Introduction to Stochastic Processes

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Stochastic processes

Stochastic (or random) process: mathematical model of a magnitude that evolves as time passes, in such a way that for any fixed instant of time one has a random variable.

Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $R_t \subseteq \mathbb{R}$.

A stochastic process X is a mapping

$$X: \quad \begin{aligned} R_t \times \Omega &\longrightarrow \mathbb{R} \\ (t, \omega) &\mapsto X(t, \omega) \end{aligned}$$

such that $X(t, \omega)$ is a random variable for all $t \in R_t$.

Stochastic processes

- Let $t \in R_t$.
  Since $X(t, \omega)$ is a random variable, we have that the subset of outcomes
  $$\{\omega \in \Omega : X(t, \omega) \leqslant x\}$$
  is an event (an element of $\mathcal{F}$) for all $x \in \mathbb{R}$.
- In many applications the parameter t corresponds to time.
- The dependence on ω is usually omitted and one writes X or $X(t)$ to denote the process.

Stochastic processes

Therefore,

- A stochastic process X is a collection of random variables indexed by a parameter t (usually, t means time):

$$\{X(t,\omega) \equiv X_t(\omega),\ t \in R_t\}$$

- X is also a collection of functions of time:

$$\{X(t,\omega) \equiv X_\omega(t),\ \omega \in \Omega\}$$

  where $X_\omega(t)$ is the realization of the process corresponding to the outcome $\omega \in \Omega$.

Discrete-time processes

When R_t is a discrete set (finite or denumerable) we say that X is a discrete-time process.

- If $R_t = \{t_1, t_2, \ldots, t_n\}$ is finite, then

$$X = (X(t_1), X(t_2), \ldots, X(t_n))$$

  is a random vector.
- If $R_t = \{t_1, t_2, \ldots, t_n, \ldots\}$ is countable, then

$$X[n] = \{X(t_1), X(t_2), \ldots, X(t_n) \ldots\}$$

  is a sequence of random variables.

Continuous-time processes

When R_t is uncountable one says that X is a continuous-time process.

Usual cases are:

$$X(t),\ \ t \in [a,b]$$

$$X(t),\ \ t \in (0,\infty)$$

$$X(t),\ \ t \in \mathbb{R}$$

State space

According to the values taken by the process we have the following classification:

- $X(t)$ is a discrete-state process if its values are countable, that is, if the image set of the mapping X is finite or denumerable.
- Otherwise, $X(t)$ is a continuous-state process.

Distribution and density functions

Consider (for any fixed $t \in R_t$) the probability distribution function of the random variable $X(t)$:

$$F_X(x;t) \equiv F_{X(t)}(x) = \mathbb{P}(X(t) \leqslant x)$$

- $F_X(x;t)$ is the first-order distribution function of the process.
- It describes the time evolution of the process probability law.

Distribution and density functions

For a continuous-state process (such that $X(t)$ is a continuous random variable for all $t \in R_t$) we can consider the first-order density:

$$f(x;t)\,\Delta x \approx \mathbb{P}\{x \leqslant X(t) \leqslant x + \Delta x\}$$

More formally,

$$f(x;t) = \lim_{\Delta x \to 0} \frac{\mathbb{P}\{x \leqslant X(t) \leqslant x + \Delta x\}}{\Delta x}$$

- $f_X(x;t)$ is, for each $t \in R_t$, the density $f_{X(t)}(x)$ of the continuous variable $X(t)$.

Distribution and density functions

The known relations between distribution and density functions hold. For instance,

$$f_{X(t)}(x) = \frac{d}{dx} F_{X(t)}(x),$$

that is to say,

$$f_X(x;t) = \frac{\partial F_X(x;t)}{\partial x}$$

Analogously,

$$F_{X(t)}(x) = \int_{-\infty}^{x} f_{X(t)}(u)\, du,$$

that is,

$$F_X(x;t) = \int_{-\infty}^{x} f_X(u;t)\, du$$

Distribution and density functions

If $X(t)$ is a discrete-state process taking the values $x_1, x_2, \ldots, x_k, \ldots$, we can define the first-order probability function:

$$p_X(x_k;t) \equiv p_{X(t)}(x_k) = \mathbb{P}(X(t) = x_k), \quad k \geqslant 1$$

We have that

$$F_X(x;t) = \sum_{x_k \leqslant x} p_X(x_k;t).$$

Distribution and density functions

Definition

The n-order distribution function of the process $X(t)$ is the joint distribution function of the r.v.

$$X(t_1), X(t_2), \ldots, X(t_n)$$

arising from $X(t)$ at arbitrary instants of time $t_1, t_2, \ldots, t_n \in R_t$:

$$\begin{aligned} &F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) \\ &\quad \equiv F_{X(t_1), X(t_2), \ldots, X(t_n)}(x_1, x_2, \ldots, x_n) \\ &\quad = \mathbb{P}(X(t_1) \leqslant x_1, X(t_2) \leqslant x_2, \ldots, X(t_n) \leqslant x_n) \end{aligned}$$

Distribution and density functions

For a continuous-state process the n-order density function is obtained as:

$$\begin{aligned} &f_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) \\ &\quad = \frac{\partial^n F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)}{\partial x_1\, \partial x_2 \cdots \partial x_n} \end{aligned}$$

It is to the joint density of the random variables $X(t_1)$, $X(t_2)$, …, $X(t_n)$.

Analogously, if $X(t)$ is a discrete-sate process, the n-order probability function is

$$\begin{aligned} &p_X(x_{k_1}, x_{k_2}, \ldots, x_{k_n}; t_1, t_2, \ldots, t_n) \\ &\quad = \mathbb{P}(X(t_1) = x_{k_1}, X(t_2) = x_{k_2}, \ldots, X(t_n) = x_{k_n}) \end{aligned}$$

Distribution and density functions

The usual relations and properties of these functions hold. For example,

$$\int_{-\infty}^{\infty} f_X(x; t)\, dx = 1$$

$$\int_{-\infty}^{\infty} f_X(x_1, x_2; t_1, t_2)\, dx_1 = f_X(x_2; t_2)$$

$$\lim_{x_2 \to \infty} F_X(x_1, x_2; t_1, t_2) = F_X(x_1; t_1)$$

$$p_X(x_{k_1}; t_1) = \sum_{k_2} p_X(x_{k_1}, x_{k_2}; t_1, t_2)$$

Mean and autocorrelation

Definition

The mean function of the process is $m_X(t) = \mathbb{E}(X(t))$

- For each $t \in R_t$, $m_X(t)$ is the expected value of the r.v. $X(t)$.
- For a continuous-state process

$$m_X(t) = \int_{-\infty}^{\infty} x\, f_X(x; t)\, dx$$

- For a discrete-state process

$$m_X(t) = \sum_k x_k\, p_X(x; t)$$

Mean and autocorrelation

For discrete-time processes the above formulas are

$$m_X[n] = \mathbb{E}(X[n])$$

$$= \begin{cases} \int_{-\infty}^{\infty} x\, f_X(x;n)\, dx, & \text{continuous-state} \\ \sum_k x_k\, p_X(x_k;n), & \text{discrete-state} \end{cases}$$

Mean and autocorrelation

Definition

The autocorrelation function of the process is defined as

$$R_X(t_1,t_2) = \mathbb{E}(X(t_1)X(t_2)).$$

- $R_X(t_1,t_2)$ is the joint moment m_{11} of the variables $X(t_1)$ and $X(t_2)$.
- It is a symmetric function, that is, $R_X(t_1,t_2) = R_X(t_2,t_1)$.
- For $t_1 = t_2 = t$ we have that $R_X(t,t) = \mathbb{E}(X^2(t))$. This function of time is called the average power of $X(t)$.

Mean and autocorrelation

- If $X(t)$ is a continuous-state process,

$$R_X(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f_X(x_1,x_2;t_1,t_2)\, dx_1 dx_2,$$

- Otherwise, if $X(t)$ a discrete-state process we have that

$$R_X(t_1,t_2) = \sum_{k_1}\sum_{k_2} x_{k_1} x_{k_2}\, p_X(x_{k_1}, x_{k_2}; t_1, t_2).$$

- For discrete-time processes we can write analogous expressions.

Autocovariance

Definition

The autocovariance function of the process is defined as

$$C_X(t_1,t_2) = \mathbb{E}\left((X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))\right).$$

- Expanding this expression we obtain

$$C_X(t_1,t_2) = R_X(t_1,t_2) - m_X(t_1)m_X(t_2)$$

- Notice that

$$C_X(t,t) = \mathbb{E}\left((X(t) - m_X(t))^2\right)$$

is the variance $\sigma^2_{X(t)}$ of the r.v. $X(t)$.

Stationary processes

$X(t)$ is a strict-sense stationary process if its statistical properties are invariant to a shift of the origin of time.

This means that the processes $X(t)$ and $X(t+c)$ have the same statistics for any $c > 0$.

Definition

The process $X(t)$ is strict-sense stationary if the families of r.v.

$$\{X(t_1), X(t_2), \ldots, X(t_n)\}$$

and

$$\{X(t_1+c), X(t_2+c), \ldots, X(t_n+c)\}$$

have the same joint distribution for all $t_1, t_2, \ldots, t_n$ and $c > 0$.

Stationary processes

Therefore, for all $n \geqslant 1$, for all $c > 0$ and for all $t_1, t_2, \ldots, t_n$, we have that

$$F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = F_X(x_1, x_2, \ldots, x_n; t_1+c, t_2+c, \ldots, t_n+c).$$

- When this condition is satisfied for $n = 1, 2, \ldots, k$ we say that $X(t)$ is k-order stationary.
- Hence, a strict-sense stationary process is k-order stationary for any $k \geqslant 1$.

Stationary processes

In particular, taking $n = 1$ we have that

$$F_X(x; t) = F_X(x; t+c) \quad \text{for all } c$$

That is, the first-order probability distribution function of a stationary process is time-invariant:

$$F_X(x; t) \equiv F_X(x)$$

Analogously, if $X(t)$ is a continuous-state process:

$$f_X(x; t) \equiv f_X(x)$$

Stationary processes

Theorem

The mean function of a stationary stochastic process is constant.

Proof:

$$m_X(t) = \mathbb{E}(X(t)) = \int_{-\infty}^{\infty} x\, f_X(x; t)\, dx = \int_{-\infty}^{\infty} x\, f_X(x)\, dx \equiv m_X.$$

- More generally, any property involving a single random variable arising from the process will be time-invariant.

Stationary processes

Taking $n = 2$,

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; t_1 + c, t_2 + c) \quad \text{for all } c$$

That is,

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; 0, t_2 - t_1),$$

Therefore

$$F_X(x_1, x_2; t_1, t_2) \equiv F_X(x_1, x_2; \tau), \quad \tau = t_2 - t_1$$

- The joint distribution of two random variables $X(t_1)$, $X(t_2)$ arising from the process only depends on the lag $\tau = t_2 - t_1$ between the two instants t_1 and t_2, but not on the absolute origin of time.

Stationary processes

Theorem

The autocorrelation function of a k-order stationary process ($k \geqslant 2$) only depends on the lag τ:

$$R_X(\tau) = \mathbb{E}(X(t)X(t + \tau)),$$

where this expectation is independent of t.

In particular, taking $t = 0$:

$$R_X(\tau) = \mathbb{E}(X(0)X(\tau)).$$

Properties of $R_X(\tau)$

- $R_X(\tau)$ is an even function:

  $$R_X(-\tau) = \mathbb{E}(X(t)X(t - \tau)) = \mathbb{E}(X(t' + \tau)X(t')) = R_X(\tau)$$

  Hence, we can take $\tau = |t_1 - t_2|$.

- The average power of $X(t)$ is constant:

  $$\mathbb{E}(X^2(t)) = R_X(0) \geqslant 0$$

- We have that $|R_X(\tau)| \leqslant R_X(0)$ for any τ.
  Indeed,

  $$0 \leqslant \mathbb{E}((X(0) \pm X(\tau))^2) = 2R_X(0) \pm 2R_X(\tau)$$

Wide-sense stationary processes

Definition

A stochastic process $X(t)$ is wide-sense stationary if

- The mean function is time-invariant:

  $$m_X(t) = m_X.$$

- The autocorrelation depends only on τ:

  $$R_X(t, t + \tau) = \mathbb{E}(X(t)X(t + \tau)) = R_X(\tau).$$

Wide-sense stationary processes

For time-discrete processes the above definition is translated as:

Definition

A random sequence $X[n]$ is wide-sense stationary if

- The mean is constant, that is $m_X[n] = m_X[0]$ for all n.
- The autocorrelation $R_X[n, n+k] \equiv R[k]$ depends only on k.