A G-invariant section $h\colon \pi^*\mathrm{T}M \longrightarrow \mathrm{T}P$, called the* tangent lift.
3. *A G-invariant retraction $\theta\colon \mathrm{T}P \longrightarrow P\times\mathfrak{g}$, called the* connection form

Remark 1.69 In many references, principal bundles are defined in terms of a left action $l\colon G\times P \longrightarrow P$ defined from the right action r in definition 1.60 as $l_g(p) = r_{g^{-1}}(p)$. Items 1 and 2 hold, but the connection form θ becomes a G-equivariant map, intertwining the tangent lift l_* and the diagonal action (Ad, l).

Throughout the rest of the chapter we will consider a fixed Ehresmann connection and represent the previous constructions in the commutative diagram (1.54).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\tau_* & \underset{\theta}{\overset{i}{\rightleftarrows}} & \mathrm{T}B & \underset{h}{\overset{\tilde\tau}{\rightleftarrows}} & \tau^*\mathrm{T}M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \Phi} & & \downarrow & & \\ 0 & \longrightarrow & \ker\tau_* & \underset{\pi_1}{\overset{i_1}{\rightleftarrows}} & \ker\tau_*\oplus\mathcal{H}_B & \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} & \mathcal{H}_B & \longrightarrow & 0 \end{array} \tag{1.54}$$

Suppose that we have a fibre bundle $\tau\colon B \longrightarrow M$ with typical fibre Q and a G-structure in B. The natural phase spaces of physical systems have been, so far, cotangent bundles of certain manifolds, which were understood to be the configuration space of the particle. In our case, the natural phase space of our particle is the bundle B, which encodes hidden symmetries in our system. For this reason, we expect the "natural" phase space of our particle to be described within T^*B. Additionally, we could also use the quotient space T^*B/G as phase space (or, rather, a symplectic leaf in the spirit of section 1.4), given that we have symmetries. This disgression leads us to consider the induced splitting of the dual Atiyah sequence, represented in the following diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tau^*\mathrm{T}^*M & \underset{h^\dagger}{\overset{\tilde\tau^\dagger}{\rightleftarrows}} & \mathrm{T}^*B & \underset{\theta^\dagger}{\overset{i^\dagger}{\rightleftarrows}} & (\ker\tau_*)^* & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle \Psi} & & \uparrow & & \\ 0 & \longrightarrow & P^\sharp & \underset{i_2^\dagger}{\overset{\pi_2^\dagger}{\rightleftarrows}} & (\ker\tau_*)^*\oplus P^\sharp & \underset{\pi_1^\dagger}{\overset{i_1^\dagger}{\rightleftarrows}} & (\ker\tau_*)^* & \longrightarrow & 0 \end{array} \tag{1.55}$$

The following theorem, which is due to Weinstein,[16] can be contextualized in the context of classical mechanics using this discussion. Theorems 1.61 and 1.67 are key in providing a meaningful interpretation of the theorem, although not explicitly, because Weinstein constructs the natural phase space of the particle using the association process with a principal bundle.

[16] Alan Weinstein. A universal phase space for particles in yang-mills fields. *Letters in Mathematical Physics*, 2(5):417–420, Sep 1978. ISSN 1573-0530. DOI: 10.1007/BF00400169. URL `https://doi.org/10.1007/BF00400169`

Theorem 1.70 *Consider a principal G-bundle $\pi\colon P \longrightarrow M$ over and a Hamiltonian G-space Q.*

1. *The product space $\mathrm{T}^*P \times Q$ is Hamiltonian with moment map $\mu_P + \mu_Q$.*

2. *The hypothesis of the reduction theorem 1.48 are satisfied and, consequently, the space $(\mathrm{T}^*P \times Q)_0$ is a symplectic manifold.*

3. *The horizontal lift $h^\dagger$ is well defined in classes of equivalence and defines a map $\alpha\colon (\mathrm{T}^*P \times Q)_0 \longrightarrow \mathrm{T}^*M$.*

Here, the reduced space $(\mathrm{T}^*P \times Q)_0$ is understood to be the natural phase space of a classical particle in a gauge theory. Assume that we have a Hamiltonian function $H \in C^\infty(\mathrm{T}^*M)$ which, in the case of classical mechanics, can be the kinetic energy of the particle. The map α is can be used to define a function $\alpha^*H \in C^\infty((\mathrm{T}^*P \times Q)_0)$. The Hamiltonian equations of motion generated by this function are known as *Wong's equations*, and describe the evolution of classical particles interacting with gauge fields.

1.6 Concluding notes

This chapter has been a brief introduction to the fundamental results in symplectic and Poisson geometry from the perspective and motivation of physics. There are, of course, many different aspects of both branches which have not been covered. We briefly review some of them.

In section 1.2, we have not mentioned the role of submanifolds in differential geometry. More specifically, we have not defined isotropic, coisotropic and lagrangian submanifolds. The latter are relevant in symplectic geometry following Weinstein's lagrangian creed.[17] In this vision of symplectic geometry one can characterize symplectomorphisms between symplectic manifolds studying lagrangian submanifolds. The method of generating functions, for instance, exploits this correspondence. Many local results, such as Darboux's theorem 1.18, can be proved using modern techniques such as Moser's path method. This approach does not only generalize Darboux's theorem to a semilocal version using Weinstein's lagrangian neighbourhood theorem, but also allows for a classification of compact symplectic surfaces (S, ω) in terms of cohomology classes $[\omega] \in H^2_{\mathrm{dR}}(S)$.

[17] Ana Cannas da Silva. *Lectures on Symplectic Geometry*. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1 edition, 2008. DOI: 10.1007/978-3-540-45330-7

Liouville's theorem 1.20 imposes certain restrictions on which maps can be considered symplectomorphisms between symplectic manifolds. There are stronger statements in this matter. For instance, Gromov's theorem shows that there is no symplectomorphism from a solid ball $\mathbf{B}^3(R)$ of radius R to a solid cylinder $\mathbf{D}^2(r) \times \mathbf{R}$ of radius r if $R > r$. This result, known as the *non-squeezing theorem*, is related to a generalization of Heisenberg's uncertainty principle and is the starting point for concepts such as symplectic capacities.[18] Moreover, the techniques used in the proof of Gromov's theorem use results from pseudoholomorphic curves, showing connections with complex geometry. It is surprising that symplectic geometry is so deeply related with complex geometry and physics, even in the quantum realm.

[18] Maurice de Gosson. How classical is the quantum universe?, 2008. URL `https://arxiv.org/abs/0808.2774`

In section 1.4, we introduced the moment map of an action in order to generalize Noether's theorem 1.50 to Lie group actions. We have not been concerned with the existence or uniqueness of moment maps. To give a satisfactory answer to these questions we have to introduce first the notion of Chevalley-Eilenberg cohomology in a Lie algebra $\mathfrak{g}$. The existence and uniqueness of moment maps are encoded in the cohomology classes of the Chevalley-Eilenberg cohomology for $\mathfrak{g}$. This is an example of a Lie algebroid cohomology, which we will briefly review after definition 2.14.

Noether's theorem 1.53 is also the starting point for the notion of integrability in the sense of Liouville. A system (M, ω) is said to be integrable if there exist smooth functions $f_1, \ldots, f_n \in C^\infty(M)$ which Poisson commute and are generically linearly independent. This functions can be understood, following the discussion before Noether's theorem, as the moment map of a Lie group action G. Given that they Poisson commute, the only possible action of a compact Lie group is $\mathbf{T}^n$. The study of integrable systems is deeply related to the study of toric actions in manifolds as a consequence of this result. Analogue results to Darboux's theorem can be proved in a neighbourhood of a compact leaf, giving the known *action-angle coordinates* in classical mechanics.

In section 1.5 we have only considered the universal phase spaces of Weinstein. There exists a symplectomorphism with a different manifold, called the *Sternberg phase space*. This isomorphism is called the *minimal coupling procedure* in the physics literature. Additionally, when we consider as fibre $Q = \mathcal{O}(\mathrm{Ad}^*)$ a coadjoint orbit in $\mathfrak{g}^*$, Montgomery showed that the phase spaces of Weinstein and Sternberg are symplectic leaves of two different Poisson manifolds, which are equivalent in the Poisson sense. We will introduce these concepts in section 2.4 and prove these claims for E-manifolds.

2

The Hamiltonian formalism of gauge theories in E-manifolds

We have seen that symplectic geometry is the tool we use to formulate the equations of motion of a classical or relativistic physical system. It is not only a language to speak about motion, but the specific shape of Hamilton's equations has further topological implications. The use of symplectic geometry in speaking about conserved quantities naturally gives rise to Poisson geometry, which generalizes symplectic geometry relaxing the non-degeneracy condition of the symplectic form.

There are, however, natural problems in physics for which the phase space is not a smooth manifold. The most obvious example is the movement of a particle inside a space with physical walls. In this case, we have to consider that a manifold with boundary is the correct mathematical description of our system. In the same way, motion of particles in manifolds with corners or with more complex stratifications can also be considered. Another example of this behaviour in classical mechanics is given by some Marsden-Weinstein reduced spaces. During section 1.4, we assumed some regularity on the group action ρ in order to ensure that the quotient space M/G is a smooth manifold. Under less restrictive conditions, M/G becomes what is known as a stratified space. This result was proved by Sjamaar and Lerman.[1]

[1] Reyer Sjamaar and Eugene Lerman. Stratified symplectic spaces and reduction. *Annals of Mathematics*, 134(2):375–422, 1991. ISSN 0003486X. URL `http://www.jstor.org/stable/2944350`

In this work we consider the setting of E-manifolds. We will think of an E-manifold as a suitable replacements of the tangent bundle. This point of view is well suited for our objective to describe a physical system. We will see that there are two basic conditions upon which classical Hamiltonian mechanics are constructed. First of all, defining Hamilton's equations of motion through a symplectic structure already implies the existence of calculus, understood as an exterior differential in a complex of forms. This idea is implicit in the definition of E-manifold, within the involutivity condition. Secondly, we need a way to connect transformations in the configuration space M with transformations in the phase space T^*M. In differential geometry, the tangent map gives a functor $\mathrm{d}\colon \mathcal{Man} \longrightarrow \mathcal{VBun}$ which is the basis of many commutative diagrams which lie at the heart of classical mechanics. By the construction of E-maps, the morphisms of E-manifolds, we will have a similar functor which is induced from the pushforward of smooth maps. These constructions will allow us to prove theorem 1.70 and its generalizations in the category of E-manifolds.

2.1 Examples of E-manifolds

Before studying the differential and symplectic geometry of E-manifolds, we give examples in mathematics and physics where E-manifolds naturally appear. We will give a brief definition of E-manifolds, which will be enough to prove that the following examples are indeed E-manifolds.

Definition 2.1 An *E-manifold* is a pair (M, E), where M is a smooth manifold and $E \subseteq \chi(M)$ is an involutive and locally finitely generated $\mathcal{C}^\infty(M)$-submodule. Sometimes we will call any such submodule an *E-structure* on M.

Notation In some cases, we will deal with several E-manifolds at the same time. In order to avoid confusion, each the E-structure of an E-manifold (M, E) will be denoted E_M.

Mathematical examples

Example 2.2 — The canonical E-structure. Consider a smooth manifold M. The module $E = \chi(M)$ is an E-structure in M, called the *canonical E-structure*. It is trivially involutive. To see that it is locally finitely generated, take a chart (U, φ) with coordinates $q_1, \ldots, q_n$. The fields

$$\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \ldots, \frac{\partial}{\partial q_n}$$

are local generators of $\chi(U)$.

This example, while quite obvious, shows that definitions such as the canonical symplectic form of T^*M and E-gauge theories are particular cases of deeper and more general constructions.

Example 2.3 — b-Manifolds and b^k-manifolds. One of the first generalizations of smooth manifolds are manifolds with boundary. Their differential calculus, called b-calculus, was developed by Melrose with the aim of generalizing the Atiyah-Patodi-Singer theorem to manifolds with boundary.[2] The differential structure of b-manifolds allows for a rich generalization of symplectic geometry to manifolds with boundary (see Guillemin, Miranda, and Pires, for example).[3]

Frejlich, Torres, and Miranda show that the diffential geometry of manifolds with boundary can be recovered from that of another similar structures, called b-manifolds, by a process of gluing.[4] A *b-manifold* is a pair (M, Z), where M is a smooth manifold and $Z \subset M$ is an embedded hypersurface. Any b-manifold has a natural E-structure associated; the submodule of tangent vector fields to the submanifold Z. Such vectors are called *b-vector fields*, and the set of all b-vector fields is denoted by ${}^{\mathrm{b}}\chi(M)$. To see that ${}^{\mathrm{b}}\chi(M)$ is an E-structure in M take a point $p \in M$ and a coordinate chart (U, φ) with coordinates $q_1, \ldots, q_n$ adapted to Z, that is, fulfilling $\varphi(U \cap Z) = \{q_1 = 0\}$. Under these assumptions, the local sections

$$q_1\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \ldots, \frac{\partial}{\partial q_n}$$

are generators of the module ${}^{\mathrm{b}}\chi(U)$. This trivially shows that ${}^{\mathrm{b}}\chi(M)$ is locally finitely generated and involutive, proving that it is an E-structure.

[2] Richard Melrose. The atiyah-patodi-singer index theorem, 1992. URL https://www.maths.ed.ac.uk/~v1ranick/papers/melrose.pdf

[3] Victor Guillemin, Eva Miranda, and Ana Rita Pires. Symplectic and poisson geometry on b-manifolds. *Advances in Mathematics*, 264:864–896, oct 2014a. DOI: 10.1016/j.aim.2014.07.032. URL https://doi.org/10.1016%2Fj.aim.2014.07.032

[4] Pedro Frejlich, David Martínez Torres, and Eva Miranda. A note on symplectic topology of b-manifolds. 2013. DOI: 10.48550/ARXIV.1312.7329. URL https://arxiv.org/abs/1312.7329

In the same way, having a smooth manifold M and an embedded hypersurface $Z \subset M$ we may consider as elements of E the vector fields which are tangent to Z with order of tangency k or more, the set of all these fields is called the space of *b^k-vector fields*. In a chart (U, φ) with coordinates $\boldsymbol{q}$ adapted to Z, a set of local generators is

$$q_1^k \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \ldots, \frac{\partial}{\partial q_n}. \tag{2.1}$$

As before, this shows that b^k-vector fields give rise to different E-structures in b-manifolds.

Example 2.4 — c-Manifolds. The previous example can be generalized to consider manifolds with intersections of higher order. We define a *c-manifold* as a pair (M, Z), where M is a smooth manifold M and $i \colon Z \longrightarrow M$ is an immersed hypersurface with self-transverse instersections (see Miranda and Scott[5] for a more detailed description of the construction). By analogy with the idea that Z accounts for the boundary of a manifold, self-transverse intersections can be understood as corners (hence the name c-manifolds). We consider the set of vector fields of M which are tangent to $i(Z)$ and call them *c-vector fields*. We can prove that the set of c-vector fields is an E-structure in M. For every point $p \in Z_k \setminus Z_{k+1}$ there exist coordinates $\boldsymbol{q}$ such that $i(Z) = \cup_{i \leqslant k}\{x_i = 0\}$. In these coordinates, we have the local generators

$$q_1 \frac{\partial}{\partial q_1}, \ldots, q_k \frac{\partial}{\partial q_k}, \frac{\partial}{\partial q_{k+1}}, \ldots, \frac{\partial}{\partial q_n}.$$

From this expression we trivially have that c-vector fields give rise to an E-structure in M.

[5] Eva Miranda and Geoffrey Scott. The geometry of e-manifolds. *Revista Matemática Iberoamericana*, 37(3):1207–1224, nov 2020. DOI: 10.4171/rmi/1232. URL `https://doi.org/10.4171%2Frmi%2F1232`

Example 2.5 — Regular foliations. Consider a smooth manifold M and a regular, smooth and involutive distribution $\mathcal{D}$ of rank k. By Frobenius' theorem, there exists a foliation $\mathcal{F}$ such that any element of $\mathcal{D}$ is tangent to a leaf of $\mathcal{F}$. The distribution $\mathcal{D}$ defines an E-structure by the involutivity condition. A choice of coordinates $\boldsymbol{q}$ adapted to the foliation $\mathcal{F}$ gives a local basis

$$\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_k}.$$

Examples in physics

The previous examples have considered mathematical structures which are examples of E-manifolds. We will see now that some of these examples appear naturally in different physical contexts.

Example 2.6 Consider a Minkowski space (M, g) of dimension $\dim M = n + 1$ and signature $(-, +, \cdots, +)$. Consider an orthonormal basis $\{E_0, \ldots, E_n\}$ and extend it to a global chart in M, giving a global inertial frame of reference. Without loss of generality, assume that E_0 is time-like (in our convention, $g(E_0, E_0) = -1$). We can consider now the singular foliation of $\mathrm{T}M$ given by level sets of the kinetic energy, $\mathcal{F}_k = \{X \in \mathrm{T}M \mid g(X, X) = k\}$. As the norm of a geodesic is preserved, the geodesic spray is tangent to the leaves of the foliation $\mathcal{F}$. This implies that we can regard the natural phase space of the physical system (M, g) as the E-manifold $\mathrm{T}\mathcal{F}$, instead of $\mathrm{T}M$.

The particular case of $M = \mathbf{R}^4$ with a Minkowski metric g is the framework of special relativity. Given a basis $\{E_0, E_1, E_2, E_3\}$ of $\mathbf{R}^4$ one can construct a global coordinate system (t, x_1, x_2, x_3) of $\mathbf{R}^4$, called *space-time coordinates*, satisfying

$$\frac{\partial}{\partial t} = E_0, \quad \frac{\partial}{\partial x_i} = E_i, \quad i = 1, 2, 3.$$

If the basis is orthonormal, the metric g is expressed as

$$g = -(\mathrm{d}t)^2 + (\mathrm{d}x_1)^2 + (\mathrm{d}x_2)^2 + (\mathrm{d}x_3)^2.$$

The geodesics of the Minkowski space are straight lines with respect to the space-time coordinates.

At any point $p \in \mathbf{R}^4$, a vector $X \in \mathrm{T}_p\mathbf{R}^4$ can be:

1. *time-like*, if $g(X, X) < 0$,

2. *null*, if $g(X, X) = 0$,

3. *space-like*, if $g(X, X) > 0$.

The foliation $\mathcal{H}$ of $\mathrm{T}\mathbf{R}^4 \cong \mathbf{R}^8$ given by the level sets $\mathcal{H}_k = \{X \in \mathrm{T}\mathbf{R}^4 \mid g(X, X) = k\}$ is singular. The leaves $\mathcal{H}_k$ are divided into three different types of hyper-surfaces, depending on the previous classification of vectors:

1. *time-like hyper-surfaces*, the two-sheet hyperboloids corresponding to $\mathcal{H}_k$ for any $k > 0$.

2. *null hyper-surface*, the (singular) double cone $\mathcal{H}_0$.

3. *space-like hyper-surfaces*, the one-sheet hyperboloids corresponding to $\mathcal{H}_k$ for any $k < 0$.

The physical implications of the postulates of special relativity (no absolute time and no travel faster than the speed of light) are encoded in the foliation $\mathcal{H}$. Any curve $\alpha : I \longrightarrow \mathbf{R}^4$ is called *time-like* if $\dot{\alpha}(t)$ is a time-like vector at $\alpha(t)$ for all t. Then, the world-line of any observer is a time-like curve (see Figure 2.1 and the motion of an inertial (non-accelerating) observer follows a time-like geodesic. Light, on the other hand, moves on null geodesics, i.e., on the cone.

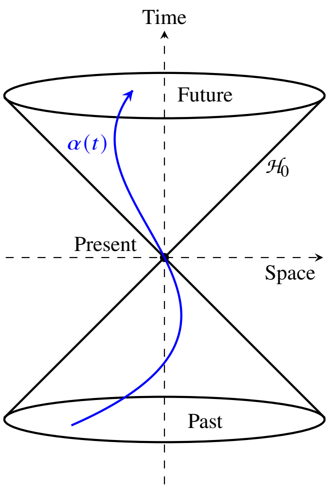


Figure 2.1: The postulates of special relativity induce a foliation $\mathcal{H}$ in the space-time coordinates of the Minkowski space $(\mathbf{R}^4, g)$. The natural phase space of the physical system is $\mathrm{T}\mathcal{H}$, an E-manifold. The blue trajectory $\alpha(t)$ in the interior of the *light cone* represents the world-line of an observer.

Example 2.7 Spin Calogero-Moser systems are formulated on the cotangent bundle of a Lie group G, T^*G. The natural phase space of a spin Calogero-Moser system is a symplectic leaf of the Poisson space T^*G/G. The cotangent bundle can be trivialized as $\mathrm{T}^*G \simeq \mathfrak{g}^*$ using right-invariant vector fields. The Poisson space T^*G/G can be identified with $\mathfrak{g}^* \times_G G \simeq \mathfrak{g}^*$, and the natural projection $\pi\colon \mathfrak{g}^* \times G \longrightarrow \mathfrak{g}^*$ is a Poisson map, where the Poisson structure Π in $\mathfrak{g}^*$ is the linear Poisson structure. In this case, the Hamiltonian equations of motion can be restricted to the E-tangent bundle generated by the orbits of the coadjoint action on $\mathfrak{g}^*$. For a survey on spin Calogero-Moser systems, see Reshetikhin.[6]

[6] Nicolai Reshetikhin. Spin calogero-moser models on symmetric spaces, 2019. URL https://arxiv.org/abs/1903.03685

Assume now that $\mathfrak{g}$ is semi-simple, and we have an identification $\mathfrak{g} \simeq \mathfrak{g}^*$ by means of the Killing form κ. Take now the adjoint representation of $\mathfrak{g}$ in $\mathfrak{g}^*$. The Hamiltonian function for the spin Calogero-Moser system is given by $H = \mathrm{tr}(x^2)$, where $x \in \mathfrak{g}^*$. In the specific case of $\mathfrak{su}^*(n)$, a computation shows that the Hamiltonian of the system can be expressed in coordinates

$p_1, \ldots, p_n, q_1, \ldots, q_n$ as

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\mu_{ij}^2}{(q_i - q_j)^2}, \qquad \mu_{ii} = 0.$$

The first term accounts for the kinetic energy of a system with n different particles in a straight line, while the second term is an interaction potential dependent on the inverse square of the distance.

Even though both examples arise from specific physical systems, they represent two different situations in mechanics which arise generally.

- The proper time foliation $\mathcal{H}$ in example 2.6 can be explicitly described as the level sets of the function $H = g(X, X)$, which is the Hamiltonian function for the equations of motion of geodesics. More generally, if the level sets of a Hamiltonian function $H \in \mathcal{C}^\infty(M)$ are the leaves of a foliation, the dynamics of the system can be expressed in terms of the E-manifold of vector fields tangent to the leaves.
- In example 2.7, the definition of a spin Calogero-Moser system uses the Marsden-Weinstein reduction described in section 1.4. More generally, the Hamiltonian dynamics described by X_H in an arbitrary Poisson manifold (M, Π) can be restricted to the symplectic foliation $\mathcal{F}_\Pi$. The E-structure to be considered in this case is $E = \mathrm{T}\mathcal{F}_\Pi$.

  This result gives a different approach to the reduction of the Hamiltonian dynamics in a symplectic manifold (M, ω) by a symplectic group action. In theorem 1.48 we have to fix a point $p \in M$ to consider the symplectic leaf $\mathcal{F}_\Pi(\pi(p))$ in which the reduced dynamics take place. Taking the E-manifold $(M/G, \mathrm{T}\mathcal{F}_\Pi)$ amounts to a global choice of the base point; in this sense, the E-manifold arising from the symplectic foliation of a Poisson manifold can be understood as both the natural phase space of Hamiltonian dynamics and a global approach to the reduction procedure of Marsden and Weinstein.

Gauge theories in E-manifolds

Finally, we show some examples of different phenomena which fall under the more general setting of gauge theories over E-manifolds. Both of them will be generalized in section 2.4.

Example 2.8 Miranda and Oms generalize geodesic flows to b-manifolds and b^k-manifolds.[7] The geodesic equations of motion can be framed under a gauge theory, where the principal bundle is taken to be the orthonormal bundle $\mathrm{O}(M)$. Any principal connection induces an affine connection in ${}^{\mathrm{b}}\mathrm{T}M$, which gives the equations for the geodesic flow. There exists a straightforward generalization of this result to arbitrary E-manifolds. Following theorem 1.10, we will show that the kinetic energy is the Hamiltonian function to describe such generalized geodesic flows.

[7] Eva Miranda and Cédric Oms. The geometry and topology of contact structures with singularities, 2018. URL `https://arxiv.org/abs/1806.05638`

Example 2.9 Braddell, Kiesenhofer, and Miranda study the minimal coupling procedure of Montgomery for b-Lie groups.[8] These groups are pairs (G, H), where G is a Lie group and H is a closed codimension one subgroup. This structure is an example of a b-manifold in the sense of example 2.3, where H is taken to be the critical set. In this case, the natural projection

[8] Roisin Braddell, Anna Kiesenhofer, and Eva Miranda. b-structures on lie groups and poisson reduction. *Journal of Geometry and Physics*, 175: 104471, 2022. ISSN 0393-0440. DOI: https://doi.org/10.1016/j.geomphys.2022.104471. URL `https://www.sciencedirect.com/science/article/pii/S0393044022000213`

$\pi\colon G \longrightarrow G/H$ is an H-principal bundle, and the quotient group is a b-manifold taking as singular hypersurface the class $Z_H = [1]$. Observe that $[1]$ has codimension 1 given that $\dim(G/H) = 1$. This is the unique choice of hypersurface which makes π a b-map. Some highlighted examples of b-Lie groups are the Galilean group or the Heisenberg group, where the subgroup H is identified with the set of time-preserving transformations. We generalize the minimal coupling procedure of Braddell, Kiesenhofer, and Miranda in 2.57.

2.2 *Fundamentals of E-manifolds*

Having motivated E-manifolds through different examples in mathematics and physics, we begin presenting the basic results on the geometry of E-manifolds. Throughout this section we will develop the tools we have used in chapter 1, which include group actions, the cochain complex of differential forms, fibre and principal bundles, and products of E-manifolds.

E-tangent bundle, E-forms and cohomology

We interpret the E-structure in an E-manifold as the natural fields describing the dynamics of a system. In smooth manifolds, vector fields are described as the sections of the tangent bundle $\mathrm{T}M$. This geometric description has proved to be very fruitful in the development of Hamiltonian mechanics. An analogous result holds in the setting of E-manifolds.

Lemma 2.10 *Let (M, E) be an E-manifold. There exists a vector bundle $\mathcal{E}$, called the E-*tangent bundle*, such that local sections of $\mathcal{E}$ are in one-to-one correspondence with local sections of E.*

Proof. As E is locally finitely generated, the lemma is a straightforward corollary of Serre-Swan's theorem. ■

The previous lemma shows that the E-structure of an E-manifold can be equivalently characterized in terms of a vector bundle $\mathcal{E}_M$. Throughout the rest of the chapter we will think about $\mathcal{E}_M$ as a replacement of the tangent bundle $\mathrm{T}M$. For example, in b-manifolds (example 2.3) the motion of a physical system can only be described in terms of b-fields. The elements of E_M will be the meaningful geometric objects in the formulation of classical mechanics over E-manifolds.

There are many constructions in differential geometry which are defined algebraically using the tangent bundle $\mathrm{T}M$. We present analogous definitions for E-manifolds.

Definition 2.11 Let (M, E) be an E-manifold and let $\mathcal{E}_M$ be its E-tangent bundle.

1. The *E-cotangent bundle* is the vector bundle $\mathcal{E}_M^*$.
2. We define an *E-vector field* or *E-field* as a section $X \in \Gamma(\mathcal{E}_M)$. Analogously, an *$E$-multivector field* of degree $k \in \mathbf{N}$ is a section $X \in \Gamma(\bigwedge^k \mathcal{E}_M)$.
3. We define an *E-differential form* of degree $k \in \mathbf{N}$ as a section $\omega \in \Gamma(\bigwedge^k \mathcal{E}_M^*)$.

Notation In an E-manifold $\mathcal{E}_M$, the set of all multivector fields of degree k is denoted by $\chi^k(\mathcal{E}_M)$. Similarly, the set of all forms of degree k is denoted by $\Omega^k(\mathcal{E}_M)$

Much of the differential geometry in smooth manifolds, understood to be contained in the cochain complex of differential forms, is encoded in the Lie bracket of vector fields.[9] Given the involutivity of the submodule E, the natural bracket of vector fields admits a restriction to the E-tangent bundle $\mathcal{E}_M$.

[9] In fact, the exterior differential $\mathrm{d}\colon \Omega^\bullet(M) \longrightarrow \Omega^{\bullet+1}(M)$ is completely determined by the Lie bracket of vector fields, and that the converse statement holds. There is a third equivalent characterization of the differential structure of smooth manifolds, which is represented in the Schouten-Nijenhuis bracket. We will use these characterizations interchangeably.

Proposition 2.12 *Let (M, E) be an E-manifold and let $\mathcal{E}_M$ be its E-tangent bundle. The Lie bracket of vector fields admits a restriction to $\mathcal{E}_M$. Moreover, the following equation is satisfied,*

$$[X, fY] = f[X, Y] + (\mathcal{L}_X f)Y, \tag{2.2}$$

where $X, Y \in \chi(\mathcal{E}_M)$ and $f \in C^\infty(M)$. As a consequence, $(\mathcal{E}_M, i, [\cdot,\cdot])$ is a Lie algebroid, where $i\colon \mathcal{E}_M \longrightarrow \mathrm{T}M$ is the inclusion of vector fields.

Proof. Take an open set $U \subset M$ for which $\chi_U(\mathcal{E}_M) \simeq E_M(U)$. Given that the submodule E_M is involutive, the Lie bracket of vector fields admits a restriction to $E_M(U)$ and, consequently, to $\chi_U(\mathcal{E}_M)$. As $\chi_U(\mathcal{E}_M) = \Gamma_U(\mathcal{E}_M)$ by definition, we conclude that the Lie bracket of vector fields induces a Lie bracket in $\mathcal{E}_M$. Equation (2.2) is direct because it is fulfilled for smooth vector fields $X, Y \in \chi(M)$. ■

Following the observation of Weinstein, this result allows for a definition of differential calculus over any E-manifold. The following definitions are direct restatements well-known definitions in the literature of Lie algebroids and can be found, for example, in the book of Cannas da Silva and Weinstein.[10]

[10] Ana Cannas da Silva and Alan Weinstein. *Geometric Models for Noncommutative Algebras*. Berkeley Mathematics Lecture Notes. American Mathematical Society, 1999. ISBN 978-0-8218-0952-5

Definition 2.13 Let (M, E) be an E-manifold and let $\mathcal{E}_M$ be its E-tangent bundle. The contraction of a form $\omega \in \Omega^k(\mathcal{E}_M)$ with an E-field $X \in \chi(\mathcal{E}_M)$ is defined by its action on elements $X_1, \ldots, X_{k-1} \in \chi(\mathcal{E}_M)$ as

$$(\iota_X\omega)(X_1, \ldots, X_{k-1}) = \omega(X, X_1, \ldots, X_{k-1}). \tag{2.3}$$

Definition 2.14 Let (M, E) be an E-manifold and let $\mathcal{E}_M$ be its E-tangent bundle. Given a form $\omega \in \Omega^k(\mathcal{E}_M)$ we define its exterior differential by its action on any elements $X_0, \ldots, X_k \in \chi(\mathcal{E}_M)$ as

$$\begin{aligned}\mathrm{d}\omega(X_0, \ldots, X_p) = &\sum_{i=0}^{p}(-1)^i \mathcal{L}_{X_i}\omega\Big(X_0, \ldots, \widehat{X}_i, \ldots, X_p\Big) \qquad (2.4)\\ &+ \sum_{0\leqslant i<j\leqslant p}(-1)^{i+j}\omega\Big([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_p\Big).\end{aligned}$$

It is a computation to check that $\mathrm{d}^2 = 0$ and, as in the smooth case, the differential d induces a cochain complex structure in $\Omega^\bullet(\mathcal{E}_M)$. The cohomology groups of degree k of this cochain are written $H^k(\mathcal{E}_M)$.

There is an analogous notion of Lie derivative of E-differential forms along E-fields. With the previous definitions we can give a straightforward definition using Cartan's formula.

Definition 2.15 Let (M, E) be an E-manifold and let $\mathcal{E}_M$ be its E-tangent bundle. The *Lie derivative* of an E-form $\omega \in \Omega^k(\mathcal{E}_M)$ along an E-field $X \in \chi(\mathcal{E}_M)$ is defined as

$$\mathcal{L}_X \omega = \mathrm{d}\iota_X \omega + \iota_X \,\mathrm{d}\omega. \tag{2.5}$$

We present now a description of the local structure of E-manifolds. The following concepts are directly inherited from the literature of Lie algebroids.

Definition 2.16 Consider an E-manifold (M, E_M) and a coordinate set $U \subset M$ with coordinates $q_1, \ldots, q_n$. Let $E_1, \ldots, E_k$ be a set of generators of $\Gamma_U(\mathcal{E}_M)$. The *structure functions* are the smooth functions $\rho_{ij} \in C^\infty(U)$ satisfying

$$E_i = \sum_{j=1}^{n} \rho_{ij} \frac{\partial}{\partial q_j}, \quad i = 1, \ldots, p, \tag{2.6}$$

and the elements $C_{ij}^k \in C^\infty(U)$ such that

$$[E_i, E_j] = \sum_{k=1}^{p} C_{ij}^k E_k, \quad i, j = 1, \ldots, p. \tag{2.7}$$

Remark 2.17 The structure functions are well defined by the involutivity of the E-structure, $[E, E] \subset E$. Moreover, they are skew-symmetric, $C_{ij}^k = -C_{ji}^k$, by the skew-symmetry of the Lie bracket.

E-maps

Having defined E-manifolds and their differential structure, the next natural step is to define morphisms between E-manifolds. As in many other branches of mathematics, the morphisms between two different objects are structure-preserving maps. We will see that this guiding principle also applies to E-manifolds.

Definition 2.18 Let (M, E_M) and (N, E_N) be E-manifolds. A smooth map $f\colon M \longrightarrow N$ is said to be an *E-map* if $E_M \subset f_*^{-1}(E_N)$.

$$\begin{CD} \mathcal{E}_M @>{f_*}>> \mathcal{E}_N \\ @V{\tau_{\mathcal{E}_M}}VV @VV{\tau_{\mathcal{E}_N}}V \\ M @>>{f}> N \end{CD} \tag{2.8}$$

Lemma 2.19 *Let (M, E_M), (N, E_N) and (L, E_L) be E-manifolds.*

1. *The identity map* $\mathrm{id}_M\colon M \longrightarrow M$ *is an E-map.*
2. *If $f\colon M \longrightarrow N$ and $g\colon N \longrightarrow L$ are E-maps, $gf\colon M \longrightarrow L$ is an E-map. Moreover, the composition of E-maps is associative.*

Proof. The identity map id_M satisfies the property $(\mathrm{id}_M)_* = \mathrm{id}_{\chi(M)}$ and, as a consequence, $(\mathrm{id}_M)_*(E_M) = E_M$. This proves that id_M is an E-map.

Regarding the composition of E-maps, by the chain rule we have $(gf)_*(E_M) = g_*(f_*(E_M))$. By definition, $E_M \subset f_*^{-1}(E_N)$ and $E_N \subset g_*^{-1}(E_L)$, so $E_M \subset (gf)_*^{-1}(E_L)$, as we wanted to prove. The composition of E-maps is associative because it is inherited from the composition of maps between sets. ■

Lemma 2.10 shows that the structure of an E-manifold is characterized by the vector bundle $\mathcal{E}_M$. Similarly, E-maps can be defined in terms of the

E-tangent bundle.

Proposition 2.20 *Let (M, E_M) and (N, E_N) be E-manifolds and let $\mathcal{E}_M$, $\mathcal{E}_N$ be their E-tangent bundles, respectively. A map $f\colon M \longrightarrow N$ is an E-map if and only if $f_*\colon \mathrm{T}M \longrightarrow \mathrm{T}N$ admits a restriction to the E-tangent bundles, $f_*\colon \mathcal{E}_M \longrightarrow \mathcal{E}_N$.*

Proof. Both implications are straightforward from the local equivalence between sections of $\mathcal{E}_M$ and elements of E_M given by Serre-Swan's theorem. ■

The previous proposition shows that E-maps can be defined as smooth applications whose tangent map admits a restriction to the relevant E-structures. Moreover, pushforwards commute with the Lie bracket of vector fields. This property can be used to show that E-maps are also Lie algebroid morphisms with the canonical Lie algebroid structure of any E-manifold.

Proposition 2.21 *Let (M, E_M) and (N, E_N) be E-manifolds. For every E-map $f\colon M \longrightarrow N$ and any form $\omega \in \Omega^k(\mathcal{E}_N)$ we have $\mathrm{d}^{\mathcal{E}_M}(f^*\omega) = f^*(\mathrm{d}^{\mathcal{E}_N}\omega)$. As a consequence, E-maps are Lie algebroid morphisms in the sense of de León, Marrero, and Martínez.* [11]

[11] Manuel de León, Juan Carlos Marrero, and Eduardo Martínez. Lagrangian submanifolds and dynamics on lie algebroids. *Journal of Physics A: Mathematical and General*, 38(24):R241–R308, jun 2005. DOI: 10.1088/0305-4470/38/24/r01. URL https://doi.org/10.1088/0305-4470/38/24/r01

Proof. We know that, for any smooth map $f\colon M \longrightarrow N$, the pushforward f_* satisfies that $[f_*X, f_*Y] = f_*[X, Y]$ for any smooth fields $X, Y \in \chi(M)$. As a consequence, the same property is satisfied for any sections $X, Y \in E_M$. Using the definition of exterior differential (2.4), we see that

$$\begin{aligned}
f^*(\mathrm{d}^{\mathcal{E}_N}\omega)(X_0,\dots,X_p) &= \sum_{i=0}^{p}(-1)^i \mathcal{L}_{f_*X_i}\omega\Big(f_*X_0,\dots,f_*\widehat{X}_i,\dots,f_*X_p\Big)\\
&\quad+\sum_{0\leqslant i<j\leqslant p}(-1)^{i+j}\omega\Big([f_*X_i,f_*X_j],f_*X_0,\dots,f_*\widehat{X}_i,\dots,f_*\widehat{X}_j,\dots,f_*X_p\Big).\\
&= \sum_{i=0}^{p}(-1)^i \mathcal{L}_{f_*X_i}\omega\Big(f_*X_0,\dots,f_*\widehat{X}_i,\dots,f_*X_p\Big)\\
&\quad+\sum_{0\leqslant i<j\leqslant p}(-1)^{i+j}\omega\Big(f_*[X_i,X_j],f_*X_0,\dots,f_*\widehat{X}_i,\dots,f_*\widehat{X}_j,\dots,f_*X_p\Big)\\
&= \mathrm{d}^{\mathcal{E}_M}(f^*\omega)(X_0,\dots,X_p)
\end{aligned}$$

for any arbitrary sections $X_0, \dots, X_k \in \chi(\mathcal{E}_M)$. ■

Flows of E-vector fields

So far, the E-tangent bundle has been a replacement of the standard tangent bundle $\mathrm{T}M$ used to describe constrained or singular dynamics in a smooth manifold M. The equations of motion in classical mechanics are specified in the form of vector fields, and their flow gives the dynamical evolution of the system. It can be shown that the flow of any vector field $X \in E$ is an E-map, proving that the mechanics induced by an E-field restrict to the E-tangent bundle.

Proposition 2.22 *If (M, E) is an E-manifold, the flow φ_t of any section $X \in E$ is an E-map whenever it is defined, that is, $(\varphi_t)_*(E) \subset E$.*

Proof. Let us consider the flow φ_t generated by an E-field $X \in \chi(\mathcal{E}_M)$. Given that $[E, E] \subset E$ by involutivity, we can conclude that $(\varphi_t)_*(E) = E$. ■

Remark 2.23 A generalization of this statement to general Lie algebroids was presented by Loja Fernandes, who showed how to integrate a section of any Lie algebroid to produce a one-parameter group of transformations [12]. This construction is not necessary for E-manifolds given that we identify sections of E as vector fields in M.

[12] Rui Loja Fernandes. Lie algebroids, holonomy and characteristic classes. *Advances in Mathematics*, 170(1):119–179, 2002. ISSN 0001-8708. DOI: https://doi.org/10.1006/aima.2001.2070. URL `https://www.sciencedirect.com/science/article/pii/S0001870801920705`

Let us consider now an E-field $X \in \chi(\mathcal{E}_M)$. The previous proposition allows us to consider the lift (φ_t), which is well-defined an a local diffeomorphism. This condition is represented in diagram (2.9). As a consequence of this fact, we can define the Lie derivative of a local section $Y \in \chi_U(\mathcal{E}_M)$ along a section $X \in \chi(\mathcal{E}_M)$ using the lift as

$$\begin{array}{ccc} \mathcal{E}_M & \xrightarrow{(\varphi_t)_*} & \mathcal{E}_M \\ {\scriptstyle \tau_{\mathcal{E}_M}}\downarrow & & \downarrow{\scriptstyle \tau_{\mathcal{E}_M}} \\ M & \xrightarrow[\varphi_t]{} & M \end{array} \tag{2.9}$$

$$\mathcal{L}_X Y = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\varphi_{-t})_*(X). \tag{2.10}$$

In particular, we have that $[X, Y] = \mathcal{L}_X Y$ because both operations are inherited from those of smooth vector fields.

This result connects two different worlds: the algebraic one, with the Lie bracket of sections, and the geometric one, where the Lie derivative measures the rate of change of Y along the flow of X. Given that φ_t is a local diffeomorphism, we can consider a similar lift to the exterior bundles $\wedge^k \mathcal{E}_M^*$, represented in diagram (2.11). Similarly to equation (2.10), we can define the Lie derivative of a form $\omega \in \Omega^k(\mathcal{E}_M)$ along an E-field $X \in \chi(\mathcal{E}_M)$ as

$$\begin{array}{ccc} \bigwedge^k \mathcal{E}_M & \xrightarrow{(\varphi_t)_*} & \bigwedge^k \mathcal{E}_M \\ {\scriptstyle \tau_{\bigwedge^k \mathcal{E}_M}}\downarrow & & \downarrow{\scriptstyle \tau_{\bigwedge^k \mathcal{E}_M}} \\ M & \xrightarrow[\varphi_t]{} & M \end{array} \tag{2.11}$$

$$\mathcal{L}_X \omega = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\varphi_{-t})_*(\omega). \tag{2.12}$$

As $[X, Y] = \mathcal{L}_X Y$, from equation (2.12) we have that many classical results in classical geometry hold for E-manifolds. In particular, the Lie derivative defined in (2.5) algebraically is equivalent to its geometric counterpart (2.10). This digression justifies the use of the Lie derivative $\mathcal{L}_X$ as a tool to measure the rate of change of a differential form along an E-field X.

Prolongations and local coordinates

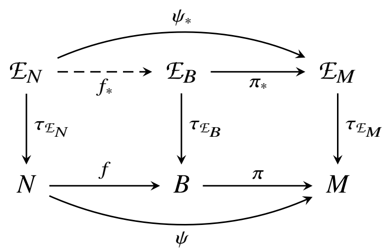
Given an E-manifold, which we assume to be the natural phase space of our physical system with degeneracies, we expect to describe Hamiltonian mechanics in the dual bundle $\mathcal{E}_M^*$. In order to reflect the degeneracy of E in the dynamics on $\mathcal{E}_M^*$, it is necessary to introduce the notion of *prolongation*. This procedure induces an E-manifold structure in the E-cotangent bundle $\mathcal{E}_M^*$. More importantly, this idea will be used in section 2.4 for a similar purpose in a principal G-bundle. Instead of using the definition for Lie algebroids directly, we will present a more specific definition which will be equivalent to that in de León, Marrero, and Martínez[13] by showing that both of them satisfy the same universal property.

[13] Manuel de León, Juan Carlos Marrero, and Eduardo Martínez. Lagrangian submanifolds and dynamics on lie algebroids. *Journal of Physics A: Mathematical and General*, 38(24):R241–R308, jun 2005. DOI: 10.1088/0305-4470/38/24/r01. URL `https://doi.org/10.1088/0305-4470/38/24/r01`

Definition 2.24 Let (M, E_M) be an E-manifold and consider a fibre bundle $\pi\colon B \longrightarrow M$. The *prolongation* of E by B is the pullback submodule $E_B = \pi_*^{-1}(E_M)$.

Proposition 2.25 *Let (M, E_M) be an E-manifold and consider a fibre bundle*

$\pi\colon B \longrightarrow M$. The prolongation E_B satisfies the following pullback universal property: for any E-manifold (N, E_N) and any E-map $\psi\colon N \longrightarrow M$, if $\psi = \pi f$ for some $f\colon N \longrightarrow B$ then f is an E-map. This property is represented in diagram (2.13).



(2.13)

Proof. By the fact that ψ is an E-map, $E_N \subset \psi_*^{-1}(E_M)$. Now, if $\psi = \pi f$ for some f, a direct computation shows that $E_N \subset \psi_*^{-1}(E_M) = (\pi f)_*^{-1}(E_M) = f_*^{-1}(\pi_*^{-1}(E_M))$. This implies, by definition, that $E_N \subset f_*^{-1}(E_B)$ and, as a consequence, f is an E-map. ■

Remark 2.26 The universal property shows that the pullback structure described here agrees with the structure of pullback Lie algebroid, which is more general.

We describe now the structure of any prolongation in local coordinates. The following procedure can be understood as a generalization of the induced coordinates in the tangent bundle TM from a coordinate chart (U, φ). As a consequence, we will refer to the local coordinates in the prolongation of a bundle as *natural coordinates*.

Proposition 2.27 *Let (M, E_M) be an E-manifold and let $\tau\colon B \longrightarrow M$ be a principal bundle with typical fibre Q endowed with the pullback structure E_B. Let $U \subset M$ be a local chart satisfying the trivialization property $\tau^{-1}(U) \simeq U \times Q$ with coordinates $\boldsymbol{q}$, let $E_1, \ldots, E_p$ be local generators of $\mathcal{E}_M(U)$ with structure functions ρ_{ij} and C^k_{ij} and consider a local chart $V \subset Q$ with coordinates $\boldsymbol{r}$. If $\boldsymbol{p}$ are the coordinates induced by the sections $E_1, \ldots, E_p$ and $\boldsymbol{s}$ are the natural coordinates from $\boldsymbol{r}$, then the open set $U \times V$ with coordinates $\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{s}$ is a chart of $\mathcal{E}_B(U \times V)$ and $E_1, \ldots, E_p, V_1, \ldots, V_q$ are the local sections associated to $\boldsymbol{p}$ and $\boldsymbol{s}$, respectively. Moreover, we have*

$$\mathrm{d}f = \sum_{i=1}^{p}\sum_{j=1}^{n} \rho_{ij}\frac{\partial f}{\partial q_j}E_i^* + \sum_{i=1}^{q}\frac{\partial f}{\partial r_i}V_i^*. \tag{2.14}$$

Additionally, the differential of the dual sections is given by

$$\mathrm{d}E_i^* = -\frac{1}{2}\sum_{j,k=1}^{p} C^i_{jk}E_j^* \wedge E_k^*, \quad \mathrm{d}V_i^* = 0. \tag{2.15}$$

Proof. The product $U \times V$ is a local chart of B with coordinates $\boldsymbol{q}, \boldsymbol{r}$. In local coordinates, the bundle projection $\pi\colon U \times V \longrightarrow U$ is expressed as $\pi(\boldsymbol{q}, \boldsymbol{r}) = \boldsymbol{q}$. As a consequence of the definition of pullback structure, $E_B(U \times V) = E_M(U) \times \chi(V)$ and, as a consequence, we have locally $\mathcal{E}_B(U \times V) \simeq \mathcal{E}_M(U) \times \mathrm{T}V$.

Take now local generators $\langle E_1, \ldots, E_p\rangle$ of $\mathcal{E}_M(U)$, which induce coordinates $\boldsymbol{q}$, $\boldsymbol{p}$ in $\mathcal{E}_M(U)$. Consider now the local coordinates $\boldsymbol{r}$, $\boldsymbol{s}$ in TV and let $X_1, \ldots, X_q$ be a basis of sections associated to these coordinates. As a consequence of the preceding discussion, the coordinates $\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{p}, \boldsymbol{s}$ are associated to a local chart of $\mathcal{E}_B(U \times V)$ with local sections $E_1, \ldots, E_p, V_1, \ldots, V_q$.

From this definition we have

$$E_i = \sum_{j=1}^{n} \rho_{ij} \frac{\partial}{\partial q_j}, \qquad [E_i, E_j] = \sum_{k=1}^{p} C_{ij}^k E_k, \tag{2.16}$$

$$V_i = \frac{\partial}{\partial r_i}, \qquad [V_i, V_j] = 0. \tag{2.17}$$

The crossed Lie brackets $[E_i, V_j]$ vanish. We compute the contraction of the differential $\mathrm{d}f$ with the local generators, obtaining

$$\langle \mathrm{d}f, E_i \rangle = \mathcal{L}_{\sum_{j=1}^n \rho_{ij}\partial_{q_j}} f = \sum_{j=1}^{n} \rho_{ij} \frac{\partial f}{\partial q_j},$$
$$\langle \mathrm{d}f, V_i \rangle = \mathcal{L}_{\partial_{r_i}} f = \frac{\partial f}{\partial r_i}.$$

Equation (2.14) is a direct consequence of these computations,

$$\mathrm{d}f = \sum_{i=1}^{p} \sum_{j=1}^{n} \rho_{ij} \frac{\partial f}{\partial q_j} E_i^* + \sum_{i=1}^{q} \frac{\partial f}{\partial r_i} V_i^*.$$

To obtain equations (2.15) we have to compute the contraction of $\mathrm{d}E_i^*$ following equation (2.4). A direct computation shows

$$\begin{aligned} \mathrm{d}E_i^*(E_j, E_k) &= \mathcal{L}_{E_j}\langle E_i^*, E_k\rangle - \mathcal{L}_{E_j}\langle E_i^*, E_k\rangle - \langle E_i^*, [E_j, E_k]\rangle \\ &= -C_{jk}^i, \\ \mathrm{d}E_i^*(E_j, V_k) &= \mathcal{L}_{E_j}\langle E_i^*, V_k\rangle - \mathcal{L}_{V_k}\langle E_i^*, E_j\rangle - \langle E_i^*, [E_j, V_k]\rangle \\ &= 0, \\ \mathrm{d}E_i^*(V_j, V_k) &= \mathcal{L}_{V_j}\langle E_i^*, V_k\rangle - \mathcal{L}_{V_k}\langle E_i^*, V_j\rangle - \langle E_i^*, [V_j, V_k]\rangle \\ &= 0. \end{aligned}$$

As a consequence, we have

$$\mathrm{d}E_i^* = -\frac{1}{2} \sum_{j,k=1}^{p} C_{jk}^i E_j^* \wedge E_k^*. \tag{2.18}$$

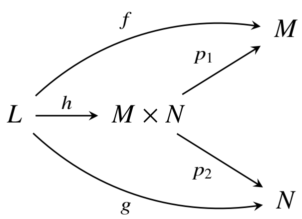
A similar computation shows that $\mathrm{d}V_i^* = 0$. ■

Products of E-manifolds

We will eventually face the necessity of considering the product of an E-manifold with a smooth manifold or the product of two E-manifolds.[14] For this purpose, we show that products exist in the category of E-manifolds, and that their E-structure is what should be expected.

[14] Actually, the first case fits into the second one, for any smooth manifold is an E-manifold taking the canonical structure 2.2.

Proposition 2.28 *Let (M, E_M) and (N, E_N) be two E-manifolds. The product $E_M \times E_N$ is an E-structure in $M \times N$, and the E-manifold $(M \times N, E_M \times E_N)$ satisfies the product universal property: for every E-manifold (L, E_L) and every pair of E-maps $f\colon L \longrightarrow M$ and $g\colon L \longrightarrow N$ there exists a unique E-map $h\colon L \longrightarrow M \times N$ making diagram* (2.19) *commute.*



(2.19)

Proof. We start by showing that $E_M \times E_N$ is an E-structure. As E_M and E_N are locally finitely generated, so is $E_M \times E_N$. Regarding involutivity, the Lie bracket of two elements $X_1 \oplus Y_1, X_2 \oplus Y_2 \in E_M \times E_N$ is given by

$[X_1 \oplus Y_1, X_2 \oplus Y_2] = [X_1, X_2] \oplus [Y_1, Y_2]$. As a consequence, involutivity of $E_M \times E_N$ is directly inherited from the involutivity of E_M and E_N.

We check now that $(M \times N, E_M \times E_N)$ satisfies the product universal property. Take an E-manifold (L, E_L) and a pair of E-maps $f\colon L \longrightarrow M$, $g\colon L \longrightarrow N$. We define the map $h\colon L \longrightarrow M \times N$ in each coordinate as $h(m,n) = (f(m), g(n))$. Notice that h is the unique map making diagram (2.19) commutative. Observe now that $E_L \subset f_*^{-1}(E_M) = h_*^{-1}((p_1)_*^{-1}(E_M)) = h_*^{-1}(E_M \times \chi(N))$. Similarly, $E_L \subset h_*^{-1}(\chi(M) \times E_N)$. As a consequence, we can conclude that $E_L \subset h_*^{-1}(E_M \times \chi(N)) \cap h_*^{-1}(\chi(M) \times E_N) = h_*^{-1}((E_M \times \chi(N)) \cap (\chi(M) \times E_N)) = h_*^{-1}(E_M \times E_N)$, showing that h is an E-map. ■

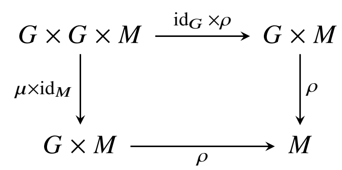
Lie group actions on E-manifolds

Lie group actions, as we have seen throughout chapter 1, have been fundamental in the definition of gauge theories. A definition of Lie group actions on E-manifolds is a necessary step in order to work with gauge theories over E-manifolds. The definition of a Lie group action on an E-manifold is straightforward. However, example shows that there exist Lie group actions on E-manifolds whose fundamental vector fields are not E-maps. In order to realize the cotangent lift of a Lie group action on an E-manifold as a Hamiltonian group action, we will have to consider a subclass of actions.

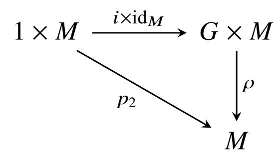
Definition 2.29 Let (M, E) be an E-manifold and consider a Lie group G. A Lie group action $\rho\colon G \times M \longrightarrow M$ is said to be an E-action if ρ is an E-map, where the E-structure in $G \times M$ is the product structure and G has the canonical E-structure $\chi(G)$.

Remark 2.30 This definition of group action follows in the categorical definition of group action.[15] The map $\rho\colon G \times M \longrightarrow M$ fits the commutative diagrams (2.20) and (2.21).

[15] Saunders Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer New York, NY, 2 edition, 1978. ISBN 978-0-387-98403-2. DOI: 10.1007/978-1-4757-4721-8



(2.20)



(2.21)

Remark 2.31 A consequence of the definition of E-action is that ρ_g is an E-map for every $g \in G$.

Remark 2.32 The definition of E-action implies that the fundamental vector fields of ρ are E-fields. The fundamental action $\hat{\rho}$ can be recovered from the action ρ as $\hat{\rho}(X) = \rho_*(X \oplus \mathbf{0})$ for any left-invariant vector field $X \in \mathrm{Lie}(G) \simeq \mathfrak{g}$. From the universal property of products 2.28, the pushforward is an E-field, $\rho_*(X \oplus \mathbf{0}) \in \chi(\mathcal{E}_M)$.

2.3 *Symplectic geometry in E-manifolds*

The formulation of Hamiltonian mechanics in chapter 1 used explicitly a symplectic form ω to define Hamilton's equations of motion. In this section we review the notion of E-symplectic form, that is, a symplectic form in an E-manifold. We will also show that the dual of any E-tangent bundle, which is understood to replace the standard cotangent bundle, carries a natural symplectic structure which will be used in the development of E-gauge theories in section 2.4.

Elements of symplectic geometry

Having at our disposal the differential calculus of E-manifolds developed in 2.2, the definition of symplectic form over an E-manifold is a straightforward generalization of the notion of symplectic form in differential geometry.

Definition 2.33 Let (M, E) be an E-manifold. An *E-symplectic form* is a two-form $\omega \in \Omega^2(\mathcal{E}_M)$ which is non-degenerate and closed, that is, $\mathrm{d}\omega = 0$. An E-manifold $\mathcal{E}_M$ equipped with an E-symplectic form $\omega \in \Omega^2(\mathcal{E}_M)$ is called an *E-symplectic manifold.*

Remark 2.34 Here, non-degeneracy means that the vector bundle morphism

$$\begin{aligned} \omega^\flat : \quad \chi(\mathcal{E}) &\longrightarrow \Omega(\mathcal{E}) \\ X &\longmapsto \iota_X \omega \end{aligned} \tag{2.22}$$

is a vector bundle isomorphism. The inverse map is denoted by $\omega^\sharp$ and, together, are called the *musical isomorphisms* induced by ω.

The non-degeneracy condition for ω, specified in the vector bundle isomorphism (2.22), is the basis for the definition of Hamiltonian vector field in an E-manifold. These fields describe the dynamics of physical systems modeled over E-manifolds.

Definition 2.35 Let (M, E) be an E-symplectic manifold with symplectic form $\omega \in \Omega^2(\mathcal{E}_M)$. The *Hamiltonian vector field* of a function $H \in \mathcal{C}^\infty(M)$, denoted by $X_H \in E$, is the unique E-field which satisfies

$$\iota_{X_H}\omega = -\,\mathrm{d}H. \tag{2.23}$$

Poisson's equation (1.23) is also valid in the setting of E-manifolds, as

$$\mathcal{L}_{X_H} f = \langle \mathrm{d}f, X_H \rangle = \omega(X_H, X_f).$$

This result motivates the definition of Poisson structures over E-manifolds. As in the smooth case, Poisson structures can be characterized from a bivector field called the Poisson tensor.

Definition 2.36 Let (M, E_M) be an E-manifold. A *Poisson tensor* is a bivector $\Pi \in \chi^2(\mathcal{E}_M)$ which satisfies the integrability condition $[\Pi, \Pi] = 0$.

The symplectic geometry of the E-cotangent bundle

Definition 2.24 of prolongation of a fibre bundle makes the natural projection $\tau \colon \mathcal{E}^*_M \longrightarrow M$ an E-map. This condition is enough to define a Liouville form in the E-manifold $\mathcal{E}^*_M$. This form gives a natural framework for Hamiltonian mechanics in E-manifolds by means of the canonical symplectic form, defined analogously to the smooth case. These constructions are particular cases of the Hamiltonian formalism for Lie algebroids of de León, Marrero, and Martínez.[16]

[16] Manuel de León, Juan Carlos Marrero, and Eduardo Martínez. Lagrangian submanifolds and dynamics on lie algebroids. *Journal of Physics A: Mathematical and General*, 38(24):R241–R308, jun 2005. DOI: 10.1088/0305-4470/38/24/r01. URL https://doi.org/10.1088/0305-4470/38/24/r01

Definition 2.37 Let (M, E_M) be an E-manifold and consider the prolongation $(\mathcal{E}^*_M, \tau_*^{-1}(E_M))$. The one-form $\lambda \in \Omega(\tau^1\mathcal{E}_M)$, defined by its action on a section $X \in \chi(\tau^! \mathcal{E}_M)$ as

$$\langle \lambda, X \rangle = \left\langle \tau_{\tau^! \mathcal{E}_M}(X), (\tau_{\mathcal{E}_M^*})_*(X) \right\rangle, \tag{2.24}$$

is called the *Liouville form* of $\mathcal{E}_M^*$.

Lemma 2.38 *Let (M, E_M) be an E-manifold and consider the E-cotangent bundle $(\mathcal{E}_M^*, \tau_*^{-1}(E_M))$. Take an open set U, let $E_1, \ldots, E_p$ be a set of local generators of $\chi_U(\mathcal{E}_M)$ and consider the dual basis $F_1, \ldots, F_p$ in $\mathcal{E}_M^*$. In natural coordinates 2.27, the Liouville one-form* (2.24) *is expressed as*

$$\lambda = \sum_{i=1}^{n} r_i E_i^*. \tag{2.25}$$

Proof. In natural coordinates 2.27, the bundle projections are given by $\tau_{\tau^! \mathcal{E}_M}(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{s}) = (\boldsymbol{q}, \boldsymbol{r})$ and $(\tau_{\mathcal{E}_M^*})_*(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{r}, \boldsymbol{s}) = (\boldsymbol{q}, \boldsymbol{p})$. Moreover, as $\{F_i\}$ is the dual basis of $\{E_i\}$, the natural pairing of $\mathcal{E}_M$ and $\mathcal{E}_M^*$ in these coordinates reads as $\langle \boldsymbol{r}, \boldsymbol{p} \rangle = \sum_{i=1}^{p} r_i p_i$.

Consider now a local E-field $X \in \chi(\tau^! \mathcal{E}_M)$. By the definition of pullback bundle, there are functions $p_i(\boldsymbol{q}, \boldsymbol{r})$ and $s_i(\boldsymbol{q}, \boldsymbol{r})$ such that

$$X = \sum_{i=1}^{p} p_i(\boldsymbol{q}, \boldsymbol{r}) E_i + \sum_{i=1}^{p} s_i(\boldsymbol{q}, \boldsymbol{r}) V_i.$$

As a consequence of the local expressions for $\tau_{\tau^! \mathcal{E}_M}$ and $(\tau_{\mathcal{E}_M^*})_*$, we have that

$$\langle \lambda, X \rangle = \left\langle \tau_{\tau^! \mathcal{E}_M}(X), (\tau_{\mathcal{E}_M^*})_*(X) \right\rangle = \langle \boldsymbol{r}, \boldsymbol{p} \rangle = \sum_{i=1}^{p} r_i p_i(\boldsymbol{q}, \boldsymbol{r}).$$

Equation (2.25) follows from this result. ■

Definition 2.39 Let (M, E_M) be an E-manifold and consider the cotangent bundle $(\mathcal{E}_M^*, \tau_*^{-1}(E_M))$. The *canonical symplectic form* is defined in terms of the Liouville form as

$$\omega = \mathrm{d}\lambda. \tag{2.26}$$

Lemma 2.40 *Let (M, E_M) be an E-manifold and consider the E-cotangent bundle $(\mathcal{E}_M^*, \tau_*^{-1}(E_M))$. Take an open set U, let $E_1, \ldots, E_p$ be a set of local generators of $\chi_U(\mathcal{E}_M)$ and consider the dual basis $F_1, \ldots, F_p$ in $\mathcal{E}_M^*$. In natural coordinates 2.27, the canonical symplectic form* (2.26) *is expressed as*

$$\omega = \sum_{i=1}^{p} V_i^* \wedge E_i^* - \frac{1}{2} \sum_{i,j,k=1}^{p} r_i C_{jk}^i E_j^* \wedge E_k^*. \tag{2.27}$$

As a consequence, the symplectic form ω is non-degenerate.

Proof. Let us consider the natural coordinates 2.27. Applying equations (2.14) and (2.15) to the local expression for the Liouville form (2.25) we have

$$\omega = \sum_{i=1}^{p} \mathrm{d}r_i \wedge E_i^* + \sum_{i=1}^{p} r_i \,\mathrm{d}V_i^* = \sum_{i=1}^{p} V_i^* \wedge E_i^* - \frac{1}{2} \sum_{i,j,k=1}^{p} r_i C_{jk}^i E_j^* \wedge E_k^*.$$

This expression implies the non-degeneracy of ω. A straightforward computation shows that

$$\omega^p = V_1^* \wedge \cdots \wedge V_p^* \wedge E_1^* \wedge \cdots \wedge E_p^*$$

and, as a consequence, ω^p is a volume form. ■

Besides the canonical symplectic form in the E-cotangent bundle, we will also need the notion of E-cotangent lift of E-maps $f \in \mathrm{Diff}(\mathcal{E}_M)$. We will see that these lifts can be defined and that, in fact, they are E-maps with the prolongation structure in the E-cotangent bundle $\mathcal{E}_M^*$. We will afterwards prove that any cotangent lift preserves the Liouville one-form and, as a consequence, the canonical symplectic form as well.

Proposition 2.41 *Let (M, E_M) be an E-manifold and consider the E-cotangent bundle with the pullback structure $(\mathcal{E}_M^*, \tau_*^{-1}(E_M))$. Any E-diffeomorphism $f \in \mathrm{Diff}(\mathcal{E}_M)$ defines an E-diffeomorphism $\hat{f} \in \mathrm{Diff}(\tau^!\mathcal{E}_M)$, called the* E-cotangent lift.

Proof. Consider an E-diffeomorphism $f \in \mathrm{Diff}(\mathcal{E}_M)$ and define, similarly to the smooth case, the E-cotangent bundle as $\hat{f} = (f^*)^{-1}$. This map satisfies $\widehat{fg} = \hat{f}\hat{g}$ and gives rise to the commutative diagram 2.28.

$$\begin{array}{ccc} \mathcal{E}_M^* & \xrightarrow{\hat{f}} & \mathcal{E}_M^* \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} \\ M & \xrightarrow[f]{} & M \end{array} \qquad (2.28)$$

To check that the cotangent lift is a map we observe that, by the commutativity of diagram (2.28), $f_*\tau_* = \tau_*\hat{f}_*$ and, as a consequence, $\tau_*^{-1} f_*^{-1}(E_M) = \hat{f}_*^{-1}\tau_*(E_M)$. From the definition of E-map and the condition $E_{\mathcal{E}_M^*} = \tau_*^{-1}(E_M)$ we conclude $E_{\mathcal{E}_M^*} = \tau_*^{-1}(E_M) \subset \tau_*^{-1} f_*^{-1}(E_M) = \hat{f}_*^{-1}\tau_*^{-1}(E_M) = \hat{f}_*^{-1}(E_{\mathcal{E}_M^*})$, which concludes the proof. ■

Proposition 2.42 *Let (M, E_M) be an E-manifold and consider the E-cotangent bundle with the pullback structure $(\mathcal{E}_M^*, \tau_*^{-1}(E_M))$. The cotangent lift $\hat{f}$ of any E-diffeomorphism $f \in \mathrm{Diff}(\mathcal{E}_M)$ preserves the Liouville one-form* (2.24).

$$\begin{array}{ccc} \tau^!\mathcal{E}_M^* & \xrightarrow{\hat{f}_*} & \tau^!\mathcal{E}_M^* \\ \downarrow{\scriptstyle (\tau_{\mathcal{E}_M^*})_*} & & \downarrow{\scriptstyle (\tau_{\mathcal{E}_M^*})_*} \\ \mathcal{E}_M & \xrightarrow[f_*]{} & \mathcal{E}_M \end{array} \qquad (2.29)$$

$$\begin{array}{ccc} \tau^!\mathcal{E}_M & \xrightarrow{\hat{f}_*} & \tau^!\mathcal{E}_M \\ \downarrow{\scriptstyle \tau_{\tau^!\mathcal{E}_M}} & & \downarrow{\scriptstyle \tau_{\tau^!\mathcal{E}_M}} \\ \mathcal{E}_M^* & \xrightarrow[\hat{f}]{} & \mathcal{E}_M^* \end{array} \qquad (2.30)$$

Proof. As in the smooth case, the proof follows from definition (2.24) and the commutativity of diagrams (2.29) and (2.30). A direct computation shows

$$\begin{aligned} \langle \hat{f}^*\lambda, X\rangle &= \langle \lambda, \hat{f}_* X\rangle \\ &= \left\langle \tau_{\tau^!\mathcal{E}_M}(\hat{f}_* X), (\tau_{\mathcal{E}_M^*})_*\hat{f}_* X\right\rangle \\ &= \left\langle \hat{f}\tau_{\tau^!\mathcal{E}_M}(X), f_*(\tau_{\mathcal{E}_M^*})_* X\right\rangle \\ &= \left\langle \tau_{\tau^!\mathcal{E}_M}(X), (\tau_{\mathcal{E}_M^*})_* X\right\rangle \\ &= \langle \lambda, X\rangle. \end{aligned}$$ ■

In the case where we are considering an E-action $\rho\colon G \times M \longrightarrow M$ of a Lie group G, the previous result implies that the cotangent lift is a Hamiltonian group action.[17]

[17] A precise definition of Hamiltonian group action will be given in the following subsection.

Corollary 2.43 *Let (M, E_M) be an E-manifold and consider the E-cotangent bundle with the pullback structure $(\mathcal{E}_M^*, \tau_*^{-1}(E_M))$. Given an E-action $\rho\colon G \times M \longrightarrow M$, its cotangent lift σ is Hamiltonian with comoment map $\mu^X(p) = \langle \lambda_p, \hat{\sigma}_p(X)\rangle$.*

Proof. From proposition 2.41, we have that $\sigma_\bullet^*\lambda = \lambda$ and, as a consequence,

$\mathcal{L}_{\hat{\sigma}(X)}\lambda = 0$. As a consequence of Cartan's formula (2.5),

$$\iota_{\hat{\sigma}(X)}\, \mathrm{d}\lambda = \iota_{\hat{\sigma}(X)}\omega = -\,\mathrm{d}\langle \lambda, \hat{\sigma}(X)\rangle. \qquad \blacksquare$$

Marsden-Weinstein reduction of E-symplectic manifolds

We will briefly review the notion of Marsden-Weinstein reduction of an E-manifold by a Hamiltonian group action over an E-manifold. The reduction of Marsden and Weinstein for Lie algebroids was proved by Marrero, Padrón, and Rodríguez-Olmos.[18] We state here an analogous version over E-manifolds.

[18] Juan Carlos Marrero, Edith Padrón, and Miguel Rodríguez-Olmos. Reduction of a symplectic-like lie algebroid with momentum map and its application to fiberwise linear poisson structures. *Journal of Physics A: Mathematical and Theoretical*, 45(16), apr 2012. DOI: 10.1088/1751-8113/45/16/165201. URL https://doi.org/10.1088%2F1751-8113%2F45%2F16%2F165201

Definition 2.44 Let (M, E) be an E-manifold with symplectic form $\omega \in \Omega^2(\mathcal{E}_M)$ and consider an E-Lie group action $\rho\colon G\times M \longrightarrow M$. Let $\mu\colon M \longrightarrow \mathfrak{g}^*$ be a smooth map which is equivariant with respect to the coadjoint action,

$$\mathrm{Ad}^*_g\, \mu = \mu\rho_g.$$

We say that ρ is *Hamiltonian* with moment map μ if $\rho^*\omega = \omega$ and

$$\iota_{\hat{\rho}(X)}\omega = -\,\mathrm{d}\mu^X.$$

Here, μ^X is the smooth function defined as $\mu^X(p) = \langle X, \mu(p)\rangle$, called the *comoment map* of the action.

An E-symplectic manifold $(\mathcal{E}_M, \omega)$ with a Hamiltonian action $\rho\colon G \times M \longrightarrow M$ is an *E-Hamiltonian manifold.*

Theorem 2.45 *Let (M, E) be an E-symplectic manifold with symplectic form $\omega \in \Omega^2(\mathcal{E}_M)$. Consider a proper, free and Hamiltonian group action $\rho\colon G \times M \longrightarrow M$ with moment map $\mu\colon M \longrightarrow \mathfrak{g}^*$. Assume that $\alpha \in \mathfrak{g}^*$ is a regular value of μ. Then, $\mu^{-1}(\alpha)/G$ is an E-symplectic manifold with symplectic form ω_{red} given by*

$$\pi^*\omega_{\mathrm{red}} = i^*\omega. \tag{2.31}$$

Here, $\pi\colon \mu^{-1}(\alpha) \longrightarrow \mu^{-1}(\alpha)/G$ is the canonical projection to the quotient and $i\colon \mu^{-1}(\alpha) \longrightarrow M$ is the natural inclusion.

During section 2.4 we will need to consider the product of E-Hamiltonian manifolds. The following lemma is a technical result which ensures that this product is well defined and its construction is the natural one.

Lemma 2.46 *Let (M, E_M) and (N, E_N) be two E-symplectic manifolds with symplectic forms ω_M and ω_N, respectively and let G be a Lie group. Given two Hamiltonian G-actions $\rho_M\colon G \times M \longrightarrow M$ and $\rho_N\colon G \times N \longrightarrow N$ with respective moment maps μ_M and μ_N, the induced G-action in $M \times N$ with symplectic form $p_1^*\omega_M + p_2^*\omega_N$ is Hamiltonian with moment map $\mu = p_1^*\mu_M + p_2^*\mu_N$.*

Proof. It is a straightforward computation to check that the induced G-action on $M \times N$ acts by complete lifts. We start checking the equivariance of μ. A direct computation shows that for a point $p \in M \times N$ with $p_1(p) = m$ and

$p_2(p) = n$,

$$\begin{aligned}\mathrm{Ad}_g^* \mu(p) &= \mathrm{Ad}_g^* \mu_M(m) + \mathrm{Ad}_g^* \mu_N(n) \\ &= \mu_M\big((\rho_M)_g(m)\big) + \mu_N\big((\rho_N)_g(m)\big) \\ &= \mu(\rho_g(p)).\end{aligned}$$

Regarding the fundamental vector fields of the action, from the definition of product action we have that $\hat{\rho}(X) = \hat{\rho}_M(X) \oplus \hat{\rho}_N(X)$. Now, the comoment map of the action is given by $\mu^X = p_1^*\mu_M^X + p_2\mu_N^X$. A straightforward computation with these results yields

$$\begin{aligned}\iota_{\hat{\rho}(X)}\omega &= \iota_{\hat{\rho}_M(X)\oplus\hat{\rho}_N(X)}(p_1^*\omega_M + p_2^*\omega_N) \\ &= p_1^*(\iota_{\hat{\rho}_M(X)}\omega_M) + p_2^*(\iota_{\hat{\rho}_N(X)}\omega_N) \\ &= -p_1^*\,\mathrm{d}\mu_M^X - p_2^*\,\mathrm{d}\mu_N^X \\ &= -\,\mathrm{d}(p_1^*\mu_M^X + p_2^*\mu_N^X).\end{aligned}$$

This chain shows that $\iota_{\hat{\rho}(X)}\omega = -\,\mathrm{d}\mu^X$, completing the proof. ■

2.4 Gauge theory of E-manifolds

Connections in E-principal bundles and properties

Let us consider a principal G-bundle $\pi\colon P \longrightarrow M$ over an E-manifold (M, E). The pullback structure in P gives an E-manifold structure in P which encodes the degeneracies in the phase space present in M. Principal connections, which are assumed to be the source terms for the equations of motion, have to be replaced by suitable splittings of a new short exact sequence. This connections were already described by Nest and Tsygan.[19]

[19] Ryszard Nest and Boris Tsygan. Formal deformations of symplectic manifolds with boundary. *Journal für die reine undangewandte Mathematik*, 1996(481):27–54, 1996. DOI: doi:10.1515/crll.1996.481.27. URL https://doi.org/10.1515/crll.1996.481.27

As in the smooth case, we will trivialize the vertical bundle of P by the fundamental vector fields of the action. The following result ensures that the construction is well-defined.

Lemma 2.47 *Let $\pi\colon P \longrightarrow M$ be a principal G-bundle over an E-manifold (M, E_M).*

1. *The map $r\colon G \times P \longrightarrow P$ is an E-action.*
2. *The vector bundle map*

$$\begin{array}{rccc}\varphi\colon & P\times\mathfrak{g} & \longrightarrow & \ker\pi_* \\ & (p,\alpha) & \longmapsto & (\psi_p)_*(\alpha)\end{array} \tag{2.32}$$

is an isomorphism. Here, ψ_p is the orbit map at a point $p \in P$.

Proof. Consider the E-structure $\chi(G) \times E_P$ in $G \times P$. The proof of the first item follows trivially from the G-invariance condition. The proof of the second item, knowing now that $\ker\pi_* \subset E_P$, follows from the smooth case. ■

Definition 2.48 Consider an E-manifold (M, E_M) and a principal G-bundle $\pi\colon P \longrightarrow M$ with the pullback structure $\mathcal{E}_P$. An *E-principal connection* is a splitting of the following Atiyah-like sequence,

$$0 \longrightarrow P\times_G \mathfrak{g} \xrightarrow{\ \iota\ } \mathcal{E}_P/G \xrightarrow{\ \pi_*\ } \mathcal{E}_M \longrightarrow 0, \tag{2.33}$$

called the *E-Atiyah sequence*.

Remark 2.49 For a general fiber bundle $\pi\colon B \longrightarrow M$, an Ehresmann connection is defined as a splitting of the sequence

$$0 \longrightarrow \ker \pi_* \xrightarrow{\iota} \mathrm{T}B \xrightarrow{\pi_*} \pi^*\mathrm{T}M \longrightarrow 0.$$

For principal connections, the fact that we consider splittings in the quotient sequence implies an invariance condition by the action of G on the splitting. It is easy to show that any splitting of the Atiyah sequence in the smooth case gives a G-invariant Ehresmann connection on $\pi\colon P \longrightarrow M$. Similarly, G-invariant Ehresmann connections factorize to a splitting of the Atiyah sequence. The same properties hold for the E-Atiyah sequence, given that changing the fiber of the spaces does not affect the constructions from the smooth case.

Once again, we will need the splitting lemma to formulate and prove key results in gauge theories over E-manifolds. The statement and proof are analogous to those for smooth manifolds.

Proposition 2.50 *Consider a principal G-bundle $\pi\colon P \longrightarrow M$ over an E-manifold (M, E_M) with the pullback structure $\mathcal{E}_P$. The following objects are in one-to-one correspondence.*

1. *A G-invariant splitting of the short exact sequence* (2.33).
2. *A G-invariant retraction $\theta\colon \mathcal{E}_P \longrightarrow P \times \mathfrak{g}$.*
3. *A G-invariant section $h\colon \pi^*\mathcal{E}_M \longrightarrow \mathcal{E}_P$.*

Notation Consider a principal connection (2.33) and the induced connection in $\mathcal{E}_P$. The induced retraction $\theta\colon \mathcal{E}_P \longrightarrow P \times \mathfrak{g}$ is called the *connection form*. It is an element of $\Omega^1(P \times \mathfrak{g}; \mathcal{E}_P)$. The section h is called the *horizontal lift* of the connection.

We will consider from now and onward the natural left action induced from $r\colon P \times G \longrightarrow P$, defined as $l_g(p) = r_{g^{-1}}(p)$ for any $p \in P$ and $g \in G$. The splitting Φ of the E-Atiyah sequence and the section h remain invariant by the action of l, while the connection form θ intertwines l and the adjoint action in $P \times \mathfrak{g}$. This conceptual difference will not be important in the rest of the section, but it is important to keep in mind when performing computations with coordinates.

Geodesics on riemannian E-manifolds

In classical mechanics, the base manifold of a physical system is called the *configuration space*, and is assumed to be a riemannian manifold. Metrics can be similarly defined for arbitrary E-manifolds, and their construction is analogue to that of E-symplectic forms.

Definition 2.51 Consider an E-manifold (M, E). A *metric* on (M, E) is a section $g \in \Gamma(\mathcal{S}^2\mathcal{E}_M)$ which is positive-definite and non-degenerate. An E-manifold $\mathcal{E}_M$ together with a metric g is called a *riemannian E-manifold*.

Analogously to remark 2.34, the non-degeneracy of the metric g gives rise to vector bundle isomorphisms $g^\flat, g^\sharp$, called the musical isomorphisms of the metric. Given a riemannian E-manifold, one can consider the frame bundle $\mathrm{GL}(\mathcal{E}_M)$. In the presence of a metric g, there exists an analogue notion of the orthonormal frame bundle, denoted by $\mathrm{O}(\mathcal{E}_M)$. In this setting, any principal connection on $\mathrm{O}(\mathcal{E}_M)$ is defined, by analogy with the smooth case, as an *affine connection*. Any such connection induces a splitting of the sequence

$$0 \longrightarrow \ker \tau_* \xrightarrow{\ \iota\ } \tau^!\mathcal{E}_M \xrightarrow{\ \tau_*\ } \mathcal{E}_M \longrightarrow 0. \tag{2.34}$$

The lift $X_\nabla \colon \mathcal{E}_M \longrightarrow \tau^!\mathcal{E}_M$ is an E-field in $\mathcal{E}_M$, called the *geodesic flow*. It is a well-known result that geodesic flows in the cotangent bundle T^*M of a smooth manifold M are Hamiltonian. To construct such a flow in the dual bundle $\mathcal{E}_M^*$ we consider the musical isomorphism given by the metric g, giving the following isomorphism of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\tau_* & \xrightarrow{\ \iota\ } & \tau^!_{\mathcal{E}_M}\mathcal{E}_M & \underset{X_\nabla}{\overset{(\tau_{\mathcal{E}_M})_*}{\rightleftarrows}} & \mathcal{E}_M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g^\flat_*} & & \downarrow{\scriptstyle g^\flat_*} & & \downarrow{\scriptstyle g^\flat} & & \\
0 & \longrightarrow & \ker\tau_* & \xrightarrow[\ \iota\ ]{} & \tau^!_{\mathcal{E}^*_M}\mathcal{E}_M & \underset{(\tau_{\mathcal{E}^*_M})_*}{\overset{Y_\nabla}{\leftrightarrows}} & \mathcal{E}^*_M & \longrightarrow & 0
\end{array} \tag{2.35}$$

The same result holds for E-manifolds. The following proof, which does not involve coordinates, is also valid for smooth manifolds.

Proposition 2.52 *Consider an E-manifold (M, E) with a metric $g \in \Gamma(\mathcal{S}^2\mathcal{E}^*_M)$ and an affine connection $X_\nabla \colon \mathcal{E}^*_M \longrightarrow \tau^!\mathcal{E}^*_M$. The induced flow φ_t of $Y_\nabla = g^\flat_* X_\nabla$ is Hamiltonian, with Hamiltonian function*

$$H(\alpha) = g(g^\sharp\alpha, g^\sharp\alpha).$$

Proof. For the first part of the proof we only have to observe that the flow φ_t gives an isomorphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\tau_* & \xrightarrow{\ \iota\ } & \tau^!\mathcal{E}_M & \underset{Y_\nabla}{\overset{(\tau_{\mathcal{E}^*_M})_*}{\rightleftarrows}} & \mathcal{E}^*_M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle (\varphi_t)_*} & & \downarrow{\scriptstyle \varphi_t} & & \\
0 & \longrightarrow & \ker\tau_* & \xrightarrow[\ \iota\ ]{} & \tau^!\mathcal{E}_M & \underset{(\tau_{\mathcal{E}^*_M})_*}{\overset{Y_\nabla}{\leftrightarrows}} & \mathcal{E}^*_M & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccc}
\tau^!\mathcal{E}_M & \xrightarrow{\ (\varphi_t)_*\ } & \tau^!\mathcal{E}_M \\
{\scriptstyle \tau_{\tau^!\mathcal{E}_M}}\downarrow & & \downarrow{\scriptstyle \tau_{\tau^!\mathcal{E}_M}} \\
\mathcal{E}^*_M & \xrightarrow[\ \varphi_t\ ]{} & \mathcal{E}^*_M
\end{array} \tag{2.36}$$

Notice that the splitting Y_∇ remains unchanged under the action of φ_t. This is because $[Y_\nabla, Y_\nabla] = 0$ and, as a consequence, $(\varphi_t)_* Y_\nabla = Y_\nabla$. From this diagram we have the commutative diagram (2.36). Repeating the proof of proposition 2.41 from diagram (2.29), the flow φ_t preserves the Liouville one-form. As a consequence of this fact, $\mathcal{L}_{Y_\nabla}\lambda = 0$ and a straightforward application of Cartan's formula gives

$$\iota_{Y_\nabla}\,\mathrm{d}\lambda = -\,\mathrm{d}\iota_{Y_\nabla}\lambda.$$

This proves that Y_∇ is Hamiltonian.

Regarding the Hamiltonian function, from the previous result $H(\alpha) = \langle \lambda_\alpha, Y_\nabla(\alpha)\rangle$, thus, by definition (1.5), equals $\langle \alpha, (\tau_{\mathcal{E}^*_M})_* Y_\nabla(\alpha)\rangle$. Take now

$X = g^\sharp(\alpha)$ or, equivalently, $\alpha = g^\flat(X)$. Given that the musical isomorphisms are vector bundle morphisms, we have that $(\tau_{\mathcal{E}_M^*})_* g_*^\flat = (\tau_{\mathcal{E}_M})_*$. Recall that $Y_\nabla = g_*^\flat X_\nabla$. As a consequence,

$$\begin{aligned} \langle \lambda_\alpha, Y_\nabla(\alpha)\rangle &= \langle \alpha, (\tau_{\mathcal{E}_M^*})_* g_*^\flat X_\nabla(X))\rangle \\ &= \langle \alpha, (\tau_{\mathcal{E}_M})_* X_\nabla(X)\rangle \\ &= \langle \alpha, X\rangle \\ &= g(X, X). \end{aligned}$$

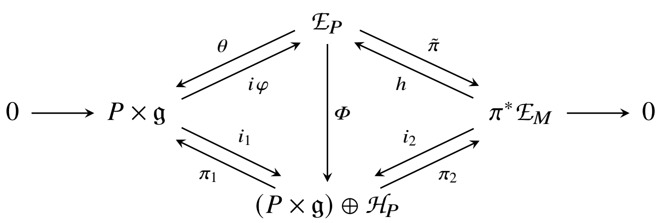
Therefore, we conclude that the Hamiltonian function of the geodesic flow φ_t is $g(g^\sharp\alpha, g^\sharp\alpha)$. ■

The Hamiltonian formalism of gauge theories

Having seen that geodesic flows on the E-cotangent bundle are Hamiltonian, our goal is to generalize this result to splittings of principal bundles, which are the basis for the formulation of gauge theories. The main results in the smooth setting were obtained by Weinstein.[20] We shall give an analogous proof with the formalism for E-manifolds.

[20] Alan Weinstein. A universal phase space for particles in yang-mills fields. *Letters in Mathematical Physics*, 2(5):417–420, Sep 1978. ISSN 1573-0530. DOI: 10.1007/BF00400169. URL https://doi.org/10.1007/BF00400169

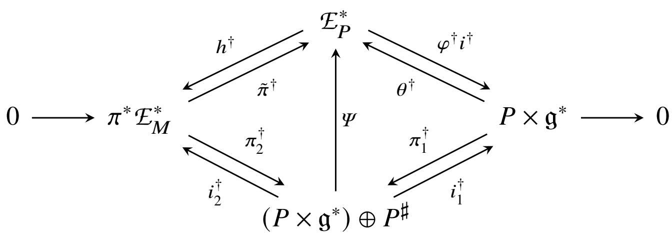
For the rest of the section we will consider a fixed E-connection over $\pi\colon P \longrightarrow M$, which is represented by a map $\Phi\colon \mathcal{E}_P \longrightarrow (P\times\mathfrak{g}) \oplus \mathcal{H}_P$. Sometimes we will work with the isomorphism $(P\times\mathfrak{g})\oplus\mathcal{H}_P \simeq \mathfrak{g}\times\mathcal{H}_P$. The splitting gives rise to the following commutative diagram



(2.37)

In this diagram, the splitting gives an identification $\mathcal{H}_P \simeq \pi^*\mathcal{E}_M$. As a consequence, $\mathcal{H}_P$ is an E-manifold and the splitting Φ is an E-map. The isomorphism Φ also induces a G-action in $(P\times\mathfrak{g})\oplus\mathcal{H}_P$.

Given that we will consider the symplectic spaces $\mathcal{E}_P^*$, $\mathcal{E}_M^*$ with their canonical symplectic forms (2.26) as phase spaces, we consider the dual splitting induced from Φ in the dual E-Atiyah sequence, represented in diagram (2.38).



(2.38)

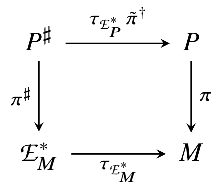
We have denoted $P^\sharp = \mathcal{H}_P^*$. Moreover, it can be noted that the induced splitting is given by $\Psi = \Phi^\dagger$. As before, the isomorphism Ψ induces an action of G in $(P\times\mathfrak{g}^*)\oplus P^\sharp$ and a symplectic form by pushforward.

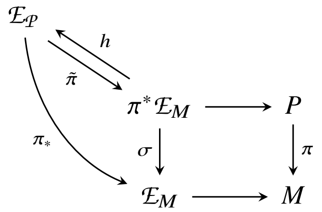
Before presenting the main result, we will show that the vector bundle

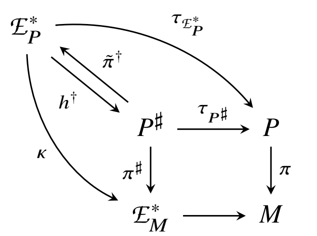
$P^\sharp$ is a specific realization of the pullback bundle of $\mathcal{E}_M^*$ by the projection $\pi\colon P \longrightarrow M$.

Lemma 2.53 *Let $\pi\colon P \longrightarrow M$ be an E-principal G-bundle over an E-manifold (M, E_M). In the notation of diagrams* (2.37) *and* (2.38)*, the vector bundle $P^\sharp$ fits the pullback bundle diagram* (2.40)*. The map $\pi^\sharp\colon P^\sharp \longrightarrow \mathcal{E}_M^*$ is defined by its action on elements $X \in \mathfrak{g} \times \mathcal{E}_P$ and $\beta \in \mathfrak{g}^* \times P^\sharp$ as*

$$\langle \beta, X\rangle = \langle \kappa\tilde{\pi}^\dagger\beta, \pi_* hX\rangle. \tag{2.39}$$


(2.40)


(2.41)


(2.42)

Proof. To begin the proof we realize that, by definition, $\pi^*\mathcal{E}_M^*$ is the pullback bundle of $\mathcal{E}_M^*$ by the submersion $\pi\colon P \longrightarrow M$. We take the identification $P^\sharp \simeq \pi^*\mathcal{E}_M^*$ and construct the commutative diagrams (2.41) and (2.42), which express the pullback property of $\pi^*\mathcal{E}_M$ and $\pi^*\mathcal{E}_M^*$.

The proof of the commutativity of these diagrams now follows from the same elemental idea: the maps in the splittings (2.37) and (2.38) are vector bundle morphisms. Firstly, as $\pi^\dagger$ is a vector bundle morphism it is straightforward to see that $\tau_{P^\sharp} = \tau_{\mathcal{E}_P^*}\tilde{\pi}^\dagger$. We are defining the projection $\kappa\colon \mathcal{E}_P^* \longrightarrow \mathcal{E}_M^*$ by means of the dual splitting $h^\dagger$.

To prove equation (2.39) we use the fact that $\mathcal{E}_M$ and $P^\sharp$ are pullback bundles and, as a consequence, $\langle \beta, X\rangle = \langle \pi^\sharp\beta, \sigma X\rangle$. Following diagram (2.41) we see that $\sigma = \pi_* h$, which proves the result. ∎

Remark 2.54 To prove equation (2.39) we have used the inclusions $h, \tilde{\pi}^\dagger$ to obtain representatives of X and β in $\mathcal{E}_P$ and $\mathcal{E}_P^*$, respectively. We can show that this choice of representatives is not relevant for expression (2.39), that is, for any $Y \in \mathcal{E}_P$ and $\eta \in \mathcal{E}_P^*$ such that $\tilde{\pi}Y = X$ and $h^\dagger\eta = \beta$ we have $\langle \beta, X\rangle = \langle \kappa\eta, \pi_* Y\rangle$.

To prove this fact we only have to see, given equation (2.39), that $\langle \kappa\eta, \pi_* Y\rangle = \langle \kappa\tilde{\pi}^\dagger\beta, \pi_* hX\rangle$. A direct computation shows

$$\begin{aligned}\langle \kappa\eta, \pi_*Y\rangle - \langle \kappa\tilde{\pi}^\dagger\beta, \pi_* hX\rangle &= \langle \kappa\eta, \pi_*Y\rangle - \langle \kappa\tilde{\pi}^\dagger\beta, \pi_*Y\rangle \\ &\quad + \langle \kappa\tilde{\pi}^\dagger\eta, \pi_*Y\rangle - \langle \kappa\tilde{\pi}^\dagger\beta, \pi_* hX\rangle\rangle \\ &= \langle \kappa(\eta - \tilde{\pi}^\dagger\beta), \pi_*Y\rangle - \langle \kappa\tilde{\pi}^\dagger\beta, \pi_*(Y - hX)\rangle.\end{aligned}$$

We use now the fact that $\tilde{\pi}h = \mathrm{id}_{\pi^*\mathcal{E}_M}$ from the properties of the splitting lemma 2.50. From diagram (2.41) and using $\tilde{\pi}Y = X$, $\pi_*(Y - hX) = \sigma\tilde{\pi}Y - \sigma\tilde{\pi}hX = \sigma X - \sigma X = 0$. Similarly, $h^\dagger\tilde{\pi}^\dagger = \mathrm{id}_{P^\sharp}$. Using condition $h^\dagger\eta = \beta$ we can show that $\kappa(\eta - \tilde{\pi}^\dagger\beta) = \pi^\sharp h^\dagger\eta - \pi^\sharp h^\dagger\tilde{\pi}^\dagger\beta = \pi^\sharp\beta - \pi^\sharp\beta = 0$. This proves the desired result.

Theorem 2.55 *Consider a principal G-bundle $\pi\colon P \longrightarrow M$ over an E-manifold M and a Hamiltonian G-space Q.*

1. *The product space $\mathcal{E}_P^* \times Q$ is Hamiltonian with moment map $\mu_P + \mu_Q$.*
2. *The hypothesis of the reduction theorem 2.45 are satisfied and, consequently, the space $(\mathcal{E}_P^* \times Q)_0$ is an E-symplectic manifold.*
3. *The horizontal lift $h^\dagger$ is well defined in classes of equivalence and defines a map $\alpha\colon (\mathcal{E}_P^* \times Q)_0 \longrightarrow \mathcal{E}_M^*$.*

Proof. The first item is straightforward from lemma 2.46. We simply take the natural E-manifold structure $(Q, \chi(Q))$.

To see that we can define the Marsden-Weinstein reduced space we have to check that we are in the conditions of theorem 2.45. Given that $0 \in \mathfrak{g}^*$ is a regular value of μ_P, it is also a regular value of $\mu_P + \mu_Q$. As G acts transitively in $\mathcal{E}_P^*$, it also acts transitively in $\mathcal{E}_P^* \times Q$. This shows that the reduced space $\mu^{-1}(0)/G$ is well defined and carries the symplectic structure (2.31) induced from that of $\mathcal{E}_P^* \times Q$.

Regarding the surjection $\alpha\colon (\mathcal{E}_P^* \times Q)_0 \longrightarrow \mathcal{E}_M^*$, we consider the composition of the surjection $h^\dagger\colon \mathcal{E}_P^* \longrightarrow P^\sharp$ induced from the connection and the natural projection $\tilde{p}_1\colon \mathcal{E}_P^* \times Q \longrightarrow \mathcal{E}_P^*$, giving the map $h^\dagger \tilde{p}_1\colon \mathcal{E}_P^* \times Q \longrightarrow P^\sharp$. We can consider the restriction of this map to the submanifold $j\colon \mu^{-1}(0) \longrightarrow \mathcal{E}_P^* \times Q$, giving the map $h^\dagger \tilde{p}_1 j$. As $h^\dagger$, $\tilde{p}_1$, and j intertwine the G-actions in their respective manifolds, we can take the map defined in classes $\alpha = [h^\dagger \tilde{p}_1 j]$, which completes the proof. ■

Given a Hamiltonian function $H \in C^\infty(\mathcal{E}_M^*)$, the pullback function $\alpha^* H \in C^\infty\big((\mathcal{E}_P^* \times Q)_0\big)$ is taken to be the Hamiltonian function for the equations of motion in the symplectic space $(\mathcal{E}_P^* \times Q)_0$. These equations are called, by analogy with the smooth case, *Wong's equations of motion*, which describe the motion of a point-mass particle interacting with a gauge field.

Weinstein also gives an equivalence of his result with the so-called Sternberg space, originally introduced in Sternberg.[21] An analogue statement, which we present here, holds in the context of E-manifolds.

[21] Shlomo Sternberg. Minimal coupling and the symplectic mechanics of a classical particle in the presence of a yang-mills field. *Proceedings of the National Academy of Sciences*, 74(12):5253–5254, 1977. DOI: 10.1073/pnas.74.12.5253. URL `https://www.pnas.org/doi/abs/10.1073/pnas.74.12.5253`

Theorem 2.56 *Let $\pi\colon P \longrightarrow M$ be an E-principal G-bundle and consider a G-Hamiltonian space Q. There exists a diffeomorphism*

$$\mu^{-1}(0) \simeq P^\sharp \times Q. \tag{2.43}$$

As a consequence, there exists a symplectomorphism of spaces $(\mathcal{E}_P^* \times Q)_0 \simeq P^\sharp \times_G Q$.

Proof. Notice that, from diagram (2.38), the horizontal lift induces a map $h^\dagger\colon \mathcal{E}_P^* \longrightarrow P^\sharp$. In order to obtain the isomorphism with Sternberg's space we notice that, because we have chosen the trivialization $\ker \pi_* \simeq P \times \mathfrak{g}^*$ by the infinitesimal action of G on $\mathcal{E}_P$, the moment map of the action is simply $\mu_P = -p_2$ (see Weinstein[22]). As a consequence, the moment map $\mu\colon \mathfrak{g}^* \times P^\sharp \times Q \longrightarrow \mathfrak{g}^*$ is given by $\mu = -p_1 + \mu_Q$. With this expression, we can readily show that $\mu^{-1}(0) \simeq P^\sharp \times Q$. Of course, if $\mu(p, X, q) = -X + \mu_Q(q) = 0$ we have that $p \in P^\sharp$ is arbitrary and that $X = \mu_Q(q)$. As a consequence, $q \in Q$ is arbitrary and $X \in \mathfrak{g}^*$ is completely determined by q, so the claim is proved. ■

[22] Alan Weinstein. A universal phase space for particles in yang-mills fields. *Letters in Mathematical Physics*, 2(5):417–420, Sep 1978. ISSN 1573-0530. DOI: 10.1007/BF00400169. URL `https://doi.org/10.1007/BF00400169`

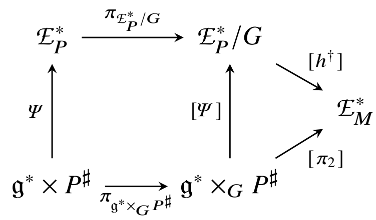
The Montgomery procedure

Following Weinstein, Montgomery exhibited the natural phase space of a classical particle under a Yang-Mills field as a symplectic leaf of a Poisson space.[23] In fact, Montgomery does not consider Poisson universal phase spaces of particles in the sense of Weinstein, but rather an isomorphism of the "natural" Poisson phase space of a particle, given by $\mathcal{E}_P^*/G$, with a Sternberg-like Poisson space.

[23] Richard Montgomery. Canonical formulations of a classical particle in a Yang-Mills field and Wong's equations. *Letters in Mathematical Physics*, 8(1):59–67, January 1984. DOI: 10.1007/BF00420042

The following statement presents and proves an analogue statement in the context of E-manifolds.

Theorem 2.57 *Consider an E-principal G-bundle $\pi\colon P \longrightarrow M$. Any E-principal connection gives rise to the commutative diagram* (2.44). *Moreover, the map $[\Psi]$, called the* minimal coupling*, is a Poisson isomorphism.*



(2.44)

Proof. Consider the induced Poisson structure $\Pi_{\mathcal{E}_P^*} \in \chi^2(\tau^!\mathcal{E}_P^*)$ from the canonical symplectic form ω_P. This structure induces, by pushforward, a Poisson structure $\Pi_{\mathfrak{g}^*\times P^\sharp} = \Psi_*^{-1}\Pi_{\mathcal{E}_P^*} \in \chi^2(\mathcal{E}_{\mathfrak{g}^*\times P^\sharp})$. As the action of G in $\mathcal{E}_P^*$ is symplectic, because it is a cotangent lift, the quotient $\mathcal{E}_P^*/G$ can be endowed with a Poisson structure $\Pi_{\mathcal{E}_P^*/G} = (\pi_{\mathcal{E}_P^*/G})_*\Pi_{\mathcal{E}_P^*}$. The induced G-action in $\mathfrak{g}^* \times P^\sharp$ defines a surjection $\pi_{\mathfrak{g}^*\times_G P^\sharp}$ which can be used to define a Poisson structure $\Pi_{\mathfrak{g}^*\times_G P^\sharp} = (\pi_{\mathfrak{g}^*\times_G P^\sharp})_*\Pi_{\mathfrak{g}^*\times P^\sharp}$. The fact that this structure exists also follows from the fact that the G-action in $\mathfrak{g}^*\times P^\sharp$ is symplectic. The map Ψ intertwines the orbits of G in $\mathcal{E}_P^*$ and $\mathfrak{g}^* \times P^\sharp$ and, as a consequence, there exists a map $[\Psi]\colon \mathfrak{g}^* \times_G P^\sharp \longrightarrow \mathcal{E}_P^*/G$ which is well-defined on orbits and makes diagram (2.44) commute. Given that the projections to the quotients $\pi_{\mathcal{E}_P^*/G}$, $\pi_{\mathfrak{g}^*\times_G P^\sharp}$, and Ψ are Poisson maps, the induced map $[\Psi]$ is a Poisson morphism as

$$\begin{aligned}[\Psi]_*\Pi_{\mathfrak{g}^*\times_G P^\sharp} &= [\Psi]_*(\pi_{\mathfrak{g}^*\times_G P^\sharp})_*\Pi_{\mathfrak{g}^*\times P^\sharp}\\ &= (\pi_{\mathcal{E}_P^*/G})_*\Psi_*\Pi_{\mathfrak{g}^*\times P^\sharp}\\ &= \Pi_{\mathcal{E}_P^*/G}.\end{aligned}$$ ■

The following result is a direct generalization of the expression for the induced symplectic form given in Montgomery.[24] This computation is also present in Guillemin and Sternberg.[25]

[24] Richard Montgomery. *The Bundle Picture in Mechanics*. PhD thesis, University of California, Berkeley, 1986

[25] Victor Guillemin and Shlomo Sternberg. *Symplectic Techniques in Physics*. Cambridge University Press, 1990. ISBN 9780521389907. URL https://books.google.es/books?id=07Rbx4ptxqsC

Proposition 2.58 *Let $\pi\colon P \longrightarrow M$ be an E-principal G-bundle. Denote by λ_M, λ_P the canonical Liouville forms in $\mathcal{E}_M^*$ and $\mathcal{E}_P^*$, respectively, and by ω_M and ω_P their canonical symplectic forms.*

1. *We have the equality $(\theta^\dagger)^*\lambda_P = \langle\alpha, \theta^\sharp\rangle$. Here, $\theta^\sharp$ is the pullback connection $\theta^\sharp = \tau^*_{P\times\mathfrak{g}^*}\theta$ and $\langle\alpha,\theta^\sharp\rangle$ is defined as the one-form $\langle\alpha,\theta^\sharp\rangle \in \Omega^1(\mathfrak{g}^*\times P^\sharp)$ whose action on elements is $(\alpha, X) \longmapsto \langle\alpha, \theta^\sharp(X)\rangle$.*
2. *We have the equality $(\tilde{\pi}^\dagger)^*\lambda_P = (\pi^\sharp)^*\lambda_M$.*
3. *The induced Liouville form in $(P\times\mathfrak{g}^*)\oplus P^\sharp$ by the isomorphism Ψ is*

$$\Psi^*\lambda_P = (\pi^\sharp)^*\lambda_M + \langle\alpha,\theta^\sharp\rangle. \tag{2.45}$$

As a consequence, the induced symplectic form is expressed as

$$\Psi^*\omega_P = (\pi^\sharp)^*\omega_M - \mathrm{d}\langle\alpha,\theta^\sharp\rangle. \tag{2.46}$$

Proof. To prove the first item we show by a direct computation, taking $(p,\alpha) \in P\times\mathfrak{g}^*$ and $X \in \mathrm{T}_\alpha(P\times\mathfrak{g}^*)$, that

$$\begin{aligned}(\theta^\dagger)^*(\lambda_P)_{(p,\alpha)}(X) &= \left\langle(\lambda_P)_{\theta^\sharp(p,\alpha)}, \theta^\dagger_*X\right\rangle\\ &= \langle\theta^\dagger(p,\alpha), (\tau_{\mathcal{E}_P^*})_*\theta^\dagger_*X\rangle\\ &= \langle\alpha, \theta(\tau_{\mathcal{E}_P^*})_*\theta^\dagger_*X\rangle\end{aligned}$$

Now, we can use that $\theta^\dagger$ is a vector bundle morphism to get that $\tau_{\mathcal{E}_P^*}\theta^\dagger = \tau_{P\times\mathfrak{g}^*}$. As a consequence, $\theta(\tau_{\mathcal{E}_P^*})_*\theta_*^\dagger = \tau_{P\times\mathfrak{g}^*}^*\theta$, concluding the proof.

Regarding the second item, we compute both terms separately. Let us choose a point $\beta \in \mathcal{E}_P^*$ and a tangent vector $X \in (\tau^!\mathcal{E}_P^*)_\beta$. A direct computation using the definition of Liouville form and the conclusion of remark 2.54 yields

$$\begin{aligned}(\tilde{\pi}^\dagger)^*(\lambda_P)_\beta(X) &= \left\langle (\lambda_P)_{\tilde{\pi}^\dagger\beta}, \tilde{\pi}_*^\dagger X\right\rangle \\ &= \langle \tilde{\pi}^\dagger\beta, (\tau_{\mathcal{E}_P})_*\tilde{\pi}_*^\dagger X\rangle \\ &= \langle \pi^\sharp\beta, \pi_*(\tau_{\mathcal{E}_P})_*\pi_*^\dagger X\rangle.\end{aligned}$$

An analogous computation yields

$$\begin{aligned}(\pi^\sharp)^*(\lambda_M)_\beta(X) &= \left\langle (\lambda_M)_{\pi^\sharp\beta}, \pi_*^\sharp X\right\rangle \\ &= \langle \pi^\sharp\beta, (\tau_{\mathcal{E}_M})_*\pi_*^\sharp X\rangle.\end{aligned}$$

Both expressions agree by the commutativity of diagram (2.40).

Finally, to prove the third item we use that, from the splitting $(P\times\mathfrak{g}^*)\oplus P^\sharp$, any one-form α can be decomposed as $\alpha = (\pi_2^\dagger)^*\alpha + (\pi_1^\dagger)^*\alpha$. Consequently, $\Psi^*\lambda_P = (\Psi\pi_2^\dagger)^*\lambda_P + (\Psi\pi_1^\dagger)^*\lambda_P$ and, as $\Psi\pi_2^\dagger = \tilde{\pi}^\dagger$ and $\Psi\pi_1^\dagger = \theta^\dagger$ from diagram (2.38), we conclude that $\Psi^*\lambda_P = (\pi^\sharp)^*\lambda_P + (\theta^\dagger)^*\lambda_P$. Equation (2.45) follows from the first and second items.

Regarding the expression of the induced symplectic form, we only have to use equation (2.45) and the commutativity of the pullback of an E-map with the exterior differential, as was proved in proposition 2.21. As all the maps in diagram (2.38) are E-manifolds, we have

$$\begin{aligned}\Psi^*\omega_P &= -\Psi^*\,\mathrm{d}\lambda_P \\ &= -\,\mathrm{d}\Psi^*\lambda_P \\ &= -\,\mathrm{d}(\pi^\sharp)^*\lambda_M - \mathrm{d}\langle\alpha, \theta^\sharp\rangle \\ &= (\pi^\sharp)^*\omega_M - \mathrm{d}\langle\alpha, \theta^\sharp\rangle. \qquad \blacksquare\end{aligned}$$

Example 2.59 — General minimal coupling in b-manifolds. Consider a b-manifold (M, Z) and a principal G-bundle $\pi\colon P \longrightarrow M$. We take now a local chart (U, φ) adapted to Z with coordinates $q_1, \ldots, q_n$. We construct natural induced coordinates $\boldsymbol{q}, \boldsymbol{v}$ in the b-tangent bundle ${}^b\mathrm{T}M$. Consider the local trivialization $V = \pi^{-1}(U) \simeq U \times G$ of P. Taking the identification $\mathrm{T}G \simeq G \times \mathfrak{g}$ by right-invariant vector fields, we have that ${}^b\mathrm{T}V \simeq {}^b\mathrm{T}U \times G \times \mathfrak{g}$, inducing coordinates $(\boldsymbol{q}, \boldsymbol{v}, g, \boldsymbol{Q})$. The tangent lift $(l_\bullet)_*$ is expressed as $(l_h)_*(\boldsymbol{q}, \boldsymbol{v}, g, \boldsymbol{Q}) = (\boldsymbol{q}, \boldsymbol{v}, l_h g, \mathrm{Ad}_g\,\boldsymbol{Q})$. As a consequence, a local expression of the b-cotangent bundle is ${}^b\mathrm{T}^*V \simeq {}^b\mathrm{T}^*U \times G \times \mathfrak{g}^*$ and we have local coordinates $(\boldsymbol{q}, \boldsymbol{p}, g, \boldsymbol{O})$. The cotangent lift in these coordinates is $\hat{l}_h(\boldsymbol{q}, \boldsymbol{p}, g, \boldsymbol{O}) = (\boldsymbol{q}, \boldsymbol{p}, l_h g, \mathrm{Ad}_g^*\,\boldsymbol{O})$. The reduced space is ${}^b\mathrm{T}^*V/G \simeq {}^b\mathrm{T}^*U \times \mathfrak{g}^*$.

The set of coordinates (U, φ) also induces a local trivialization $P_U^\sharp \simeq {}^b\mathrm{T}^*U \times G$. Therefore, the adjoint bundle is locally expressed as $P_U^\sharp \times_G \mathfrak{g}^* = {}^b\mathrm{T}^*U \times \mathfrak{g}^*$ and we have coordinates $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{Q})$. This coordinatization is actually the same used for $\mathcal{E}_P(V)/G$. Montgomery[26] uses the name $\boldsymbol{p}^{\mathrm{can}}$ for the momenta in $P^\sharp \times_G \mathfrak{g}^*$; we will not follow this convention.

[26] Richard Montgomery. *The Bundle Picture in Mechanics*. PhD thesis, University of California, Berkeley, 1986

A connection can be locally specified, by the splitting lemma, as linear map $h\colon {}^{b}TU \longrightarrow {}^{b}TV/G$ such that $\pi_* h = \mathrm{id}_{{}^{b}TU}$. In coordinates $(\boldsymbol{q}, \boldsymbol{v})$ of ${}^{b}TU$ and coordinates $(\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{Q})$ of ${}^{b}TV/G$ the most general expression for such a map is $h(\boldsymbol{q}, \boldsymbol{v}) = (\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{Q} + A \cdot \boldsymbol{v})$. Here, A_i^j are smooth functions of $\boldsymbol{q}$ and $\boldsymbol{v}$. As a consequence, the splitting of the Atiyah sequence has local expression $[\Phi](\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{Q}) = (\boldsymbol{q}, \boldsymbol{v}, \boldsymbol{Q} + A \cdot \boldsymbol{v})$. A straightforward computation using this result yields $\langle \partial_{v_i}^*, [\Phi]_* \partial_{v_j} \rangle = \delta_{ij}$, $\langle \partial_{v_i}^*, [\Phi]_* \partial_{Q_j} \rangle = A_i^j$, $\langle \partial_{Q_i}^*, [\Phi]_* \partial_{v_j} \rangle = 0$, $\langle \partial_{Q_i}^*, [\Phi]_* \partial_{Q_j} \rangle = \delta_{ij}$. As a consequence, the minimal coupling is given as $[\Psi](\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{O}) = (\boldsymbol{q}, \boldsymbol{p} + A \cdot \boldsymbol{O}, \boldsymbol{O})$.

The local expression of the canonical Poisson structure in ${}^{b}TP/G$ is

$$\Pi_P = q_1 \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \sum_{i=2}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j,k=1}^{n} O_k c_{ij}^k \frac{\partial}{\partial O_i} \wedge \frac{\partial}{\partial O_j}.$$

From the expression of Ψ in local coordinates and the fact that it is a Poisson map, the induced structure is

$$\begin{aligned} [\Psi]_* \Pi_P = q_1 \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} &+ \sum_{i=2}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} \\ &+ \frac{1}{2} \sum_{i,j,k=1}^{n} O_k F_{ij}^k \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_j} \\ &- \frac{1}{2} \sum_{i,j,k,l=1}^{n} O_l c_{jk}^l A_i^k \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial O_j} \\ &+ \frac{1}{2} \sum_{i,j,k=1}^{n} O_k c_{ij}^k \frac{\partial}{\partial O_i} \wedge \frac{\partial}{\partial O_j}. \end{aligned}$$

This example extends the computations of Montgomery to b-manifolds.[27] It also generalizes Braddell, Kiesenhofer, and Miranda.[28]

[27] Richard Montgomery. *The Bundle Picture in Mechanics*. PhD thesis, University of California, Berkeley, 1986

[28] Roisin Braddell, Anna Kiesenhofer, and Eva Miranda. b-structures on lie groups and poisson reduction. *Journal of Geometry and Physics*, 175: 104471, 2022. ISSN 0393-0440. DOI: https://doi.org/10.1016/j.geomphys.2022.104471. URL `https://www.sciencedirect.com/science/article/pii/S0393044022000213`

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Conclusions and future work

In this work we have developed a setting for gauge theories over E-manifolds. The proofs of many well known properties follow from the commutativity of certain diagrams; these, in turn, have been obtained from elementary categorical constructions such as the pullback of an E-manifold and the product of two E-manifolds. Similar constructions exist for Lie algebroids, which generalize E-manifolds. In fact, Lie algebroids are the most general objects satisfying the two requirements pointed out in the beginning of this chapter. The differential structure is completely determined by the Lie bracket of the algebroid, in a similar definition to 2.4; the lifting of fields is known as the complete lift. The symplectic geometry developed in section 2.3 has been extracted from de León, Marrero, and Martínez,[1] where it is developed for Lie algebroids. We have good reasons to believe that the classical theorems of Weinstein and Montgomery extend without problems to the setting of Lie algebroids.

[1] Manuel de León, Juan Carlos Marrero, and Eduardo Martínez. Lagrangian submanifolds and dynamics on lie algebroids. *Journal of Physics A: Mathematical and General*, 38(24):R241–R308, jun 2005. DOI: 10.1088/0305-4470/38/24/r01. URL https://doi.org/10.1088/0305-4470/38/24/r01

There are many open questions to be solved which arise from our digression, not necessarily related to the formalism of gauge theories. We will briefly outline the symplectic stratification arising from a Marsden-Weinstein quotient taking spin Calogero-Moser systems as an example.

The stratification of spin Calogero-Moser systems

During the development of section 1.4 we assumed that symplectic Lie group actions $\rho\colon G \times M \longrightarrow M$ on a symplectic manifold (M, ω) are proper and free. This condition is imposed with the objective of ensuring that the quotient space M/G is a smooth manifold. If G is Hamiltonian with moment map $\mu\colon M \longrightarrow \mathfrak{g}^*$ and the previous assumptions are dropped, the quotient space M/G is a *stratified manifold*, whose decomposition in disjoint strata is given as

$$M = \coprod_{H<G} \big(M_{(H)} \cap \mu^{-1}(0)\big)/G. \tag{3.1}$$

Here, the manifold $M_{(H)}$ is the stratum of orbit type (H), defined as the set of points $p \in H$ whose stabilizer is conjugated to H. Each leaf can be given a symplectic form induced from the ambient symplectic form ω.

We will briefly describe the Sjamaar-Lerman stratification and the relationship with Hamiltonian dynamics for a spin Calogero-Moser system taking $G = \mathfrak{su}^*(3)$. Under a trivialization by translations we can regard $\mathrm{T}^*\mathfrak{su}^*(3) \simeq \mathfrak{su}^*(3) \times \mathfrak{su}^*(3)$. The symplectic form is the natural symplectic form in the cotangent bundle. As $\mathfrak{su}^*(3)$ is semisimple, we have an identification $\mathfrak{su}(3) \simeq \mathfrak{su}^*(3)$ by means of the Killing form κ. With this identification and the coadjoint representation, $\mathfrak{su}^*(3)$ can be regarded as the vector space of traceless hermitian matrices. The diagonal coadjoint action of SU(3) on $\mathfrak{su}^*(3) \times \mathfrak{su}^*(3)$ is given as $d_U(A, X) = (c_U A, c_U X)$, where c_U is the conjugation action of standard matrices. This action is Hamiltonian, and its moment map is given by $\mu(A, X) = [A, X]$. The motion is generated by the Hamiltonian function $H = \mathrm{tr}(X^2)$. Direct integration of these equations yields the flow $\varphi_t(A, X) = (A + tX, X)$.

We will study the relationship of the Hamiltonian dynamics X_H with the biggest stratum of the Sjamaar-Lerman decomposition (3.1), given by the biggest orbit type of the diagonal action,

$$Z = \left\{ A \in \mathfrak{su}^*(3) \mid \mathrm{ChP}(A) = (t-\alpha)^2(t-\beta),\ \alpha, \beta \in \mathbf{R} \right\}, \tag{3.2}$$

that is, elements conjugated to the Cartan algebra of $\mathfrak{su}^*(3)$.

Besides this coordinate-free definition of the stratum Z, we will need to have a characterization in coordinates. We take the parametrization of $\mathfrak{su}^*(3)$ by real coordinates as

$$A = \begin{pmatrix} c_1 & a_1 + \mathrm{i}b_1 & a_2 + \mathrm{i}b_2 \\ a_1 - \mathrm{i}b_1 & c_2 & a_3 + \mathrm{i}b_3 \\ a_2 - \mathrm{i}b_2 & a_3 - \mathrm{i}b_3 & -c_1 - c_2 \end{pmatrix}. \tag{3.3}$$

Given that every element of $A \in \mathfrak{su}^*(3)$ is conjugated to some element in the Cartan (that is, these matrices diagonalize with real eigenvalues), we can obtain the eigenvalues as the roots of the characteristic polynomial of A. Therefore, an element A belongs to the set Z if and only if its characteristic polynomial has two coinciding roots. This is an algebraic condition that is easier to impose. As an observation, the matrices in $\mathfrak{su}^*(3)$ are traceless and hermitian; therefore, the characteristic polynomial $\mathrm{ChP}(A) \in \mathbf{R}[\lambda]$ has real coefficients and is depressed.

Proposition 3.1 *Let $p(\lambda) = \lambda^3 + p\lambda + q \in \mathbf{R}[\lambda]$ be a depressed polynomial of degree* 3*. Then, p has a double root if and only if*

$$\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = 0. \tag{3.4}$$

Proof. We rely on the computations of Janson.[2] Equations (2.28), (2.29) and (2.30) imply that two roots agree if and only if $u^3 = v^3$. Equations (2.32) and (2.33),

$$u^3 = -\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}, \quad v^3 = -\frac{q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}.$$

imply that a necessary and sufficient condition is

$$\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = 0. \qquad \blacksquare$$

[2] Svante Janson. Roots of polynomials of degrees 3 and 4, 2010. URL `https://arxiv.org/abs/1009.2373`

With expression (3.3), a direct substitution in expression (3.4) yields the an implicit function for Z in global coordinates. This expression, which is polynomial in parameters (3.3), is far too complicated and meaningless to be explicitly written. The interested reader is encouraged to perform a symbolic computation to obtain the complete equation.

We focus now in the study of local intersections of the flow. We can directly use the expression of the flow $\varphi_t(A, X) = (A + tX, X)$ and the implicit function f we to find an explicit condition for φ_t to remains in Z or not as a parameter of t. In order to perform this computation we will use the following technical observation.

Remark 3.2 As the Hamiltonian H is invariant by the action of SU(3) and such action is also Hamiltonian in $\mathfrak{su}^*(3) \times \mathfrak{su}^*(3)$, we can see that $\varphi_t(d_U(A, X)) = d_U(\varphi_t(A, X))$. Given that Z is invariant by the group action $d_\bullet$, we conclude that $\varphi_t(d_U(A, X)) \in Z$ if and only if $\varphi_t(A, X) \in Z$. As a consequence, we may diagonalize A or X when studying the condition $f(\varphi_t(A, X)) = 0$ without loss of generality.

A key remark is that a direct substitution of φ_t in f yields a polynomial equation in the time parameter t of degree 6. A conclusion of this fact is that a trajectory that intersects Z does it six times at most or it always remains in Z. In order to see which trajectories remain inside Z we have a system of polynomial equations in the coefficients of the powers of t.

- The condition for the power t^0 is equivalent to asking $A \in Z$.
- By symmetry of the equations, the condition for the power t^6 is exactly $X \in Z$.
- As a result of the traceless condition for A and X, the coefficients of t and t^5 automatically vanish.
- Taking $X \in Z$ parametrized as

$$X = \begin{pmatrix} z_1 & x_1 + \mathrm{i}y_1 & x_2 + \mathrm{i}y_2 \\ x_1 - \mathrm{i}y_1 & z_2 & x_3 + \mathrm{i}y_3 \\ x_2 - \mathrm{i}y_2 & x_3 - \mathrm{i}y_3 & -z_1 - z_2 \end{pmatrix}$$

  and $A = \mathrm{diag}(a, a - 2a)$, the coefficients of the powers t^4, t^3 and t^2 are, respectively,

$$\begin{aligned} 0 = {} & \frac{2}{3}a^2x_1^4 - \frac{5}{3}a^2x_1^2x_2^2 - \frac{5}{3}a^2x_1^2x_3^2 + \frac{4}{3}a^2x_1^2y_1^2 - \frac{5}{3}a^2x_1^2y_2^2 - \frac{5}{3}a^2x_1^2y_3^2 \\ & - \frac{25}{6}a^2x_1^2z_1^2 - \frac{29}{3}a^2x_1^2z_1z_2 - \frac{25}{6}a^2x_1^2z_2^2 - 3a^2x_1x_2x_3z_1 - 3a^2x_1x_2x_3z_2 \\ & - 3a^2x_1y_2y_3z_1 - 3a^2x_1y_2y_3z_2 - \frac{1}{12}a^2x_2^4 - \frac{1}{6}a^2x_2^2x_3^2 - \frac{5}{3}a^2x_2^2y_1^2 \\ & - \frac{1}{6}a^2x_2^2y_2^2 - \frac{1}{6}a^2x_2^2y_3^2 - \frac{7}{6}a^2x_2^2z_1^2 + \frac{5}{6}a^2x_2^2z_1z_2 + \frac{1}{3}a^2x_2^2z_2^2 \\ & + 3a^2x_2y_1y_3z_1 + 3a^2x_2y_1y_3z_2 - \frac{1}{12}a^2x_3^4 - \frac{5}{3}a^2x_3^2y_1^2 - \frac{1}{6}a^2x_3^2y_2^2 \\ & - \frac{1}{6}a^2x_3^2y_3^2 + \frac{1}{3}a^2x_3^2z_1^2 + \frac{5}{6}a^2x_3^2z_1z_2 - \frac{7}{6}a^2x_3^2z_2^2 - 3a^2x_3y_1y_2z_1 \\ & - 3a^2x_3y_1y_2z_2 + \frac{2}{3}a^2y_1^4 - \frac{5}{6}a^2y_1^2y_2^2 - \frac{5}{3}a^2y_1^2y_3^2 - \frac{25}{6}a^2y_1^2z_1^2 \\ & - \frac{29}{3}a^2y_1^2z_1z_2 - \frac{25}{6}a^2y_1^2z_2^2 - \frac{1}{12}a^2y_2^4 - \frac{1}{6}a^2y_2^2y_3^2 - \frac{7}{6}a^2y_2^2z_1^2 \\ & + \frac{5}{6}a^2y_2^2z_1z_2 + \frac{1}{3}a^2y_2^2z_2^2 - \frac{1}{12}a^2y_3^4 + \frac{1}{3}a^2y_3^2z_1^2 + \frac{5}{6}a^2y_3^2z_1z_2 \\ & - \frac{7}{6}a^2y_3^2z_2^2 - \frac{13}{12}a^2z_1^4 - \frac{1}{6}a^2z_1^3z_2 + \frac{5}{2}a^2z_1^2z_2^2 - \frac{1}{6}a^2z_1z_2^3 - \frac{13}{12}a^2z_2^4, \end{aligned}$$

$$
\begin{aligned}
0 = &-6a^3x_1^2z_1 - 6a^3x_1^2z_2 - 2a^3x_1x_2x_3 - 2a^3x_1y_2y_3 - \frac{1}{2}a^3x_2^2z_1 \\
&+ \frac{1}{2}a^3x_2^2z_2 + 2a^3x_2y_1y_3 + \frac{1}{2}a^3x_3^2z_1 - \frac{1}{2}a^3x_3^2z_2 - 2a^3x_3y_1y_2 \\
&- 6a^3y_1^2z_1 - 6a^3y_1^2z_2 - \frac{1}{2}a^3y_2^2z_1 + \frac{1}{2}a^3y_2^2z_2 + \frac{1}{2}a^3y_3^2z_1 - \frac{1}{2}a^3y_3^2z_2 \\
&- \frac{3}{2}a^3z_1^3 + \frac{3}{2}a^3z_1^2z_2 + \frac{3}{2}a^3z_1z_2^2 - \frac{3}{2}a^3z_2^3, \\
0 = &-3a^4x_1^2 - 3a^4y_1^2 - \frac{3}{4}a^4z_1^2 + \frac{3}{2}a^4z_1z_2 - \frac{3}{4}a^4z_2^2.
\end{aligned}
$$

As a consequence of these results, we see that the set of vectors $X \in \mathfrak{su}^*(3)$ for which the trajectory $\varphi_t(A, X)$ remains inside Z are solutions to a set of polynomial equations. Assuming that $(0, 0, 0)$ is a regular value of the previous system of polynomial equations, the dimension of the space of solutions $S \subset \mathfrak{su}^*(3)$ is, at least, four. The results of the previous computations are reflected in the following propositions.

Proposition 3.3 *In the previous assumptions, let $A \notin Z$. Then, for every $X \in \mathfrak{su}^*(3)$ we have*

$$\operatorname{Card}\{t \in \mathbf{R} \mid \varphi_t(A, X) \in Z\} \leqslant 6. \tag{3.5}$$

Proposition 3.4 *In the previous assumptions, let $A \in Z$. Under regularity assumptions, there exists a smooth manifold $S \subset \mathfrak{su}^*(3)$ of dimension 4 such that $\varphi_t(A, X) \in Z$ for every $t \in \mathbf{R}$ and $X \in S$.*

Finally, we study the global counterpart of this problem; that is, we are interested in giving a global description of the set of points $(A, X)J \subset\in \mathfrak{su}^*(3) \times \mathfrak{su}^*(3)$ such that $\varphi_t(A, X) \in Z$ for some $t \in \mathbf{R}$ and $\varphi_t(A, X)$ is not fully contained in Z. This problem is formally equivalent to finding a parametrization for Z from the implicit function determined by equation (3.4). Consequently, we do not expect this goal to be feasible. We can, however, give a theoretical description of J and find some geometric properties.

Proposition 3.5 *In the previous assumptions, we have $J = (Z \times (\mathfrak{su}^*(3) \setminus Y) \times \mathbf{R})/\sim$, where $\sim$ is the equivalence relation given by $(A, X, t) \sim (A', X', t')$ if $(\varphi_{t-t'})_*(A, X) = (A', X')$.*

Proof. The idea of the proof is to construct part of the set J by using the Hamiltonian equations of motion as source of parameters. Consider a point $A \in Z$ and a vector $X \in \mathfrak{su}^*(3) \setminus S$, which ensures that the map $\varphi_t(A, X)$ is not strictly contained in Z. We claim that the map

$$
\begin{array}{rccc}
\beta\colon & Z \times (\mathfrak{su}^*(3) \setminus S) \times \mathbf{R} & \longrightarrow & J \\
& (A, X, t) & \longmapsto & (\varphi_t)_*(A, X)
\end{array} \tag{3.6}
$$

is surjective. Take a point $(B, Y) \in J \subset \mathfrak{su}^*(3) \times \mathfrak{su}^*(3)$. By definition of J, there exists a scalar $s \in \mathbf{R}$ such that $\varphi_s(B, Y) \in Z$. Let us denote $(A, X) = (\varphi_s)_*(B, Y)$. Notice that $X \in \mathfrak{su}^*(3) \setminus S$; otherwise, $\varphi_t(A, X)$ is fully contained in Z which, from the fact that $\varphi_t(A, X) = \varphi_{t+s}(B, Y)$, contradicts the definition of J. By the properties of the flow of a vector field, $(B, Y) = (\psi_{-t})_*(A, X)$, proving surjectivity.

Application (3.6), even though surjective, is not injective. Define now the equivalence relationship $(A, X, t) \sim (A', X', t')$ as in the statement. We can check that it is indeed an equivalence relationship.

- It is *reflexive*, as $(\varphi_{t-t})_*(A, X) = (\varphi_0)_*(A, X) = \mathrm{id}_*(A, X) = (A, X)$.
- To prove symmetry we consider two elements $(A, X, t) \sim (A', X', t')$. By definition, $(A', X') = (\varphi_{t-t'})_*(A, X)$. Applying the group properties of φ and the chain rule we obtain $(A, X) = (\varphi_{t'-t})_*(A', X')$.
- To prove *transitivity* consider $(A, X, t) \sim (A', X', t')$ and $(A', X', t') \sim (A'', X'', t'')$. Using once again the group properties of the flow and the chain rule,

$$\begin{aligned}(\varphi_{t-t''})_*(A, X) &= (\varphi_{t-t'+t'-t''})_*(A, X) \\ &= (\varphi_{t'-t''}\varphi_{t-t'})_*(A, X) \\ &= (\varphi_{t'-t''})_*(\varphi_{t-t'})_*(A, X) \\ &= (\varphi_{t'-t''})_*(A', X') \\ &= (A'', X''),\end{aligned}$$

  showing that $(A, X, t) \sim (A'', X'', t'')$.

We can show that the function (3.6) is well defined in classes of equivalence by $\sim$. Take two elements (A, X, t), (A', X', t') with $(A, X, t) \sim (A', X', t')$. A straightforward computation with the definition of J shows that

$$\begin{aligned}(\varphi_{t'})_*(A', X') &= (\varphi_t\varphi_{t'-t})_*(A', X') \\ &= (\varphi_t)_*(\varphi_{t'-t})_*(A', X') \\ &= (\varphi_t)_*(A, X),\end{aligned}$$

showing that it is well defined. Moreover, it is an injective function when considered over classes of equivalence. Consider two elements (A, X, t) and (A', X', t') such that $(\varphi_t)_*(A, X) = (\varphi_{t'})_*(A', X')$. As a consequence of the properties of flows, $(A', X') = (\varphi_{t-t'})_*(A, X)$, showing that $(A, X, t) \sim (A', X', t')$. ■

Remark 3.6 Observe now that any given trajectory $\varphi_t(A, X)$ which is not contained in Z intersects at most six times the critical surface Z by proposition 3.3. As a consequence, the classes of the equivalence relation $\sim$ have six elements at most, and the quotient $Z \times (\mathfrak{su}^*(3) \setminus Y) \times \mathbf{R}/\!\sim$ has dimension

$$\begin{aligned}\dim\left(Z \times (\mathfrak{su}^*(3) \setminus Y) \times \mathbf{R}/\!\sim\right) &= \dim\left(Z \times (\mathfrak{su}^*(3) \setminus Y) \times \mathbf{R}\right) \\ &= \dim Z + \dim(\mathfrak{su}^*(3) \setminus Y) + \dim \mathbf{R} \\ &= 7 + 8 + 1 \\ &= 16.\end{aligned}$$

We conclude this section with some remarks on the problem of this stratification for different spin Calogero-Moser systems.

- Many results in this section have been obtained from the explicit expression of hypersurface Z and the fact that it is a polynomial equation. Such a closed expression is deeply related with the roots of the characteristic polynomial of a matrix; as a consequence, finding an analogue to equation (3.4) might not be possible in general.

  In particular, the local behaviour of the flow φ, only intersecting Z once in a neighbourhood, as well as the global one, only intersecting six times at most, are consequences of the fact that expression (3.4) is polynomial. Even if generalizations of equation (3.4) were found, it is not clear if analogue results hold.

- During the study of the space of solutions S we were able to discard polynomial equations in t^1 and t^5 thanks to an additional symmetry due to the traceless condition for hermitian matrices. This result shows that, in general, the local behaviour of intersections cannot be studied with transversality results. We hope that more powerful techniques, such as intersection homology,[3] will be fruitful in studying this problem.

[3] Mark Goresky and Robert MacPherson. Intersection homology theory. *Topology*, 19(2):135–162, 1980. ISSN 0040-9383. DOI: https://doi.org/10.1016/0040-9383(80)90003-8. URL https://www.sciencedirect.com/science/article/pii/0040938380900038

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Bibliography

Roger Penrose. *The Emperor's New Mind*. Oxford landmark science. Oxford University Press, 2016. ISBN 9780198784920.

Daniel S. Freed. Classical field theory and supersymmetry, 2001. URL https://web.ma.utexas.edu/users/dafr/pcmi.pdf.

John Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer New York, NY, 2 edition, 2012. ISBN 978-1-4419-9981-8. DOI: 10.1007/978-1-4419-9982-5.

Ana Cannas da Silva. *Lectures on Symplectic Geometry*. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1 edition, 2008. DOI: 10.1007/978-3-540-45330-7.

Ivan Kolář, Slovák Jan, and Peter W. Milchor. *Natural Operations in Differential Geometry*. Springer Berlin, Heidelberg, 1 edition, 1993. ISBN 978-3-540-56235-1. DOI: 10.1007/978-3-662-02950-3.

Zuoqin Wang. The moment map. URL http://staff.ustc.edu.cn/~wangzuoq/Courses/15S-Symp/Notes/Lec08.pdf.

Alan Weinstein. A universal phase space for particles in yang-mills fields. *Letters in Mathematical Physics*, 2(5):417–420, Sep 1978. ISSN 1573-0530. DOI: 10.1007/BF00400169. URL https://doi.org/10.1007/BF00400169.

Maurice de Gosson. How classical is the quantum universe?, 2008. URL https://arxiv.org/abs/0808.2774.

Reyer Sjamaar and Eugene Lerman. Stratified symplectic spaces and reduction. *Annals of Mathematics*, 134(2):375–422, 1991. ISSN 0003486X. URL http://www.jstor.org/stable/2944350.

Richard Melrose. The atiyah-patodi-singer index theorem, 1992. URL https://www.maths.ed.ac.uk/~v1ranick/papers/melrose.pdf.

Victor Guillemin, Eva Miranda, and Ana Rita Pires. Symplectic and poisson geometry on b-manifolds. *Advances in Mathematics*, 264:864–896, oct 2014a. DOI: 10.1016/j.aim.2014.07.032. URL https://doi.org/10.1016%2Fj.aim.2014.07.032.

Pedro Frejlich, David Martínez Torres, and Eva Miranda. A note on symplectic topology of b-manifolds. 2013. DOI: 10.48550/ARXIV.1312.7329. URL https://arxiv.org/abs/1312.7329.

Eva Miranda and Geoffrey Scott. The geometry of e-manifolds. *Revista Matemática Iberoamericana*, 37(3):1207–1224, nov 2020. DOI: 10.4171/rmi/1232. URL https://doi.org/10.4171%2Frmi%2F1232.

Nicolai Reshetikhin. Spin calogero-moser models on symmetric spaces, 2019. URL https://arxiv.org/abs/1903.03685.

Eva Miranda and Cédric Oms. The geometry and topology of contact structures with singularities, 2018. URL https://arxiv.org/abs/1806.05638.

Roisin Braddell, Anna Kiesenhofer, and Eva Miranda. b-structures on lie groups and poisson reduction. *Journal of Geometry and Physics*, 175:104471, 2022. ISSN 0393-0440. DOI: https://doi.org/10.1016/j.geomphys.2022.104471. URL https://www.sciencedirect.com/science/article/pii/S0393044022000213.

Ana Cannas da Silva and Alan Weinstein. *Geometric Models for Noncommutative Algebras*. Berkeley Mathematics Lecture Notes. American Mathematical Society, 1999. ISBN 978-0-8218-0952-5.

Manuel de León, Juan Carlos Marrero, and Eduardo Martínez. Lagrangian submanifolds and dynamics on lie algebroids. *Journal of Physics A: Mathematical and General*, 38(24):R241–R308, jun 2005. DOI: 10.1088/0305-4470/38/24/r01. URL https://doi.org/10.1088/0305-4470/38/24/r01.

Rui Loja Fernandes. Lie algebroids, holonomy and characteristic classes. *Advances in Mathematics*, 170(1):119–179, 2002. ISSN 0001-8708. DOI: https://doi.org/10.1006/aima.2001.2070. URL https://www.sciencedirect.com/science/article/pii/S0001870801920705.

Saunders Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer New York, NY, 2 edition, 1978. ISBN 978-0-387-98403-2. DOI: 10.1007/978-1-4757-4721-8.

Juan Carlos Marrero, Edith Padrón, and Miguel Rodríguez-Olmos. Reduction of a symplectic-like lie algebroid with momentum map and its application to fiberwise linear poisson structures. *Journal of Physics A: Mathematical and Theoretical*, 45(16), apr 2012. DOI: 10.1088/1751-8113/45/16/165201. URL https://doi.org/10.1088%2F1751-8113%2F45%2F16%2F165201.

Ryszard Nest and Boris Tsygan. Formal deformations of symplectic manifolds with boundary. *Journal für die reine undangewandte Mathematik*, 1996 (481):27–54, 1996. DOI: doi:10.1515/crll.1996.481.27. URL https://doi.org/10.1515/crll.1996.481.27.

Shlomo Sternberg. Minimal coupling and the symplectic mechanics of a classical particle in the presence of a yang-mills field. *Proceedings of the National Academy of Sciences*, 74(12):5253–5254, 1977. DOI: 10.1073/pnas.74.12.5253. URL https://www.pnas.org/doi/abs/10.1073/pnas.74.12.5253.

Richard Montgomery. Canonical formulations of a classical particle in a Yang-Mills field and Wong's equations. *Letters in Mathematical Physics*, 8(1):59–67, January 1984. DOI: 10.1007/BF00420042.

Richard Montgomery. *The Bundle Picture in Mechanics*. PhD thesis, University of California, Berkeley, 1986.

Victor Guillemin and Shlomo Sternberg. *Symplectic Techniques in Physics*. Cambridge University Press, 1990. ISBN 9780521389907. URL https://books.google.es/books?id=07Rbx4ptxqsC.

Svante Janson. Roots of polynomials of degrees 3 and 4, 2010. URL https://arxiv.org/abs/1009.2373.

Mark Goresky and Robert MacPherson. Intersection homology theory. *Topology*, 19(2):135–162, 1980. ISSN 0040-9383. DOI: https://doi.org/10.1016/0040-9383(80)90003-8. URL https://www.sciencedirect.com/science/article/pii/0040938380900038.

Eva Miranda and Nguyen Tien Zung. A note on equivariant normal forms of poisson structures. 2005. DOI: 10.48550/ARXIV.MATH/0510523. URL https://arxiv.org/abs/math/0510523.

James Waldron. Lie algebroids over differentiable stacks, 2015. URL https://arxiv.org/abs/1511.07366.

Victor Guillemin, Eva Miranda, Ana Rita Pires, and Geoffrey Scott. Toric actions on b-symplectic manifolds. *International Mathematics Research Notices*, 2015(14):5818–5848, jul 2014b. DOI: 10.1093/imrn/rnu108. URL https://doi.org/10.1093%2Fimrn%2Frnu108.

Victor Guillemin, Eva Miranda, and Jonathan Weitsman. Desingularizing b^m-Symplectic Structures. *International Mathematics Research Notices*, 2019(10):2981–2998, 06 2017. ISSN 1073-7928. DOI: 10.1093/imrn/rnx126. URL https://doi.org/10.1093/imrn/rnx126.

Victor Guillemin, Eva Miranda, and Jonathan Weitsman. Convexity of the moment map image for torus actions on b^m-symplectic manifolds. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 376(2131):20170420, 2018. DOI: 10.1098/rsta.2017.0420. URL https://royalsocietypublishing.org/doi/abs/10.1098/rsta.2017.0420.

Eva Miranda and Arnau Planas. Equivariant classification of bm-symplectic surfaces. *Regular and Chaotic Dynamics*, 23(4):355–371, Jul 2018. ISSN 1468-4845. DOI: 10.1134/S1560354718040019. URL https://doi.org/10.1134/S1560354718040019.

Marco Gualtieri and Songhao Li. Symplectic Groupoids of Log Symplectic Manifolds. *International Mathematics Research Notices*, 2014(11):3022–3074, 03 2013. ISSN 1073-7928. DOI: 10.1093/imrn/rnt024. URL https://doi.org/10.1093/imrn/rnt024.

Marco Gualtieri, Songhao Li, Álvaro Pelayo, and Tudor Ratiu. The tropical momentum map: a classification of toric log symplectic manifolds. *Mathematische Annalen*, 367(3):1217–1258, Apr 2017. ISSN 1432-1807. DOI: 10.1007/s00208-016-1427-9. URL https://doi.org/10.1007/s00208-016-1427-9.

Anna Kiesenhofer and Eva Miranda. Cotangent models for integrable systems. *Communications in Mathematical Physics*, 350(3):1123–1145, jul 2016. DOI: 10.1007/s00220-016-2720-x. URL https://doi.org/10.1007%2Fs00220-016-2720-x.

Jacques Dixmier. Champs de vecteurs adjoints sur les groupes et algèbres de lie semi-simples. 1979(309):183–190, 1979. DOI: doi:10.1515/crll.1979.309.183. URL `https://doi.org/10.1515/crll.1979.309.183`.

S. K. Wong. Field and particle equations for the classical yang-mills field and particles with isotopic spin. *Il Nuovo Cimento A (1965-1970)*, 65(4): 689–694, Feb 1970. ISSN 1826-9869. DOI: 10.1007/BF02892134. URL `https://doi.org/10.1007/BF02892134`.