Markov chains: limiting distributions

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Periodicity

Definition

The period of a state i is the number

$$d_i = \gcd\{n \geqslant 1 : r_{ii}(n) > 0\}$$

If $d_i = 1$ we say that i is an aperiodic state.

- $r_{ii}(n) = 0$ if n is not divisible by d_i. (Conditional on $X_0 = i$, the probability of being in state i at time n is 0 is n is not a multiple of d_i.)
- d_i is the greatest integer with such property.
- Notice that $p_{ii} > 0$ is a sufficient condition for $d_i = 1$.

Periodicity

Theorem

If $i \leftrightarrow j$, then $d_i = d_j$.

- One can speak of the period of a whole class of states.
- For an irreducible chain we can speak of the period of the chain. If this period is 1 we say that the chain is aperiodic.

Periodicity

Proof:

Condition $i \leftrightarrow j$ implies $r_{ij}(k) > 0$ and $r_{ji}(m) > 0$ for some integers k and m. So we have that

$$r_{ii}(k+m) \geqslant r_{ij}(k) r_{ji}(m) > 0$$

and d_i divides $k+m$.

On the other hand, let n be any integer such that $r_{jj}(n) > 0$. Then

$$r_{ii}(k+n+m) \geqslant r_{ij}(k) r_{jj}(n) r_{ji}(m) > 0$$

Hence d_i also divides $k+n+m$ and therefore it divides n.

Thus d_i divides any n such that $r_{jj}(n) > 0$. This implies $d_j \geqslant d_i$. Interchanging i by j we deduce that $d_i \geqslant d_j$. Then $d_i = d_j$.

Structure of a periodic chain

In an irreducible chain with period $d > 1$ the transitions between states show a cyclic pattern.

Theorem

An irreducible chain has period $d > 1$ if and only if there exist d subsets S_j, $0 \leqslant j \leqslant d-1$, of the state space S such that:

- $S_i \cap S_j = \emptyset$ *if* $i \neq j$ *and* $S = \bigcup_{j=0}^{d-1} S_j$.
- *If* $k \in S_j$ *and* $p_{kt} > 0$, *then* $t \in S_{j+1}$ *(the subindices are reduced modulo* d*).*
- d *is the greatest integer for which such a decomposition of* S *is possible.*

Aperiodic chains

Theorem

For a finite and irreducible chain the following propositions are equivalent.

- *The chain is aperiodic.*
- *The chain can not be decomposed into a periodic structure (as in the previous theorem).*
- *There exists an integer* $l \geqslant 1$ *and a state* j *such that* $r_{ij}(l) > 0$ *for all* $i \in S$.
- *There exists an integer* $m \geqslant 1$ *such that* $r_{ij}(m) > 0$ *for all* $i, j \in S$.
- *For all* $i \in S$, *the greatest common divisor of the integers* n *such that* $r_{ii}(n) > 0$ *is 1.*

Aperiodic chains

- The existence of an integer $l \geqslant 1$ and a state j such that $r_{ij}(l) > 0$ for all $i \in S$ means that all the elements of the j-th column of $\boldsymbol{P}^l$ are > 0 for some l and j.

  That is, state j can be reached from any $i \in S$ in precisely l steps.
- Analogously, the existence of an integer $m \geqslant 1$ such that $r_{ij}(m) > 0$ for all $i, j \in S$ implies that all the entries of $\boldsymbol{P}^m$ are > 0 for some m (and, hence, for any $m' > m$).

  That is, there exists an m such that for any $m' \geqslant m$ any state j can be reached from any state i in precisely m' steps.

Ergodic theorem

Theorem

For an irreducible aperiodic chain we have that, for all $i, j \in S$,

$$\lim_{n\to\infty} r_{ij}(n) = \frac{1}{\mu_j}, \quad \text{independently of } i.$$

Therefore

- *If the states are non-null recurrent and $(\pi_j = 1/\mu_j : j \in S)$ is the stationary distribution, then*

$$\lim_{n\to\infty} r_{ij}(n) = \pi_j, \quad \text{independently of } i.$$

- *If the states are transient or null recurrent, then*

$$\lim_{n\to\infty} r_{ij}(n) = 0$$

Ergodic theorem

Theorem

For an irreducible and periodic chain with period $d > 1$ we have that the chain given by

$$Y_n = X_{nd}, \quad n \geqslant 0$$

is aperiodic and

$$r_{jj}(nd) = \mathbb{P}(Y_n = j \mid Y_0 = j) \to \frac{d}{\mu_j} \quad \text{as } n \to \infty.$$

Doeblin's proof

Let us consider the proof of the ergodic theorem for the non-null recurrent case. That is, for any initial state i we have that

$$\lim_{n\to\infty} r_{ij}(n) = \pi_j$$

Doeblin's proof:

Let X be our irreducible, aperiodic, and non-null recurrent chain and let Y be another Markov chain, independent of X, with the same transition probabilities.

Doeblin's proof

Consider the composite chain $Z = (X, Y)$ defined by

$$Z_n = (X_n, Y_n), \quad n \geqslant 0$$

It can be checked that:

- Z is an irreducible chain with transition probabilities

$$\mathbb{P}\left(Z_{n+1} = (j, \beta) \mid Z_n = (i, \alpha)\right) = p_{ij} p_{\alpha\beta}$$

- Z is non-null recurrent and aperiodic.
- The stationary distribution of Z is given by

$$\pi(j, k) = \pi_j \, \pi_k$$

Doeblin's proof

Consider an state s and let

$$T_s = \min\{n \geqslant 1 : Z_n = (s,s)\}$$

Moreover, let

$$T = \min\{n \geqslant 1 : X_n = Y_n\}$$

Since $T \leqslant T_s$ and Z is recurrent, we have that $\mathbb{P}(T < \infty) = 1$. Then T is not a defective r.v. and

$$\mathbb{P}(T > n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

Doeblin's proof

By the strong Markov property we have that

$$\mathbb{P}(X_n = j \mid T \leqslant n) = \mathbb{P}(Y_n = j \mid T \leqslant n)$$

Therefore

$$P(X_n = j, T \leqslant n) = \mathbb{P}(Y_n = j, T \leqslant n)$$

Doeblin's proof

Then

$$\begin{aligned} \mathbb{P}(X_n = j) &= P(X_n = j, T \leqslant n) + \mathbb{P}(X_n = j, T > n) \\ &= \mathbb{P}(Y_n = j, T \leqslant n) + \mathbb{P}(X_n = j, T > n) \\ &\leqslant \mathbb{P}(Y_n = j) + \mathbb{P}(T > n) \end{aligned}$$

Analogously,

$$\mathbb{P}(Y_n = j) \leqslant \mathbb{P}(X_n = j) + \mathbb{P}(T > n)$$

Then, as $n \to \infty$,

$$|\mathbb{P}(X_n = j) - \mathbb{P}(Y_n = j)| \leqslant \mathbb{P}(T > n) \longrightarrow 0$$

Doeblin's proof

Take $X_0 = i$ and let Y_0 have the stationary distribution $(\pi_j = 1/\mu_j : j \in S)$. We have that

$$\mathbb{P}(X_n = j) = r_{ij}(n), \qquad \mathbb{P}(Y_n = j) = \pi_j$$

Therefore

$$|r_{ij}(n) - \pi_j| \longrightarrow 0, \quad \text{as } n \to \infty$$

that is,

$$r_{ij}(n) \longrightarrow \pi_j, \quad \text{independently of } i$$

and the proof is finished.

Doeblin's proof

Notice that for any initial probability distribution we have that

$$\mathbb{P}(X_n = j) = \sum_{i \in S} r_{ij}(n)\, \mathbb{P}(X_0 = i)$$
$$\longrightarrow \sum_{i \in S} \pi_j\, \mathbb{P}(X_0 = i) = \pi_j \sum_{i \in S} \mathbb{P}(X_0 = i) = \pi_j$$

Limiting and stationary distribution

Theorem

If the Markov chain is finite and for all $i, j \in S$ we have that

$$\lim_{n \to \infty} r_{ij}(n) = \pi_j, \text{ independently of } i,$$

then $(\pi_j : j \in S)$ is a stationary distribution and it is unique.

Limiting and stationary distribution

Proof:

First, notice that $\pi_j \geqslant 0$ because $r_{ij}(n) \geqslant 0$.

Let $n \to \infty$ in the Chapman-Kolmogorov equations

$$r_{ij}(n+1) = \sum_{k \in S} r_{ik}(n)\, p_{kj}$$

to get

$$\pi_j = \sum_{k \in S} \pi_k\, p_{kj}$$

Moreover, since $\sum_{j \in S} r_{ij}(n) = 1$ we have that $\sum_{j \in S} \pi_j = 1$ (again, let $n \to \infty$).

Hence $(\pi_j : j \in S)$ is a stationary distribution.

Limiting and stationary distribution

To prove uniqueness suppose that $(u_j : j \in S)$ is also a stationary distribution. Thus $u_j = \sum_{k \in S} u_k p_{kj}$ and, more generally,

$$u_j = \sum_{k \in S} u_k\, r_{kj}(n), \quad n \geqslant 1$$

Letting $n \to \infty$ we obtain

$$u_j = \sum_{k \in S} u_k\, \pi_j = \pi_j \sum_{k \in S} u_k = \pi_j$$

Limiting and stationary distribution

For general Markov chains the previous theorem is as follows:

Theorem

If for all $i, j \in S$ we have that

$$\lim_{n\to\infty} r_{ij}(n) = \pi_j, \text{ independently of } i,$$

then

- $\sum_{k\in S} \pi_k \leqslant 1$ *and* $\pi_j = \sum_{k\in S} \pi_k\, p_{kj}$ *for all* $j \in S$.
- *Either* $\pi_j = 0$ *for all* $j \in S$*, or else* $\sum_{j\in S} \pi_j = 1$.
- *If all* $\pi_j = 0$*, there is no stationary distribution.*
  If $\sum_{j\in S} \pi_j = 1$*, then* $(\pi_j : j \in S)$ *is the unique stationary distribution.*

A birth and death process

Example:

Consider a Markov chain such that $S = \mathbb{N}$ and

$$p_{ij} = \begin{cases} 1-b, & j = i = 0 \\ b, & j = i+1,\ i \geqslant 0 \\ d, & j = i-1,\ i \geqslant 1 \\ 1-b-d, & j = i,\ i \geqslant 1 \end{cases}$$

(We suppose that $b > 0$, $d > 0$, $b + d \leqslant 1$.)

- The chain is irreducible and aperiodic because $p_{00} > 0$.

A birth and death process

With $\rho = b/d$ the detailed balance equations give

$$\pi_{j+1} = \rho\, \pi_j, \quad j \geqslant 0$$

Hence

$$\pi_j = \pi_0\, \rho^j, \quad j \geqslant 0$$

A birth and death process

- Case $\rho < 1$.
  Condition $\sum_{j\in S} \pi_j = 1$ implies

$$1 = \pi_0 \sum_{j=0}^{\infty} \rho^j = \frac{\pi_0}{1-\rho}$$

Therefore $\pi_0 = 1 - \rho$ and

$$\pi_j = (1-\rho)\,\rho^j, \quad j \geqslant 0$$

There exists a stationary distribution and, so, the chain is non-null recurrent. Moreover,

$$\lim_{n\to\infty} r_{ij}(n) = (1-\rho)\,\rho^j, \text{ independently of } i$$

A birth and death process

- Case $\rho \geqslant 1$.

  If $\pi_0 = 0$, then $\pi_j = 0$ for all $j \geqslant 0$.
  On the other hand, if $\pi_0 > 0$, then $\pi_0 \sum_{j=0}^{\infty} \rho^j = \infty$.

  In any case there is no stationary distribution and all the states are either transient or null recurrent. Moreover,

  $$\lim_{n\to\infty} r_{ij}(n) = 0, \quad i, j \geqslant 0$$

  It can be proved that for $\rho = 1$ the chain is null recurrent, whereas for $\rho > 1$ it is transient.