

SINGULAR COTANGENT MODELS AND COMPLEXITY IN FLUIDS WITH DISSIPATION

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ABSTRACT. In this article we analyze several mathematical models with singularities where the classical cotangent model is replaced by a b -cotangent model. We provide physical interpretations of the singular symplectic geometry underlying in b -cotangent bundles featuring two models: the canonical (or non-twisted) model and the twisted one. The first one models systems on manifolds with boundary and the twisted model represents Hamiltonian systems where the singularity of the system is in the fiber of the bundle. The twisted cotangent model includes (for linear potentials) the case of fluids with dissipation. We relate the complexity of the fluids in terms of the Reynolds number and the (non)-existence of cotangent lift dynamics. We also discuss more general physical interpretations of the twisted and non-twisted b -symplectic models. These models offer a Hamiltonian formulation for systems which are dissipative, extending the horizons of Hamiltonian dynamics and opening a new approach to study non-conservative systems.

1. INTRODUCTION

Complexity of fluids has been revealed through several reincarnations (see the recent article [CMPSP21, CMPS21] where new facets of complexity are analyzed). In particular, the Reynolds number provides a first approach to measure its complexity. High Reynolds numbers flows tend to be turbulent. In particular, for the Euler flow the Reynolds number is infinity.

In this article we give a mathematical interpretation of the physics of fluids obeying the Stokes' Law using the Hamiltonian formalism on a singular cotangent model. However, the inherent complexity of the systems does not let identify these models as twisted cotangent lifts in general.

Symplectic geometry provides the landscape where classical mechanics take place. The binomial of position and momenta is the physical manifestation of the existence of a cotangent bundle underlying this picture. The role of base and fibers of the cotangent bundle is an important landmark that fixes and precises the Hamiltonian dynamics. However, this perfect symplectic picture is often insufficient to describe the complexity of physical phenomena. Poisson geometry provides a more general scenery appropriate to capture the complexity of physical systems. Nevertheless, Poisson geometry is, in general, too involved and even the existence of appropriate local coordinates is a difficult battlefield. From this perspective, singular symplectic manifolds provide a much more controlled terrain to fulfill some of these needs. In this article we explore some physical systems that can be described as singular symplectic manifolds. For the sake of simplicity, we focus on the case of b -symplectic manifolds and identify two models: a canonical and a twisted one. We associate relevant physical systems to these two models.

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In the context of symplectic geometry, singular forms have been an important object of study in the last years. A main class of such singular forms is the class of b -symplectic forms, formally introduced in [GMP11] and [GMP14]. They provide a way to model systems with boundary and to study manifolds through compactification.

Among the variety of applications of b -symplectic geometry (and its b -contact counterpart), there have been obtained remarkable results on general integrable systems, celestial mechanics and fluid dynamics (see, for instance, [MO18], [MO21], [PA19], [BDM⁺19], [CMPS19] [DKM17]).

The phase space of a physical problem can be associated with the cotangent bundle of the configuration space. Therefore, it is automatically symplectic and this is one of the main reasons that makes symplectic geometry the natural language of mechanics. In the general setting, the physical Hamiltonian is the sum of a kinetic term depending on the momentum and a potential term depending only on the position. It provides an associated Hamiltonian flow which describes exactly the physical dynamics and yields the usual Newton's laws.

At the crossroads of b -symplectic techniques and cotangent models for physical systems, *singular cotangent models* supply the techniques to generalize procedures such as the cotangent lift from symplectic manifolds to b -symplectic manifolds. These techniques were explored in [KM17] in the integrable case and following two different approaches. In the first approach, the singularity of the integrable system defined in a cotangent bundle of a smooth manifold is placed at the base manifold. In the second case, the so-called *twisted case*, the singularity is placed at the fiber. In both cases, the singularity permeates the geometric structure and the b -symplectic form carries the characterization of the singularity.

Singular cotangent models have also been considered in [CMPS19], [MO18] and [MO21]. Connections to other singularities in physical systems are explored in [YM20] (see also [NT01] and [MS21] for the geometrical study of E -symplectic structures).

In this article, we give a new application of the twisted cotangent model. In particular, we present the Stokes' Law of motion for free-falling particles in fluids with viscosity. We prove that, in general, one-dimensional dissipation which is proportional to the velocity can be modeled by a twisted b -symplectic form.

The fact that b -symplectic techniques can be used to model fluid systems is interesting because, classically, the study of the evolution of moving fluids, has been tackled via partial differential equations. As the most important example, consider the following Navier-Stokes equations:

$$(1) \quad \rho \left(\frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p + \mu \nabla^2 u + \frac{1}{3} \mu \nabla (\nabla \cdot u) + \rho g,$$

where ρ is the density of the fluid, u is the flow velocity, μ is the dynamic viscosity, p is the pressure and g represents body accelerations, for instance gravity.

A set of PDEs models viscous Newtonian fluids expressing their mass and momentum conservation. This is one of the most used tools in fluid dynamics. Usually, solutions to these PDEs can only be found numerically and it is difficult to prove if, for some initial conditions, they are smooth or even continuous. This complexity has led to other approaches to model the behaviour of fluids. Among the alternative formulations of fluid dynamics, there are the Hamiltonian and Lagrangian formulations, which are used naturally in a wide collection of mechanics problems. In this respect, the b -symplectic approach given in this paper contributes to this alternative approach.

The main problem with these Hamiltonian procedures is that viscous fluids suffer energy dissipation and hence there is not such a natural conserved quantity. Still, there are techniques that can be applied for situations with friction and loss of energy. The work done by Morrison on the Hamiltonian viewpoint of fluid mechanical systems is one of the examples of these distinct procedures. In [Mor84a], Morrison introduces the *metriplectic formalism* as an extension of the Hamiltonian

formalism in such a way that it includes a dissipation while the essence of a conserved quantity is not lost. It couples Poisson brackets, coming from the Hamiltonian symplectic formalism, with metric brackets, coming from out-of-equilibrium thermodynamics (see also [Mor84b], [Mor98]) and [CM20]). This formalism is able to describe systems with both Hamiltonian and dissipative components, and to model friction, electric resistivity, collisions and more.

In [MM17], Materassi and Morrison use the metriplectic formalism to effectively represent dissipative models in various contexts such as biophysics and geophysics. Their construction builds in asymptotic convergence to a pre-selected equilibrium state. In practice, this means that the angular velocity of the particle affected by dissipation is driven to a fixed angle, while the kinetic energy is conserved in the process.

Following these ideas, in this article we make use of Hamilton's equations to model a system which is dissipative in the classical sense. The original idea is that we do not rearrange the conservative Hamilton's equations but, instead, we introduce a singularity at the level of the symplectic structure of the manifold, which we equip with a twisted b -symplectic form.

Organization of this article. In Section 2 we give a crash course on b -symplectic geometry. In Section 3 we introduce the new model for fluids with dissipation based on the twisted b -symplectic structure. We start with the one-dimensional case and the linear potential, which provides an analogue of Stokes' Law, and we extend it to higher dimensions and more general potentials. We see that the Reynolds number can be related with the modular weight of the b -symplectic form and we prove that the dynamics of our model does not come from a cotangent lift of a group action. In Section 4 we consider time-dependent singular models in which friction arises from a re-scaling of time.

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2. PRELIMINARIES

2.1. b -Symplectic geometry. A symplectic manifold is a manifold M which admits a *symplectic 2-form* ω which is closed and non-degenerate. Given a function H over a symplectic manifold, called Hamiltonian, it is useful to consider its associated *Hamiltonian flow*, which is the flow of the vector field X defined by $\iota_X \omega = -dH$. The existence and uniqueness of X and its flow is a consequence non-degeneracy of the symplectic form.

In physics, the usual and more general formalism used to study dynamics is Poisson geometry ([JM99]). Poisson manifolds are generalizations of symplectic manifolds in which the symplectic form ω is replaced by a bivector Π . Indeed, a symplectic form ω in a symplectic manifold (M, ω) may be seen as a smooth map from the space of vector fields $\mathfrak{X}(M)$ to the space of 1-forms $\Omega^1(M)$. Among the large class of Poisson manifolds we find b -symplectic manifolds, that can also be considered a wider class of manifolds that contains symplectic manifolds.

The basic definitions of b -symplectic geometry start with the notions of *b -manifold* (a pair (M, Z) where Z is a hypersurface in a manifold M) and *b -map* (a map $f : (M_1, Z_1) \rightarrow (M_2, Z_2)$ between b -manifolds with f is transverse to Z_2 and $Z_1 = f^{-1}(Z_2)$) and *b -vector field* (a vector field on M which is tangent to Z at all points of Z).

Let (M^n, Z) be a b -manifold. If x is a local defining function for Z on an open set $U \subset M$ and (x, y_1, \dots, y_{n-1}) is a chart on U , then the set of b -vector fields on U is a free $C^\infty(M)$ -module with

basis

$$\left(x \frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right).$$

There exists a vector bundle associated with this module called the *b-tangent bundle* and denoted by bTM . The *b-cotangent bundle* ${}^bT^*M$ of M is defined to be the vector bundle dual to bTM .

For each $k > 0$, let ${}^b\Omega^k(M)$ denote the space of sections of the vector bundle $\Lambda^k({}^bT^*M)$, called *b-de Rham k-forms*. For any defining function f of Z , every b-de Rham k -form can be written as

$$(2) \quad \omega = \alpha \wedge \frac{df}{f} + \beta, \quad \text{with } \alpha \in \Omega^{k-1}(M) \text{ and } \beta \in \Omega^k(M).$$

A special class of closed b-de Rham 2-forms is the class of *b-symplectic forms* as defined in [GMP14]. It contains forms with singularities and can be introduced formally for b-symplectic manifolds, making it possible to extend the symplectic structure from $M \setminus Z$ to the whole manifold M .

Definition 2.1 (*b-symplectic manifold*). Let (M^{2n}, Z) be a b-manifold and $\omega \in {}^b\Omega^2(M)$ a closed b-form. We say that ω is *b-symplectic* if ω_p is of maximal rank as an element of $\Lambda^2({}^bT_p^*M)$ for all $p \in M$. The triple (M, Z, ω) is called a *b-symplectic manifold*.

2.2. b-cotangent lifts. The cotangent bundle of a smooth manifold M is naturally equipped with a symplectic structure, since there is always an intrinsic canonical linear form λ on T^*M defined by

$$\langle \lambda_p, v \rangle = \langle p, d\pi_p v \rangle, \quad p = (m, \xi) \in T^*M, v \in T_p(T^*M),$$

where $d\pi_p : T_p(T^*M) \rightarrow T_mM$ is the differential of the canonical projection at p . In local coordinates (q_i, p_i) , the form is written as $\lambda = \sum_i p_i dq_i$ and is called the *Liouville 1-form*. Its differential $\omega = d\lambda = \sum_i dp_i \wedge dq_i$ is a symplectic form on T^*M .

For b-symplectic manifolds, there are two natural choices for the singular Liouville form, each of them giving a different symplectic form.

$$(1) \text{ Non-twisted 1-form, } \lambda = \frac{c}{q_1} p_1 dq_1 + \sum_{i=2}^n p_i dq_i \text{ and } \omega = \frac{c}{q_1} dp_1 \wedge dq_1 + \sum_{i=2}^n dp_i \wedge dq_i$$

$$(2) \text{ Twisted 1-form, } \lambda = c \log(p_1) dq_1 + \sum_{i=2}^n p_i dq_i \text{ and } \omega = \frac{c}{p_1} dp_1 \wedge dq_1 + \sum_{i=2}^n dp_i \wedge dq_i$$

The non-twisted symplectic form assumes the discontinuity is at the base (the transversal hypersurface Z is given by $q_1 = 0$), while the twisted symplectic form assumes the discontinuity is at the fiber (the transversal hypersurface Z is given by $p_1 = 0$). The constant c is called the *modular weight*.

The cotangent lift to the cotangent bundle is defined in the following way.

Definition 2.2. Let $\rho : G \times M \rightarrow M$ be a group action of a Lie group G on a smooth manifold M . For each $g \in G$, there is an induced diffeomorphism $\rho_g : M \rightarrow M$. The *cotangent lift* of ρ_g , denoted by $\hat{\rho}_g$, is the diffeomorphism on T^*M given by

$$\hat{\rho}_g(q, p) := (\rho_g(q), ((d\rho_g)_q^*)^{-1}(p)), \quad (q, p) \in T^*M$$

which makes the following diagram commute (see also Figure 1):

$$\begin{array}{ccc} T^*M & \xrightarrow{\hat{\rho}_g} & T^*M \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{\rho_g} & M \end{array}$$

Given a diffeomorphism $\rho : M \rightarrow M$, its cotangent lift preserves the Liouville 1-form λ . As a consequence the cotangent lift $\hat{\rho}_g$ also preserves the symplectic form of T^*M . In the twisted case, the twisted b -cotangent lift preserves the twisted 1-form $\lambda = c \log(p_1) dq_1 + \sum_{i=2}^n p_i dq_i$ and the twisted b -symplectic form $\omega = \frac{c}{p_1} dp_1 \wedge dq_1 + \sum_{i=2}^n dp_i \wedge dq_i$.

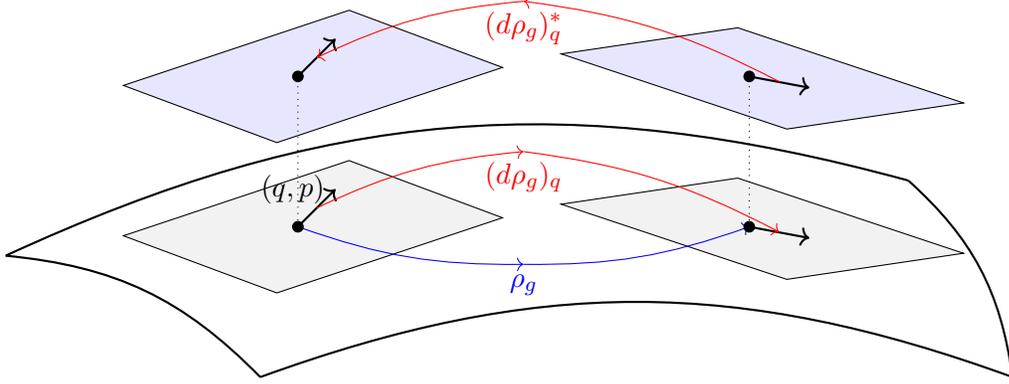


FIGURE 1. The cotangent lift of an action ρ_g acts on the cotangent bundle T^*M .

Proposition 2.3 (Kiesenhofer-Miranda, [KM17]). *Given a group action $\rho : G \times M \rightarrow M$ on a smooth manifold, the twisted b -cotangent lift $\hat{\rho}$ is b -Hamiltonian with moment map $\mu : T^*M \rightarrow \mathfrak{g}^*$ given by*

$$\langle \mu(p), X \rangle := \langle \lambda_p, X^\#|_p \rangle = \langle p, X^\#|_{\pi(p)} \rangle,$$

where $p \in T^*M$, X is an element of the Lie algebra \mathfrak{g} and $X^\#$ denotes the fundamental vector field of X generated by the action on T^*M on M .

Moreover, for a toric action the moment map of the lifted action with respect to the twisted b -symplectic form $\omega = \frac{c}{p_1} d\theta_1 \wedge dp_1 + \sum_{i=2}^n d\theta_i \wedge dp_i$ is given by $\mu = (c \log |p_1|, p_2, \dots, p_n)$.

2.3. Cotangent models for integrable systems. The dynamics of an integrable system $F = (f_1, \dots, f_n)$ is explained by the classical Arnold-Liouville-Mineur Theorem at the regular points, namely, at the points of the manifold where the differential $dF = (df_1, \dots, df_n)$ is not singular. This theorem was restated by Kiesenhofer and Miranda in [KM17] revealing that at a semilocal level the regular leaves are equivalent to a completely toric cotangent lift model.

Theorem 2.4 (Kiesenhofer-Miranda, [KM17]). *Let $F = (f_1, \dots, f_n)$ be an integrable system on a symplectic manifold (M, ω) . Then, semilocally around a regular Liouville torus, the system is equivalent to the cotangent model $(T^*\mathbb{T}^n)_{can}$ restricted to a neighbourhood of the zero section $(T^*\mathbb{T}^n)_0$ of $T^*\mathbb{T}^n$.*

Cotangent lifts arise naturally in physical problems, and the link between integrable systems and cotangent models is clear in view of Theorem 2.4. For the singular cases, and, in particular, for the b -integrable systems, the cotangent models can be made explicit in terms of action-angle coordinates.

Theorem 2.5 (Kiesenhofer-Miranda-Scott [KMS16]). *Suppose (M, Z, ω, F) is a b -integrable system, and let $m \in Z$ be a regular point of F for which the integral manifold containing m is compact, i.e. a Liouville torus. Then, there exists an open neighbourhood U of the torus and coordinates $(\theta_1, \dots, \theta_n, p_1, \dots, p_n) : U \rightarrow \mathbb{T}^n \times B^n$ such that*

$$(3) \quad \omega|_U = \sum_{i=1}^{n-1} dp_i \wedge d\theta_i + \frac{c}{p_n} dp_n \wedge d\theta_n,$$

where the coordinates p_1, \dots, p_n depend only on F and the number c is the modular period of the component of Z containing m .

3. THE TWISTED b -SYMPLECTIC MODEL FOR DISSIPATION

In this section, we see that b -symplectic geometry offers a way to model, in a Hamiltonian fashion, a particle moving in a fluid with viscosity. In particular, we construct an example that uses the twisted b -symplectic form in the cotangent bundle of \mathbb{R}^2 . This example gives the precise equation of the friction force applied to a small spherical particle moving through a viscous fluid in one dimension, the so-called Stokes' Law. We also generalize this model to higher dimensions and to configuration spaces different from \mathbb{R}^n .

Take in $M = \mathbb{R}$ and $T^*M \cong \mathbb{R}^2$ and consider the Hamiltonian

$$(4) \quad H(q, p) = \frac{p^2}{2} + f(q),$$

which corresponds to the energy of a massive particle subject to some potential $f(q)$. The Hamilton's equations derived from $\iota_{X_H}\omega = -dH$, i.e., from the standard symplectic form $\omega = dp \wedge dq$, would provide the following system, which models the main toy models in classical mechanics:

$$(5) \quad \begin{cases} \dot{q} = p \\ \dot{p} = -\frac{\partial f}{\partial q} \end{cases} .$$

But, more interestingly, the Hamilton's equations derived from the twisted b -symplectic form

$$\omega = \frac{1}{p} dp \wedge dq,$$

namely, from $\iota_{X_H}\omega = -dH$ are:

$$(6) \quad \begin{cases} \dot{q} = p^2 \\ \dot{p} = -p \frac{\partial f}{\partial q} \end{cases} .$$

For $p = 0$, Hamilton's equations coming from the twisted b -symplectic form, namely, system (6), give no dynamics. Hence, we can reduce the study to $p > 0$, since for $p < 0$ it is symmetric.

Differentiating the first equation and substituting into the second one, we find

$$(7) \quad \ddot{q} = -2\dot{q} \frac{\partial f}{\partial q},$$

which is a second order ODE depending on q . Notice that, although we have associated q to the position coordinate, \dot{q} is not equal to p but to p^2 . However, we can still think of $p = \sqrt{\dot{q}}$ as a modified physical momentum, since it is an increasing function of \dot{q} . Taking into account this point of view, we can obtain different models of dynamics for different types of potential $f(q)$.

3.1. A new model of Hamiltonian dissipation. Stokes' Law as a twisted model. A natural choice for the potential $f(q)$ is a function of linear type. This simple model already gives an original way of considering dissipation as a b -symplectic model, as the following result proves.

Theorem 3.1 (Dissipation as a twisted singular cotangent model). *Consider the twisted b -symplectic model in $T^*\mathbb{R}^2$, given by Equation 6. The particular case $f(q) = \frac{\lambda}{2}q$ is precisely the model of a spherical particle moving in a fluid with viscosity and suffering a friction proportional to its velocity, i.e., the Stokes' Law.*

Proof. Consider the particular case $f(q) = \frac{\lambda}{2}q$, with $\lambda > 0$, in the case of the Hamilton's equations coming from the twisted b -symplectic form, namely, system (6). In this case, Hamilton's equations are

$$(8) \quad \begin{cases} \dot{q} = p^2 \\ \dot{p} = -\frac{\lambda}{2}p \end{cases} .$$

And the corresponding second order ODE becomes

$$(9) \quad \ddot{q} = -\lambda\dot{q}.$$

This is exactly the situation of a free massive particle moving in one dimension and affected by viscous friction. In fact, Stokes' Law (Equation 10) is precisely describing the same case, appearing in the study of non-ideal fluids:

$$(10) \quad F = 6\pi\mu Rv,$$

where F is the frictional force, μ is the dynamic viscosity, R is the radius of the particle and v is the flow velocity relative to the object (or minus the object velocity relative to the flow). Stokes' Law essentially computes the magnitude of the drag force that acts against the particle motion and slows it. It says that this force is proportional to the velocity of the particle with respect to the fluid and with opposite sign.

Combining physical constants, assuming that the force F is proportional to the acceleration ($F \propto \ddot{q}$) and that the velocity v is \dot{q} , we deduce that equation 9 is equivalent to Stokes' Law. \square

Remark 3.2. In the classical symplectic setting, the particular case $f(q) = \frac{\lambda}{2}q$ in 6, with $\lambda > 0$, gives rise to the dynamics of a linear motion with constant acceleration (of $\frac{\lambda}{2}$). It is, for instance, the model for the free fall of a particle subject to a one-dimensional constant gravity field. Notice that there is no loss of energy of the system.

3.1.1. *Description of the dynamics.* From the point of view of dynamical systems, the phase portrait in the (p, q) -plane of (6) is highly similar to the phase portrait of the standard system (5), since the vectors (\dot{q}, \dot{p}) of the two systems are proportional by a p factor at each point. The essential difference between both systems is found at $p = 0$, where in (6) new punctual orbits appear and "break" the orbits that were crossing the horizontal axis $p = 0$ transversally in (5). Besides, orbits in the lower plane $p < 0$ change direction in (6). See Figure 2 for the phase space representation of the particular case $f(q) = \frac{\lambda}{2}q$.

The dynamical evolution of a physical system driven by the Hamiltonian $H(q, p) = \frac{1}{2}p^2 + \frac{\lambda}{2}q$ and the standard symplectic form $\omega = dp \wedge dq$ is really different from the dynamical evolution of a physical system governed by the same Hamiltonian but taking the twisted b -symplectic form $\omega = \frac{1}{p}dp \wedge dq$.

In the standard case orbits are parabolic everywhere (see the phase portrait on the left of Figure 2). They are of the form $q = -p^2 + c$, with c a constant. The evolution of a particle in this system is unbounded and for any initial conditions we have that $q, p \rightarrow_{t \rightarrow \infty} -\infty$. This is the model of a massive particle moving in an infinite one-dimensional well, affected by a constant force field and with no friction.

In the twisted b -symplectic case there are two types of orbits. On the one hand, there are fixed points for $p = 0$ and any q . On the other hand, there are half-parabolas of the same form $q = -p^2 + c$ at each side of the horizontal axis. The evolution of a particle starting either in the upper or in

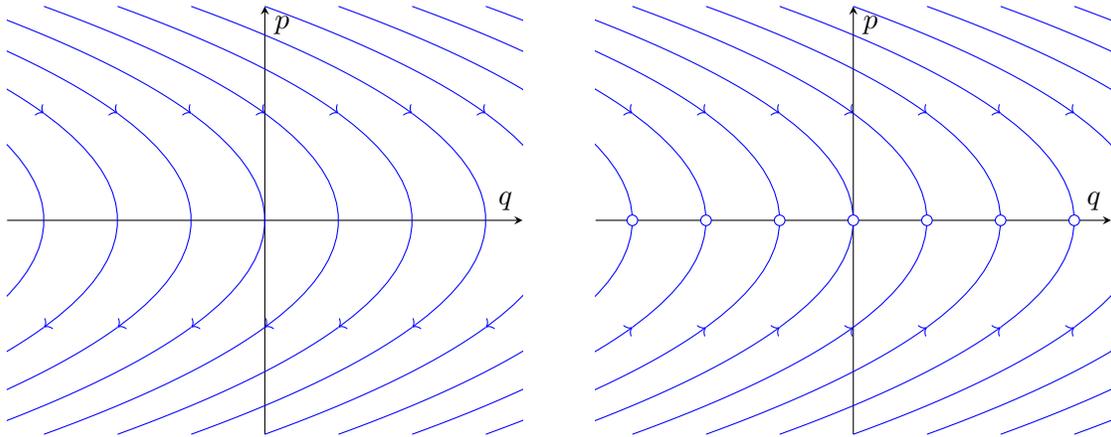


FIGURE 2. Some orbits in the phase spaces of (5) On the left and (6) on the right for $f(q) = \frac{\lambda}{2}q$.

the lower plane is similar, in both cases it will approach as much as we want to $p = 0$ and to a fixed $q = q_0$ greater than the initial q . Nevertheless, at finite time the only particles that can be found at $p = 0$ are the ones that already started there. This has physical sense, since the force is acting proportionally to the speed of the particle and in the opposite direction. Then, a particle with non-zero initial velocity will always be slowing down but it will never stop completely because the acting force also decreases in proportion.

The nature of the trajectories in both systems is also very different. In Figure 3 we can see some trajectories corresponding to both the classical Hamilton's equations and the twisted b -symplectic Hamilton's equations coming from the same Hamiltonian $H(q, p) = \frac{1}{2}p^2 + \frac{\lambda}{2}q$.

We can see that the trajectory of a particle under the classical model (5) is driven by $q(t) = -\frac{\lambda}{4}t^2 + c_1t + c_0$. It depends on the constants c_0, c_1 (equivalently, on the starting q and p) but, for any initial conditions, it ends with $q(t), p(t) \rightarrow_{t \rightarrow \infty} -\infty$. This corresponds to the aforementioned one-dimensional "free fall" of a particle in a constant force field.

On the other hand, the trajectory $q(t)$ of a particle under the twisted model, i.e., that solves system (6) is of the form $q(t) = d_0 - \frac{d_0^2}{\lambda}e^{-\lambda t}$. Hence, the particle does not go to $-\infty$ but has a limit at a fixed $q = d_0$ greater than the initial $q(0)$, no matter which initial conditions are chosen.

What we have observed is exceptional because friction is a non-conservative force and, while it cannot be described by the usual basic Hamiltonian setup, it can be described with the twisted b -symplectic setting. In this setting, the critical hyper-surface in our example corresponds to zero velocity or momentum. This is physically consistent with the fact that viscous friction alone cannot bring a particle to zero velocity in finite time.

This model, using a general viscous integral f could bring complex behaviours. Now, it is possible to use Hamiltonian tools to study it, for instance for Hamiltonian simulations. The limit of this model is that it is one-dimensional and that no additional force can be added. These general models seem, up to now, too viscous to work in such a setup.

3.2. The higher-dimensional linear case and the Reynolds number. We have introduced the one-dimensional model of the Stokes' Law using the twisted b -symplectic setting, but we can consider higher-dimensional models. The most direct generalization is to extend the Hamiltonian to $T^*\mathbb{R}^n$ in the following way:

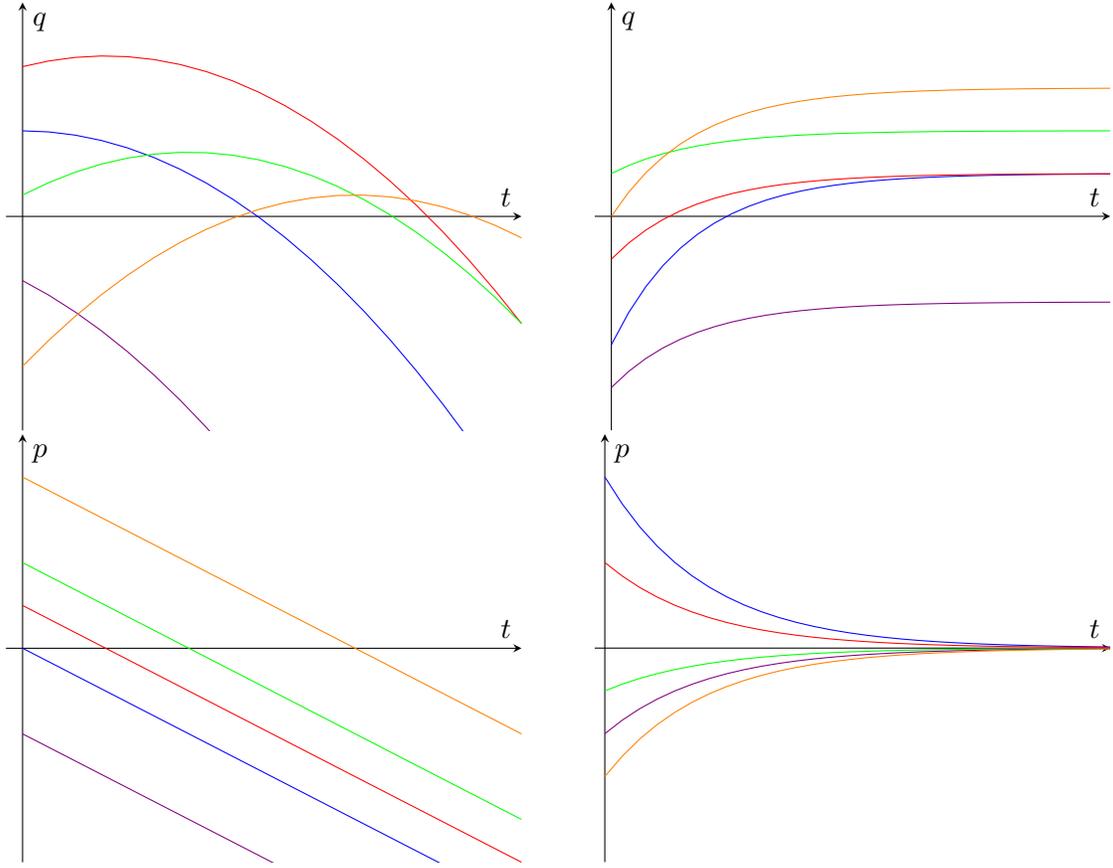


FIGURE 3. On the left, some trajectories $q(t)$ and $p(t)$ given by the classical Hamilton's equations (5). On the right, some trajectories $q(t)$ and $p(t)$ given by the twisted b -symplectic Hamilton's equations (6). Both are for $f(q) = \frac{\lambda}{2}q$.

$$H(q_1, \dots, q_n, p_1, \dots, p_n) = \frac{1}{2} \sum_{i=1}^n p_i^2 + \frac{\lambda}{2} q_1.$$

The dynamics governed by this Hamiltonian together with the twisted b -symplectic form $\omega = c \frac{1}{p_1} dp_1 \wedge dq_1 + \sum_{i=2}^n dp_i \wedge dq_i$ are the following. In the direction of q_1 , the motion suffers dissipation and velocity tends to zero, as it behaves by Stokes' Law. In the other directions, the motion corresponds to that of a free particle. As a consequence, the evolution of the trajectory is a curve that starts with some initial direction given by a velocity (v_1, \dots, v_n) in \mathbb{R}^n and tends to a motion restricted to the directions $(0, v_2, \dots, v_n)$.

In this case, the modular weight c appearing in the twisted b -symplectic form is giving a measure of the predominance of the singular term over the regular terms. In a way, the modular weight is also a weight of the importance of the direction in which there is dissipative friction relative to the rest of directions. In the fluid context, the Reynolds number, which is the ratio of inertial forces to viscous forces within a fluid, accounts for a similar concept. It is a scalar used in fluid systems in which the viscosity affects the velocities or the flow pattern.

Definition 3.3 (Reynolds number). The *Reynolds number* Re is the ratio of inertial forces to viscous forces in a fluid. It is defined as

$$Re = \frac{\rho v d}{\mu},$$

where ρ is the density of the fluid, v the velocity of the fluid, d the diameter or characteristic length of the system and μ the dynamic viscosity of the fluid.

The Reynolds number quantifies the relative importance of these two types of forces and a lower Reynolds number viscous forces are dominant. In practice, it is used to determine whether a fluid is in laminar or turbulent flow. It is accepted that a Reynolds number less than 2100 indicates laminar flow while a Reynolds number greater than 2100 indicates turbulent flow.

Remark 3.4. The modular weight c is showing the importance of the dissipative component compared to the other directions in which there is free motion. Then, it can be associated with an analogue of the Reynolds number Re . A high modular weight c implies that there is a big influence of the dissipation by viscosity in the overall system, which is equivalent to a low Re .

Another option would be to consider dissipation in one direction of motion in manifolds which are not \mathbb{R}^n , such as a particle moving over a cylinder $S^1 \times \mathbb{R}$. Figure 4 shows the trajectories of a particle in $S^1 \times \mathbb{R}$ under the Hamiltonians

$$H_1(\theta, q, p_\theta, p_q) = \frac{p_\theta^2 + p_q^2}{2} + \frac{\lambda}{2}q$$

and

$$H_2(\theta, q, p_\theta, p_q) = \frac{p_\theta^2 + p_q^2}{2} + \frac{\lambda}{2}\theta,$$

together with the twisted b -symplectic forms $\omega_1 = c \frac{1}{p_q} dp_q \wedge dq + dp_\theta \wedge d\theta$ and $\omega_2 = c \frac{1}{p_\theta} dp_\theta \wedge d\theta + dp_q \wedge dq$ respectively.

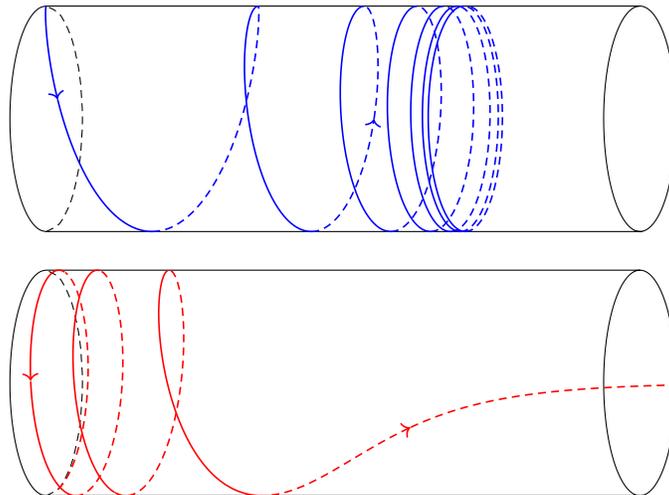


FIGURE 4. Two kinds of dissipating trajectories in the cylinder. In the first one, the singularity is in the \mathbb{R} component of the fiber, while in the second it is in the S^1 component of the fiber.

3.3. The pure quadratic potential. Consider now a quadratic potential of the type $f(q) = \frac{\lambda}{4}q^2$. The dynamical evolution of a physical system driven by the Hamiltonian $H(q, p) = \frac{1}{2}p^2 + \frac{\lambda}{4}q^2$ and the standard symplectic form $\omega = dp \wedge dq$ corresponds to an harmonic oscillator. Explicitly, we obtain the Hamilton's equations corresponding to the classical simple harmonic oscillator, namely:

$$(11) \quad \begin{cases} \dot{q} = p \\ \dot{p} = -\frac{\lambda}{2}q \end{cases} .$$

In this standard case, orbits are circles in the phase space everywhere (see the phase portrait on the left of Figure 5). They are of the form $2p^2 + \lambda q^2 = c$, with c a constant. The position of a particle in this system is bounded and so is its velocity for any initial conditions, since $q(t)$ and $p(t)$ are sinusoids. This corresponds exactly to the classical model of an harmonic oscillator, which is natural given a quadratic potential.

However, and more interestingly, the same Hamiltonian together with the twisted b -symplectic form $\omega = \frac{1}{p}dp \wedge dq$ gives another dynamics. In particular, the twisted b -symplectic form gives the following motion equations:

$$(12) \quad \begin{cases} \dot{q} = p^2 \\ \dot{p} = -\frac{\lambda}{2}pq \end{cases} .$$

On the right of Figure 5 we can see the phase space representation of the orbits of this system. On the left of Figure 6 we can see some trajectories $q(t)$ and $p(t)$.

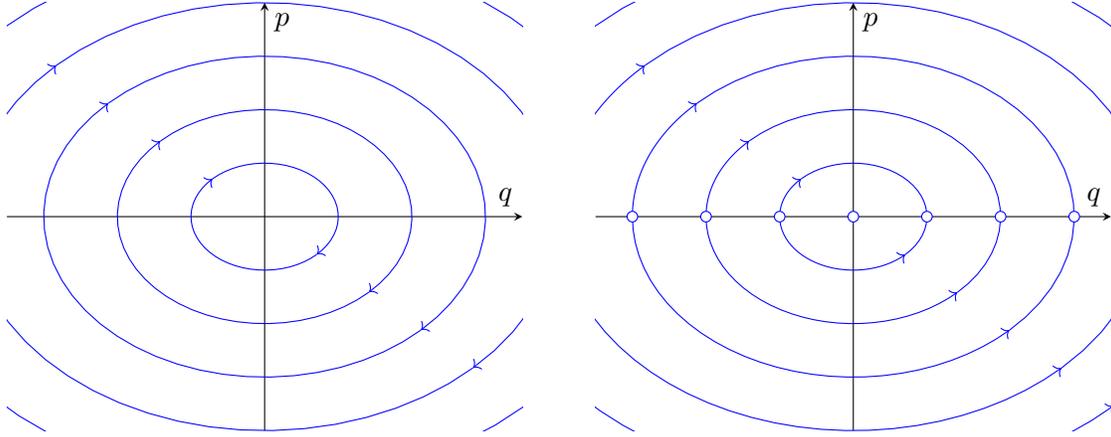


FIGURE 5. Some orbits in the phase spaces of (11) On the left and (12) On the right for $f(q) = \frac{\lambda}{4}q^2$.

The equivalent second order ODE is:

$$(13) \quad \ddot{q} = -\lambda\dot{q}q.$$

Equation 13 is a highly non-linear equation which has the following solution for the trajectory:

$$q(t) = \frac{c_1}{\sqrt{\lambda}} \tanh \frac{c_1\sqrt{\lambda}}{2}t + c_2,$$

with c_1 and c_2 depending on the initial conditions. On the right of Figure 6, we can see some trajectories $q(t)$ and $p(t)$ for different values of c_1 and c_2 .

Again, as in the linear case, $p(t) \rightarrow_{t \rightarrow \pm\infty} 0$. On the other hand, the position $q(t)$ of a particle under this potential is bounded on the range $(-c_1, c_1)$, which makes it different from the linear potential case.

In this setting, we can think the particle is really enclosed in a uni-dimensional container and goes from one end to the other as time passes. It does so by starting to separate slowly from one border, then accelerating fast to pass over the mid space of the container, and then slowing again to arrive at the other border.

This quadratic potential, then, models a particle crossing the interior of some box at a slow speed when it is near each edge and at a high speed in the middle.

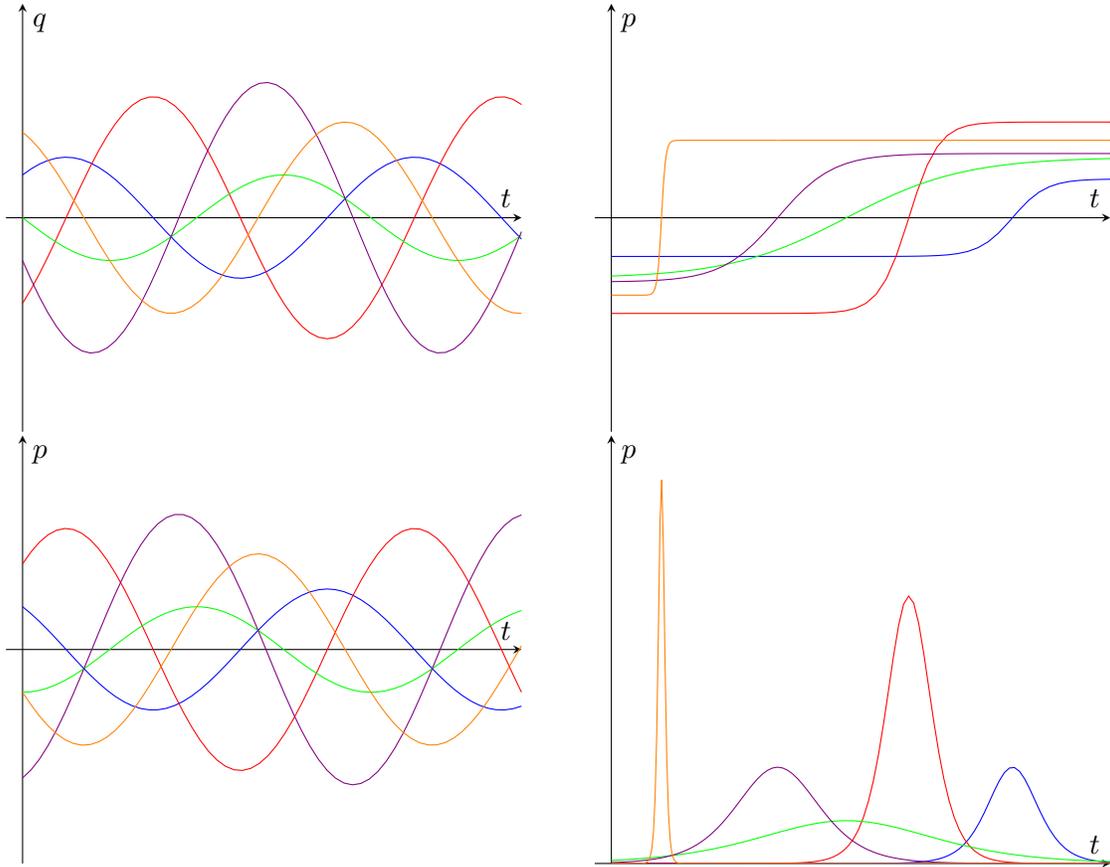


FIGURE 6. On the left, some trajectories $q(t)$ and $p(t)$ given by the classical Hamilton's equations (5). On the right, some trajectories $q(t)$ and $p(t)$ given by the twisted b -symplectic Hamilton's equations (6). Both are for $f(q) = \frac{\lambda}{4}q^2$.

3.4. The general quadratic potential. It is natural to couple the pure quadratic potential $f(q) = \frac{\lambda}{4}q^2$ with the linear case studied previously. Indeed, consider a physical particle moving in a fluid and then getting a Stokes force. Now, suppose that your fluid has a non-uniform viscosity. This is for instance the case whenever you have a gradient of temperature as the viscosity usually depends on the temperature. For small fluctuations, to first order in position, the viscosity writes $\eta = \eta_0(1 + \alpha q)$. Therefore, potential $f(q)$ accounting for the drag coefficient, becomes $f(q) = \lambda(1 + \alpha q)$ and the equation of motion is

$$\ddot{q} = -\lambda(1 + \alpha q)\dot{q}.$$

This equation includes both the linear regime (which is expected to be dominant) and the quadratic regime as a perturbation. The associated Hamiltonian is

$$H(p, q) = \frac{p^2}{2} + \frac{\lambda}{2}q \left(1 + \alpha \frac{q}{2}\right)$$

and it is the most natural generalization of the linear regime from the physical point of view.

3.5. General dynamics of the twisted b -symplectic model. With the previous illustrative examples in mind, the interpretation of the twisted b -symplectic model in the general case is straightforward. Recall that the Hamilton's equations derived from the twisted b -symplectic form $\omega = \frac{1}{p}dp \wedge dq$ are:

$$\begin{cases} \dot{q} = p^2 \\ \dot{p} = -p \frac{\partial f}{\partial q} \end{cases} .$$

If (q, p) are assumed to be the coordinates of the phase space of a mechanical system, the behaviour of a particle under H is clearly conditioned by the singularity of the system at $p = 0$.

If a particle starts at any $p \neq 0$, it will follow a trajectory that either tends to $p = 0$ or escapes from $p = 0$. In the first case, the most direct physical interpretation of this model is that of a decelerating motion, for instance the one encountered in a dissipative system. The second case can be interpreted just as the first one but reversed in time.

The implications of having the singularity at the fibers of the cotangent bundle extend further than it seems at first glance. The singularity determines an unreachable location in the fiber, i.e., that zero momentum is unreachable. But the momentum of the particle will tend there (or escape from there). As a consequence, the position of the particle is also indirectly conditioned by the singularity, since tending to zero momentum will cause a stabilization of the position.

The twisted b -symplectic model with the singular fiber at zero is, then, a physical model that can explain systems in which velocity decays and so does the change in position of the particle as a consequence.

Observe, however, that the dynamics of this model can not come from the cotangent lift of an action by the following argument. In the lowest dimensional case, suppose that there is an action $\rho : G \times \mathbb{R} \rightarrow \mathbb{R}$ with cotangent lift $\hat{\rho} : G \times T^*\mathbb{R} \rightarrow T^*\mathbb{R}$ and with moment map $H = \frac{p^2}{2} + f(q)$. By definition, the $\hat{\rho}_g(q, p)$ restricted to the base \mathbb{R} of $T^*\mathbb{R} \ni (q, p)$ is equal to $\rho_g(q)$ for any $g \in G, q \in \mathbb{R}$. Then, the restriction does not depend on the fiber coordinate p and the same happens for the $\frac{\partial}{\partial q}$ component of the infinitesimal generator of $\hat{\rho}_g(q, p)$. But this infinitesimal generator has to be of the form

$$X = p^2 \frac{\partial}{\partial q} - \frac{\partial f(q)}{\partial q} p \frac{\partial}{\partial p}$$

in order to satisfy $\iota_X \omega = -dH$, where ω is the twisted b -symplectic form $\frac{1}{p}dp \wedge dq$. The term $p^2 \frac{\partial}{\partial q}$ depends on the fiber coordinate p , which is a contradiction.

Another way to see it is the following. By Proposition 2.3, the twisted b -cotangent lift $\hat{\rho}$ is b -Hamiltonian and its moment map $\mu : T^*M \rightarrow \mathfrak{g}^*$ contains a logarithm term associated to the toric component of the action ρ . In Proposition 26 of [BKM18], it is proved that the action of G on the mapping torus Z always lifts to an action of a product group $S^1 \times H$ on a finite trivializing cover of Z , where H is compact and connected. G is necessarily of the form $G = (S^1 \times H)/\Gamma$ for a finite cyclic subgroup Γ . Hence, the moment map of the lifted action of a group action with non-vanishing modular weight [BKM18, MM22] has to include a term of the form $\mu = c \log |p|$, which is not compatible with the Hamiltonian $H = \frac{p^2}{2} + f(q)$. In higher dimensions, the model can not be a cotangent lift for the same reason.

4. TIME-DEPENDENT SINGULAR MODELS

In order to generalize this friction model to multiple dimensions, the key idea is to extend the configuration space Q to $Q \times \mathbb{R}$. The \mathbb{R} component in $Q \times \mathbb{R}$ describes the real time t while the dynamics inside the phase space is computed according to a curvilinear time s . This is conceptually the idea of the well-known method of characteristics in PDEs. After calculation of the trajectories, one only needs to project the trajectory on the space Q and read the time on the real axis. We require $\dot{t} > 0$ to be consistent and we denote q the position in Q and p the associated momentum. We also denote by E the conjugated variable associated with t , since the energy is the natural conjugate of time in physics.

To start, consider the Hamiltonian

$$(14) \quad H(p, q, t, E) = \frac{p^2}{2} + V(q, t) - E.$$

Assuming E is the energy of the system, one expects the preservation of the Hamiltonian (the conservation of $H = 0$) along the physical trajectory. We use the canonical symplectic form

$$(15) \quad \omega = \sum_i dq_i \wedge dp_i - dt \wedge dE.$$

The associated dynamics writes

$$(16) \quad \dot{q}_i = p_i \quad \dot{p}_i = -\frac{\partial V(q, t)}{\partial q_i}$$

$$(17) \quad \dot{t} = 1 \quad \dot{E} = \frac{\partial V(q, t)}{\partial t}$$

Therefore, the curvilinear coordinate is the real time: $s = t$. The particle follows the expected dynamics with a potential which may depend on time.

Now, to model friction, it is natural to consider adding to the Hamiltonian a factor depending on a friction coefficient λ . The friction will slow down the dynamics and thus t compared with s . However, the potential remains associated with real time and thus appears accelerated compared with the curvilinear time. Then, we consider the following Hamiltonian

$$(18) \quad H(p, q, t, E) = \frac{p^2}{2} + \frac{e^{2\lambda t}}{\lambda^2} V(q, t) - \frac{e^{\lambda t}}{\lambda} E$$

with the same canonical symplectic form. The associated dynamics writes

$$(19) \quad \dot{q}_i = p_i \quad \dot{p}_i = -\frac{e^{2\lambda t}}{\lambda^2} \frac{\partial V(q, t)}{\partial q_i}$$

$$(20) \quad \dot{t} = \frac{e^{\lambda t}}{\lambda} \quad \dot{E} = \frac{e^{2\lambda t}}{\lambda^2} \frac{\partial V(q, t)}{\partial t} + \frac{2e^{2\lambda t}}{\lambda} V(q, t) - e^{\lambda t} E$$

The two first terms describe the energy linked with the time-dependence of the potential. The last term describes the loss of energy caused by the viscous dissipation. The equation for t can be solved exactly: $t(s) = -\frac{\ln(s)}{\lambda}$. In particular, $ds = \lambda e^{-\lambda t} dt$. Let us now reconstruct the particle dynamics in real time:

$$(21) \quad \frac{dq_i}{dt} = \lambda e^{-\lambda t} \dot{q}_i = \lambda e^{-\lambda t} p_i \quad \frac{dp_i}{dt} = \lambda e^{-\lambda t} \dot{p}_i = -\frac{e^{\lambda t}}{\lambda} \frac{\partial}{\partial q_i} V(q, t)$$

and therefore

$$(22) \quad \frac{d^2 q_i}{dt^2} = -\lambda \frac{dq_i}{dt} - \frac{\partial}{\partial q_i} V(q, t)$$

which is the equation of a particle in a n -dimensional space with a viscous friction of coefficient λ and in a time-dependent potential $V(q, t)$.

The friction arises from an exponential re-scaling of time. Such a re-scaling is actually the source of singularity, and then singular geometry arises if we perform a change of variables from t to s in the symplectic form, using $s(t) = e^{-\lambda t}$ and $dt = -\frac{ds}{\lambda s}$. For convenience, we also redefine $E_s = E/\lambda$. We then obtain,

$$(23) \quad \omega = \sum_i dq_i \wedge dp_i + \frac{1}{s} ds \wedge dE_s$$

which is the non-twisted canonical b -symplectic form. In these coordinates the Hamiltonian becomes

$$(24) \quad H(p, q, t, E) = \frac{p^2}{2} + \frac{V(q, t(s))}{(\lambda s)^2} - \frac{E_s}{s}$$

which has a singularity of higher order. Indeed, it is a b^2 -function and not a b -function. Such a discrepancy of the degree of the singularity at the symplectic form and the total energy of the system (Hamiltonian) is not new (see [DKM17, MP22] for other examples).

Summing up, the Hamiltonian is simpler in these coordinates. But the main advantage is that the intrinsic time (the curvilinear coordinate) now corresponds to coordinate s . Indeed, the equations of motion now read as follows:

$$(25) \quad \dot{q}_i = p_i \quad \dot{p}_i = -\frac{1}{(\lambda s)^2} \frac{\partial V(q, t(s))}{\partial q_i}$$

$$(26) \quad \dot{s} = 1 \quad \dot{E} = \frac{\partial}{\partial s} \left(\frac{1}{(\lambda s)^2} V(q, t(s)) \right) + \frac{E_s}{s^2}$$

The coordinate s is now trivial and we may omit this dimension, leaving a standard Hamiltonian dynamics with a modified time-dependent potential. The real-time solution is finally obtained by undoing the change of variables $s(t) = e^{-\lambda t}$.

Remark 4.1 (Connection with magnetism). One could think to extend the singular models considered in this article to include the effects of an electromagnetic field on a charge. In that case, the configuration space would be \mathbb{R}^3 , with the electric potential function ϕ and a magnetic vector potential A . The corresponding electric and magnetic fields would be $E = \nabla\phi$ and $B = \nabla \times A$ respectively, both depending on the position $q \in \mathbb{R}^3$ of the particle. The force F acting on the particle would be the Lorentz force $F = e(E + v \times B)$, a function of both position and velocity.

The problem could then be studied in the Hamiltonian setting by identifying the tangent and the cotangent vectors via a fixed Riemannian metric. The vector potential A can be associated with the 1-form $\langle A, \cdot \rangle$, and the magnetic field B can be associated with the 2-form dA . The Maxwell equation $\nabla \cdot B = 0$ becomes the condition $dB = 0$ on the phase space (T^*Q, ω_B) , where ω_B is the sum of the canonical symplectic form ω_Q on T^*Q and the pullback of the 2-form B by the natural projection $\pi : T^*Q \rightarrow Q$, i.e., $\omega_B = \omega_Q + \pi^*B$.

The discussion on the multidimensional case would be still valid under the presence of a time-dependent magnetic potential $B = B_{ij}(q, t) dq_i \wedge dq_j$, and the subtlety in the case of dissipation is to adjust the speed of the magnetic field. Similarly to what has been done in this section, the recipe in this case would be given by the change $B \rightarrow \frac{e^{\lambda t}}{\lambda} B$. Nevertheless, since the magnetic field would no longer be closed, the method presented here would need further development to be convenient for magnetic dynamics.

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