The Galton-Watson process

February 15, 2022

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A simple branching process

Let us consider a simple stochastic model for the evolution of the size of a population.

(Assumptions)

1. *The population evolves in generations. Let Z_k, $k \geqslant 0$, be the number of members of the k-th generation. By assumption, $Z_0 = 1$.*
2. *Each member of the k-th generation gives birth to a family (possibly empty) of members of the $(k+1)$-th generation.*

A simple branching process

- The number of descendants of a given individual (family size) is a random variable X with probability function

$$\mathbb{P}(X = i) = p_i, \quad i \geqslant 0$$

- The family sizes form a collection of independent random variables identically distributed as X.

Such a branching process

$$\{Z_0, Z_1, \ldots, Z_k, \ldots\}$$

is called a Galton-Watson process.

Example

For instance, suppose that

$$p_0 = \frac{1}{2}, \qquad p_1 = \frac{1}{4}, \qquad p_2 = \frac{1}{4}$$

Since $Z_0 = 1$, we have that Z_1 is distributed as X, that is,

$$\mathbb{P}(Z_1 = 0) = \frac{1}{2}, \qquad \mathbb{P}(Z_1 = 1) = \frac{1}{4}, \qquad \mathbb{P}(Z_1 = 2) = \frac{1}{4}$$

Example

If, for instance, there are two individuals in the first generation, i.e. $Z_1 = 2$, then

$$\mathbb{P}(Z_2 = 0 \mid Z_1 = 2) = \mathbb{P}(X_1 + X_2 = 0) = p_0^2 = \frac{1}{4}$$
$$\mathbb{P}(Z_2 = 1 \mid Z_1 = 2) = \mathbb{P}(X_1 + X_2 = 1) = p_0 p_1 + p_1 p_0 = \frac{1}{4}$$
$$\mathbb{P}(Z_2 = 2 \mid Z_1 = 2) = \mathbb{P}(X_1 + X_2 = 2) = p_0 p_2 + p_1 p_1 + p_2 p_0 = \frac{5}{16}$$
$$\mathbb{P}(Z_2 = 3 \mid Z_1 = 2) = \mathbb{P}(X_1 + X_2 = 3) = p_1 p_2 + p_2 p_1 = \frac{1}{8}$$
$$\mathbb{P}(Z_2 = 4 \mid Z_1 = 2) = \mathbb{P}(X_1 + X_2 = 4) = p_2^2 = \frac{1}{16}$$

Notice that

$$\sum_{n \geqslant 0} \mathbb{P}(Z_2 = n \mid Z_1 = 2) = 1$$

Example

Performing similar calculations and taking into account that

$$\mathbb{P}\left(Z_2 = n\right) = \sum_{r=0}^{2} \mathbb{P}(Z_2 = n \mid Z_1 = r)\mathbb{P}(Z_1 = r)$$

we obtain the probability function of Z_2:

$$\mathbb{P}(Z_2 = 0) = \frac{11}{16}, \quad \mathbb{P}(Z_2 = 1) = \frac{2}{16}, \quad \mathbb{P}(Z_2 = 2) = \frac{9}{64},$$
$$\mathbb{P}(Z_2 = 3) = \frac{1}{32}, \quad \mathbb{P}(Z_2 = 4) = \frac{1}{64}$$

Probability distribution of Z_{k+1}

In general, if $r \geqslant 1$, then

$$\begin{aligned}\mathbb{P}(Z_{k+1} = n \mid Z_k = r) &= \mathbb{P}(X_1 + X_2 + \cdots + X_r = n \mid Z_k = r)\\ &= \mathbb{P}(X_1 + X_2 + \cdots + X_r = n),\end{aligned}$$

where the random variables X_i are independent and identically distributed as X.

Moreover, if $r = 0$, then $\mathbb{P}(Z_{k+1} = 0 \mid Z_k = 0) = 1$.

Then, by the TPT,

$$\mathbb{P}\left(Z_{k+1} = n\right) = \sum_{r \geqslant 0} \mathbb{P}(Z_{k+1} = n \mid Z_k = r)\mathbb{P}(Z_k = r)$$

Extinction probabilities

Let us consider the probability d_m that the population disappears in a number of generations $\leqslant m$, that is to say,

$$d_m \equiv \mathbb{P}(Z_m = 0)$$

In the previous example,

$$d_0 = 0$$

$$d_1 = \frac{1}{2}$$

$$d_2 = \frac{11}{16}$$

$$\cdots$$

Extinction probabilities

Let $D_m = \{Z_m = 0\}$. Then

$$\begin{aligned} d_m &= \mathbb{P}(D_m) \\ &= \sum_{k\geqslant 0} \mathbb{P}(D_m \mid Z_1 = k)\,\mathbb{P}(Z_1 = k) = \sum_{k\geqslant 0} (d_{m-1})^k\, \mathbb{P}(X = k) \end{aligned}$$

Hence, if $G_X(s) = \sum_{k\geqslant 0} s^k\,\mathbb{P}(X = k)$ is the probability generating function of X, then

$$d_m = G_X(d_{m-1})$$

Extinction probabilities

In our example,

$$G_X(s) = \frac{1}{2} + \frac{1}{4}s + \frac{1}{4}s^2$$

Hence

$$d_0 = 0$$

$$d_1 = G_X(d_0) = G_X(0) = \frac{1}{2}$$

$$d_2 = G_X(d_1) = G_X\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 = \frac{11}{16}$$

$$d_3 = G_X(d_3) = G_X\left(\frac{11}{16}\right) = \frac{1}{2} + \frac{1}{4}\cdot\frac{11}{16} + \frac{1}{4}\cdot\left(\frac{11}{16}\right)^2 = \frac{809}{1024}$$

Ultimate extinction

Notice that $D_0 \subseteq D_1 \subseteq \cdots \subseteq D_m \subseteq D_{m+1} \subseteq \cdots$, and so

$$d_0 \leqslant d_1 \leqslant \cdots \leqslant d_m \leqslant d_{m+1} \leqslant \cdots \leqslant 1$$

Hence the following limit exists:

$$d \equiv \lim_{m\to\infty} d_m$$

What is its meaning?

Ultimate extinction

Since $D_0 \subseteq D_1 \subseteq \cdots \subseteq D_m \subseteq D_{m+1} \subseteq \cdots$, ite makes sense to consider the limit event

$$\lim_{m\to\infty} D_m \equiv \bigcup_{k=0}^{\infty} D_k,$$

which corresponds to the ultimate extinction of the population.

Therefore,

$$d = \lim_{m\to\infty} d_m = \lim_{m\to\infty} \mathbb{P}(D_m) = \mathbb{P}\left(\lim_{m\to\infty} D_m\right)$$

is the probability of that ultimate extinction.

Ultimate extinction

(Limit of an increasing sequence of events)

Let

$$\emptyset = D_0 \subseteq D_1 \subseteq \cdots \subseteq D_m \subseteq D_{m+1} \subseteq \cdots$$

be an increasing sequence of events and define

$$\lim_{m\to\infty} D_m \equiv \bigcup_{k=0}^{\infty} D_k.$$

Then,

$$\mathbb{P}\left(\lim_{m\to\infty} D_m\right) = \lim_{m\to\infty} \mathbb{P}(D_m)$$

Ultimate extinction

Proof:

$$\mathbb{P}\left(\lim_{m\to\infty} D_m\right) = \mathbb{P}\left(\bigcup_{k=0}^{\infty} D_k\right) = \mathbb{P}\left(\bigcup_{k=1}^{\infty} (D_k \setminus D_{k-1})\right),$$

where the last union is of mutually disjoint events

Therefore,

$$\begin{aligned}\mathbb{P}\left(\lim_{m\to\infty} D_m\right) &= \sum_{k=1}^{\infty} (P(D_k) - \mathbb{P}(D_{k-1})) \\ &= \lim_{m\to\infty} \sum_{k=1}^{m} (P(D_k) - \mathbb{P}(D_{k-1})) \\ &= \lim_{m\to\infty} \mathbb{P}(D_m)\end{aligned}$$

Ultimate extinction

Letting $m \to \infty$ we get from $d_m = G_X(d_{m-1})$ that

$$d = G_X(d)$$

- So, the extinction probability d is a solution of the equation

$$s = G_X(s)$$

- Notice that $s = 1$ is always a root of $s = G_X(s)$.

Ultimate extinction

Returning to our example, but considering arbitrary values of the probabilities p_0, p_1, and p_2, such that $p_0 + p_1 + p_2 = 1$, we have

$$s = G_X(s) \iff s = p_0 + p_1\, s + p_2\, s^2$$

Hence, $p_2\, s^2 - (p_0 + p_2)\, s + p_0 = 0$. The two roots of this equation are

$$s_1 = 1, \quad s_2 = \frac{p_0}{p_2}$$

Let m be the expected number of descendants per individual:

$$m = \mathbb{E}(X) = p_1 + 2p_2 = 1 - p_0 + p_2$$

Ultimate extinction

(1) Case $m < 1 \iff p_0 > p_2 \iff s_2 > 1$

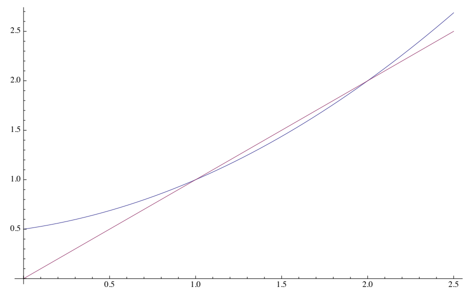

Figure: $p_0 = 1/2, \quad p_1 = 1/4, \quad p_2 = 1/4$

The iterations $d_m = G_X(d_{m-1})$ converge to $d = 1$.

Ultimate extinction

(2) Case $m = 1 \iff p_0 = p_2 \iff s_1 = s_2 = 1$

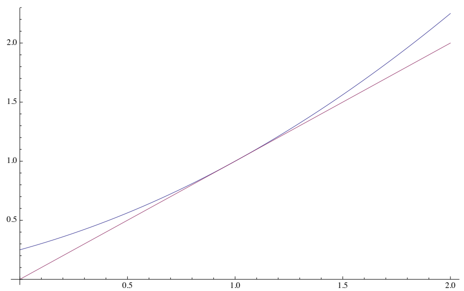

Figure: $p_0 = 1/4, \quad p_1 = 1/2, \quad p_2 = 1/4$

The iterations $d_m = G_X(d_{m-1})$ also converge to $d = 1$.

Ultimate extinction

(3) Case $m > 1 \iff p_0 < p_2 \iff s_2 < 1$

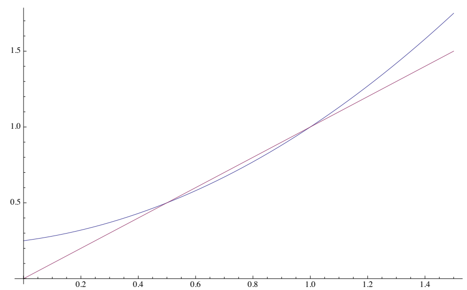

Figure: $p_0 = 1/4, \quad p_1 = 1/4, \quad p_2 = 1/2$

The iterations $d_m = G_X(d_{m-1})$ converge to $d = s_2 < 1$.

General case

The same analysis can be performed in the case of a more general probability generating function $G_X(s)$.

Notice that if $s \geqslant 0$, then

$$G_X(s) = p_0 + p_1 s + p_2 s^2 + p_3 s^3 + \cdots \geqslant 0$$

$$G_X'(s) = p_1 + 2p_2 s + 3p_3 s^2 + \cdots \geqslant 0$$

$$G_X''(s) = 2p_2 + 6p_3 s + \cdots \geqslant 0$$

So, the plot of $G_X(s)$ is as in the case of a polynomial of degree 2.

General case

- In general, if $s \geqslant 0$, the plot of $G_X(s)$ will intersect in two points the line (with equation) s. So, the equation $s = G_X(s)$ will have two solutions.
- The iterations $d_m = G_X(d_{m-1})$ will converge to the smallest of the two solutions.
- Moreover, $m = \mathbb{E}(X) = G_X'(1)$ is the slope of the tangent line of $G_X(s)$ at $s = 1$.
- Hence, we have again the three cases considered previously for $G_X(s)$ a polynomial of degree 2.

Ultimate extinction

Theorem

Let d be the smallest nonnegative root of the equation $s = G_X(s)$ and let $m = \mathbb{E}(X)$. Then d is the probability of ultimate extinction of the population. Moreover,

- *If $m < 1$, then $d = 1$. The extinction is sure.*
- *$m = 1$, then $d = 1$. The line s is tangent at $s = 1$ to $G_X(s)$ (double root $d = 1$). The extinction is sure.*
- *$m > 1$, then $d < 1$. In this case, there is a non-zero probability, $1 - d$, of non-extinction.*

Probability generating function of Z_n

Let $G_n(s)$ be the probability generating function of Z_n.

$$G_{n+1}(s) = \mathbb{E}\left(s^{Z_{n+1}}\right) = \mathbb{E}\left(\mathbb{E}\left(s^{Z_{n+1}} \mid Z_n\right)\right)$$

Since

$$\mathbb{E}\left(s^{Z_{n+1}} \mid Z_n = k\right) = \mathbb{E}\left(s^{X_1 + \cdots + X_k} \mid Z_n = k\right)$$
$$= \mathbb{E}\left(s^{X_1} \cdots s^{X_k}\right) = \left(\mathbb{E}\left(s^X\right)\right)^k = (G_X(s))^k$$

we have

$$G_{n+1}(s) = \mathbb{E}\left((G_X(s))^{Z_n}\right) = G_n\left(G_X(s)\right)$$

Probability generating function of Z_n

So, we have

$$G_1(s) = G_X(s)$$
$$G_2(s) = G_1(G_X(s)) = G_X(G_X(s)) = G_X^{(2)}(s)$$
$$G_3(s) = G_2(G_X(s)) = G_X(G_X(G_X(s))) = G_X^{(3)}(s)$$
$$\cdots\cdots\cdots$$

The probability generating function of Z_n is the n-folded composition of $G_X(s)$ with itself:

$$G_n(s) = G_{n-1}(G_X(s)) = G_X^{(n)}(s), \quad n \geqslant 1.$$

Probability generating function of Z_n

In our example,

$$G_X(s) = \frac{1}{2} + \frac{1}{4}s + \frac{1}{4}s^2$$

Then

$$\begin{aligned} G_2(s) &= G_X(G_X(s)) = \frac{1}{2} + \frac{1}{4}G_X(s) + \frac{1}{4}(G_X(s))^2 \\ &= \frac{1}{2} + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{4}s + \frac{1}{4}s^2\right) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{4}s + \frac{1}{4}s^2\right)^2 \\ &= \frac{11}{16} + \frac{1}{8}s + \frac{9}{64}s^2 + \frac{1}{32}s^3 + \frac{1}{64}s^4 \end{aligned}$$

Expected number of individuals

Let $m_n = \mathbb{E}(Z_n)$.

The derivative of $G_{n+1}(s) = G_n(G_X(s))$ is

$$G'_{n+1}(s) = G'_n(G_X(s))\; G'_X(s)$$

Taking $s = 1$ we have

$$G'_{n+1}(1) = G'_n(G_X(1))\; G'_X(1) = G'_n(1)\; G'_X(1)$$

Hence $m_{n+1} = m_n\, m$. Therefore,

$$m_n = m^n$$