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Rings of differential operators on singular varieties

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Abstract

This project will be centered in the study of rings of differential operators on singular varieties and the theory of D-modules. A D-module is defined as a module over the ring of differential operators on a $\Bbbk$-algebra.

We will start with the case of Weyl algebras which coincide with the ring of $\Bbbk$-linear differential operators on a polynomial ring $\Bbbk[x_1, \ldots, x_n]$. We are going to prove that this ring of differential operators is a simple Noetherian domain whose dimension is two times the dimension of the polynomial ring. Also, considering modules over this ring we will show Bernstein's inequality that give us a lower bound to the dimension of the module and, using this result, we will define holonomic D-modules.

Our goal is to study what good structural properties of the regular case can be extended in the singular one. We will use the results proved in the case of Weyl algebras to give a description of the ring of $\Bbbk$-linear differential operators on a finitely generated $\Bbbk$-algebra that can always be presented as a quotient $S = R/I$ of a polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$ by an ideal I. We will prove that those differential operators can be obtained in terms of the differential operators that preserve the ideal and differential operators on the Weyl algebra. And show that the ring of differential operators of a finitely generated $\Bbbk$-algebra do not have as much as good properties as Weyl algebras, however we will obtain some of them under several conditions. We will study modules over this kind of rings of differential operators and their dimension, proving a generalized Bernstein's inequality under some conditions.

Finally, we will apply these results to study the case of the ring of differential operators on a hyperplane arrangement and explain several methods to obtain a system of generators of this ring.

Keywords

Gelfand-Kirillov dimension, filtrations, Weyl Algebras, Differential Operators, Hyperplane Arrangements

Contents

Introduction

The main objects of this project are D-modules. We will denote by $D_{A|\Bbbk}$ the ring of $\Bbbk$-linear differential operators on a $\Bbbk$-algebra A. The simplest and most familiar example is given by the ring of $\Bbbk$-linear differential operators on the polynomial ring $A = \Bbbk[x_1, \ldots, x_n]$ and is known as the n-th Weyl algebra. It consists of the endomorphisms of A generated by multiplication by the coordinates x_i and taking partial derivatives ∂_{x_i}. Also in the case of Weyl algebras, we can think of D-modules as A-modules on which we can differentiate, in the sense that partial derivatives act on the modules. We will introduce Weyl algebras as our first example of the theory, but their study will also be useful later on to describe more abstract rings of differential operators.

More generally, there is a notion of ring of differential operators on any Noetherian ring A given by Grothendieck [Grothendieck, 1967]. Such ring is always, by definition, a subring of the endomorphisms of A. The key idea of this generalization is that one can define the order of a differential operator inductively using the commutator. A D-module is then defined as a module over the ring of differential operators on A.

Despite the advantage of Grothendieck's definition, working in such a general setting becomes impracticable. For example, one cannot obtain a good theory of D-modules neither an explicit description of $D_{A|\Bbbk}$ in such generality. In order to improve this situation it is wise to focus on a smaller class of rings. This was done by McConnell and Robson in the book [McConnell and Robson, 1988], where they gave a more concrete description of the ring of differential operators for the case of finitely generated $\Bbbk$-algebras. We will follow this advice in our study, and later further specialize to the case of hyperplane arrangements.

The origin of the theory of D-modules can be found in the independent works of Kashiwara [Kashiwara, 1970] and Bernstein [Bernstein, 1971]. The motivation behind Bernstein's approach was to solve a question proposed by I. M. Gelfand at the 1954 edition of the ICM regarding the analytic continuation of the complex zeta function. His solution is based on the existence of what is now known as Bernstein-Sato polynomials. This existence result is based on Bernstein's inequality which which give us a lower bound of the dimension of a module in terms of the dimension of the ring of differential operators on a polynomial ring. However, this inequality can be generalized to other $\Bbbk$-algebras such as the ring of holomorphic functions [Sato et al., 1972] and the ring of formal series [Björk, 1979]. Linked to this result one can define a central object of the theory of D-modules: the class of holonomic modules. This class of D-modules have the property that are of finite length as D-modules and finite-dimensional de Rham cohomology.

The theory of D-modules over singular rings has not been developed yet. Indeed, only a few examples of rings of differential operators can be found in the literature. For example, Stanley-Reisner rings [Eriksson, 1998, Tripp, 1997], semigroups algebras [Saito and Traves, 2004], the cubic curve

$$S = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3),$$

which was studied by Bernstein, I.M. Gelfand and S.I. Gelfand in [Bernstein et al., 1972], the ring of invariants of finite groups in characteristic zero [Levasseur and Stafford, 1989] and the case of hyperplane arrangements [Holm, 2002].

To conclude this introduction, we summarize the contents of this project. In Section 1, we will review several concepts of commutative algebra such as filtrations, gradings and different equivalent definitions of dimension of a $\mathbb{k}$-algebra or a module. We will also define the ring of $\mathbb{k}$-linear differential operators over a $\mathbb{k}$-algebra A in Section 1.3 and describe the differential operators of order zero and one over A.

In Section 2, we will center on studying Weyl algebras which is the $\mathbb{k}$-algebra generated by the variables $x_1, \ldots, x_n$ and the partial derivatives $\frac{\partial}{\partial_{x_1}}, \ldots, \frac{\partial}{\partial_{x_n}}$. We will prove in Section 2.1 that they coincide with the ring of differential operators over a polynomial ring $R = \mathbb{k}[x_1, \ldots, x_n]$. We will present in Section 2.2 two different filtrations on the Weyl algebra, one given by the order of the differential operators and another given by the total order known as Bernstein's filtration. Also, using this last filtration we will show in Section 2.3 some structural properties satisfied by the Weyl algebra. Finally, we will consider modules over the Weyl algebra in Section 2.4 and prove in Section 2.5 the Bernstein's inequality that give us a lower bound to the dimension of the module and using this result we will define holonomic D-modules.

In Section 3, we will use the results proved in Sections 1.3 and 2 to give a description of the ring $D_{S|\mathbb{k}}$ of $\mathbb{k}$-linear differential operators on a finitely generated $\mathbb{k}$-algebra that can always be presented as a quotient $S = R/I$ of a polynomial ring $R = \mathbb{k}[x_1, \ldots, x_n]$ by an ideal I. We will prove in Section 3.2 that those differential operators can be obtained in terms of the differential operators that preserve the ideal and the differential operators on the Weyl algebra. Similarly as in Section 2 we will give two filtrations on the ring of diffferential operators, one given by the order and the second one is a generalized Bernstein's filtration. We will see that the case of rings of differential operators on finitely generated $\mathbb{k}$-algebras will not have as much as good properties as Weyl algebras, however we will prove some of them under several conditions in Section 3.4. We will end this section studying modules over this ring of differential operators and its dimension in Sections 3.5 and 3.6.

In Section 4 we are going to focus on the case of hyperplane arrangements. In Section 4.1 we fix the basics of hyperplane arrangements. In Section 4.2 will study the ring of differential operators in this setting and in Section 4.3 we will explain several methods to obtain a system of generators of the ring of differential operators.

Finally, in Section 5 we give the conclusions of this work and some ideas for future work in the theory of D-modules.

1. Background in Commutative Algebra

The purpose of this project is to study (non-commutative) rings of differential operators over certain $\Bbbk$-algebras, such as polynomial rings or quotients of these. In this section we are going to review some definitions and results from commutative algebra that we will need in the rest of the project. All the results of this section are well known and can be found in full detail in [Atiyah and MacDonald, 1969], [Bourbaki, 1998] and [Eisenbud, 1995].

In Sections 1.1 and 1.2 we introduce, and sometimes generalize, concepts which are well-known in the commutative setting. More concretely, in Section 1.1 we introduce filtrations and gradings. Filtrations and gradings can be thought of as decompositions of a ring into smaller parts which are easier to understand. In particular, filtrations are useful to define the notion of dimension, which is the purpose of Section 1.2.

After that, we introduce rings of differential operators with examples in Sections 1.3 and 1.4. The simplest and best understood example of ring of differential operators is the Weyl algebra, see Definition 2.1, which consists of the usual differential operators on a polynomial ring $\Bbbk[x_1, \dots, x_n]$. We will introduce a more general version for $\Bbbk$-algebras due to Grothendieck in Section 1.3.

1.1 Filtrations and gradings

We start with the following setting: let $\Bbbk$ be a field and let A be an associative ring with unit, although it might not be commutative. We are going to give in this section a way to "decompose" a ring or a module via filtrations and gradings.

Definition 1.1. The *centre* of A is the subring $Z(A)$ of elements of A which commute with all the elements of A. In other words, $a \in Z(A)$ if and only if for every $b \in A$ we have that $ab = ba$.

Definition 1.2. A $\Bbbk$-*algebra* is a pair (A, φ) where A is a ring and $\varphi : \Bbbk \to A$ is a ring morphism such that $\varphi(\Bbbk) \subseteq Z(A)$. We can think of A as a left vector space over $\Bbbk$, and we shall define the *dimension of A over* $\Bbbk$, $\dim_{\Bbbk}(A)$, to be the dimension of A as a $\Bbbk$-vector space.

Definition 1.3. Let A and B be $\Bbbk$-algebras, $f : A \to B$ is a *morphism of* $\Bbbk$-*algebras* if $f(a \cdot a') = f(a) \cdot f(a')$, $f(a + a') = f(a) + f(a')$ and $f(\lambda a) = \lambda f(a)$ for all $a, a' \in A$ and $\lambda \in \Bbbk$.

Definition 1.4. An *endomorphism* of a $\Bbbk$-algebra A is a homomorphism $f : A \to A$. We will denote by $End_{\Bbbk}(A) \coloneqq \{f : A \to A \text{ homomorphism of } \Bbbk\text{-algebras}\}$.

Definition 1.5. A *filtration* $\mathcal{F} = (F_n)_{n \in \mathbb{Z}_{\geq 0}}$ of a $\Bbbk$-algebra A is a sequence of abelian groups

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots$$

indexed by the non-negative integers, such that

1. $1 \in F_0$.
2. $F_i F_j \subseteq F_{i+j}$ for each $i, j \in \mathbb{Z}_{\geq 0}$.
3. $\bigcup_{n\in\mathbb{Z}_{\geq 0}} F_n = A$.

If $\mathcal{F}$ is a filtration on A, we say that $(A, \mathcal{F})$ is a *filtered* $\Bbbk$-*algebra.*

Remark 1.6. In this work we are going to be interested in filtrations given either by a sequence of finitely dimensional $\Bbbk$-vector spaces or a sequence of finitely generated A-modules.

In the case of finitely generated $\Bbbk$-algebras we can consider a special filtration:

Definition 1.7. Let A be a finitely generated $\Bbbk$-algebra. We say that a filtration $\mathcal{F}$ on A is *standard* if there exists a generating set $v_1, \dots, v_l$ of A such that $F_1 = \Bbbk\langle v_1, \dots, v_l\rangle$ and for each $i \in \mathbb{Z}_{\geq 0}$, $F_i = (F_1)^i$, the $\Bbbk$-subspace generated by monomials in $v_1, \dots, v_l$ of degree $\leq i$.

Definition 1.8. Given a filtered $\Bbbk$-algebra $(A, \mathcal{F})$ and a left A-module M, a filtration $\mathcal{G} = (G_m)_{m\in\mathbb{Z}_{\geq 0}}$ on M compatible with $\mathcal{F}$ is a sequence of abelian groups

$$G_0 \subseteq G_1 \subseteq G_2 \subseteq \dots$$

indexed by the non-negative integers, and such that

1. $F_i G_j \subseteq G_{i+j}$ for every $i, j \in \mathbb{Z}_{\geq 0}$.
2. $\bigcup_{n\in\mathbb{Z}_{\geq 0}} G_n = M$.

If $\mathcal{G}$ is a filtration on M compatible with $\mathcal{F}$, we say that $(M, \mathcal{G})$ is an $(A, \mathcal{F})$-module.

Remark 1.9. As before, we are going to consider the case that we have a sequence of finitely dimensional $\Bbbk$-vector subspaces of M or finitely generated A-submodules of M, depending on the situation.

Definition 1.10. Let $(A, \mathcal{F})$ be a filtered $\Bbbk$-algebra and M a finitely generated left A-module. We say that a filtration $\mathcal{G}$ on M compatible with $\mathcal{F}$ is *standard* if there exists a generating set $m_1, \dots, m_l$ of M such that $G_i = F_i\langle m_1, \dots, m_l\rangle$ for all $i \in \mathbb{N}$.

By simplicity we are going to write $A = (A, \mathcal{F})$ and $M = (M, \mathcal{G})$ when there is no risk of confusion.

Definition 1.11. If M is a filtered A-module and $N \subseteq M$ is a submodule then

- $N_n \coloneqq M_n \cap N$ defines a filtration that we call the *submodule filtration* on N.
- $(M/N)_n \coloneqq (M_n + N)/N$ defines a filtration that we call the *quotient filtration* on M/N.

Definition 1.12. A *grading* on a ring A is a sequence of additive subgroups $\mathcal{A} = (A_n)_{n\in\mathbb{Z}_{\geq 0}}$ such that $A = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} A_n$ and $A_m A_n \subset A_{m+n}$ for all $m, n \in \mathbb{Z}_{\geq 0}$. When such a grading has been chosen we say that the ring A is *graded*, often without mentioning the grading $\mathcal{A}$.

Example 1.13. A polynomial ring $A = \Bbbk[x_1, \dots, x_n]$ is graded by taking the grading $\mathcal{A} = (A_m)_{m\in\mathbb{Z}_{\geq 0}}$ as the $\Bbbk$-submodules generated by the monomials of degree m:

$$A_m = \{x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n} \mid \alpha_1 + \dots + \alpha_n = m\}.$$

Definition 1.14. Given a graded ring $(A, \mathcal{A})$ and an A-module M, we say that M is (compatibly) *graded* if there is a sequence of additive subgroups $\mathcal{M} = (M_n)_{n\in\mathbb{Z}_{\geq 0}}$ with $M = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} M_n$ and $A_m M_n \subset M_{m+n}$ for all $m, n \in \mathbb{Z}_{\geq 0}$ť. We call M_n the n-th *homogeneous component* and we say its elements are *homogeneous* of degree n. Obviously, M_n is an A_0-module.

Definition 1.15. Let A be a ring and $\mathcal{F} = (F_n)_{n\in\mathbb{N}}$ be a filtration. We define the *graded ring* associated to the pair $(A, \mathcal{F})$ as

$$gr_{\mathcal{F}}(A) = \bigoplus_{n\geq 0} F_n/F_{n+1}.$$

It is graded by $\mathcal{A} = (A_n)_{n\in\mathbb{Z}_{\geq 0}} = (F_n/F_{n-1})_{n\in\mathbb{Z}_{\geq 0}}$ and it has a natural multiplication. Namely, the product of $x_n + A_{n+1}$ and $x_m + A_{m+1}$ with $n, m \in \mathbb{N}$ is

$$(x_n + A_{n+1})(x_m + A_{m+1}) = x_n x_m + A_{n+m+1}.$$

Note that $gr_{\mathcal{F}}(A)$ is an A-module because every homogeneous piece is.

Analogously, we can define the graded module $gr_{\mathcal{G}}(M)$ associated to an A-module M and a compatible filtration $\mathcal{G} = (G_n)_{n\in\mathbb{N}}$ as

$$gr_{\mathcal{G}}(M) = \bigoplus_{n\geq 0} G_n/G_{n+1}.$$

Note that $gr_{\mathcal{G}}(M)$ is naturally a $gr_{\mathcal{F}}(A)$-module.

1.2 Dimension Theory

Classically, one can study the dimension of a ring from two different points of view, computing its Krull dimension or using Hilbert functions (see [Atiyah and MacDonald, 1969] and [Eisenbud, 1995]). During this section we are going to review these two concepts and also introduce the Gelfand-Kirillov dimension of a $\Bbbk$-algebra, see [McConnell and Robson, 1988], which coincides with the Krull dimension when the $\Bbbk$-algebra is finitely generated. Finally, we will present a new notion of dimension which generalizes the previous ones but which makes sense for non finitely generated $\Bbbk$-algebras. It can be found in the work of [Bavula, 2009] and [Àlvarez Montaner et al., 2021].

Definition 1.16. Let A be a ring and M an A-module. We call M *simple* if it is nonzero and its only proper submodule is 0. We call a chain of submodules,

$$M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_m = 0$$

a *composition series of length* m if each successive quotient M_{i-1}/M_i is simple. Finally, we define the *length* $\ell(M)$ to be the infimum of all those lengths:

$$\ell(M) \coloneqq \inf\{m \mid M \text{ has a composition series of length } m\}.$$

By convention, if M has no composition series, then $\ell(M) \coloneqq \infty$. Note that $\ell(M) = 0$ if and only if $M = 0$ and that $\ell(M) = 1$ if and only if M is simple.

Definition 1.17. Let A be a commutative ring, its *Krull dimension*, $\dim(A)$, is the supremum of the lengths of all strictly ascending chains of prime ideals, if it exists, or otherwise infinite. This is

$$\dim(A) \coloneqq \sup\{r \mid \text{there is a chain of prime ideals } \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r \text{ in } A\}.$$

Generally, during this project we will consider gradings on modules given by sequences of $\Bbbk$-vector spaces of finite dimension.

Definition 1.18. Let A be a commutative graded ring, with $A_0 = \Bbbk$ a field, and assume that A is finitely generated as a $\Bbbk$-algebra. Let M be a finitely generated graded A-module with grading $\mathcal{M} = (M_n)_{n\in\mathbb{Z}_{\geq 0}}$. We define the *Hilbert function* of M as

$$\begin{aligned} H_M : \mathbb{Z}_{\geq 0} &\longrightarrow \mathbb{Z}_{\geq 0}. \\ n &\longmapsto \dim_{\Bbbk}(M_n) \end{aligned}$$

We define the *Hilbert series* of M to be

$$S(M,t) = \sum_{n\geq 0} H_M(n)t^n.$$

It is well known (see [Atiyah and MacDonald, 1969]) that there exists a unique polynomial $H(M,t)$ which agrees with $H_M(n)$ for $n \gg 0$. We call $H(M,t)$ the *Hilbert polynomial* of M.

Remark 1.19. It may happen that choosing a different grading on M produces a different Hilbert function and thus a different Hilbert polynomial. However, the degree of the Hilbert polynomial of M is an intrinsic value which does not depend on the chosen grading.

Definition 1.20. Let M be a finitely generated A-module. The dimension $\dim(M)$ of M is the degree of $H(M,t)$. Let a_d be the leading coefficient of $H(M,t)$, where $d = \dim(M)$. The multiplicity of M is defined as

$$e(M) = \lim_{n\to\infty} \frac{\dim_{\Bbbk} M_n}{n^d} = \lim_{n\to\infty} \frac{H_M(n)}{n^d} = a_d.$$

Remark 1.21. Observe that this definition of multiplicity of a module M differs from the classical one, $e(M) = a_d \cdot d!$, which is always an integer. This is only a convention and it is in our interest to consider $e(M) = a_d$ in order to compare it with the next equivalent definitions of multiplicity.

Example 1.22. Let $R = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring, graded as in Example 1.13. Since R_d is the $\Bbbk$-vector space generated by the monomials of degree d, it has rank $\binom{n-1+d}{n-1}$. Note that $R_0 = \Bbbk$, so the factor $\dim_\Bbbk R_0$ equals 1.

The Hilbert function of $M = R$ has value $\dim_\Bbbk(R_d) = \dim_\Bbbk(\Bbbk)\binom{n-1+d}{n-1}$ at $d \geq 0$. It is therefore a polynomial itself, so it agrees with the Hilbert polynomial:

$$H(R,t) = \binom{n-1+t}{n-1} = \frac{(t+1)\dots(t+n-1)}{(n-1)!}.$$

Using the binomial identity $\binom{n-1+d}{n-1} = (-1)^d\binom{-n}{d}$ we can compute the Hilbert Series of R:

$$S(R,t) = \sum_{d\geq 0} \dim_\Bbbk(\Bbbk)\binom{n-1+d}{n-1}t^d = \sum_{d\geq 0} 1\binom{-n}{d}(-t)^d = \frac{1}{(1-t)^n}.$$

Now, we are going to introduce the Gelfand-Kirillov dimension, which works for non necessarily commutative rings.

Definition 1.23. Let A be a finitely generated $\Bbbk$-algebra, let $V \subseteq A$ be a subspace containing a set of generators of A. Consider the filtration $\mathcal{F} = (F_n)_{n\in\mathbb{Z}_{\geq 0}}$ with $F_n = \sum_{i=1}^n V^i \subseteq A$ and $V^0 = \Bbbk$. The *Gelfand-Kirillov dimension* of A is defined as

$$GK(A) := \inf\{\nu \mid \dim_\Bbbk F_n \leq n^\nu \text{ for } n \gg 0\}.$$

For a non finitely generated $\Bbbk$-algebra A, the Gelfand-Kirillov dimension of A is defined as

$$GK(A) = \sup\{GK(A') \mid A' \text{ is a finitely generated subalgebra of } A\}.$$

For a left finitely generated A-module M with a standard filtration $\mathcal{G} = (G_n)_{n\in\mathbb{Z}_{\geq 0}}$, the Gelfand Kirillov dimension of M is given by

$$GK(M) = \inf\{\nu \mid \dim_\Bbbk G_n \leq n^\nu \text{ for } n \gg 0\}.$$

We can also define the multiplicity of M as

$$\mathrm{e}(M) = \lim_{n\to\infty} \frac{\dim_\Bbbk G_n}{n^d} \in \mathbb{R}_{\geq 0} \cup \{\infty\}, \text{ where } d = GK(M).$$

In the case when the $\Bbbk$-algebra A is not finitely generated and we have a filtration $(A, \mathcal{F})$ of finite-dimensional vector spaces we have another kind of dimension that coincide with the Gelfand-Kirillov dimension in the finite case. Observe that we will not consider generally in this case a standard filtration on the $\Bbbk$-algebra neither on the module. The definition can be found on [Bavula, 2009] and [Àlvarez Montaner et al., 2021]:

Definition 1.24. Let $\mathcal{G}$ be an ascending sequence of finite-dimensional $\Bbbk$-vector spaces. We define

$$\mathrm{Dim}(\mathcal{G}) = \inf\left\{t \in \mathbb{R}_{\geq 0} \mid \dim_\Bbbk G_i \leq i^t \quad \forall i \gg 0\right\}$$

If $d = \mathrm{Dim}(\mathcal{G})$ is finite, then the multiplicity of $\mathcal{G}$ is the extended real number:

$$\mathrm{e}(\mathcal{G}) = \limsup_{i\to\infty} \frac{\dim_{\Bbbk} G_i}{i^d} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

By convention, if $\mathrm{Dim}(\mathcal{G})$ is infinite, then we set $\mathrm{e}(\mathcal{G}) = \infty$.

Remark 1.25. In fact, in the case when we are computing the dimension of a finitely generated $\Bbbk$-algebra we do not need the limit superior only the limit. One example that illustrate that the limit does not exist can be found in detail in Section 4 of [Katzman et al., 2012].

1.3 Rings of Differential Operators

The origin of the definition of differential operator over a Noetherian ring can be found in the work of Groethendieck [Grothendieck, 1967], however the modern theory of D-modules was developed at the same time by Kashiwara in his master thesis on algebraic analysis [Kashiwara, 1970] and by J. Bernstein who developed a very similar theory in the algebraic setting on [Bernstein, 1971], where he introduced the concept of holonomic modules.

Let $\Bbbk$ be a field and A a commutative $\Bbbk$-algebra. Most of the results from this section are well known and can be found in [Coutinho, 1995] and [McConnell and Robson, 1988].

Definition 1.26. Let D be an extension of the ring A (or a $\Bbbk$-algebra), the *commutator* of two elements a, b of D is defined by $[a, b] = a \circ b - b \circ a$.

Proposition 1.27. *Let D be an extension of the ring A, α, β, γ operators in D, and $\lambda \in A$. Some useful results on the commutator are:*

(1) $[\alpha\beta, \gamma] = \alpha[\beta, \gamma] + [\alpha, \gamma]\beta$.

(2) $[\lambda\alpha, \beta] = [\alpha, \lambda\beta] = \lambda[\alpha, \beta]$.

Proof. (1) $\alpha[\beta, \gamma] + [\alpha, \gamma]\beta = \alpha(\beta\gamma - \gamma\beta) + (\alpha\gamma - \gamma\alpha)\beta = \alpha\beta\gamma - \gamma\alpha\beta = [\alpha\beta, \gamma]$.

(2) For the first identity: $[\lambda\alpha, \beta] = \lambda\alpha\beta - \beta\lambda\alpha = \alpha\lambda\beta - \lambda\beta\alpha = [\alpha, \lambda\beta]$. The other identity can be proven similarly from the first term: $[\lambda\alpha, \beta] = \lambda\alpha\beta - \beta\lambda\alpha = \lambda(\alpha\beta - \beta\alpha) = \lambda[\alpha, \beta]$. □

Clearly, the commutator of any two commuting elements is zero. In general, it is a measure of the non-commutativity of a ring. For us, the commutator is key to define differential operators.

Definition 1.28. The *ring of* $\Bbbk$*-linear differential operators* of a $\Bbbk$-algebra A is defined, inductively, as a subring of $End_{\Bbbk}(A)$. We say that a differential operator

- $\delta \in End_{\Bbbk}(A)$ has order zero if $[\delta, a] = 0$ for every $a \in A$.
- $\delta \in End_{\Bbbk}(A)$ has order n if $[\delta, a]$ has order $\leq n - 1$ for every $a \in A$.

We are going to denote by $D^n_{A|\Bbbk}$ the set of all differential operators of order $\leq n$ of the $\Bbbk$-algebra A. The *ring of $\Bbbk$-linear differential operators* is defined as

$$D_{A|\Bbbk} = \bigcup_n D^n_{A|\Bbbk} \subseteq End_{\Bbbk}(A)$$

Notation 1.29. In order to have a clearer notation we do not distinguish between $a \in A$ and the endomorphism multiplication by a: $L_a \in End_{\Bbbk} A$ defined as $L_a : A \to A : b \mapsto a \cdot b$. Also, when we write the commutator we are going to leave the symbol $\circ$ out.

Let us study the first two pieces of $D_{A|\Bbbk} = \bigcup_n D^n_{A|\Bbbk}$:

Proposition 1.30. *Let A be a $\Bbbk$-algebra. The set of differential operators of order zero coincides with the $\Bbbk$-algebra A, this is, $D^0_{A|\Bbbk} = A$.*

Proof. We have that $A \subset D^0_{A|\Bbbk}$ because A is a commutative ring and A is a subset of $End_{\Bbbk}(A)$.

On the other hand, if $\delta \in D^0_{A|\Bbbk}$ and $a, b \in A$, then

$$[\delta, a](b) = (\delta \circ a - a \circ \delta)(b) = \delta(ab) - a\delta(b) = 0.$$

Therefore $\delta \in End_A(A) = A$. □

Remark 1.31. Observe that it is tautological that $End_A(A) = A : L_a \mapsto a$. Because if $\delta \in End_A A$ then $\delta(a) = a\delta(1)$ is determined by $\delta(1)$.

The second term $D^1_{R|\Bbbk}$ can be understood in terms of derivations.

Definition 1.32. A *derivation* of A over $\Bbbk$ is a $\Bbbk$-linear differential operator $\delta \in End_{\Bbbk}(A)$ which satisfies Leibniz's rule:

$$\delta(rs) = r\delta(s) + s\delta(r)$$

for every $r, s \in A$. We denote the set of derivations of A over the field $\Bbbk$ as $Der_{\Bbbk}(A)$.

Let $\delta \in Der_{\Bbbk}(A)$ and $a \in A$, we can define the derivation $a\delta$ by $(a\delta)(b) = a\delta(b)$ for every $b \in A$ therefore by this action $Der_{\Bbbk}(A)$ is a left A-module.

Proposition 1.33. *Let $D_{A|\Bbbk}$ be the ring of $\Bbbk$-linear differential operators of a $\Bbbk$-algebra A. Any differential operator of order one can be expressed in terms of derivations and elements of the $\Bbbk$-algebra, i.e., $D^1_{A|\Bbbk} = Der_{\Bbbk}(A) + A$.*

Proof. Clearly $Der_{\Bbbk}(A) + A \subseteq D^1_{A|\Bbbk}$. To prove the other inclusion, let $\gamma \in D^1_{A|\Bbbk}$ and $\delta = \gamma - \gamma(1)$. Note that $\delta(1) = 0$ and $\delta \in D^1_{A|\Bbbk}$. Then, for any $a, b \in A$ we have that $[[\delta, a], b] = 0$. Indeed,

$$\delta(ab) - a\delta(b) = (\delta a - a\delta)(b) = (\delta a - a\delta)(b1) = b(\delta a - a\delta)(1) = b\delta(a),$$

so it satisfies Leibniz's rule and we have that $\delta \in Der_{\Bbbk}(A)$. Therefore, we have that $\gamma = \delta - \gamma(1) \in Der_{\Bbbk}(A) + A$. □

Let us prove that we have a filtration given by the order of the differential operators in $D_{A|\Bbbk}$.

Proposition 1.34. *For all $n \in \mathbb{N}$, the set of $\Bbbk$-linear differential operators of order n, $D^n_{A|\Bbbk}$, is an A-module.*

Proof. By induction on n. For $n = 0$, as we have that $D^0_{A|\Bbbk} = A$, then $D^0_{A|\Bbbk}$ is an A-module.

Now, assume that the result holds for n and lets prove it for $n+1$. Let $\delta_1, \delta_2 \in D^{n+1}_{A|\Bbbk}$ and $a, b \in A$, so by definition of differential operator of order n, we have that $[\delta_1, a], [\delta_2, a] \in D^n_{A|\Bbbk}$.

Observe that $[b\delta_1, a] = b[\delta_1, a]$ and $[\delta_1 + \delta_2, a] = [\delta_1, a] + [\delta_2, a]$ by Proposition 1.27. As $D^n_{A|\Bbbk}$ is an A-module by induction hypothesis, we have finished. □

Proposition 1.35. *Let $n, m \in \mathbb{N}$, $\delta_1 \in D^m_{A|\Bbbk}$ and $\delta_1 \in D^l_{A|\Bbbk}$ be two differential operators. Then $\delta_1\delta_2 \in D^{m+l}_{A|\Bbbk}$.*

Proof. We are going to prove it by induction on $m + l$. If $m + l = 0$, it is obvious. Suppose it is true whenever $m + l < k$. If $m + l = k$ and $a \in A$, we have by Proposition 1.27 that

$$[\delta_1\delta_2, a] = \delta_1 [\delta_2, a] + [\delta_1, a] \delta_2.$$

By the definition of the order of a differential operator we have that $[\delta_1, a] \in D^{m-1}_{A|\Bbbk}$ and $[\delta_2, a] \in D^{l-1}_{A|\Bbbk}$. Thus, by the induction hypothesis $[\delta_1, a] \delta_2,\ \delta_1 [\delta_2, a] \in D^{m+l-1}_{A|\Bbbk}$, therefore $[\delta_1\delta_2, a] \in D^{m+l-1}_{A|\Bbbk}$. □

By Definition 1.28 we have that $D_{A|\Bbbk} = \bigcup_{n\in\mathbb{Z}_{\geq 0}} D^n_{A|\Bbbk}$ and by Proposition 1.30, we get that $1 \in D^0_{A|\Bbbk} = A$. Using also Propositions 1.34 and 1.35, it follows that $\mathcal{F} = (D^n_{A|\Bbbk})_{n\in\mathbb{Z}_{\geq 0}}$ is a filtration given by a sequence of A-modules of the ring $D_{A|\Bbbk}$ because $D^m_{A|\Bbbk} D^l_{A|\Bbbk} \subseteq D^{m+l}_{A|\Bbbk}$ and the induction in the definition of the ring of differential operators give us that $D^m_{A|\Bbbk} \subseteq D^{m+1}_{A|\Bbbk}$ for all m and l.

1.4 Examples of rings of differential operators

In the literature one can find two main examples of rings of differential operators over a $\Bbbk$-algebra. Those are

The Weyl algebra

The ring $D_{R|\Bbbk}$ of differential operators over a polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$ also known as the n-th Weyl algebra. In this case, we will see in Section 2 that $D_{R|\Bbbk}$ has a lot of good properties, for example, it is Noetherian, finitely generated as a $\Bbbk$-algebra and, considering a certain filtration on $D_{R|\Bbbk}$ known as Bernstein's filtration, one can prove that

$$\dim(D_{R|\Bbbk}) = GK(D_{R|\Bbbk}) = 2n.$$

Also, one can study $D_{R|\Bbbk}$-modules M in this setting and obtain the well known Bernstein's inequality that tell us, considering a special kind of filtrations on M compatible with the filtration on $D_{R|\Bbbk}$, that

$$\dim(M) \geq \frac{1}{2}\dim(D_{R|\Bbbk}).$$

The ring of differential operators of finitely generated $\Bbbk$-algebras

We can go one step beyond and consider the quotient of a polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$ by an ideal I, this is $S = R/I$, which in general is a non-regular ring, and study the ring of differential operators over S. The main result in this setting is that we can obtain the differential operators over S in terms of the differential operators in R that preserve the ideal I, this is, there is the following correspondence

$$D_{S|\Bbbk} = \frac{\{\delta \in D_{R|\Bbbk} \mid \delta(I) \subseteq I\}}{ID_{R|\Bbbk}}.$$

We will prove this result in Section 3.

This ring may not be Noetherian or finitely generated as a $\Bbbk$-algebra (see examples in Section 3.4). However, under certain assumptions we will see that $GK(D_{S|\Bbbk}) = 2\dim S$. This will be one of our main results in Section 3.

When S is a graded ring we can endow $D_{S|\Bbbk}$ with a generalized version of the Bernstein's filtration. Under certain circumstances (see Section 3.5) we can obtain a Bernstein's type inequality. Namely, let M be a finitely generated $D_{S|\Bbbk}$-module with a filtration compatible with the generalized Bernstein filtration $\mathcal{B}_S$. If $\mathcal{B}_S$ is linearly simple (see Definition 3.16) then

$$\mathrm{Dim}(M) \geq \frac{1}{2}\mathrm{Dim}(D_{S|\Bbbk}).$$

2. The ring of differential operators of a polynomial ring

Historically, the first example of a ring of differential operators is the Weyl algebra which, as we will see, coincides with the ring of differential operators $D_{R|\Bbbk}$ (see Definition 1.28) of a polynomial ring $R = \Bbbk[x_1, \ldots, x_n]$. The Weyl algebra was presented as the subalgebra of endomorphisms of R generated by multiplication by the variables x_i and partial derivatives ∂_{x_i}. We will show that both presentations are equivalent in Theorem 2.6.

The Weyl algebra first appeared in quantum mechanics in 1925 as an algebra generated by position and momentum operators. Littlewood [Littlewood, 1933] introduced the canonical form of a differential operator, which we will see in Section 2.1 and proved that the Weyl algebra is a domain, see Section 2.3. The name "Weyl algebra" comes from the book [Weyl, 1950] where all these ideas were presented.

Later work in this setting can be found in the works of Kashiwara [Kashiwara, 1970] and I.N. Bernstein [Bernstein, 1971]. In particular, the last author studied the dimension of finitely generated modules over the Weyl algebra and proved the so-called Bernstein's inequality. We will show this result in Section 2.5.

All these results (among others) will be useful in Sections 3 and 4, where we focus on the ring $D_{S|\Bbbk}$ of differential operators on a quotient S of a polynomial ring by an ideal.

2.1 The Weyl algebra

Throughout this section, let $\Bbbk$ be a field of characteristic zero and $R = \Bbbk[x_1, \ldots, x_n]$ be the ring of polynomials in n variables with coefficients in $\Bbbk$.

Let $End_{\Bbbk}(R)$ be the $\Bbbk$-algebra of endomorphisms of the $\Bbbk$-vector space R. As the product in this algebra is just the composition of endomorphisms then $End_{\Bbbk}(R)$ is a noncommutative ring with unit. The Weyl algebra will be defined as a subalgebra of $End_{\Bbbk}(R)$.

Each of the variables x_i in R defines two elements of $End_{\Bbbk}(R)$. The first one, also denoted x_i, is "multiplication by x_i": $x_i(f) = x_i \cdot f$. The second one is taking partial derivative with respect to x_i: $\partial_i(f) = \frac{\partial f}{\partial x_i}$.

Definition 2.1. The n-th *Weyl algebra* is the $\Bbbk$-algebra generated by the variables $x_1, \ldots, x_n$ and the partial derivatives $\partial_1, \ldots, \partial_n$, this is,

$$A_n(\Bbbk) = \Bbbk\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle.$$

Notation 2.2. By convention $x_i^0 = 1$ and $\partial_i^0 = 1$ for $i = 1, \ldots, n$.

By definition, any element in the n-th Weyl algebra can be written as a linear combination with coefficients in $\Bbbk$ of products of $x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$. However, we will see next that there are relations among these generators, meaning that one element of the Weyl algebra may me written in terms of the generators in several different ways. Below we introduce the canonical form to avoid this ambiguity.

For every $f \in R$, we have

$$(\partial_i \circ x_i)(f) = \partial_i(x_i \cdot f) = \partial_i(x_i)f + x_i\partial_i(f) = f + x_i\partial_i(f) = (1 + x_i \circ \partial_i)(f)$$

So, the identity $\partial_i \circ x_i - x_i \circ \partial_i = 1$ holds in $End_{\Bbbk}(R)$ and therefore in the n-th Weyl algebra. This shows that $D_{R|\Bbbk}$ is a noncommutative ring.

More generally, for any $f, g \in R$ we have

$$(\partial_i \circ g)(f) = \partial_i(g \cdot f) = \partial_i(g)f + g\partial_i(f) = \partial_i(g) \cdot (f) + g \circ \partial_i(f)$$

so the identity $\partial_i \circ g + g \circ \partial_i = \partial_i(g)$ holds for every $g \in R$. This is known as Leibniz's rule.

However, there are some cases where these elements commute:

(1) $\partial_i \circ x_j = x_j \circ \partial_i$ for all $i \neq j$.

(2) $x_i \circ x_j = x_j \circ x_i$ for all i, j.

(3) $\partial_i \circ \partial_j = \partial_j \circ \partial_i$ for all i, j.

Canonical form

We are going to construct a basis for the Weyl algebra as a $\Bbbk$-vector space, known as the canonical basis. An element of $A_n(\Bbbk)$ written in this basis is said to be in canonical form. Of course, comparing two elements is the same as comparing their coefficients of their canonical forms, and that is easily done.

We are going to use a multi-index notation. Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ the corresponding monomial in R and the corresponding endomorphism in the Weyl algebra. Similarly, we write $\partial^\beta = \partial_1^{\beta_1} \ldots \partial_n^{\beta_n}$ for $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. An element of the form $x^\alpha\partial^\beta \in A_n(\Bbbk)$ for $\alpha, \beta \in \mathbb{N}^n$ is also called a *monomial*.

As we know, we can obtain any element of $A_n(\Bbbk)$ as sums of compositions of x_i's and ∂_i's in more than one way. Our goal is to prove that we can write all such compositions in terms of monomials.

Notation 2.3. Observe that $\partial^\beta \circ f = \partial^\beta f$ is different than $\partial^\beta(f)$, in this second case we are applying the monomial ∂^β to the polynomial f.

Proposition 2.4. *Let $R = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring and consider the Weyl algebra $A_n(\Bbbk) = \Bbbk\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$.*

(1) Let $f \in R$ and $\beta \in \mathbb{N}^n$. The product $\partial^\beta f$ in the n-th Weyl algebra can be written as

$$\partial^\beta f = \sum_{\sigma << \beta} \binom{\beta}{\sigma} \partial^\sigma(f)\partial^{\beta-\sigma}$$

where $\sigma << \beta$ stands for $\sigma_i \leq \beta_i$ for $i = 1, \ldots, n$, $\binom{\beta}{\sigma} = \frac{\beta!}{\sigma!(\beta-\sigma)!}$ and $\beta! = \beta_1! \cdots \beta_n!$.

(2) Let $\beta, \gamma \in \mathbb{N}^n$ then we have that $\partial^\beta(x^\gamma) = \beta!\binom{\gamma}{\beta}x^{\gamma-\beta}$ where $\binom{\gamma}{\beta} = 0$ if the relation $\beta << \gamma$ doesn't hold.

(3) Let $\alpha, \alpha', \beta, \beta' \in \mathbb{N}^n$ then we have

$$x^\alpha \partial^\beta x^{\alpha'} \partial^{\beta'} = x^{\alpha+\alpha'}\partial^{\beta+\beta'} + \sum_{\sigma<<\beta, \sigma<<\alpha', \sigma\neq 0} \sigma! \binom{\beta}{\sigma}\binom{\alpha'}{\sigma} x^{\alpha+\alpha'-\sigma}\partial^{\beta+\beta'-\sigma}.$$

Proof. (1) The sketch of this proof can be found in [Castro Jiménez, 2010], here we give a complete one. It follows from the case $n = 1$ and the distributive property of the product with respect to the sum in the Weyl algebra. If we do a change of variable: $x = x_1$ and $\partial_x = \partial_1$. We are going to prove that

$$\partial_x^j f = \sum_{k=0}^{j} \binom{j}{k} \partial_x^k(f)\partial_x^{j-k}$$

by induction on j. The case when $j = 1$ is the chain rule: $\partial_x f = f\partial_x + \partial_x(f)$. Suppose it is true for $j > 1$, lets prove it for $j + 1$:

$$\partial_x^{j+1} f = \partial_x(\partial_x^j f) = \partial_x \left(\sum_{k=0}^{j} \binom{j}{k} \partial_x^k(f)\partial_x^{j-k} \right) = \sum_{k=0}^{j} \binom{j}{k} \left[\partial_x^{k+1}(f)\partial_x^{j-k} + \partial_x^k(f)\partial_x^{j-k+1} \right],$$

now, we are going to separate in two summations

$$\begin{aligned} &\sum_{k=0}^{j} \binom{j}{k} \partial_x^{k+1}(f)\partial_x^{j-k} + \sum_{k=0}^{j} \binom{j}{k} \partial_x^k(f)\partial_x^{j-k+1} \\ =&\sum_{k=0}^{j-1} \binom{j}{k} \partial_x^{k+1}(f)\partial_x^{j-k} + \partial_x^{j+1}(f) + f\partial_x^{j+1} + \sum_{k=1}^{j} \binom{j}{k} \partial_x^k(f)\partial_x^{j-k+1} \end{aligned}$$

and do the change of variable $m = k - 1$ in the second summation:

$$\sum_{k=0}^{j-1} \binom{j}{k} \partial_x^{k+1}(f)\partial_x^{j-k} + \partial_x^{j+1}(f) + f\partial_x^{j+1} + \sum_{m=0}^{j-1} \binom{j}{m+1} \partial_x^{m+1}(f)\partial_x^{j-m}.$$

So, we have obtained that

$$\begin{aligned} &\partial_x^{j+1}(f) + f\partial_x^{j+1} + \sum_{k=0}^{j-1} \left[\binom{j}{k} + \binom{j}{k+1} \right] \partial_x^{k+1}(f)\partial_x^{j-k} \\ &= \partial_x^{j+1}(f) + f\partial_x^{j+1} + \sum_{k=0}^{j-1} \binom{j+1}{k+1} \partial_x^{k+1}(f)\partial_x^{j-k}. \end{aligned}$$

Doing the change of variable $m = k + 1$ and writing all in one summation we have what we wanted:

$$\partial_x^{j+1} f = \sum_{m=0}^{j+1} \binom{j+1}{m} \partial_x^m(f)\partial_x^{j-m+1}$$

(2) The formula follows by induction on n. The case $n = 1$ can be proved by induction on β_1:

- $\beta_1 = 1$: Its only the derivative of a polynomial, $\partial_x(x^\alpha) = \alpha x^{\alpha-1}$ where $\alpha \in \mathbb{N}$.
- $\beta_1 = \beta > 1$. Lets compute $\partial_x^\beta(x^\alpha)$ where $\alpha, \beta \in \mathbb{N}$, supposing true the case $\beta - 1$, by induction:

$$\partial_x^\beta(x^\alpha) = \partial_x(\partial_x^{\beta-1}(x^\alpha)) = \partial_x\left((\beta-1)!\binom{\alpha}{\beta-1}x^{\alpha-\beta+1}\right) = (\beta-1)!\binom{\alpha}{\beta-1}\partial_x(x^{\alpha-\beta+1})$$
$$= (\beta-1)!\binom{\alpha}{\beta-1}(\alpha-\beta+1)x^{\alpha-\beta} = \beta!\frac{\alpha!}{\beta!(\alpha-\beta)!}x^{\alpha-\beta} = \beta!\binom{\alpha}{\beta}x^{\alpha-\beta}.$$

Which is what we wanted.

(3) It follows from (1) and (2). □

Proposition 2.5. *The set $\{x^\alpha\partial^\beta : \alpha, \beta \in \mathbb{N}^n\}$ is a basis of the Weyl algebra $A_n(\Bbbk)$ as a vector space over $\Bbbk$. Each nonzero element $\delta \in A_n(\Bbbk)$ can be written in an unique way as a finite sum*

$$\delta = \sum_{\alpha,\beta} p_{\alpha\beta}x^\alpha\partial^\beta$$

for some nonzero element $p_{\alpha\beta} \in \Bbbk$. Moreover, $\delta = \sum_\beta p_\beta(x)\partial^\beta$ with $p_\beta(x) = \sum_\alpha p_{\alpha\beta}x^\alpha$.

Proof. Every monomial $x^\alpha\partial^\beta$, where $\alpha, \beta \in \mathbb{N}^n$, is a product of the elements $x_1, \dots, x_n, \partial_1, \dots, \partial_n$. Also, we know by Proposition 2.4 (3) that any product $x^{\alpha_1}\partial^{\beta_1}x^{\alpha_2}\partial^{\beta_2}$ with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}^n$ can be written as a linear combination of monomials with coefficients in $\Bbbk$. So the elements of $\{x^\alpha\partial^\beta : \alpha, \beta \in \mathbb{N}^n\}$ are a generating system of $A_n(\Bbbk)$.

Let us prove that they are linearly independent. Suppose that $\delta = \sum_{\alpha,\beta} p_{\alpha\beta}x^\alpha\partial^\beta$ is a non trivial differential operator. We need to see that if some $p_{\alpha\beta}$ are different than zero then $\delta \neq 0$. As δ is a linear differential operator of R, then $\delta \neq 0$ if and only if there exists a polynomial f such that $\delta(f) \neq 0$. We will construct such f.

Let $\beta \in \mathbb{N}^n$ such that $p_{\alpha\beta} \neq 0$ for some index α, but $p_{\alpha\sigma} = 0$ for all indices σ such that $|\sigma| < |\beta|$. Using Proposition 2.4 (2), we obtain $\delta(x^\beta) = \beta! \sum_\alpha p_{\alpha\beta}x^\alpha$, which is nonzero because at least one of $p_{\alpha\beta}$ is different that zero. So, taking $f = x^\beta$ we finish the proof. □

We are going to see that the definition of the ring of differential operators $D_{R|\Bbbk}$ stated in Section 1.3 coincides with the definition of the Weyl algebra:

Theorem 2.6. *Let R be the polynomial ring in n variables with coefficients in a field $\Bbbk$. The ring $D_{R|\Bbbk}$ of $\Bbbk$-linear differential operators of R coincides with the n-th Weyl algebra $A_n(\Bbbk)$.*

Proof. Observe that $\tilde{D}^i_{R|\Bbbk} = \bigoplus_{\beta_1+\dots+\beta_n \leq i} R\partial_1^{\beta_1}\dots\partial_n^{\beta_n}$ is a filtration of $A_n(\Bbbk)$. We are going to prove that $\tilde{D}^i_{R|\Bbbk} = D^i_{R|\Bbbk}$.

First, we show $\tilde{D}^i_{R|\Bbbk} \subseteq D^i_{R|\Bbbk}$. For $i = 0$, it is clear. We have that

$$[\partial_i, x_j] = \delta_{ij}$$

where δ_{ij} denotes the Kronecker delta. So, to prove the inclusion when $i > 1$, it suffices to check that any element of $\tilde{D}^i_{R|\Bbbk}$ hold the general definition of differential operator over the generators of R only. As the partial derivatives $\partial_i \in D^1_{R|\Bbbk}$ for each i, then $\partial_1^{\beta_1} \dots \partial_n^{\beta_n} \in D^i_{R|\Bbbk}$ by Proposition 1.34. Therefore $\tilde{D}^i_{R|\Bbbk} \subseteq D^i_{R|\Bbbk}$.

For the other inclusion, we are going to prove that the map $\tilde{D}^i_{R|\Bbbk} \to \mathrm{Hom}_{\Bbbk}(R_{\leq i}, R)$ is an isomorphism.

We are going to show first that it is injective. By induction on i:

- Case $i = 0$: $\tilde{D}^0_{R|\Bbbk}$ is R and $R_{\leq 0} = \Bbbk$. Given a polynomial p, we claim we can recover p from the morphism $\Bbbk \to R : a \mapsto p \cdot a$. Simply take $a = 1$.

- Case $i = 1$. We can write an element of $\tilde{D}^1_{R|\Bbbk}$ in its canonical form as $\delta = p(x) + \sum_{i=1} p_i(x)\partial_i$. We can recover $p(x)$ as in case $i = 0$. To recover the polynomials $p_i(x)$, evaluate $\delta - p(x)$ at the monomial x_i.

- Case $i > 1$. As before, since applying ∂^α for $|\alpha| = i$ vanishes on all monomials of degree $< i$, we can recover the coefficients (polynomials) of the components of order $|\alpha| \leq i - 1$. To recover the polynomials of the components of order $|\alpha| = i$, consider the morphisms $R_{\leq i} \to R : x^\beta \mapsto \delta'(x^\beta)$ for $|\beta| = i$, where $\delta' = \frac{1}{\beta!}(\delta - \sum_{\alpha \leq i-1} p_\alpha(x)\partial^\alpha)$.

As $\tilde{D}^i_{R|\Bbbk} \subseteq D^i_{R|\Bbbk}$, we have it is injective.

Let us prove now that it is surjective. As $R_{\leq i}$ is a finitely generated vector space with basis the monomials of degree $\leq i$. Then, giving a morphism of $\mathrm{Hom}_{\Bbbk}(R_{\leq i}, R)$ is the same as freely choosing for every monomial of degree $\leq i$ a polynomial to be its image. Thus, surjectivity is equivalent to finding for any family of polynomials $\{s_\alpha\}_{\alpha_1+\cdots+\alpha_n \leq i} \subseteq R$ a $\delta \in \tilde{D}^i_{R|\Bbbk}$ such that $\delta(x^{\alpha_1} \dots x^{\alpha_n}) = s_\alpha$.

This is similar to the proof of injectivity. In concrete terms, given $\{s_\alpha\}_{\alpha_1+\cdots+\alpha_n \leq i}$ inductively we can find $\delta' \in \tilde{D}^{i-1}_{R|\Bbbk}$ with $\delta'(x^{\alpha_1} \dots x^{\alpha_n}) = s_\alpha$ for $|\alpha| \leq i - 1$. Take

$$\delta = \delta' + \sum_{|\alpha|<i} \left[\frac{s_\alpha - \delta'(x^{\alpha_1} \dots x^{\alpha_n})}{\beta!} \partial_1^{\beta_1} \dots \partial_n^{\beta_n} \right].$$

Now, for any $\delta \in D^i_{R|\Bbbk}$, there exist $\tilde{\delta} \in \tilde{D}^i_{R|\Bbbk}$ such that $\delta|_{R_{\leq i}} = \tilde{\delta}|_{R_{\leq i}}$. By the injectivity, we have $\delta = \tilde{\delta}$. Thus $D^i_{R|\Bbbk} \subseteq \tilde{D}^i_{R|\Bbbk}$ □

2.2 Filtrations on the Weyl algebra

In this section we will introduce two different filtrations on $D_{R|\Bbbk}$: the order filtration and the so-called Bernstein's filtration given by the total order. The associated graded rings with respect to these filtrations are isomorphic to the polynomial ring in $2n$ variables and we will use this fact to prove that $D_{R|\Bbbk}$ is a Noetherian domain.

From now on we will denote by $\Bbbk[x,\xi] = \Bbbk[x_1,\dots,x_n,\xi_1,\dots,\xi_n]$ the polynomial ring in the set of variables x_i's and ξ_i's with coefficients in $\Bbbk$. And, let us denote by $|\beta| = \sum_i \beta_i$ for each $\beta \in \mathbb{N}^n_{\geq 0}$.

Definition 2.7. For an nonzero differential operator

$$\delta = \sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \partial^\beta = \sum_\beta p_\beta(x)\partial^\beta \in D_{R|\Bbbk}.$$

We define the *order* of the differential operator $\delta \in D_{R|\Bbbk}$, $ord(\delta)$, as the maximum of $|\beta|$ for $p_\beta(x) \neq 0$. Similarly, we define the total order, $ord^T(\delta)$, as the maximum of $|\alpha|+|\beta|$ for $p_{\alpha\beta} \neq 0$. By convention, we say that $ord^T(0) = -\infty$.

Recall that $(D_{R|\Bbbk}, \mathcal{F})$ is a filtered ring with the filtration given by the order $F_k = \{\delta \in D_{R|\Bbbk} \mid ord(\delta) \leq k\} = D^k_{R|\Bbbk}$ where each F_k is an R-module. In the polynomial ring case it is sometimes more useful to consider the filtration given by the total order.

Definition 2.8. The filtration $\mathcal{B} = (B_k)_{k\in\mathbb{Z}_{\geq 0}}$ on $D_{R|\Bbbk}$ defined by the total order

$$B_k = \{\delta \in D_{R|\Bbbk} : ord^T(\delta) \leq k\}$$

is called the *Bernstein's filtration*.

This is indeed a filtration on $D_{R|\Bbbk}$ because $\mathcal{B} = (B_k)_{k\in\mathbb{Z}_{\geq 0}}$ satisfies all the properties of the definition:

- $B_k \subset B_{k+1}$ for every k. This is straightforward from the definition of B_k.
- $B_k B_l \subset B_{k+l}$ for $k, l \in \mathbb{Z}_{\geq 0}$. It suffices to prove it in the case of monomials, let $\delta = x^\alpha\partial^\beta \in B_k$ and $\delta' = x^{\alpha'}\partial^{\beta'} \in B_l$, the product of both differential operators is $\delta\delta' = x^\alpha\partial^\beta x^{\alpha'}\partial^{\beta'}$ and expressing this product in canonical form we are going to prove later in Proposition 2.12 that $ord^T(\delta\delta') = |\alpha+\beta+\alpha'+\beta'| = k+l$.
- $D_{R|\Bbbk} = \bigcup_{k\in\mathbb{Z}_{\geq 0}} B_k$ by definition.
- Finally, each B_k is a $\Bbbk$-vector space of dimension $\binom{2n+k}{k}$: we need to compute the number of elements in the canonical basis of $ord^T \leq k$. This agrees with the number of monomials in $2n$ variables of degree less or equal than k. Homogenizing, this is the number of monomials in $2n+1$ variables of degree equal to k, which is $\binom{2n+k}{k}$.

From the Bernstein's filtration we can also construct its associated graded ring:

$$gr_{\mathcal{B}}(D_{R|\Bbbk}) \coloneqq \bigoplus_{n\in\mathbb{N}} B_k/B_{k-1}.$$

Definition 2.9. The *principal symbol* of a differential operator $\delta \in D_{R|\Bbbk}$ is the polynomial

$$\sigma(\delta) = \sum_{|\beta|=ord(\delta)} p_\beta(x)\xi^\beta \in \Bbbk[x,\xi].$$

Similarly, we define the *total principal symbol* of $\delta \in D_{R|\Bbbk}$ as the polynomial

$$\sigma^T(\delta) = \sum_{|\alpha|+|\beta|=\deg(\delta)} p_{\alpha\beta}x^\alpha\xi^\beta \in \Bbbk[x,\xi].$$

Remark 2.10. Observe that the polynomial $\sigma^T(\delta)$ is homogeneous of degree $ord^T(\delta)$. In general, we have that $\sigma(\delta) \neq \sigma^T(\delta)$. Also, in general $\sigma^T(\delta_1+\delta_2) \neq \sigma^T(\delta_1)+\sigma^T(\delta_2)$. However, we have the following identities.

Proposition 2.11. *Let $D_{R|\Bbbk}$ be the ring of $\Bbbk$-linear differential operators of the polynomial ring R. For two differential operators $\delta_1, \delta_2 \in D_{R|\Bbbk}$ one has*

1. $ord(\delta_1\delta_2) = ord(\delta_1)+ord(\delta_2)$ *and* $\sigma(\delta_1\delta_2) = \sigma(\delta_1)\sigma(\delta_2)$.
2. $ord(\delta_1\delta_2 - \delta_2\delta_1) \leq ord(\delta_1)+ord(\delta_2)-1$.
3. $ord(\delta_1+\delta_2) \leq \max\{ord(\delta_1), ord(\delta_2)\}$.
4. *If* $ord(\delta_1) = ord(\delta_2)$ *and* $\sigma(\delta_1)+\sigma(\delta_2) \neq 0$ *then* $\sigma(\delta_1+\delta_2) = \sigma(\delta_1)+\sigma(\delta_2)$.

We have similar identities for the degree and the principal total symbol:

Proposition 2.12. *Let $D_{R|\Bbbk}$ be the ring of $\Bbbk$-linear differential operators of the polynomial ring R. For two differential operators $\delta_1, \delta_2 \in D_{R|\Bbbk}$ one has*

1. $ord^T(\delta_1\delta_2) = ord^T(\delta_1)+ord^T(\delta_2)$ *and* $\sigma^T(\delta_1\delta_2) = \sigma^T(\delta_1)\sigma^T(\delta_2)$.
2. $ord^T(\delta_1\delta_2 - \delta_2\delta_1) \leq ord^T(\delta_1)+ord^T(\delta_2)-2$.
3. $ord^T(\delta_1+\delta_2) \leq \max\{ord^T(\delta_1), ord^T(\delta_2)\}$.
4. *If* $ord^T(\delta_1) = ord^T(\delta_2)$ *and* $\sigma^T(\delta_1)+\sigma^T(\delta_2) \neq 0$ *then* $\sigma^T(\delta_1+\delta_2) = \sigma^T(\delta_1)+\sigma^T(\delta_2)$.

Proof. The proofs of (1) and (2) follow from Proposition 2.4, it suffices to write both differential operators in canonical form and do the computations. The proofs of (3) and (4) follow from the definitions of total order and total principal symbol. □

2.3 Structural properties of the Weyl algebra

Let us now see which properties $D_{R|\Bbbk}$ has as a ring. But first, let us define:

Definition 2.13. The *total graded ideal* associated with the left (or right) ideal $I \in D_{R|\Bbbk}$ is the ideal $gr^T(I)$ of $\Bbbk[x, \xi]$ generated by the family of principal total symbols of elements in I

$$gr^T(I) = \Bbbk[x, \xi]\{\sigma^T(\delta) \mid \delta \in I\}.$$

Remark 2.14. In particular, if $I = D_{R|\Bbbk}\delta$ then $gr^T(I)$ is generated by $\sigma^T(\delta)$.

Proposition 2.15. *The ring $D_{R|\Bbbk}$ of $\Bbbk$-linear differential operators of R is a left and right Noetherian domain.*

Proof. If suffices to prove that any left ideal $I \subset D_{R|\Bbbk}$ is finitely generated (for right ideals the proof is similar). We can assume $I \neq (0)$. By definition of $gr^T(I)$ there are polynomials $\sigma^T(\delta_1), \dots, \sigma^T(\delta_r)$ with $\delta_i \in I$ generating $gr^T(I)$.

Let us denote J the left ideal in $D_{R|\Bbbk}$ generated by $\delta_1, \dots, \delta_r$. We will prove that $J = I$.

Assume there exists $\delta \in I \setminus J$ such that δ is of minimal total order. As $\sigma^T(\delta) \in gr^T(I)$ then there are homogeneous polynomials $H_1, \dots, H_r \in \Bbbk[x, \xi]$ such that

$$\sigma^T(\delta) = \sum_i H_i \ \sigma^T(\delta_i).$$

We can also assume $\deg(H_i) + ord^T(\delta_i) = ord^T(\delta)$. Denote by γ_i any element in $D_{R|\Bbbk}$ with $\sigma^T(\gamma_i) = H_i$. Then the differential operator

$$\delta' := \delta - \sum_i \gamma_i \delta_i$$

has total order strictly smaller than $ord^T(\delta)$ and then δ' should be in J. This implies $\delta \in J$ which is a contradiction. This proves $J = I$. Finally, it is a domain because of Proposition 2.12. □

Proposition 2.16. *The ring $D_{R|\Bbbk}$ of $\Bbbk$-linear differential operators of the polynomial ring $R = \Bbbk[x_1, \dots, x_n]$ is simple.*

Proof. Let I be a non-zero two-sided ideal of $D_{R|\Bbbk}$. Choose an element $\delta \neq 0$ of smallest total order in I. If δ has total order 0, it is a constant, and $I = D_{R|\Bbbk}$.

Assume that δ has total order $m > 0$ and let us get a contradiction. Suppose that (α, β) is a multi-index such that $|\alpha| + |\beta| = m$. If $x^\alpha \partial^\beta$ is a summand of δ with non-zero coefficient and $\beta_i \neq 0$, then $[x_i, \delta]$ is non-zero and has total order $m - 1$, by Proposition 2.12 (2). Since I is a two-sided ideal of $D_{R|\Bbbk}$, we have that $[x_i, \delta] \in I$. But this contradicts the minimality of δ. Thus $\beta = (0, \dots, 0)$. Since $m > 0$, we must have that $\alpha_i \neq 0$ for some $i = 1, \dots, n$. Hence $[\partial_j, \delta]$ is a non-zero element of I of total order $m - 1$ again by Proposition 2.12 (2), and once again we have a contradiction. □

Corollary 2.17. *Let $D_{R|\Bbbk}$ be the ring of $\Bbbk$-linear differential operators of a polynomial ring R. Every ring homomorphism $\phi : D_{R|\Bbbk} \to S$, where S is a possibly non-commutative ring, is injective.*

Proof. This is because $Ker(\phi) \subset D_{R|\Bbbk}$ is a two-sided ideal (as ϕ is a ring morphism) and because $D_{R|\Bbbk}$ is simple. □

Proposition 2.18. *Consider the ring $D_{R|\Bbbk}$ of $\Bbbk$-linear differential operators of a polynomial ring R with the Bernstein's filtration $\mathcal{B} = (B_i)_{i\in\mathbb{Z}_{\geq 0}}$. Then, the associated graded ring $gr_{\mathcal{B}}(D_{R|\Bbbk})$ is a commutative ring with unit.*

Proof. Consider the product of two differential operators $\delta_1 \in B_k$ and $\delta_2 \in B_l$. We have that $\delta_1\delta_2 - \delta_2\delta_1 \in B_{k+l-1}$ by Proposition 2.12, therefore, $\delta_1\delta_2 = \delta_2\delta_1 \in B_{k+l}/B_{k+l-1}$. This means that in each graded piece, the product of the classes of two elements is commutative and this is enough to prove commutativity in $gr_{\mathcal{B}}(D_{R|\Bbbk})$. More concretely:

$$\left(\sum_k \theta_k\right)\left(\sum_l \gamma_l\right) = \sum_n \sum_{k+l=n} \theta_k\gamma_l$$

where $\theta_k \in B_k/B_{k-1}$ and $\gamma_l \in B_l/B_{l-1}$. The unit is the class of 1 in $B_0/B_{-1} = B_0$. □

Remark 2.19. The family of vector spaces $(B_k/B_{k+1})_{k\in\mathbb{Z}_{\geq 0}}$ is a grading on the ring $gr_{\mathcal{B}}(D_{R|\Bbbk})$.

Proposition 2.20. *Let $D_{R|\Bbbk}$ be the ring of $\Bbbk$-linear differential operators of a polynomial ring R with the Bernstein's filtration $\mathcal{B} = (B_i)_{i\in\mathbb{Z}_{\geq 0}}$. Then, the associated graded ring $gr_{\mathcal{B}}(D_{R|\Bbbk})$ is isomorphic to the polinomial ring $\Bbbk[x,\xi]$.*

Proof. We only have to consider, for each $k \in \mathbb{N}$, the morphisms of vector spaces

$$\nu_k : B_k/B_{k-1} \longrightarrow \Bbbk[x,\xi]_k$$
$$\left(\sum_{|\alpha+\beta|=k} p_{\alpha\beta}x^\alpha\partial^\beta\right) + B_{k-1} \longmapsto \sum_{|\alpha+\beta|=k} p_{\alpha\beta}x^\alpha\xi^\beta$$

where $\Bbbk[x,\xi]_k$ is the $\Bbbk$-vector space of homogeneous polynomials of degree k.

This family of isomorphisms gives us the isomorphism $\nu : gr_{\mathcal{B}}(D_{R|\Bbbk}) \to \Bbbk[x,\xi]$ of graded rings. □

Given this identification we obtain:

Corollary 2.21. *The dimension of the ring $D_{R|\Bbbk}$ of $\Bbbk$-linear differential operators over the polynomial ring $R = \Bbbk[x_1,\dots,x_n]$ is*

$$\dim(D_{R|\Bbbk}) = 2n = \dim(\Bbbk[x,\xi]).$$

2.4 Modules over the Weyl algebra

As we have stated in the first section (see Definition 1.8), given a left finitely generated $D_{R|\Bbbk}$-module M, one can associate to M a filtration compatible with the filtration in $D_{R|\Bbbk}$. In particular, we are going to consider Bernstein's filtration $\mathcal{B} = (B_k)_{k\in\mathbb{N}}$ of the ring $D_{R|\Bbbk}$ of differential operators on $R = \Bbbk[x_1,\dots,x_n]$ and we are going to call *B-filtrations* the filtrations $\mathcal{G} = (G_k)_{k\in\mathbb{N}}$ of M compatible with the Bernstein filtration.

Example 2.22. Let us give three examples of $D_{R|\Bbbk}$-modules:

1. The ring $R = \Bbbk[x_1, \dots, x_n]$ is a $D_{R|\Bbbk}$-module. This is clear because a $D_{R|\Bbbk}$-module structure on R is the same as a ring homomorphism $D_{R|\Bbbk} \to End_{\Bbbk}R$, but since $D_{R|\Bbbk}$ is already a subalgebra of $End_{\Bbbk}R$, we can take the natural inclusion. We spell the details out for convenience. $(R, +)$ is an abelian group and to prove that $D_{R|\Bbbk}$-acts linearly on R, consider the map

$$\begin{aligned} D_{R|\Bbbk} \times R &\to R. \\ (\delta, p) &\mapsto \delta(p) \end{aligned}$$

   Trivially, for all $a, b \in \Bbbk$, $p, q \in R$ and $\delta, \delta_1, \delta_2 \in D_{R|\Bbbk} \subset End_{\Bbbk}(R)$ it holds that

   - $\delta(ap + bq) = a\delta(p) + b\delta(q)$,
   - $(\delta_1 + \delta_2)(p) = \delta_1(p) + \delta_2(p)$,
   - $(\delta_1\delta_2)(p) = \delta_1(\delta_2(p))$ and
   - $1(p) = p$.

2. Consider $\Bbbk(x_1, \dots, x_n)$ the field of rational functions. We can extend the left action of $D_{R|\Bbbk}$ on $R = \Bbbk[x_1, \dots, x_n]$ to the field of rational functions. The x_i continue to act by multiplication and ∂_i act on $p/q \in \Bbbk(x_1, \dots, x_n)$ as

$$\partial_i \left(\frac{p}{q} \right) = \frac{\partial_i(p)q - p\partial_i(q)}{q^2}.$$

   It can be shown easily that this left action of $D_{R|\Bbbk}$ acts linearly on $\Bbbk(x_1, \dots, x_n)$ (this is proving the four properties mentioned before for the extended action). Note that this $D_{R|\Bbbk}$-module is not finitely generated.

3. The localization of the polynomial ring $R = \Bbbk[x_1, \dots, x_n]$ at a nonzero element $f \in R$, generally denoted by R_f, i.e.

$$R_f = \left\{ \frac{g}{f^k} \;\mid\; g \in R \text{ and } k \geq 0 \right\}.$$

   Observe that $\partial_i(g/f^k)$ has denominator f^{2k}. Hence these rational functions are preserved by a partial differentiation and by multiplication by a polynomial. In other words, R_f is a left $D_{R|\Bbbk}$-submodule of $\Bbbk(x_1, \dots, x_n)$. And it was proven by Bernstein that R_f is finitely generated as $D_{R|\Bbbk}$-module.

Remark 2.23. Consider the Bernstein filtration on the ring $D_{R|\Bbbk}$ of $\Bbbk$-linear differential operators on a polynomial ring $R = \Bbbk[x_1, \dots, x_n]$. Let $I \subset D_{R|\Bbbk}$ be an ideal and denote $B_k(I) = B_k \cap I$ for $\Bbbk \in \mathbb{Z}_{\geq 0}$. The family $(B_k(I))_{k \in \mathbb{Z}_{\geq 0}}$ is a B–filtration on I considered as a left $D_{R|\Bbbk}$-module.

Definition 2.24. Let $\mathcal{G} = (G_k)_{k\in\mathbb{N}}$ be a filtration on a $D_{R|\Bbbk}$-module M. For each nonzero $m \in M$ we call the $\mathcal{G}$-*order* of m, $ord^{\mathcal{G}}(m)$, the integer k such that $m \in G_k/G_{k-1}$. And given the $\Bbbk$-linear map

$$\begin{aligned} \sigma_k^{\mathcal{G}} : G_k &\longrightarrow G_k/G_{k+1} \\ m &\longmapsto m + G_{k-1} \end{aligned}$$

we call $\sigma_k^{\mathcal{G}}$ the k-th $\mathcal{G}$-*symbol map* associated with the filtration $\mathcal{G}$.

Remark 2.25. In the case when $M = D_{R|\Bbbk}$ and $\mathcal{G} = (B_k)_{k\in\mathbb{N}}$ is the Bernstein's filtration on $D_{R|\Bbbk}$, the corresponding k-th $\mathcal{G}$-symbol map will be also denoted by σ_k^B. In this case, following the notations of Proposition 2.20, if $\delta \in B_k/B_{k+1}$ then $(\nu_k \circ \sigma_k^B)(\delta) = \sigma^T(\delta)$, which means that this composition coincides with the total principal symbol of δ.

As we explained before, if there exists a filtration on a module M compatible with the filtration on the ring, one can construct a graded ring associated to M. So, in this case: if M is a $D_{R|\Bbbk}$-module with a B-filtration $\mathcal{G} = (G_k)_k$, then we can construct the abelian group

$$gr_{\mathcal{G}}(M) := \bigoplus_{k\geq 0} G_k/G_{k-1}.$$

Proposition 2.26. *Let M be a $D_{R|\Bbbk}$-module and $\mathcal{G} = (G_k)_{k\in\mathbb{N}}$ a B-filtration on M. The abelian group $gr_{\mathcal{G}}(M)$ has a natural structure of $gr_{\mathcal{B}}(D_{R|\Bbbk})$-module.*

Proof. The map

$$\mu : gr_{\mathcal{B}}(D_{R|\Bbbk}) \times gr_{\mathcal{G}}(M) \to gr_{\mathcal{G}}(M)$$

defined by the bilinearity of the maps

$$\begin{aligned} \mu_k : B_k/B_{k-1} \times G_l/G_{l-1} &\longrightarrow G_{k+l}/G_{k+l-1} \\ (\delta_k, m_l) &\longmapsto \delta_k m_l \end{aligned}$$

defines on $gr_{\mathcal{G}}(M)$ a structure of $gr_{\mathcal{B}}(D_{R|\Bbbk})$-module. □

Theorem 2.27. *Let M be a $D_{R|\Bbbk}$-module and $\mathcal{G} = (G_k)_{k\in\mathbb{N}}$ a B-filtration on M. If $gr_{\mathcal{G}}(M)$ is a finitely generated $\Bbbk[x,\xi]$-module then M is Noetherian.*

Proof. As $D_{R|\Bbbk}$ is Noetherian, it suffices to prove that M is finitely generated. As $gr_{\mathcal{G}}(M)$ is a finitely generated $\Bbbk[x,\xi]$-module, let $\bar{m}_1, \ldots, \bar{m}_r$ be a homogeneous generating system of $gr_{\mathcal{G}}(M)$ where $m_i \in G_{s_i}$ for some $s_i \in \mathbb{N}$ and $i = 1, \ldots, r$. Let us prove that $\{m_1, \ldots, m_r\}$ generates M.

Let $M' = \langle m_1, \ldots, m_r \rangle$, we will prove that $M = M'$. Let $m \in M'$, by induction on $ord^{\mathcal{G}}(m)$ as defined in Definition 2.24.

- Case $ord^{\mathcal{G}}(m) = 0$. It is nothing to prove.

- Case $ord^{\mathcal{G}}(m) > 0$. Suppose that for any element $m' \in M$ such that $ord^{\mathcal{G}}(m') \leq k$ we have that $m' \in M'$ for some $k \in \mathbb{N}$. Let $m \in M$ be an element such that $ord^{\mathcal{G}}(m) = k+1$. Then its equivalent class in $gr_{\mathcal{G}}(M)$ can be written as

$$\overline{m} = \sum_i p_i(x,\xi)\overline{m_i}$$

for some homogeneous polynomials p_i such that $\deg(p_i) = k+1-s_i$. We can write now

$$m' = m - \sum_i \delta_i m_i$$

for some differential operators δ_i such that $\sigma^T(\delta_i) = s_i$. As $\overline{m'} = \bar{0}$ in G_{k+1} because $ord^{\mathcal{G}}(m') < k$, by induction hypothesis we have that $m' \in M'$. Thus, $m \in M'$. □

Definition 2.28. Let M be a $D_{R|\mathbb{k}}$-module and $\mathcal{G} = (G_k)_{k\in\mathbb{N}}$ be a B-filtration on M. We say that $\mathcal{G}$ is a *good filtration* if $gr_{\mathcal{G}}(M)$ is a finitely generated $gr_B(D_{R|\mathbb{k}})$-module.

A useful characterization of good filtrations is the following:

Proposition 2.29. *Let M be a $D_{R|\mathbb{k}}$-module and $\mathcal{G} = (G_k)_k$ be a B-filtration on M. The following conditions are equivalent:*

(i) $\mathcal{G}$ is a good B-filtration on M.

(ii) There exists $k_0 \in \mathbb{N}$ such that $G_{k+l} = B_l G_k$ for all $l \geq 0$ and for all $k \geq k_0$.

Proof.
- *i)* $\implies$ *ii)*. Let $\overline{m_1}, \ldots, \overline{m_r}$ be a homogeneous system of generators of $gr_{\mathcal{G}}(M)$. Suppose that $m_j \in G_{k_j} \setminus G_{k_j-1}$ for $i = 1, \ldots, r$. Consider $k_0 := \max k_j$.

  We will prove by induction on l that $G_{k+l} = B_l G_k$ for all $l \geq 0$ and all $k \geq k_0$.

  - The case $l = 0$ is trivial.
  - Suppose the result is true for $l-1$ for some $l \geq 0$. Consider $m \in G_{k+l}$ for $k \geq k_0$ and its equivalence class in G_{k+l}/G_{k+l-1}:

    $$\overline{m} = m + G_{k+l-1} = \sum_j f_j \overline{m_j}$$

    for some homogeneous polynomial $f_j \in \mathbb{k}[x,\xi]$ of degree $k+l-k_j$. Let us write

    $$m' = m - \sum_j \delta_j m_j$$

    for some $\delta_j \in B_{k+l-k_j}$ such that $\sigma^T(\delta_j) = f_j$.
    It is clear that $m' \in G_{k+l-1}$ and by induction $m' \in B_{l-1}G_k$. As $k - k_0 \geq 0$ we also have $B_{k+l-k_j} = B_l B_{k-k_j}$. Then

    $$m = m' + \sum_j \delta_j m_j.$$

    Since $\delta_j \in B_{k+l-k_j} = B_l B_{k-k_j}$ then $\delta_j m_j \in B_l B_{k-k_j} G_{k_j} \subseteq B_l G_k$.

- *ii)* $\implies$ *i)*. We only have to prove that $gr_{\mathcal{G}}(M)$ is generated by $G_0 \oplus G_1/G_0 \oplus \cdots \oplus G_{k_0}/G_{k_0-1}$ because each G_n is a vector space of finite dimension. To show this, if $m \in G_k$ and $k > k_0$ then write $k = k_0 + i$ for $i = k - k_0 > 0$. Since $G_k = B_i G_{k_0}$ we have

$$m = \sum_{j=0}^{r} \delta_j m_j$$

where $\delta_j \in B_i$ and $m_1, \dots, m_r$ is a basis of G_{k_0}. Then, we can write

$$m + G_{k-1} = \overline{m} = \sum_{j=0}^{r} \delta_j m_j + G_{k-1} = \sum_{j=0}^{r} (\delta_j + B_{i-1})(m_j + G_{k_0-1}) = \sum_{j=0}^{r} \overline{\delta_j}\,\overline{m_j}. \qquad \square$$

We show next that good filtrations are equivalent and thus the definition of dimension does not depend on the choice of such a good filtration. Notice also that if M is a finitely generated $D_{R|\Bbbk}$-module and $\mathcal{G}$ is the standard filtration, then $\mathcal{G}$ is good.

Proposition 2.30. *Let M be an $D_{R|\Bbbk}$-module; $\mathcal{G} = (G_k)_{\Bbbk \in \mathbb{Z}_{\geq 0}}$ and $\mathcal{G}' = (G'_k)_{\Bbbk \in \mathbb{Z}_{\geq 0}}$ be two B-filtrations on M. We have*

i) If $\mathcal{G}$ is a good filtration then there exists $k_1 \in \mathbb{N}$ such that

$$G_k \subset G'_{k+k_1}$$

for all $k \in \mathbb{N}$.

ii) If $\mathcal{G}$ and $\mathcal{G}'$ are both good filtrations then there exists $k_2 \in \mathbb{N}$ such that

$$G'_{k-k_2} \subset G_k \subset G'_{k+k_2}$$

for all $k \in \mathbb{N}$.

Proof. i) By the previous proposition there exists $k_0 \geq 0$ such that $G_{k+l} = B_l G_k$ for all $l \geq 0$ and for all $k \geq k_0$. As G_{k_0} is a $\Bbbk$-vector space of finite dimension there exists $k_1 \in \mathbb{N}$ such that $G_{k_0} \subseteq G'_{k_1}$. If $k \geq k_0$ we have

$$G_k = G_{k-k_0+k_0} = B_{k-k_0} G_{k_0} \subseteq B_{k-k_0} G'_{k_1} \subseteq G'_{k-k_0+k_1} \subseteq G'_{k+k_1}.$$

If $0 \leq k \leq k_0$ then $G_k \subseteq G_{k_0} \subseteq G'_{k_1} \subseteq G'_{k+k_1}$.

ii) It follows from *i)*. As there exists $l_1 \in \mathbb{N}$ such that

$$G_k \subset G'_{k+l_1}$$

for all $k \in \mathbb{N}$ and there exists $l_2 \in \mathbb{N}$ such that

$$G'_k \subset G_{k+l_2}$$

for all $k \in \mathbb{N}$. It suffices to define $k_2 = \max\{l_1, l_2\}$ and this k_2 satisfies the condition. $\square$

2.5 Bernstein's inequality and holonomic modules

Let $R = \Bbbk[x_1, \dots, x_n]$ be the polynomial ring in n variables with coefficients in a field $\Bbbk$ of characteristic zero. A fundamental result in the theory of D-modules, known as Bernstein's inequality, establishes a lower bound on the dimension of a finitely generated $D_{R|\Bbbk}$-module M.

This result has been proven in different settings: Bernstein proved it for the case when $R = \Bbbk[x_1, \dots, x_n]$ [Bernstein, 1971]; Sato, Kawai and Kashiwara proved the Bernstein's inequality in the case when $R = \mathbb{C}\{x_1, \dots, x_n\}$ is the ring of holomorphic functions [Sato et al., 1972]; Björk for the ring $\Bbbk[[x_1, \dots, x_n]]$ of formal power series [Björk, 1979]; and Gabber gave an algebraic proof of this result also in the case when R is the polynomial ring in n variables [Gabber, 1981].

Let $R = \Bbbk[x_1, \dots, x_n]$, and consider the Bernstein's filtration $\mathcal{B} = (B_i)_{i \in \mathbb{Z}_{\geq 0}}$ in $D_{R|\Bbbk}$. Let M be a left finitely generated $D_{R|\Bbbk}$-module with the standard B-filtration $\mathcal{G} = (G_i)_{i \in \mathbb{Z}_{\geq 0}}$, which in this setting is good. We are going to do a reinterpretation of the proof given by Joseph that can be found in [Coutinho, 1995] using the Bernstein's filtration on $D_{R|\Bbbk}$, contrary to all the cases cited before that used the filtration given by the order.

Let us first prove an upper bound:

Proposition 2.31. *Let M be a finitely generated $D_{R|\Bbbk}$-module. Then* $\dim(M) \leq 2n$.

Proof. Suppose that M is generated by r elements. Then there exists a surjective homomorphism $\phi : D^r_{R|\Bbbk} \to M$. Thus, we can consider the exact sequence

$$0 \to \ker \phi \to D^r_{R|\Bbbk} \to M \to 0.$$

By dimension theory we have that $\dim(D^r_{R|\Bbbk}) = \max\{\dim(M), \dim(\ker \phi)\}$ and $\dim(D^r_{R|\Bbbk}) = \dim(D_{R|\Bbbk} \oplus \cdots \oplus D_{R|\Bbbk}) = \dim(D_{R|\Bbbk}) = 2n$, therefore $\dim(M) \leq 2n$. $\square$

We are going to use the next lemmas in order to prove the lower bound known as Bernstein's inequality:

Lemma 2.32. *Let $D_{R|\Bbbk}$ be the ring of differential operators over a polynomial ring R and $\mathcal{B} = (B_i)_{i \in \mathbb{Z}_{\geq 0}}$ be the Bernstein filtration. Then, for each $i \in \mathbb{N}$ and each nonzero differential operator $\delta \in B_i$, we have*

$$1 \in B_i \delta B_i.$$

Proof. We are going to prove it by induction on the total order:

- For $i = 0$ as $B_0 = \Bbbk$ we have that the statement is true.
- Suppose that $i > 0$ and suppose the statement is true for $i - 1$, this is, for all $\delta' \in B_{i-1}$, we have $B_{i-1}\delta' B_{i-1}$. Now, consider a nonzero differential operator $\delta \in B_i$, and that $\delta = \sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \partial^\beta$. Since $\delta \neq 0$ there exists a monomial $p_{\alpha,\beta} x^\alpha \partial^\beta \neq 0$.

  If $ord^T(\delta) \neq 0$ there exists an exponent $\alpha_j \neq 0$ or $\beta_j \neq 0$. We are going to suppose that we have $\alpha_j \neq 0$, the other case can be reasoned similarly.

We have that the commutator $[\delta, \partial_j] \in B_{i-1}$ and, by hypothesis of induction, we have that $B_{i-1}[\delta, \partial_j]B_{i-1}$. On the other hand, $[\delta, \partial_j] = \delta\partial_j - \partial_j\delta \in B_1\delta B_1$, therefore,

$$1 \in B_{i-1}B_1\delta B_1 B_{i-1} \subseteq B_i\delta B_i.$$

If $ord^T(\delta) = 0$ then this monomial belongs to $\Bbbk$, and the result follows. □

Lemma 2.33. *Let M be a finitely generated left $D_{R|\Bbbk}$-module with a B-filtration $\mathcal{G} = (G_n)_{n\in\mathbb{Z}_{\geq 0}}$. Suppose that $G_i \neq 0$. The $\Bbbk$-linear transformation*

$$\begin{aligned} \phi : B_i &\longrightarrow Hom_k(G_i, G_{2i}) \\ \delta &\longmapsto \phi_\delta : G_i \to G_{2i} : m \mapsto \delta m \end{aligned}$$

is injective.

Proof. We are going to proof the contrapositive. Suppose that there exists a nonzero differential operator $\delta \in B_i$ such that $\phi_\delta = 0$. Then $\delta B_i G_i \subseteq \delta G_{2i} = 0$, and thus $\delta B_i G_i = 0$. By Lemma 2.32, we have $1 \in B_i\delta B_i$. Thus, multiplying by G_i in each side of the containment, we obtain $G_i \subseteq B_i\delta B_i G_i = 0$, and therefore $G_i = 0$. □

Theorem 2.34 (Bernstein's inequality)**.** *Let M be a nonzero finitely generated $D_{R|\Bbbk}$-module. Then* $\dim(M) \geq n$*.*

Proof. ([Coutinho, 1995]) Let $\{m_1, \dots m_r\}$ be a set of generators of M and let $\mathcal{G}$ be a good filtration obtained by giving each of this generators degree zero. Then $G_0 \neq 0$. Let $P(t) = H(M, \mathcal{G}, t)$ be the corresponding Hilbert polynomial.

By the Lemma 2.33, B_i can be embedded in $\mathrm{Hom}_\Bbbk(G_i, G_{2i})$. In particular,

$$\dim_\Bbbk B_i \leq \dim_\Bbbk(\mathrm{Hom}_\Bbbk(G_i, G_{2i})) = \dim_\Bbbk G_i \cdot \dim_\Bbbk G_{2i}.$$

Thus, assuming that $i >> 0$, $\dim_\Bbbk B_i \leq P(i)P(2i)$.

On the other hand, $\dim_\Bbbk B_i = \binom{i+2n}{2n}$ is a polynomial function in i of degree $2n$. Hence, as a polynomial in i, $P(i)P(2i)$ must have degree $\geq 2n$. But degree of $P(i)P(2i)$ is $2\dim(M)$. Thus, $\dim(M) \geq n$. □

Definition 2.35. A finitely generated $D_{R|\Bbbk}$-module M is said to be *holonomic* if either $M = (0)$ or $\dim(M) = n$.

The class of holonomic $D_{R|\Bbbk}$-modules satisfy many good properties, for instance they have finite length and play a central role in many aspects of the theory of D-modules.

Some examples of holonomic $D_{R|\Bbbk}$-modules are the ring R and the localization of the ring R in an nonzero element $f \in R$, R_f. Let us prove it:

Proposition 2.36. *The polynomial ring $R = \Bbbk[x_1, \dots, x_n]$ is a holonomic $D_{R|\Bbbk}$-module.*

Proof. Observe that $R = \Bbbk[x_1, \dots, x_n] \simeq \frac{D_{R|\Bbbk}}{D_{R|\Bbbk}(\partial_1,\dots,\partial_n)}$.

Recall the B-filtration on an ideal I of $D_{R|\Bbbk}$ explained in Remark 2.23, from $(B_k(I))_{k\in\mathbb{Z}_{\geq 0}}$ we can construct the associated graded module $gr_B(I)$.

Now, since $I = D_{R|\Bbbk}(\partial_1, \dots, \partial_n)$ is an ideal of $D_{R|\Bbbk}$ we can consider its graded ring $gr_B(I)$ which is contained in $gr_B(D_{R|\Bbbk}) \simeq \Bbbk[x, \xi]$. It can be proved that $gr_B(I)$ is generated by $\xi_1, \dots, \xi_n$. Thus the Hilbert polynomial of the graded module $\Bbbk[x, \xi]/\langle \xi_1, \dots, \xi_n\rangle \simeq \Bbbk[x]$ is $\binom{t+n-1}{n-1}$ which is a polynomial function of degree n as we proved on Example 1.22. Therefore $\Bbbk[x_1, \dots, x_n]$ is holonomic. □

Proposition 2.37. *The localization R_f if the polynomial ring at an element $f \in R$ is a holonomic $D_{R|\Bbbk}$-module.*

Proof. Suppose that f has degree m. Consider the B-filtration $\mathcal{G} = (G_k)_{k\in\mathbb{Z}_{\geq 0}}$ on R_f where

$$G_k = \{g/f^k \mid \deg(g) \leq (m+1)k\}$$

First, let us check that $\mathcal{G}$ is indeed a B-filtration: Let g/f^k be an element of R_f and suppose that g has degree s. Then $g/f^k = g \cdot f^s/f^{s+k}$. But gf^s has degree $s(m+1)$, which is $\leq (m+1)(s+k)$. Then $g/f^k \in G_{s+k}$. Thus, $R_f = \cup_{k\in\mathbb{Z}_{\geq 0}} G_k$.

Now, suppose that $g/f^k \in G_k$, then $\deg(g) \leq (m+1)k$. Observe that for each $i \in \{1, \dots, n\}$ we have $\deg(x_i g) = \deg(g) + 1$ so

$$x_i(g/f^k) = x_i g f/f^{k+1} \in G_{k+1}.$$

And now consider

$$\partial_i\left(\frac{g}{f^k}\right) = \frac{f\partial_i(g) - kg\partial_i(f)}{f^{k+1}},$$

as the numerator has degree $\leq (m+1)k + (m-1)$, we have

$$\partial_i(g/f^k) \in G_{k+1}.$$

Therefore, $B_1 G_k \subseteq G_{k+1}$. Since $B_i = B_1^i$ we also have that $B_i G_k \subseteq G_{i+k}$.

Finally, we have that $\dim_{\Bbbk}(G_k)$ is less than the dimension of polynomials of degree $\leq (m+1)k$. Hence, G_k is finite dimensional, which shows that $\mathcal{G}$ is a filtration on R_f and

$$\dim_{\Bbbk} G_k \leq \binom{(m+1)k+n}{n} \leq \frac{(m+1)^n}{n!}k^n + \text{ terms of lower degree on } k.$$

Therefore, $\dim(R_f) = n$ and R_f is a holonomic $D_{R|\Bbbk}$-module. □

3. The ring of differential operators of finitely generated $\Bbbk$-algebras

In this section we will focus on rings $D_{S|\Bbbk}$ of differential operators on a finitely generated $\Bbbk$-algebra S which can always be presented as $S = R/I$ for some ideal $I \subseteq R = \Bbbk[x_1, \ldots, x_n]$. In Section 3.1 we will see several results about differential operators that preserve the ideal I. These reults will be used in Section 3.2, where we will write the ring $D_{S|\Bbbk}$ of differential operators of S in terms of the differential operators of R (the Weyl algebra) and the differential operators that preserve I.

We will follow the same structure of Section 2: we will introduce two different filtrations on $D_{S|\Bbbk}$, one using the order and a generalized Bernstein's filtration, which we will use to describe the structural properties of $D_{S|\Bbbk}$ under certain assumptions. Finally, we will prove a generalized Bernstein's inequality for modules over $D_{S|\Bbbk}$ in some cases.

3.1 Ideal-preserving differential operators

Suppose that $I = \langle f_1, \ldots, f_r\rangle$ is an ideal in $R = \Bbbk[x_1, \ldots, x_n]$ and that $\delta \in D_{R|\Bbbk}$ is a differential operator.

Notation 3.1. In order to have a cleaner notation we are going to write

$$\mathcal{D}(I) = \{\delta \in D_{R|\Bbbk} \mid \delta(I) \subseteq I\}.$$

Observe that $\delta \in \mathcal{D}(I)$ if and only if $\delta(x^\alpha f_j) \in I$, for all α and $j = 1, \ldots, r$.

Lemma 3.2. *Let $I = \langle f_1, \ldots, f_r\rangle$ be an ideal in R, and let $\delta \in D_{R|\Bbbk}$ be a differential operator. Then $\delta \in \mathcal{D}(I)$ if and only if*

(i) $[\delta, x_i](I) \subseteq I$ *for* $i = 1, \ldots, n$, *and*

(ii) $\delta(f_j) \in I$ *for* $j = 1, \ldots, r$.

Proof. If $\delta \in \mathcal{D}(I)$, we trivially have (i) and (ii). To prove the converse, assume $\delta \in D_{R|\Bbbk}$ is such that (i) and (ii) hold. Since any element $f \in I$ can be expressed as $f = \sum_{i=1}^r a_i f_i$, it suffices to prove that $\delta(x^\alpha f_j) \in I$ for all α and for $j = 1, \ldots, r$. By induction on $|\alpha|$:

- If $|\alpha| = 0$, $\delta \in \mathcal{D}(I)$ by (ii).
- Assume that $\delta(x^\alpha f_j) \in I$ for $|\alpha| = m$, and let $\beta = \alpha + e_i$ where $e_i = (0, \ldots, \overset{i}{1}, \ldots, 0)$, for some $i \in \{1, \ldots, n\}$. Then

$$\delta(x^\beta f_j) = \delta(x_i x^\alpha f_j) = [\delta, x_i](x^\alpha f_j) + (x_i\delta)(x^\alpha f_j) \in I,$$

since the first term is in I by (i) and the second term is contained by hypothesis. $\square$

Proposition 3.3. *Let $I = \langle f_1, \dots, f_r \rangle$ be an ideal in R, and let $\delta \in D^m_{R|\Bbbk}$ be a differential operator of order $m \geq 1$. Then $\delta \in \mathcal{D}(I)$ if and only if $\delta(x^\alpha f_j) \in I$ for $|\alpha| \leq m-1$ and $j = 1, \dots, r$.*

Proof. It is clear that $\delta(x^\alpha f_j) \in I$ if $\delta \in \mathcal{D}(I)$. So, we only have to prove the other direction.

First, observe that any element in R preserves the ideal I, so we may assume without loss of generality that the order zero component of δ is zero.

We will now proceed by induction on m:

- If $m = 1$, then $\delta \in Der(R)$. By assumption, $\delta(f_j) \in I$ for all j, thus
$$\delta(hf_j) = \delta(h)f_j + h\delta(f_j) \in I$$
for every $h \in R$, so $\delta(I) \subseteq I$.

- Assume the condition is true for some $m \geq 1$ and that $\delta \in D_{R|\Bbbk}$ is of order $m+1$ and such that $\delta(x^\alpha f_j) \in I$ for $|\alpha| \leq m$ and $j = 1, \dots, r$. Then $\alpha = 0$ shows $\delta(f_j) \in I$ for all $j = 1, \dots, r$. For any β such that $|\beta| \leq m-1$ we have $|\beta + e_i| \leq m$, $i = 1, \dots, n$, so
$$\delta(x^{\beta+e_i} f_j) = \delta(x_i x^\beta f_j) = [\delta, x_i](x^\beta f_j) + x_i \delta(x^\beta f_j).$$
Observe that $\delta(x^{\beta+e_i} f_j) \in I$ by hypothesis, then this equality shows that $[\delta, x_i](x^\beta f_j) \in I$ for $i = 1, \dots, n$, $|\beta| \leq m-1$, and $j = 1, \dots, r$. But $[\delta, x_i]$ is a differential operator of order m, so $[\delta, x_i](I) \subseteq I$ for $i = 1, \dots, n$ by the induction hypothesis. Thus $\delta(I) \subseteq I$ by Lemma 3.2. $\square$

Lemma 3.4. *Let $\{I_j\}_{j \in J}$ be any family of ideals in R, and let $I = \cap_{j \in J} I_j$. Then*
$$\bigcap_{j \in J} \mathcal{D}(I_j) \subseteq \mathcal{D}(I).$$

Proof. Suppose $\delta \in \bigcap_{j \in J} \mathcal{D}(I_j)$. For all $j \in J$ we have $I \subseteq I_j$, so $\delta(I) \subseteq I_j$. Thus, in particular, $\delta(I) \subseteq I$. $\square$

Theorem 3.5. *Suppose $f_1, \dots, f_r \in R$ are nonconstant and pairwise relatively prime. Then*
$$\mathcal{D}(\langle f_1, \dots, f_r \rangle) = \bigcap_{j \in J} \mathcal{D}(\langle f_j \rangle).$$

Proof. If $r = 1$ there is nothing to prove, so let us assume that $r > 1$. As $f_1, \dots, f_r \in R$ are pairwise relatively prime,
$$\bigcap_{i=1}^{r} \langle f_i \rangle = \langle f_1 \dots f_r \rangle$$
so, Lemma 3.4 shows that
$$\bigcap_{j \in J} \mathcal{D}(\langle f_j \rangle) \subseteq \mathcal{D}(\langle f_1 \dots f_r \rangle).$$

Let us prove now the opposite inclusion by induction on r, observe that the polynomials

$$f = f_1 \dots f_{r-1} \text{ and } g = f_r$$

are also nonconstant and pairwise relatively prime. So, if we prove the result for these two polynomials we have the inclusion from induction hypothesis.

Therefore, assume that $f, g \in R$ are nonconstant and relatively prime, we must show that

$$\mathcal{D}(\langle fg \rangle) \subseteq \mathcal{D}(\langle f \rangle) \cap \mathcal{D}(\langle g \rangle).$$

By induction on the order of $\delta \in \mathcal{D}(\langle fg \rangle)$. If δ is of order zero, i.e., $\delta \in R$, then δ preserves any ideal in R.

Now, assume the inclusion is true for differential operators of some order $m \geq 0$, and let $\delta \in \mathcal{D}^{m+1}(\langle fg \rangle)$. Then $[\delta, g] \in D_{R|\Bbbk}$, and for every $h \in R$ we have

$$[\delta, g](hfg) = \delta(hfg^2) - g\delta(hfg) \in \langle fg \rangle.$$

Thus, $[\delta, g] \in \mathcal{D}^m(\langle fg \rangle)$, and $[\delta, g] \in \mathcal{D}^m(\langle f \rangle)$ by the induction hypothesis. Thus, for any $h \in R$,

$$\delta(hfg) - g\delta(hf) = [\delta, g](hf) \in \langle f \rangle.$$

Since $\delta(hfg) \in \langle fg \rangle \subseteq \langle f \rangle$, this implies that $g\delta(hf) \in \langle f \rangle$. But f and g are relatively prime, so we must have $\delta(hf) \in \langle f \rangle$.

If we repeat the argument changing the places of f and g, we obtain $\delta \in \mathcal{D}(\langle g \rangle)$. $\square$

3.2 The ring of differential operators

We will prove that if $I \subseteq R$ is an ideal and $S = R/I$, then

$$D_{S|\Bbbk} = \frac{\mathcal{D}(I)}{ID_{R|\Bbbk}}.$$

The proof of this result can be found on [McConnell and Robson, 1988] (Theorem 5.3.).

However, in order to obtain this result we will have to prove several lemmas before:

Lemma 3.6. *Let $I \subseteq R$ be an ideal and $S = R/I$. Let $\pi : R \to S$ denote the quotient map. Let $\theta \in D^m_{S|\Bbbk}$ and $\delta \in D^m_{R|\Bbbk}$, such that $\theta(x^\alpha) = \delta(x^\alpha)$ for every $\alpha \in \mathbb{N}^n$, $|\alpha| \leq m$. Then $\theta \circ \pi = \pi \circ \delta$.*

Proof. First, consider the map given by each differential operator $\delta \in \mathcal{D}(I)$:

$$\begin{aligned} \delta : S &\to S \\ \bar{r} &\mapsto \overline{\delta(r)} \end{aligned}$$

We are going to prove the result by induction on m.

- In the case $m = 0$, we have that $\bar{s}(\bar{1}) = \bar{s}$ and $\overline{r(1)} = \bar{r}$ for all $s, r \in R$.
- Assume the condition holds for $k \leq m$. We have,

$$[\theta, \overline{x_j}](\overline{x^\alpha}) = \theta(\overline{x_j x^\alpha}) - \overline{x_j}\theta(\overline{x^\alpha}) = \overline{\delta(x_j x^\alpha)} - \overline{x_j \delta(x^\alpha)} = \overline{[\delta, x_j](x^\alpha)}$$

for all $j \in \{1, \ldots, n\}$ and for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq m$. Since $ord^T([\theta, \overline{x_j}]), ord^T([\delta, x_j]) \leq m$, by the induction hypothesis $[\theta, \overline{x_j}] \circ \pi = \pi \circ [\delta, x_j]$ $(*)$.

Then, for every $j \in \{1, \ldots, n\}$ and $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq m$,

$$\begin{aligned}
\theta(\overline{x_j x^\alpha}) &= [\theta, \overline{x_j}] \circ \pi(\overline{x^\alpha}) + \overline{x_j}\theta(\overline{x^\alpha}) \\
&\overset{(*)}{=} \pi \circ [\delta, x_j](x^\alpha) + \overline{x_j}\theta(\overline{x^\alpha}) \\
&= \pi(\delta(x_j x^\alpha) - x_j \delta(x^\alpha)) + \overline{x_j}\theta(\overline{x^\alpha}) \\
&= \overline{\delta(x_j x^\alpha)}.
\end{aligned}$$

Therefore $\theta \circ \pi(x^\beta) = \pi \circ \delta(x^\beta)$ for every $\beta \in \mathbb{N}^n$, $|\beta| \leq m + 1$. So, by induction this holds for all $\beta \in \mathbb{N}^n$. And by linearity of these differential operators, we conclude they are equal. □

Lemma 3.7. *Let $I \subseteq R$ be an ideal, then*

$$ID_{R|\Bbbk} = \{\delta \in D_{R|\Bbbk} \mid \delta(R) \subseteq I\}$$

Proof. We are going to prove it by double inclusion:

- If $a\delta \in ID_{R|\Bbbk}$, then $(a\delta)(s) = a\delta(s) \in I$ by definition of ideal. Thus,

$$ID_{R|\Bbbk} \subseteq \{\delta \in D_{R|\Bbbk} \mid \delta(R) \subseteq I\}$$

- Let $\delta \in \{\delta \in D_{R|\Bbbk} \mid \delta(R) \subseteq I\}$. By Proposition 2.5, $D_{R|\Bbbk}$ is an R-module with basis $\{\partial^\beta \mid \beta \in \mathbb{N}^n\}$. Then we can write any element as

$$\delta = \sum_{\beta \in \mathbb{N}^n} p_\beta \partial^\beta.$$

So, we only have to prove that $p_\beta \in I$, for all $\beta \in \mathbb{N}^n$. By induction on $|\beta|$:

  - For $|\beta| = 0$. As $p_{(0,\ldots,0)} = \delta(1)$, because all the terms which have $|\beta| > 0$ vanish, we have that $p_{(0,\ldots,0)} \in I$.
  - Assume that we have $p_\beta \in I$ for all $|\beta| < m$, then

$$\delta' = \sum_{|\beta| < m} p_\beta \partial^\beta \in ID_{R|\Bbbk}.$$

Let $\beta' \in \mathbb{N}^n$ such that $|\beta'| = n + 1$. Therefore $\delta'' = \delta - \delta' \in \{\delta \in D_{R|\Bbbk} \mid \delta(R) \subseteq I\}$, and that way we cancel all the terms which has small order. By Proposition 2.4 (2) we have $p_{\beta'} = \delta''(x^{\beta'}) \in I$. □

Theorem 3.8. *Let $I \subseteq R$ an ideal and $S = R/I$, then*

$$D_{S|\Bbbk} = \frac{\mathcal{D}(I)}{ID_{R|\Bbbk}}.$$

Proof. Consider the map

$$\psi : \mathcal{D}(I) \longrightarrow D_{S|\Bbbk} \tag{1}$$

which is defined by the equation $\psi(\delta)(\bar{s}) = \overline{\delta(s)}$ $(**)$.

We have to prove that π is a surjective morphism of filtrated rings and $ker(\psi) = ID_{R|\Bbbk}$.

Let us show first that the morphism is well defined, this is, if $\delta \in \mathcal{D}(I)$ then $\psi(\delta) \in D_{S|\Bbbk}$. Note that, if $\bar{r} \in S$,

$$\begin{aligned}
[\psi(\delta), \bar{s}](\bar{r}) =& (\psi(\delta)\bar{s} - \bar{s}\psi(\delta))(\bar{r}) \\
\overset{(**)}{=}& \overline{\delta(sr)} - \bar{s}(\overline{\delta(r)}) \\
=& \overline{\delta(sr) - s\delta(r)} \\
=& \overline{[\delta, s](r)} \\
=& \psi([\delta, s])(\bar{r})
\end{aligned}$$

Therefore,

$$[\psi(\delta), \bar{s}] = \psi([\delta, s]) \tag{2}$$

By induction on the order of δ.

- δ has order 0. Then $[\delta, r] = 0$ for every $r \in R$ and $\psi([\delta, r]) = 0 \in D_{S|\Bbbk}$. From Equation (2), $[\psi(\delta), \bar{r}] = 0$. Thus, $\psi(\delta) \in D^0_{S|\Bbbk}$.
- Let $m \geq 0$ and assume that $\psi\left(D^k_{R|\Bbbk} \cap \mathcal{D}(I)\right) \subseteq D^k_{S|\Bbbk}$ for $k \leq m$. Let $\delta \in D^{m+1}_{R|\Bbbk} \cap \mathcal{D}(I)$, then we have, by Equation (2) and induction, that $[\psi(\delta), \bar{r}] = \psi[\delta, r] \in D^m_{S|\Bbbk}$. Hence, $\psi(\delta) \in D^{m+1}_{S|\Bbbk}$.

Thus, we have obtained that ψ is well defined and, furthermore, viewing $\mathcal{D}(I)$ as a filtrated ring by the order of its elements, ψ is a morphism of filtrated rings.

We will prove now that $\ker(\psi) = ID_{R|\Bbbk}$. Observe that,

$$\begin{aligned}
\delta \in \ker \psi &\iff \psi(\delta)(\bar{r}) = 0, \text{ for every } \bar{r} \in S \\
&\iff \overline{\delta(r)} = 0, \text{ for every } r \in R \\
&\iff \delta(R) \subseteq I.
\end{aligned}$$

Thus, from the previous lemma, it follows that $\ker(\psi) = ID_{R|\Bbbk}$.

It only remains to prove that ψ is surjective. Let $m \in \mathbb{N}$ and $\theta \in D^m_{S|\Bbbk}$. For every $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq m$, we choose $f_\alpha \in R$ such that $\theta(\overline{x^\alpha}) = \overline{f_\alpha}$. We know that there is a differential operator $\delta \in D^m_{R|\Bbbk}$ such that, if $|\alpha| \leq m$, $\delta(x^\alpha) = f_\alpha$ (by the same reasoning as in the proof of the surjectivity in Theorem 2.6). Since $\theta \circ \pi = \pi \circ \delta$, then

$$\pi \circ \delta(I) = \theta \circ \pi(I) = \theta(\{0\}) = \{0\}.$$

Therefore, $\delta(I) \subseteq I$, i.e., $\delta \in \mathcal{D}(I)$, furthermore $\psi(\delta) = \theta$. Thus, ψ is surjective. ∎

3.3 Filtrations on the ring of differential operators

Let $D_{S|\Bbbk}$ be the ring of $\Bbbk$-linear differential operators of a finitely generated $\Bbbk$-algebra S. Using the general definition of differential operators we can consider on $D_{S|\Bbbk}$ the filtration given by order. Moreover, if S is graded, we can consider a generalized Bernstein's filtration.

If R is a polynomial ring, consider the standard grading $\mathcal{R} = (R_i)_{i\in\mathbb{Z}_{\geq 0}}$ on R where

$$R_i = \left\{ \sum_{\alpha\in\mathbb{N}^n} \lambda_\alpha x^\alpha \colon \lambda_\alpha \in \Bbbk \text{ and } |\alpha| = i \right\}.$$

One can consider now a grading on $D_{R|\Bbbk}$ given by the degree. Let us recall that the *degree* of an operator $\delta = \sum_{\alpha,\beta} p_{\alpha\beta} x^\alpha \partial^\beta \in D_{R|\Bbbk}$ is defined as $\deg(\delta) = \max\{|\alpha| - |\beta| \mid p_{\alpha\beta} \neq 0\}$. Note that under this grading the partial derivatives ∂_i are homogeneous of degree -1. As a consequence, δ is homogeneous of *degree* d if and only if $\delta(R_i) \subseteq R_{i+d}$. This is how we generalized the notion of degree.

If S is a graded $\Bbbk$-algebra with grading $\mathcal{S} = (S_i)_{i\in\mathbb{Z}_{\geq 0}}$ we define the next grading on $D_{S|\Bbbk}$.

Definition 3.9. A differential operator $\delta : S \to S$ is *homogeneous of degree* d if $\delta(S_i) \subseteq S_{i+d}$ for each i.

Remark 3.10. Observe that the inclusion $S \subset D_{S|\Bbbk}$ preserves the grading.

In particular, if $S = R/I$ where I is an homogeneous ideal, then $D_{S|\Bbbk}$ is a graded algebra.

Filtration by order

Let R be a polynomial ring over a field $\Bbbk$ and $S = R/I$ for some ideal $I \subseteq R$. We have already seen that the ring of $\Bbbk$-linear differential operators on S can be described in terms of the $\Bbbk$-linear differential operators in R in the following way

$$D_{S|\Bbbk} \simeq \frac{\mathcal{D}(I)}{ID_{R|\Bbbk}}.$$

So, as the order of the differential operators is preserved under this isomorphism, then the order filtration on $D_{S|\Bbbk}$ is given by

$$D^i_{S|\Bbbk} \simeq \frac{\{\delta \in D^i_{R|\Bbbk} \mid \delta(I) \subseteq I\}}{ID^i_{R|\Bbbk}}.$$

Generalized Bernstein filtration

From now on we assume that $I \subseteq R$ is a homogeneous ideal so that $S = R/I$ is a finitely generated graded $\Bbbk$-algebra. A generalization of Bernstein filtration to this setting was introduced by [Bavula, 2009].

Definition 3.11. Let S be a commutative finitely generated graded $\Bbbk$-algebra, let $w = \max\{\deg(\lambda) \mid \lambda \text{ generator of } S\}$ and let a be a real number greater than w. The generalized Bernstein filtration $\mathcal{B}_{a,S}$ on $D_{S|\Bbbk}$ with slope a is given by

$$(B_{a,S})_i = \Bbbk\{\delta \in D_{S|\Bbbk} \text{ homogeneous } \mid \deg(\delta) + a\ ord(\delta) \leq i\}.$$

If the slope is clear from the context, then we simply write $\mathcal{B}_S$.

Example 3.12. Let $R = \Bbbk[x_1, \ldots, x_n]$ be the polynomial ring in n variables with coefficients in $\Bbbk$. Then the generalized Bernstein filtration with slope 2 is just the usual Bernstein filtration on the Weyl algebra. This is because $\deg(x_i) = 1,\ \deg(\partial_i) = -1,\ ord(x_i) = 0$ and $ord(\partial_i) = 1$. Thus, for any differential operator $\delta \in D_{R|\Bbbk}$ we have

$$\deg(\delta) + 2 \cdot ord(\delta) = ord^T(\delta).$$

3.4 Structural properties of the ring of differential operators

We would like to study in general the structure of the ring $D_{S|\Bbbk}$ of differential operators over a graded $\Bbbk$-algebra S. However, we may not have good properties on this ring in general, let us see two examples that illustrate how diverse our chances of obtaining good properties for $D_{S|\Bbbk}$ are.

Example 3.13. Consider the second Veronese $\mathbb{C}$-algebra

$$S = \mathbb{C}[x_1, \ldots, x_n]^{(2)} = \mathbb{C}[x_1^2, x_1x_2, \ldots, x_n^2],$$

whose ring of $\Bbbk$-linear differential operators of S is

$$D_{S|\mathbb{C}} = \mathbb{C}\langle x_1\partial_1, x_1\partial_2, \ldots, x_n\partial_n, \partial_1^2, \partial_1\partial_2, \ldots, \partial_n^2\rangle.$$

In this case, $D_{S|\mathbb{C}}$ is a finitely generated $\mathbb{C}$-algebra and Noetherian. $D_{S|\mathbb{C}}$ has differential operators of negative degree. Also, the ring S is $D_{S|\mathbb{C}}$-simple. And, if we consider the associated graded ring with the order filtration $\mathcal{F}$, we obtain that $gr_{\mathcal{F}}(D_{S|\mathbb{C}})$ is finitely generated as a $\mathbb{C}$-algebra.

Example 3.14. In [Bernstein et al., 1972], Bernstein, I.M. Gelfand and S.I. Gelfand proved that the ring of differential operators $D_{S|\mathbb{C}}$ of the cubic curve

$$S = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3),$$

is not finitely generated as $\mathbb{C}$-algebra and it is not Noetherian. $D_{S|\Bbbk}$ has no differential operators of negative degree. In this case, S is not $D_{S|\mathbb{C}}$-simple. And, finally, if we consider the associated graded ring with the order filtration $\mathcal{F}$, we obtain that $gr_{\mathcal{F}}(D_{S|\mathbb{C}})$ is not finitely generated as a $\mathbb{C}$-algebra.

3.5 Modules over the ring of differential operators

We saw in Section 2 that the Weyl algebra $D_{R|\Bbbk}$ is finitely generated, but this is not true in general. For a finitely generated graded $\Bbbk$-algebra S the $\Bbbk$-algebra $D_{S|\Bbbk}$ may not be finitely generated, therefore the generalized Bernstein's filtration $\mathcal{B}_S$ is not standard as introduced in Definition 1.7. In that case, the Krull dimension of $D_{S|\Bbbk}$ is undefined. Instead one can consider the dimension of $D_{S|\Bbbk}$ as in Definition 1.24, introduced in the paper [Bavula, 2009]. By convention we will write $\operatorname{Dim}(D_{S|\Bbbk}) \coloneqq \operatorname{Dim}(\mathcal{B}_S)$. It is not known whether the filtration $\mathcal{B}_S$ is dominated by *all* other filtrations of $D_{S|\Bbbk}$.

In the case of $D_{S|\Bbbk}$-modules, we are going to follow the same convention: if M is a finitely generated $D_{S|\Bbbk}$-module, $M = \langle m_1, \dots, m_r\rangle$, we will consider the standard filtration $\mathcal{G} = (G_i)_{i\in\mathbb{Z}_{\geq 0}}$ with $G_i = (B_S)_i\langle m_1, \dots, m_r\rangle$ and denote $\operatorname{Dim}(M) \coloneqq \operatorname{Dim}(\mathcal{G})$.

Example 3.15. The following are examples of $D_{S|\Bbbk}$-modules.

1. Any commutative finitely generated $\Bbbk$-algebra S has a natural structure of $D_{S|\Bbbk}$-module.

2. Given a finitely generated $\Bbbk$-algebra S and an element $f \in S$, the localization S_f is a left $D_{S|\Bbbk}$-module. The action of $D_{S|\Bbbk}$ on S_f is defined inductively as follows:

   - if $\delta \in D_{S|\Bbbk}$ has order zero, then $\delta(\frac{v}{f^r}) \coloneqq \frac{\delta(v)}{f^r}$,
   - if $\delta \in D^n_{S|\Bbbk}$, we define
   $$\delta\left(\frac{v}{f^r}\right) \coloneqq \frac{\delta(v) - [\delta, f^r](\frac{v}{f^r})}{f^r}.$$

   This action is well defined because $[\delta, f^r]$ has order at most $n-1$.

Now, we are going to prove the Bernstein inequality on a more general setting, provided that $D_{S|\Bbbk}$ satisfies the following condition:

Definition 3.16. We say that $(D_{S|\Bbbk}, \mathcal{B}_S)$ is *C-linearly simple* for some $C \in \mathbb{Z}_{\geq 0}$ if for each $i \in \mathbb{N}$ and each nonzero $\delta \in (B_S)_i$,

$$1 \in (B_S)_{Ci}\delta(B_S)_{Ci}.$$

We say that $(D_{S|\Bbbk}, \mathcal{B}_S)$ is *linearly simple* if it is C-simple for some $C \in \mathbb{Z}_{\geq 0}$.

Lemma 3.17. *Let $D_{S|\Bbbk}$ be the ring of differential operators over a graded $\Bbbk$-algebra S, $\mathcal{B}_S = ((B_S)_i)_{i\in\mathbb{Z}_{\geq 0}}$ a generalized Bernstein filtration, and M a left $D_{S|\Bbbk}$-module with filtration $\mathcal{G} = (G_i)_{i\in\mathbb{Z}_{\geq 0}}$ compatible with $\mathcal{B}_S$. Suppose that $(D_{S|\Bbbk}, \mathcal{B}_S)$ is C-linearly simple. Let*

$$\begin{aligned} \Psi : (B_S)_i &\longrightarrow Hom_{\Bbbk}(G_{(C+1)i}, G_{(C+2)i}). \\ \delta &\longmapsto \Psi_\delta(v) = \delta v \end{aligned}$$

If $G_i \neq 0$, then Ψ is injective.

Proof. We are going to prove the contrapositive statement. Suppose that there exists a nonzero element $\delta \in (B_S)_i$ such that $\Psi_\delta = 0$. Then $\delta(B_S)_{Ci}G_i \subseteq \delta G_{(C+1)i} = 0$, and that implies that $\delta(B_S)_{Ci}G_i = 0$. Since $1 \in (B_S)_{Ci}\delta(B_S)_{Ci}$ because $(D_{S|\Bbbk}, \mathcal{B}_S)$ is C-linearly simple, multiplying by G_i in each side, we have $G_i \subseteq (B_S)_{Ci}\delta(B_S)_{Ci}G_i = 0$, and therefore $G_i = 0$. ☐

Theorem 3.18 (Generalized Bernstein's inequality)**.** *Let $D_{S|\Bbbk}$ be the ring of differential operators over a graded $\Bbbk$-algebra S, $\mathcal{B}_S = ((B_S)_i)_{i\in\mathbb{Z}_{\geq 0}}$ be a generalized Bernstein filtration and assume $Dim(\mathcal{B}_S) < \infty$. Let M a nontrivial left $D_{S|\Bbbk}$-module with filtration $\mathcal{G} = (G_i)_{i\in\mathbb{Z}_{\geq 0}}$ compatible with $\mathcal{B}_S$. Suppose that $(D_{S|\Bbbk}, \mathcal{B}_S)$ is C-linearly simple. Then,*

$$\mathit{Dim}(M) \geq \frac{1}{2}\mathit{Dim}(D_{S|\Bbbk}).$$

Proof. If $\mathrm{Dim}(\mathcal{G}) = \infty$, then the inequality holds trivially.

Suppose $\mathrm{Dim}(\mathcal{G}) \neq \infty$. Let $t > l := \mathrm{Dim}(\mathcal{G})$. By Definition 1.24, we have that

$$\mathrm{Dim}(\mathcal{G}) = \inf\left\{t \in \mathbb{R}_{\geq 0} \mid \dim_\Bbbk G_i \leq i^t \quad \forall i \gg 0\right\}$$

and by hypothesis $G_i \neq 0$ because M is nontrivial. By Lemma 3.17, for sufficiently large i, we have that

$$\dim_\Bbbk(B_S)_i \leq ((C+1)i)^t((C+2)i)^t = (C+1)^t(C+2)^t i^{2t},$$

and it follows that $\mathrm{Dim}(M) \leq 2t$. Since this holds for all $t > l$, we conclude

$$\frac{1}{2}\mathrm{Dim}(D_{S|\Bbbk}) \leq l = \mathrm{Dim}(M).$$ ☐

Example 3.19. Some examples that satisfy the hypothesis of Theorem 3.18 are

- the ring of invariants of finite groups in characteristic zero and
- strongly F-regular finitely generated graded $\Bbbk$-algebras with finite F-representation type.

Moreover, in both cases it holds that $\mathrm{Dim}(D_{S|\Bbbk}) = 2\ \mathrm{Dim}(S)$. The proof of these results can be found in [Àlvarez Montaner et al., 2021].

In general, proving that $D_{S|\Bbbk}$ is C-linearly simple is very difficult.

We are now ready to give a more general definition of holonomic module using the condition of being C-linearly simple:

Definition 3.20. Let $(D_{S|\Bbbk}, \mathcal{B}_S)$ be a linearly simple filtered $\Bbbk$-algebra such that $\dim(D_{S|\Bbbk}) < \infty$ and $0 < e(D_{S|\Bbbk}) < \infty$. A nonzero $D_{S|\Bbbk}$-module is *holonomic* if it admits a filtration $\mathcal{G}$ of dimension $\frac{1}{2}\dim(D_{S|\Bbbk})$ and with finite multiplicity; the zero module is also holonomic by convention.

3.6 Gelfand-Kirillov dimension of the ring of differential operators

The generalized Bernstein's inequality (Theorem 3.18) provides a lower bound on $\mathrm{Dim}(M)$. An upper bound is given by $\mathrm{Dim}(D_{S|\Bbbk})$. In this section we will see an important result on the Gelfand-Kirillov dimension of $D_{S|\Bbbk}$ when S is is a commutative, Noetherian and reduced $\Bbbk$-algebra. Observe that $\mathrm{Dim}(D_{S|\Bbbk}) = GK(D_{S|\Bbbk})$ if $D_{S|\Bbbk}$ is finitely generated.

In general, let A be a $\Bbbk$-algebra and $I \subseteq A$ an ideal. Consider now the quotient $B = A/I$. Observe that a differential operator $\theta \in D_{A|\Bbbk}$ induces a well defined element $\bar{\theta} \in D_{B|\Bbbk}$ defined by

$$\bar{\theta}(\bar{a}) = \overline{\theta(a)}$$

if and only if $\theta(I) \subset I$. This gives a map

$$\mathcal{D}(I) = \{\theta \in D_{A|\Bbbk} \mid \theta(I) \subseteq I\} \longrightarrow D_{B|\Bbbk} \tag{3}$$

whose kernel is $ID_{A|\Bbbk}$. Observe that, in general, this map is not surjective.

Remark 3.21. If I is a left $D_{A|\Bbbk}$-submodule of A, i.e., $D_{A|\Bbbk}(I) \subseteq I$, then I is an ideal in A, and we have the canonical embedding

$$D_{A|\Bbbk}/ID_{A|\Bbbk} \hookrightarrow D_{A/I|\Bbbk}.$$

In general, given $\Bbbk$-algebras $A \subseteq B$, we will not have that $D_{A|\Bbbk} \subseteq D_{B|\Bbbk}$. However, we have the following result.

Lemma 3.22. *If $A \subseteq B$ are commutative $\Bbbk$-algebras, then*

$$\{\theta \in D_{B|\Bbbk} \mid \theta(A) \subseteq A\} \ \subseteq \ D_{A|\Bbbk}.$$

We have the equality when $D_{A|\Bbbk} \subseteq D_{B|\Bbbk}$.

Proof. It suffices to show that

$$\{\theta \in D^m_{B|\Bbbk} \mid \theta(A) \subseteq A\} \subseteq D^m_{A|\Bbbk}$$

for each $m \in \mathbb{N}$. We will prove it by induction on m.

- If $m = 0$, then $\theta \in B$ so $\theta(A) \subseteq A$.
- Suppose that the inclusion is true for some $m \geq 0$, and that $\theta \in D^{m+1}_{B|\Bbbk}$ satisfies $\theta(A) \subseteq A$. Then, $\theta \in End_{\Bbbk}(A)$. As $A \subseteq B$, we have that for any $a \in A$, $[\theta, a] \in D^m_{B|\Bbbk}$ and

  $$[\theta, a](A) = (\theta a - a\theta)(A) \subseteq A.$$

  And now, we can apply the induction hypothesis,

  $$\theta \in \{\delta \in End_{\Bbbk}(A) \mid [\delta, a] \in D^m_{A|\Bbbk} \text{ for all } a \in A\} = D^{m+1}_{A|\Bbbk}.$$

For the last part, if $D_{A|\Bbbk} \subseteq D_{B|\Bbbk}$, the inclusion $D_{A|\Bbbk} \subseteq \{\theta \in D_{B|\Bbbk} \mid \theta(A) \subseteq A\}$ is trivial. $\square$

Let us recall the definition of I-torsion functor Γ_I and some results related to it whose proofs can be found in [Másson, 1985]. In our case, we are going to apply it to the ring A and denote $\Gamma_I \coloneqq \Gamma_I(A)$.

Definition 3.23. Let A be a commutative $\Bbbk$-algebra and let $I \subseteq A$ be an ideal, we define

$$\Gamma_I = \{a \in A \mid I^m a = 0 \text{ for some } m > 0\}.$$

Lemma 3.24. *Let A be a commutative $\Bbbk$-algebra and let I be an ideal of A. The ideal Γ_I is a left $D_{A|\Bbbk}$-submodule of A. That is, $D_{A|\Bbbk}(\Gamma_I) \subset \Gamma_I$.*

Proof. We **claim** that $I^{n+d} D^d_{A|\Bbbk} \subset D^d_{A|\Bbbk} I^n$ for all integers $d \geq 0$ and $n \geq 0$.

The lemma follows immediately: Let $a \in \Gamma_I$. Then, by definition of Γ_I, there is an integer $n \geq 0$ such that $I^n a = 0$. Hence if $\theta \in D^d_{A|\Bbbk}$, the claim implies

$$I^{n+d}(\theta(a)) = (I^{n+d}\theta)(a) \subset (D^d_{A|\Bbbk} I^n)(a) = D^d_{A|\Bbbk}(I^n a) = 0$$

so by definition $\theta(a) \in \Gamma_I$.

Proof of the claim. Let us prove it by a double induction on n and d.

- In the case that $n = 0$ and $d = 0$ the statement is trivial.
- We prove the claim for a pair $(n, d) \in \mathbb{Z}^2_{\geq 0}$. By induction, we may assume that the statement is true for some (n, d) with $n, d > 0$.

  Let $x \in I^{d+n-1}$, $y \in I$ and $\delta \in D^d_{A|\Bbbk}$. We have to show that $xy\delta \in D^d_{A|\Bbbk} I^n$. However, by our induction hypothesis we have that $x\delta \in D^d_{A|\Bbbk} I^{n-1}$ and $x[\delta, y] \in D^{d-1}_{A|\Bbbk} I^n$ (since $[\delta, y] \in D^{d-1}_{A|\Bbbk}$), and then

$$xy\delta = x\delta y - x[\delta, y] \in D^d_{A|\Bbbk} I^{n-1} I + D^{d-1}_{A|\Bbbk} I^n = D^d_{A|\Bbbk} I^n. \qquad \square$$

Remark 3.25. Let A be a commutative ring and let I be an ideal of A. From algebraic geometry we can recall the definition of the variety corresponding to I, $V(I) \coloneqq \{p \in \operatorname{Spec} A \mid I \subset p\}$. If A is Noetherian, then Γ_I is the ideal of regular functions on $\operatorname{Spec} A$ that are supported on $V(I)$.

Hence, in terms of geometry, this lemma says that differential operators preserve supports of regular functions.

From now on, let us assume that A is Noetherian and reduced with minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. Consider the quotients $A_i = A/\mathfrak{p}_i$ for $i = 1, \ldots, r$, and let us denote $A' = A_1 \oplus \cdots \oplus A_r$. We will denote by $[a]_i$ the equivalent class of $a \in A$ in A_i. Since A is reduced, i.e., $\bigcap_{i=1}^r \mathfrak{p}_i = 0$, we have a canonical embedding

$$\begin{aligned} A &\hookrightarrow A' = A_1 \oplus \cdots \oplus A_r, \\ a &\mapsto ([a]_1, \ldots, [a]_r) \end{aligned} \tag{4}$$

and we can identify A with its image in A'. Let $e_i = (0, \dots, 1, \dots, 0) \in A'$, then we have that any element of A can be written as $a = \sum_i ae_i$ and any element $a' \in A'$ as $a' = \sum_i a_i e_i$, with $a_i \in A$.

Proposition 3.26. *Let A be a commutative, Noetherian and reduced ring, and let $\mathfrak{p}_1, \dots \mathfrak{p}_r$ be its minimal primes. Then*

(1) $\Gamma_{\mathfrak{p}_i} = \bigcap_{j \neq i} \mathfrak{p}_j$,

(2) $\Gamma_{\Gamma_{\mathfrak{p}_i}} = \mathfrak{p}_i$ *for each minimal prime* $\mathfrak{p}_i$ *of* A.

Proof. (1) Let $x \in \Gamma_{\mathfrak{p}_i}$. Then $\mathfrak{p}_i^n x = 0$ for some $n \in \mathbb{Z}$. Since A is reduced, for all $\mathfrak{p}_j \neq \mathfrak{p}_i$ for $i \neq j$, we can find an element $y \in \mathfrak{p}_i \setminus \mathfrak{p}_j$. Then $y^n x = 0 \in \mathfrak{p}_j$, so $x \in \mathfrak{p}_j$.

Conversely, if x is contained in $\bigcap_{j \neq i} \mathfrak{p}_j$, then $\mathfrak{p}_i x$ is contained in $\bigcap_{i=1}^r \mathfrak{p}_i$. Since A is reduced, it follows that $\mathfrak{p}_i x = 0$. Hence $x \in \Gamma_{\mathfrak{p}_i}$.

(2) Since $\bigcap_{i=1}^r \mathfrak{p}_i = 0$, part (1) implies immediately that $\mathfrak{p}_i \subseteq \Gamma_{\Gamma_{\mathfrak{p}_i}}$.

Conversely, assume that $x \in \Gamma_{\Gamma_{\mathfrak{p}_i}}$. Then for each $\mathfrak{p}_j \neq \mathfrak{p}_i$ choose an element $y_j \in \mathfrak{p}_j \setminus \mathfrak{p}_i$ (possible since all the associated primes are minimal), and let y be the multiple of these elements. Then by (1) we have that $y \in \mathfrak{p}_i$, and hence it follows that $y^n x = 0 \in \mathfrak{p}_i$ for some integer $n \geq 0$. Since $y \notin \mathfrak{p}_i$ and $\mathfrak{p}_i$ is a prime ideal, this implies that $x \in \mathfrak{p}_i$. ☐

We will need the next proposition to prove several results that relates $D_{A|\Bbbk}$ and $D_{A'|\Bbbk}$. Its proof can be found in [Muhasky, 1988].

Proposition 3.27. *Let A and B be commutative $\Bbbk$-algebras, then there is a $\Bbbk$-algebra isomorphism $D_{A\oplus B|\Bbbk} = D_{A|\Bbbk} \oplus D_{B|\Bbbk}$.*

Proposition 3.28. *Suppose that A is commutative, Noetherian and reduced $\Bbbk$-algebra, with minimal primes $\mathfrak{p}_1, \dots \mathfrak{p}_r$. Let $A' = A_1 \oplus \cdots \oplus A_r$, where $A_i = A/\mathfrak{p}_i$. Then $D_{A'|\Bbbk} = D_{A_1|\Bbbk} \oplus \cdots \oplus D_{A_r|\Bbbk}$, and*

$$D_{A|\Bbbk} = \{\theta \in D_{A'|\Bbbk} \mid \theta(A) \subseteq A\}.$$

Proof. We observe that if $a \in A$, then $a \in \mathfrak{p}_i$ if and only if $ae_i = 0$. Thus $a \in \Gamma_{\mathfrak{p}_i}$ if and only if $ae_j = 0$ for all $j \neq i$. Since Γ_I is a $D_{A|\Bbbk}$-submodule of A for any ideal I, and since $\mathfrak{p}_i = \Gamma_{\Gamma_{\mathfrak{p}_i}}$, it also follows that

$$D_{A|\Bbbk}(\mathfrak{p}_i) \subseteq \mathfrak{p}_i \quad \text{for } i = 1, \dots, r.$$

Thus each $\theta \in D_{A|\Bbbk}$ induces an operator, $[\theta]_i$, in $D_{A_i|\Bbbk}$. So, by Proposition 3.27, we have

$$D_{A'|\Bbbk} \simeq D_{A_1|\Bbbk} \oplus \cdots \oplus D_{A_r|\Bbbk},$$

as $\Bbbk$-algebras. So, now, we can identify $D_{A'|\Bbbk}$ with $D_{A_1|\Bbbk} \oplus \cdots \oplus D_{A_r|\Bbbk}$, this means that we have the homomorphism of $\Bbbk$-algebras

$$\begin{aligned} \phi : D_{A|\Bbbk} &\longrightarrow D_{A'|\Bbbk} \\ \theta &\longmapsto ([\theta]_1, \dots, [\theta]_r) \end{aligned}$$

which is well defined.

We have that

$$\ker\phi = \bigcap_{i=1}^{r}(\mathfrak{p}_i D_{A|\Bbbk}) = \left(\bigcap_{i=1}^{r}\mathfrak{p}_i\right) D_{A|\Bbbk} = 0D_{A|\Bbbk} = 0,$$

so ϕ is an injection, and the embedding given in Equation (4) extends to a canonical embedding

$$D_{A|\Bbbk} \hookrightarrow D_{A'|\Bbbk}.$$

Thus we have $D_{A|\Bbbk} \subseteq D_{A'|\Bbbk}$ and the second part of the proposition follows from Lemma 3.22. □

Proposition 3.29. *Assume that A is a commutative, Noetherian and reduced $\Bbbk$-algebra, with minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. Let $A' = A_1 \oplus \cdots \oplus A_r$, where $A_i = A/\mathfrak{p}_i$, and let*

$$\mathcal{D}(A', A) = \{\theta \in D_{A'|\Bbbk} \mid \theta(A') \subseteq A\}.$$

Then $\mathcal{D}(A', A) \subseteq D_{A|\Bbbk}$. Furthermore, $\mathcal{D}(A', A)$ is a right ideal in $D_{A'|\Bbbk}$ and a two-sided ideal in $D_{A|\Bbbk}$.

Proof. Notice that $\mathcal{D}(A', A)$ is a right ideal in $D_{A'|\Bbbk}$. Let us see the second part, if $\delta \in \mathcal{D}(A', A)$ then, in particular we have that $\delta(A) \subseteq A$ because $A \subseteq A'$. So, from the previous proposition, we have that $\mathcal{D}(A', A) \subseteq D_{A|\Bbbk}$. □

The differential operators in $\mathcal{D}(A', A)$ of order zero are

$$\{a \in A \mid aA' \subseteq A\} = (A :_A A')$$

the conductor of A' into A.

Remark 3.30. The conductor of A' into A, $(A :_A A')$, is a measurement of how far apart a commutative ring and an extension ring are.

Definition 3.31. For a commutative, Noetherian and reduced $\Bbbk$-algebra A, with minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, we define

$$\Gamma_A = \sum_{i=1}^{r} \Gamma_{\mathfrak{p}_i}.$$

Then $\Gamma_A = \sum_{i=1}^{r} \cap_{j\neq i}\mathfrak{p}_j$ is a left $D_{A|\Bbbk}$-submodule of A by Lemma 3.24. In particular, it is an ideal in A.

Remark 3.32. The geometry behind this element is that the variety corresponding to Γ_A (this is, the zeros of this ideal) is the closed subset of $\operatorname{Spec} A$ consisting of all points that lie in at least two different irreducible components of $\operatorname{Spec} A$.

Proposition 3.33. *Let A be a commutative, Noetherian and reduced $\Bbbk$-algebra, with minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$. Let $A' = A_1 \oplus \cdots \oplus A_r$, where $A_i = A/\mathfrak{p}_i$. Then*

$$\Gamma_A = (A :_A A') = \mathcal{D}(A', A)(A') := \{\theta(a') \mid a' \in A' \text{ and } \theta \in \mathcal{D}(A', A)\}$$

and

$$\mathcal{D}(A', A) = \Gamma_A D_{A'|\Bbbk} = \Gamma_A D_{A|\Bbbk}.$$

Proof. Suppose that $a \in \Gamma_{\mathfrak{p}_j}$ and $a' = \sum_i a_i e_i \in A'$ with $a_i \in A$ for $i = 1, \dots, r$. Then

$$a'a = \sum_i a_i e_i a = a_j a e_j = \sum_i a_j a e_i \in A,$$

so $A'\Gamma_{\mathfrak{p}_j} \subseteq A$. It follows that $\Gamma_A \subseteq (A : A')$.

Conversely, let $a \in A$ such that $aA' \subseteq A$. Then, in particular, $ae_i \in A$ for $i = 1, \dots, r$. But for each i, $ae_i e_j = a0 = 0$ if $j \neq i$, so $ae_i \in \Gamma_{\mathfrak{p}_i}$ for $i = 1, \dots, r$. Hence

$$a = \sum_i ae_i \in \sum_i \Gamma_{\mathfrak{p}_i} = \Gamma_A,$$

so $(A :_A A') \subseteq \Gamma_A$.

Since

$$(A :_A A') = (A :_A A')(1) \subseteq \mathcal{D}(A', A)(A'),$$

we have that $(A :_A A') \subseteq \mathcal{D}(A', A)(A')$.

Let us show the other containment: $\mathcal{D}(A', A)(A') \subseteq (A :_A A')$. Observe that the differential operators of $D_{A'|\Bbbk}$ commute with the e_i, i.e., $\theta(a'e_i) = (\theta e_i)(a') = e_i\theta(a')$ for $a' \in A'$ and $\theta \in D_{A'|\Bbbk}$. Thus, if $a' = \sum_i a_i e_i \in A'$, with $a_i \in A$ and $i = 1, \dots, r$, then

$$\begin{aligned}(a'\mathcal{D}(A', A))(A') &= \sum_i (a_i e_i \mathcal{D}(A', A))(A') = \sum_i (a_i \mathcal{D}(A', A) e_i)(A') \\ &\subseteq \sum_i (a_i \mathcal{D}(A', A))(A') \subset \mathcal{D}(A', A)(A') \subseteq A.\end{aligned}$$

To prove the second part of the statement. Showing that $\Gamma_A D_{A'|\Bbbk} \subseteq \mathcal{D}(A', A)$ and $\Gamma_A D_{A'|\Bbbk} \subseteq \Gamma_A D_{A|\Bbbk}$ is trivial. That $\mathcal{D}(A', A) \subset \Gamma_A D_{A'|\Bbbk}$ follows since $\mathcal{D}(A', A)(A') = \Gamma_A$ by the previous part. To see that $\Gamma_A D_{A|\Bbbk} \subseteq \Gamma_A D_{A'|\Bbbk}$, let $\theta \in \Gamma_A D_{A|\Bbbk}$ and $a' = \sum_i a_i e_i \in A'$ as before. Then

$$\theta(a') = \sum_i \theta(a_i e_i) = \sum_i e_i \theta(a_i) \in \sum_i e_i \Gamma_A \subseteq \Gamma_A. \qquad \square$$

Now, with the same notations, we will show that $GK(D_{A|\Bbbk}) = 2 \dim A$ if each A_i is regular. For that we will need the following lemma, whose proof can be found in [Krause et al., 2000] and [McConnell and Robson, 1988].

Lemma 3.34. *Let S and S' be any $\Bbbk$-algebras, then*

1. $GK(S \oplus S') = \max\{GK(S), GK(S')\}$,
2. *if $S \subseteq S'$, and if S' is finitely generated as a right or left S-module, then $GK(S) = GK(S')$.*

If a commutative $\Bbbk$-algebra S is a regular domain, then

3. *$D_{S|\Bbbk}$ is a simple domain.*

4. $GK(D_{S|\Bbbk}) = 2\dim S$*.*

Lemma 3.35. *There is an element in* Γ_A *which is nonzero in each direct summand.*

Proof. For each $i = 1, \dots, r$, choose an element $a_i \in \Gamma_{\mathfrak{p}_i} \setminus \mathfrak{p}_i$, so that $a_i e_j = \delta_{ij} a_i$. Let $a = a_1 + \cdots + a_r$. Then
$$a = ae_1 + \cdots + ae_r = a_1 e_1 + \cdots + a_r e_r \in \Gamma_A,$$
and $a_i e_i \neq 0$, for each i. □

Theorem 3.36. *Suppose that* A *is a commutative, Noetherian and reduced* $\Bbbk$*-algebra, with minimal primes* $\mathfrak{p}_1, \dots, \mathfrak{p}_r$*, such that* $A/\mathfrak{p}_i$ *is regular for* $i = 1, \dots, r$*. Then*

$$GK(D_{A|\Bbbk}) = 2\dim A.$$

Proof. Let $A' = A_1 \oplus \cdots \oplus A_r$, where $A_i = A/\mathfrak{p}_i$ for $i = 1, \dots, r$. By Lemma 3.34 (4) it follows that $GK(D_{A_i|\Bbbk}) = 2\dim A_i$, so from Lemma 3.34 (1), we get

$$\begin{aligned} GK(D_{A'|\Bbbk}) &= \max\{GK(D_{A_1|\Bbbk}), \dots, GK(D_{A_r|\Bbbk})\} \\ &= \max\{2\dim A_1, \dots, 2\dim A_r\} \\ &= 2\max\{\dim A_1, \dots, \dim A_r\} \\ &= 2\dim A. \end{aligned}$$

By Lemma 3.34 (2), we only have to prove that $D_{A'|\Bbbk}$ is finitely generated as a right $D_{A|\Bbbk}$-module.

By Proposition 3.29, $\mathcal{D}(A', A)$ is a right ideal in $D_{A'|\Bbbk}$. Now, consider the set $D_{A'|\Bbbk}\mathcal{D}(A', A)$. This is a two sided ideal in $D_{A'|\Bbbk}$, but $\Gamma_A \subseteq \mathcal{D}(A', A)$ by Proposition 3.33 and Γ_A contains an element as in Lemma 3.35. By Lemma 3.34 (3), $D_{A'|\Bbbk}$ is a direct sum of simple domains, so $D_{A'|\Bbbk}\mathcal{D}(A', A) = D_{A'|\Bbbk}$. Hence, there is a finite sum

$$1 = \sum_{i=1}^{m} \theta_i \psi_i,$$

with $\theta_i \in D_{A'|\Bbbk}$ and $\psi_i \in \mathcal{D}(A', A)$. By Proposition 3.28 and Proposition 3.29, it follows that

$$D_{A'|\Bbbk} = \sum_{i=1}^{m} \theta_i \psi_i D_{A'|\Bbbk} \subseteq \sum_{i=1}^{m} \theta_i \mathcal{D}(A', A) \subseteq \sum_{i=1}^{m} \theta_i D_{A|\Bbbk} \subseteq D_{A'|\Bbbk},$$

so

$$D_{A'|\Bbbk} = \sum_{i=1}^{m} \theta_i D_{A|\Bbbk}.$$ □

Corollary 3.37. *If* A *is a finitely generated* $\Bbbk$*-algebra and its ring* $D_{A|\Bbbk}$ *of* $\Bbbk$*-linear differential operators is also finitely generated, then*

$$Dim(D_{A|\Bbbk}) = 2\dim A.$$

4. Ring of differential operators of a hyperplane arrangement

In Section 4.1 we are going to introduce the notion of a hyperplane arrangement. Hyperplane arrangements correspond to finitely generated graded $\Bbbk$-algebras of the form R/I with I generated by a product of linear polynomials. In particular, there is a notion of differential operators on them as we saw in Section 1.3. Interestingly, specializing to hyperplane arrangements allows us to apply what we saw in Section 3. In particular, we will use all the results from Section 3.1 about differential operators that preserve a given ideal. Later, in Section 4.2, we will explain how to compute the ring of differential operators on a hyperplane arrangement. In order to do this, we will follow the work of Holm in his thesis [Holm, 2002].

4.1 Hyperplane Arrangements

Let $\Bbbk$ be an algebraically closed field and let V be a vector space of dimension n. All the results and definitions in this section can be found in [Orlik and Terao, 2013] and [Holm, 2002].

Definition 4.1. A hyperplane H in V is an affine subspace of dimension $n-1$. A *hyperplane arrangement* $\mathcal{A} = (\mathcal{A}, V)$ is a finite set of hyperplanes in V.

Remark 4.2. We call $\mathcal{A}$ an n-arrangement when we want to emphasize the dimension of V.

We are going to denote by V^* the dual space of V, by $S = S(V^*)$ the symmetric algebra of V^*. We can choose a basis of vectors $\{e_1, \dots, e_n\}$ in V which has an associated dual basis $\{x_1, \dots, x_n\}$ in V^* such that $x_i(e_j) = \delta_{i,j}$. We can identify $S(V^*)$ with the polynomial ring $S = \Bbbk[x_1, \dots, x_n]$. Each hyperplane $H \in \mathcal{A}$ is the kernel of a polynomial p_H of degree 1 defined up to a constant.

Definition 4.3. The product

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} p_H$$

is called a *defining polynomial* of $\mathcal{A}$. We agree that $Q(\varnothing_n) = 1$ is the defining polynomial of the empty arrangement.

Definition 4.4. We say that $\mathcal{A}$ is *centerless* if $\cap_{H \in \mathcal{A}} H = \varnothing$. If $T = \cap_{H \in \mathcal{A}} H \neq \varnothing$, we say that $\mathcal{A}$ is *centered* with center T. If $\mathcal{A}$ is centered, then coordinates may be chosen so that each hyperplane contains the origin. In this case we call $\mathcal{A}$ *central*.

4.2 Ring of differential operators

We already now that we can write any differential operator of the polynomial ring in n variables $R = \Bbbk[x_1, \dots, x_n]$ as a sum of homogeneous differential operators, so if $\theta \in D_{R|\Bbbk}$ of order m we are going to denote by θ_i its homogeneous component of order i, thus

$$\theta = \theta_m + \cdots + \theta_0, \text{ where } \theta_i = \sum_{|\alpha|=i} p_\alpha(x)\partial^\alpha.$$

Lemma 4.5. *Suppose $p \in R$ is a polynomial of degree one, and that $\theta \in D_{R|\Bbbk}$. Then*

$$\theta \in \mathcal{D}(\langle p \rangle) \textit{ if and only if } \theta_i \in \mathcal{D}(\langle p \rangle), \textit{ for } i = 0, \ldots, m,$$

where

$$\theta = \theta_m + \cdots + \theta_0,$$

is the decomposition of θ into components of homogeneous order.

Proof. If $\theta_i \in \mathcal{D}(\langle p \rangle)$ for each i, then it is obvious that $\theta \in \mathcal{D}(\langle p \rangle)$.

We are going to prove the other implication studying each homogeneous component of θ:

- For $i = 0$. Since $\theta_0 \in R$, we have that it preserves any ideal.
- For $i = 1$. Since $\theta \in \mathcal{D}(\langle p \rangle)$ by hypothesis and using the case $i = 0$, we have that $\theta_m + \cdots + \theta_1 \in \mathcal{D}(\langle p \rangle)$. Thus

$$(\theta_m + \cdots + \theta_1)(p) \in \langle p \rangle,$$

  but $\deg(p) = 1$, so

$$(\theta_m + \cdots + \theta_1)(p) = \theta_1(p) \in \langle p \rangle.$$

  By Proposition 3.3, this means that $\theta_1 \in \mathcal{D}(\langle p \rangle)$.
- From the last step, we also have that $\theta_m + \cdots + \theta_2 \in \mathcal{D}(\langle p \rangle)$. In particular,

$$(\theta_m + \cdots + \theta_2)(x_j p) \in \langle p \rangle,$$

  for each $j = 1, \ldots, n$. In this case, $\deg(x_j p) = 2$, so

$$(\theta_m + \cdots + \theta_2)(x_j p) = \theta_2(x_j p) \in \langle p \rangle,$$

  and again by Proposition 3.3, we have that $\theta_2 \in \mathcal{D}(\langle p \rangle)$.
- Continuing like this, we see that $\theta_i \in \mathcal{D}(\langle p \rangle)$ for each $i = 1, \ldots, m$. □

Notation 4.6. If I is an ideal in R we will write $\mathcal{D}^{(m)}(R)$ and $\mathcal{D}^{(m)}(I)$ respectively to denote the differential operators in $D_{R|\Bbbk}$ and $\mathcal{D}(I)$ of homogeneous order m. If $A = R/I$, we say that a differential operator in $D_{A|\Bbbk}$ if homogeneous of order m if it is induced by an element in $\mathcal{D}^{(m)}(I)$. Thus, writing $\mathcal{D}^{(m)}(A)$ for the differential operators in $D_{A|\Bbbk}$ of homogeneous order m, we have

$$\mathcal{D}^{(m)}(A) \simeq \mathcal{D}^{(m)}(I)/I\mathcal{D}^{(m)}(R).$$

Example 4.7. Consider $I = \langle x^r \rangle \subseteq \Bbbk[x]$. Then we have, for example, that

$$x\partial_x^2 - (r-1)\partial_x \in \mathcal{D}(\langle x^r \rangle)$$

but neither $x\partial_x^2$ nor ∂_x belong to $\mathcal{D}(\langle x^r \rangle)$ if $r > 1$, so

$$\bigoplus_{s \leq 2} \mathcal{D}^{(s)}(I) \subsetneq \mathcal{D}^2(I)$$

in this case.

This kind of examples are what makes the following proposition important:

Proposition 4.8. *Let I be an ideal of the polynomial ring R. If $I = \langle p_1 \dots p_r \rangle$ defines an affine arrangement, then*

$$\mathcal{D}(I) = \bigoplus_{m \geq 0} \mathcal{D}^{(m)}(I)$$

as an R-module.

Proof. We only have to prove that $\mathcal{D}(I) \subseteq \bigoplus_{m \geq 0} \mathcal{D}^{(m)}(I)$. Suppose that $\theta = \theta_m + \cdots + \theta_0 \in \mathcal{D}(I)$ with the same notations as before. We have to show that each $\theta_i \in \mathcal{D}(I)$.

By Theorem 3.5, $\theta \in \mathcal{D}(\langle p_j \rangle)$ for $j = 1, \dots, r$. Using Lemma 4.5, we have $\theta_i \in \mathcal{D}(\langle p_j \rangle)$ for each i and j. Hence, by Theorem 3.5 again, $\theta_i \in \mathcal{D}(I)$ for each i. □

Corollary 4.9. *If A is the coordinate ring of an affine arrangement, then*

$$D_{A|\Bbbk} = \bigoplus_{m \geq 0} \mathcal{D}^{(m)}(A)$$

as an A-module.

Lemma 4.10. *Let $p \in R$ be a polynomial of degree one. Then $\mathcal{D}^{(m)}(\langle p \rangle)$ is generated, as an R-module, by the homogeneous components of highest order of products of m derivations on $Der(\langle p \rangle)$, together with the set $\{p\partial^\alpha \mid |\alpha| = m\}$.*

Proof. As $A = R/\langle p \rangle$ is regular. By Proposition 1.33, $D_{A|\Bbbk}$ is generated as a $\Bbbk$-algebra by A and $Der(A)$, i.e. $D^m_{A|\Bbbk}$ is generated by $(Der(A))^j$, $j \leq m$, as an A-module.

By Theorem 3.8, the isomorphism $D_{A|\Bbbk} \simeq \mathcal{D}(\langle p \rangle)/\langle p \rangle D_{R|\Bbbk}$ preserves the order filtration on $D_{A|\Bbbk}$, so

$$Der(A) \simeq Der(\langle p \rangle)/\langle p \rangle Der(R).$$

Hence $\mathcal{D}^m(\langle p \rangle)$ is generated by $(Der(\langle p \rangle))^j + \langle p \rangle D^m_{R|\Bbbk}$ for $j \leq m$ as an R-module. Using now Proposition 4.8, we have that $\mathcal{D}^{(m)}(\langle p \rangle)$ is generated, as we wanted, by the homogeneous components of highest order of products of m derivations on $Der(\langle p \rangle)$, together with the set $\{p\partial^\alpha \mid |\alpha| = m\}$. □

Lemma 4.11. *Let f be any polynomial in R such that there is an element $\delta \in Der(R)$ with $\delta(f) = af$, for some $a \in \Bbbk \setminus \{0\}$. Let $\mathcal{N}$ be the R-module of derivations annihilating f. Then*

$$Der(\langle f \rangle) = R\delta + \mathcal{N}.$$

Proof. If $\delta \in Der(R)$, we know by Proposition 3.3 that $\delta \in Der(\langle f\rangle)$ if and only if $\delta(f) \subseteq \langle f\rangle$. Since this holds for δ, we have that $R\delta + \mathcal{N} \subseteq Der(\langle f\rangle)$.

Let us prove the opposite inclusion, assume that $\theta \in Der(\langle f\rangle)$ such that $\theta(f) = gf$ for some $g \in R$. Then

$$\theta(f) = gf = \frac{g}{a}\delta(f),$$

so,

$$\left(\theta - \frac{g}{a}\delta\right) \in \mathcal{N}.$$

Which shows that $\theta \in R\delta + \mathcal{N}$. $\square$

Definition 4.12. Let V be the $\Bbbk$-vector space

$$V = \sum_{i=1}^{n} \Bbbk\partial_i$$

and, if H is a hyperplane through the origin in $\Bbbk^n$, define $V_H \subseteq V$ to be the $(n-1)$-dimensional subspace

$$V_H = \left\{\sum_{i=1}^{n} b_i\partial_i \mid (b_1,\dots,b_n) \in H\right\}.$$

Lemma 4.13. *Let p be any polynomial of degree one, and let $\mathcal{N}$ be the R-module of derivations annihilating p. Let H be the hyperplane through the origin defined by $p - p(0)$. Then*

$$\mathcal{N} = RV_H.$$

Proof. As p is a polynomial of degree one, we can write it as $p = a_0 + \sum a_i x_i$ where $a_i \in \Bbbk$ for each i. A vector $(b_1,\dots,b_n) \in \Bbbk^n$ lies in the hyperplane H if and only if $\sum a_i b_i = 0$, this is, if and only if $(b_1,\dots,b_n)$ is a syzygy of $(a_1,\dots,a_n)$. In fact, all the vectors contained in H generate the whole R-module of syzygies of $(a_1,\dots,a_n)$.

On the other hand, a derivation $\delta = \sum p_i\partial_i \in Der(R)$ lies in $\mathcal{N}$ if and only if $0 = \delta(p) = \sum p_i a_i$, i.e., if and only if $(f_1,\dots,f_n)$ is a syzygy of $(a_1,\dots,a_n)$. Hence $\mathcal{N} = RV_H$. $\square$

Remark 4.14. We are going to use this result to construct a basis of V. Let p be a polynomial of degree one, and H the hyperplane defined by $p - p(0)$. We can choose a basis $\{\delta_1,\dots,\delta_{n-1}\}$ for V_H and extend this to a basis $\{\delta_1,\dots,\delta_n\}$ for V.

As $\delta_n \notin \langle\delta_1,\dots,\delta_{n-1}\rangle$, it cannot annihilate p. Therefore, $p\delta_n$ meets the requirements of 4.11, and it follows that $Der(\langle p\rangle)$ is generated by $\{\delta_1,\dots,\delta_{n-1}, p\delta_n\}$.

Lets do an improvement of this observation:

Proposition 4.15. *Suppose p is a polynomial of degree one. Let H be the hyperplane through the origin defined by $p - p(0)$, and let $M = \{\delta_1,\dots,\delta_n\}$ be a basis for V such that $\{\delta_1,\dots,\delta_{n-1}\}$ is a basis for the subspace V_H from Definition 4.12. Then*

$$\mathcal{D}^{(m)}(\langle p\rangle) = \sum_{\substack{|\alpha|=m\\ \alpha_n=0}} R\delta^\alpha + \sum_{\substack{|\alpha|=m\\ \alpha_n>0}} Rp\delta^\alpha,$$

as an R-module.

Proof. We have already seen that $Der(\langle p \rangle)$ is generated by $\{\delta_1, \dots, \delta_{n-1}, p\delta_n\}$. Thus, by Lemma 4.10, $\mathcal{D}^{(m)}(\langle p \rangle)$ is generated by the homogeneous components of highest order of products of m of these derivations, together with the set $\{p\partial^\alpha \mid |\alpha| = m\}$.

Observe that the homogeneous component of $(p\delta_n)^l$ is $p^l \delta_n^l$. Since the derivations $\delta_1, \dots, \delta_n$ belong to V they all commute with each other and since $\delta_i(p) = 0$ for $i = 1, \dots, n-1$, it follows that $\delta_1, \dots, \delta_{n-1}$ commute with $p\delta_n$. Thus the homogeneous component of highest order of a product of the derivations $\delta_1, \dots, \delta_{n-1}, p\delta_n$ is of the form $p^{\alpha_n}\delta^\alpha$. Therefore any such element is clearly a multiple of one of the proposed generators.

Since $\{\delta_1, \dots, \delta_n\}$ is a basis of V it follows that any ∂^β with $|\beta| = m$ can be expressed as a $\Bbbk$-linear combination of $\{\delta^\alpha \mid |\alpha| = m\}$, so any element of the form $p\delta^\beta$, $|\beta| = m$, is an R-linear combination of the proposed generators. □

Theorem 4.16. *If A is the coordinate ring of an affine arrangement,*

$$GK(D_{A|\Bbbk}) = 2 \dim A.$$

Proof. This is a direct result applying Theorem 3.36, because A and $D_{A|\Bbbk}$ hold all the hypothesis of the theorem. □

However, if we have a module M over $D_{A|\Bbbk}$, we do not know if the generalized Bernstein's filtration in this setting is C-linearly simple, so neither we know if we can obtain a generalized Bernstein's inequality.

4.3 Computation of the ring of differential operators on a generic hyperplane arrangement

Let $R = \Bbbk[x_1, \dots, x_n]$, $\mathcal{A}$ be an affine arrangement and $A = R/I$ the coordinate ring of of $\mathcal{A}$, with the same notations as before, we are going to review some methods of calculating generators of the R-module $\mathcal{D}^{(m)}(A)$, as the "Jacobian" method, the method of intersecting modules and Holm's method, all of them can be found in [Snellman, 2005] and [Holm, 2002].

4.3 The Jacobian Method

Let us first recall a result that we saw before:

Lemma 4.17. *Let $\delta = \sum_{|\beta|=m} p_\beta(x)\partial^\beta \in D^m_{S|\Bbbk}$ be homogeneous of order m. Then $\delta \in D^m_{\mathcal{A}|\Bbbk}$ if and only if $\delta(x^\alpha p) \in \langle p \rangle$ for all $|\alpha| \leq m$.*

Whose immediate consequence is

Corollary 4.18. *Let m be a positive integer. Let G be a row matrix whose entries are*

$$\{\partial^\beta \mid |\beta| = m\},$$

let H be a column matrix whose entries are

$$\{x^\alpha \mid |\alpha| < m\},$$

and let A be the matrix indexed by G and H where the $(\partial^\beta, x^\alpha)$ entry is

$$\partial^\beta(x^\alpha p).$$

Let $B = [A|pI]$, where I is the identity matrix of appropriate dimension. Then, the syzygy module of the columns of B correspond to $D^m_{\mathcal{A}|\Bbbk}$. More precisely, if

$$[A|pI]\begin{bmatrix} u \\ w \end{bmatrix} = 0,$$

then

$$\sum_\beta u_\beta \partial^\beta \in D_{\mathcal{A}|\Bbbk},$$

and this is an isomorphism.

4.3 The Method of Intersecting Modules

Now we are going to explain the method of intersection modules. It is based on these results shown in Section 4.2:

Definition 4.19. Let V be the $\Bbbk$-vector space $V = \sum_{i=1}^n \Bbbk\partial_i$ and define

$$V_H = \left\{ \sum_{i=1} b_i \partial_i \mid (b_1, \ldots, b_n) \in H \right\}.$$

Then, V_H is a codimension-one subspace of V.

Lemma 4.20. *Let $\mathcal{N}$ be the module of derivations annihilating a polynomial p of degree one. Then, $\mathcal{N} = SV_H$, and if δ is any derivation such that $\delta p = ap$, $a \in \Bbbk^*$, then $D^{(1)}(p) = \mathcal{N} + S\delta$.*

Proposition 4.21. *Suppose p is a polynomial of degree one. Let H be the hyperplane through the origin defined by $p - p(0)$, and let $M = \{\delta_1, \ldots, \delta_n\}$ be a basis for V such that $\{\delta_1, \ldots, \delta_{n-1}\}$ is a basis for the subspace V_H from Definition 4.12. Then, $N = \{\delta_1, \ldots, \delta_{n-1}, p\delta_n\}$ generates $\mathcal{D}^{(1)}(\langle p \rangle)$, and*

$$\mathcal{D}^{(m)}(\langle p \rangle) = \sum_{\substack{|\alpha|=m \\ \alpha_n=0}} R\delta^\alpha + \sum_{\substack{|\alpha|=m \\ \alpha_n>0}} Rp\delta^\alpha.$$

Together with the next theorem proven in Section 3.1:

Theorem 4.22. *$D^m_{A|\Bbbk} = \bigcap_{i=1}^r D^{(m)}(p_i)$.*

We will now translate this results to explain a method to compute the pieces $D^m_{\mathcal{A}|\Bbbk}$, let us introduce the notations and steps to follow:

Let q be a polynomial of degree one, and denote by $H \in \Bbbk^n$ the hyperplane defined by q. With the same notations as in Proposition 4.21. Let A be the $n \times n$ coordinate matrix of N, i.e., the matrix formed by the coordinate vectors of elements of N. And let $S^m A$ be the m-th symmetric power of A. And finally, let B^m be the matrix which is a result of replacing any occurrence of q^i with $i > 1$ in the matrix $S^m A$, by q.

Let us translate it, as A is a matrix it can be seen as a matrix of an endomorphism

$$\phi : R^n \to R^n,$$

then $S^m A$ is the matrix of the endomorphism

$$S^m \phi : S^m R^n \to S^m R^n.$$

But as this is a symmetric product we will obtain a matrix with less entries. And B^m_q is the matrix of the associated endomorphism on $S^m T$ where

$$T = R/(q - q^2).$$

Once we have computed the matrices $B^j_{\mathfrak{p}_i}$ for each p_i, we can use them to compute $\mathcal{D}^{(j)}(A)$ for each $j \in \mathbb{N}$. The algorithm to do that is to construct a matrix M_j for each j in the following way

$$M_j = \begin{bmatrix} I_j & B^j_{\mathfrak{p}_1} & 0 & \cdots & 0 \\ I_j & 0 & B^j_{\mathfrak{p}_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ I_j & 0 & 0 & \cdots & B^j_{\mathfrak{p}_r} \end{bmatrix},$$

where I_R is the identity matrix of dimension $\dim(\mathcal{D}^{(j)}(R))$.

Then, compute the generators of the syzygies of this matrix, and from those we can obtain the generators of $\mathcal{D}^{(j)}(A)$ for each $j \in \mathbb{N}$ from the truncations of the syzygies of the matrix. This method of computing the intersection of submodules of free modules can be found in [Loustaunau, 1994].

4.3 Holm's Method

In the first paper of his thesis [Holm, 2002], Holm also gives another method for constructing a (non necessarily minimal) generating set of $\mathcal{D}^{(m)}(A)$, when $\mathcal{A}$ is a generic n-arrangement. Let us introduce some notations and results without proof in order to arrive to this method.

Suppose that $r \geq 1$ and that the polynomials of degree one $p_1, \ldots, p_r$ define a central arrangement. Let H_i be the hyperplane defined by p_i, for $i = 1, \ldots, r$, and let $V_i = V_{H_i} \subseteq V$ be the subspace generating module of derivations annihilating p_i.

Lemma 4.23. *Let $p_1, \ldots, p_r$ be linear forms defining a generic arrangement.*

Choose a subset M of V as follows.

- *If $r < n$: choose a basis $\{\delta_{r+1}, \ldots, \delta_n\}$ for $\bigcap_{i=1}^r V_i$. Then, for each $i = 1, \ldots, r$, choose an element δ_i so that $\{\delta_i, \delta_{r+1}, \ldots, \delta_n\}$ is a basis for $\bigcap_{j \neq i} V_j$, and let $M = \{\delta_1, \ldots, \delta_n\}$.*
- *If $r \geq n$: choose a basis element for each intersection of $n-1$ of the V_i's, and let $M = \{\delta_1, \ldots, \delta_t\}$ be the set of all of these.*

A set M chosen in this way contains a basis for each intersection of V_i's, including the whole of V.

Definition 4.24. Suppose $p_1, \ldots, p_r$ be linear forms defining a generic arrangement. We define a subset D of $Der(R)$ as

$$D = \{P_1\delta_1, \ldots, P_s\delta_s\},$$

where $\{\delta_1, \ldots, \delta_s\}$ is the set M from Lemma 4.23, and where each P_i is the product of those p_i's which are not annihilated by δ_i. If all p_j's are annihilated by δ_i, we put $P_i = 1$.

Definition 4.25. Suppose $I = \langle p_1 \ldots p_r \rangle$ defines a generic arrangement and that $\{\delta_1, \ldots, \delta_s\}$ is the set M associated to I according to Lemma 4.23.

For each $\alpha \in \mathbb{N}^s$, $s > n$ if $r > n$, define P_α as the product of those p_j's such that some δ_i with $\alpha_i \neq 0$ does not annihilate p_j. If every δ_i with $\alpha_i \neq 0$ annihilate all p_j's, we define $P_\alpha = 1$.

Definition 4.26. The Euler derivation is defined as

$$\epsilon = \sum_{i=1}^{n} x_i \partial_i.$$

This derivation preserves any homogeneous ideal and it is easily seen to have the property

$$\epsilon(f) = rf$$

if f is an homogeneous polynomial of degree r.

Once we have seen all this definitions and results, we are ready to understand all the notations of the next theorem that give us a complete description of the generating set of $\mathcal{D}(I)$.

Theorem 4.27. *Suppose that $I = \langle p_1 \ldots p_r \rangle$ defines a generic arrangement. Then $\mathcal{D}(I)$ is finitely generated as a $\Bbbk$-algebra. More precisely:*

- *If $r \leq n$, then $\mathcal{D}(I)$ is generated by*

$$\{x_1, \ldots, x_n, \delta_{r+1}, \ldots, \delta_n\} \cup \{p_i\delta_i, p_i\delta_i^2, p_i\delta_i^3 \mid i = 1, \ldots, r\},$$

  where $D = \{p_1\delta_1, \ldots, p_r\delta_r, \delta_{r+1}, \ldots, \delta_n\}$ is the set from Definition 4.24.

- *If $r > n$, then $\mathcal{D}(I)$ is generated by*

$$\{x_1, \ldots, x_n, \epsilon\} \cup \{P_\alpha\delta^\alpha \mid \alpha_i \leq 2(r-n) + 3 \text{ for } i = 1, \ldots, s\},$$

  with P_α as in Definition 4.25 for each $\alpha \in \mathbb{N}^s$.

In particular, he gave a concrete description of the generating set in the case of hyperplane arrangements in $\Bbbk[x, y]$:

Proposition 4.28. *Let $\mathcal{A}$ be an arrangement in $\Bbbk^2$, with defining polynomial $p = p_1 p_2 \dots p_r \in \Bbbk[x, y]$, and let m be a positive integer. Let $P_i = p/p_i$ for $1 \le i \le r$ and define*

$$\delta_i = \begin{cases} \partial_y, & \text{if } p_i = ax, a \in \Bbbk^*, \\ \partial_x + a_i \partial_y, & \text{if } p_i = a(y - a_i x), a, a_i \in \Bbbk^*. \end{cases}$$

Then, $\mathcal{D}^{(m)}(A)$ is free, minimally generated by

$$\begin{cases} \{\epsilon_m, P_1\delta_1^m, \dots, P_r\delta_r^m\}, & \text{if } 1 \le m \le r-2, \\ \{P_1\delta_1^{r-1}, \dots, P_r\delta_r^{r-1}\}, & \text{if } m = r-1, \\ \{P_1\delta_1^m, \dots, P_r\delta_r^m, pq_r, \dots, pq_m\}, & \text{if } m \ge r. \end{cases}$$

4.3 Example

We are going to compute $\mathcal{D}^{(1)}(A)$ and $\mathcal{D}^{(2)}(A)$ when $p = Q(\mathcal{A}) = x(x-y)$ with the three methods.

The Jacobian method

The Corollary 4.18 give us a straight forward method to compute each $\mathcal{D}^{(m)}(A)$. Let us first start by computing $\mathcal{D}^{(1)}(A)$, where $p = x(x-y) = x^2 - xy$. Following the same notations of the corollary the entries of the matrix A are

$$\partial_x(1 \cdot (x^2 - xy)) = 2x - y, \ \partial_y(1 \cdot (x^2 - xy)) = -x$$

So, our matrix A is this case is

$$A = \begin{pmatrix} 2x - y & -x \end{pmatrix}$$

then, the system we have to solve is

$$[A|pI] \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} 2x - y & -x & x^2 - xy \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ w \end{bmatrix} = 0$$

Multiplying, we obtain

$$(2x - y)u_1 - xu_2 + (x^2 - xy)w = 0.$$

Therefore, we have that

$$\mathcal{D}^{(1)}(A) = \langle x(\partial_x + \partial_y), (x - y)\partial_y \rangle.$$

Let us compute now the generators of $D^2_{\mathcal{A}|\Bbbk}$. In this case we have:

$$[A|pI]\begin{bmatrix}u\\w\end{bmatrix}=\begin{bmatrix}2 & -1 & 0 & x^2-xy & 0 & 0\\ 6x-2y & -2x & 0 & 0 & x^2-xy & 0\\ 2y & 2x-2y & -2x & 0 & 0 & x^2-xy\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\w_1\\w_2\\w_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

which give us the system of equations:

$$\begin{cases} 2u_1 - u_2 = -(x^2-xy)w_1\\ (6x-2y)u_1 - 2xu_2 = -(x^2-xy)w_2\\ 2yu_1 + (2x-2y)u_2 - 2xu_3 = -(x^2-xy)w_3\end{cases}$$

So, the generators of $\mathcal{D}^{(2)}(A)$ are

$$\mathcal{D}^{(2)}(A)=\langle\, x\partial_x^2+2x\partial_x\partial_y+(2x-y)\partial_y^2,\ x^2\partial_x^2+(-3x^2+xy)\partial_x\partial_y+(-3x^2+3xy-y^2)\partial_y^2,\ (x-y)\partial_y^2\,\rangle.$$

Method of intersecting modules

Let see how this method work on our example. Recall that our situation is $R=\Bbbk[x,y]$ and $p=Q(\mathcal{A})=x(x-y)$. We want to compute

$$\mathcal{D}^{(1)}(A)=\mathcal{D}^{(1)}(\langle x\rangle)\cap\mathcal{D}^{(1)}(\langle x-y\rangle)\ \text{ and }\ \mathcal{D}^{(2)}(A)=\mathcal{D}^{(2)}(\langle x\rangle)\cap\mathcal{D}^{(2)}(\langle x-y\rangle).$$

Let us compute first the pieces for each linear polynomial and the compute the intersection in each case:

- For $q=x$. We have that V_H is spanned by ∂_y. Hence, we can take $M=\{\partial_y,\partial_x\}$ a base of $V=\Bbbk\partial_x+\Bbbk\partial_y$ and then $N=\{\partial_y,x\partial_x\}$. Where we have used the notations of Proposition 4.21, then

  $$\mathcal{D}^{(1)}(\langle x\rangle)=\langle\ \partial_y,\ x\partial_x\ \rangle.$$

  Now, if we order the monomials of degree two as x^2,xy,y^2, then

  $$A=\begin{bmatrix}0 & x\\ 1 & 0\end{bmatrix},\ S^2A=\begin{bmatrix}0&0&x^2\\0&x&0\\1&0&0\end{bmatrix}\ \text{ and }\ B^2_{(x)}=\begin{bmatrix}0&0&x\\0&x&0\\1&0&0\end{bmatrix}$$

  Thus, we can use $B^2_{(x)}$ to recover the generators of $\mathcal{D}^{(2)}(\langle x\rangle)$:

  $$[\partial_x^2,\ \partial_x\partial_y,\ \partial_y^2]B^2_{(x)}=[\partial_y^2,\ x\partial_x\partial_y,\ x\partial_x^2]$$

  Therefore,

  $$\mathcal{D}^{(2)}(\langle x\rangle)=\langle\ \partial_y^2,\ x\partial_x\partial_y,\ x\partial_x^2\ \rangle.$$

- For $q = x - y$. In this second case, V_H is spanned by $\partial_x + \partial_y$. Thus, $M = \{\partial_x + \partial_y, \partial_y\}$ is a base of V and then $N = \{\partial_x + \partial_y,\ (x-y)\partial_y\}$. Therefore,

$$\mathcal{D}^{(1)}(\langle x - y\rangle) = \langle\ \partial_x + \partial_y,\ (x-y)\partial_y\ \rangle.$$

  Now, if we order the monomials of degree two as x^2, xy, y^2, then

$$A = \begin{bmatrix} 1 & 0 \\ 1 & x-y \end{bmatrix},\ S^2A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & x-y & 0 \\ 1 & x-y & (x-y)^2 \end{bmatrix} \text{ and } B^2_{(x-y)} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & x-y & 0 \\ 1 & x-y & x-y \end{bmatrix}$$

  Thus, using $B^2_{(x-y)}$ to recover the generators of $\mathcal{D}^{(2)}(\langle x-y\rangle)$:

$$[\partial_x^2,\ \partial_x\partial_y,\ \partial_y^2]B^2_{(x-y)} = [(\partial_x + \partial_y)^2,\ (x-y)(\partial_x\partial_y + \partial_y^2),\ (x-y)\partial_y^2]$$

  Therefore,

$$\mathcal{D}^{(2)}(\langle x-y\rangle) = \langle\ (\partial_x + \partial_y)^2,\ (x-y)(\partial_x\partial_y + \partial_y^2),\ (x-y)\partial_y^2\ \rangle.$$

We are now ready to compute $\mathcal{D}^{(1)}(A)$ and $\mathcal{D}^{(2)}(A)$. Let us start with $\mathcal{D}^{(1)}(A) = \mathcal{D}^{(1)}(\langle x\rangle) \cap \mathcal{D}^{(1)}(\langle x-y\rangle)$, to do so consider the matrices of both submodules in the basis $[\partial_x, \partial_y]$ of $\mathcal{D}^{(1)}(R)$:

$$B^1_{(x)} = \begin{bmatrix} 0 & x \\ 1 & 0 \end{bmatrix};\ B^1_{(x-y)} = \begin{bmatrix} 1 & 0 \\ 1 & x-y \end{bmatrix}.$$

With these two matrices we are going to obtain the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & x & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & x-y \end{bmatrix}.$$

The generators of the syzygies of the matrix M are

$$\begin{bmatrix} -x \\ -y \\ y \\ 1 \\ x \\ -1 \end{bmatrix},\ \begin{bmatrix} 0 \\ -x+y \\ x-y \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, the intersection of $\mathcal{D}^{(1)}(\langle x-y\rangle)$ and $\mathcal{D}^{(1)}(\langle x\rangle)$ is generated by

$$\begin{bmatrix} -x \\ -y \end{bmatrix},\ \begin{bmatrix} 0 \\ -x+y \end{bmatrix}$$

Therefore, the generators of $\mathcal{D}^{(1)}(A)$ are precisely

$$\mathcal{D}^{(1)}(A) = \langle x\partial_x + y\partial_y,\ (x-y)\partial_y\ \rangle.$$

Now, let us compute the intersection of $\mathcal{D}^{(2)}(\langle x-y\rangle)$ and $\mathcal{D}^{(2)}(\langle x\rangle)$. In the basis $[\partial_x^2, \partial_x\partial_y, \partial_y^2]$ for $\mathcal{D}^{(2)}(R)$, the matrices of $\mathcal{D}^{(2)}(\langle x-y\rangle)$ and $\mathcal{D}^{(2)}(\langle x\rangle)$ are

$$B^2_{(x)} = \text{‘} \begin{bmatrix} 0 & 0 & x \\ 0 & x & 0 \\ 1 & 0 & 0 \end{bmatrix};\quad B^2_{(x-y)} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & x-y & 0 \\ 1 & x-y & x-y \end{bmatrix}$$

The zyzygies of the matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & x-y & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & x-y & x-y \end{bmatrix}$$

are generated by

$$\begin{bmatrix} -x \\ -2x \\ -x \\ 1 \\ 2 \\ 1 \\ x \\ 0 \\ 0 \end{bmatrix},\ \begin{bmatrix} 0 \\ 0 \\ x^2-xy \\ -x+y \\ 0 \\ 0 \\ 0 \\ 0 \\ -x \end{bmatrix},\ \begin{bmatrix} 0 \\ x^2-xy \\ 0 \\ 0 \\ -x+y \\ 0 \\ 0 \\ -x \\ x \end{bmatrix}$$

so the intersection of the two modules is generated by

$$\begin{bmatrix} -x \\ -2x \\ -x \end{bmatrix},\ \begin{bmatrix} 0 \\ 0 \\ x^2-xy \end{bmatrix},\ \begin{bmatrix} 0 \\ x^2-xy \\ 0 \end{bmatrix}$$

and $\mathcal{D}^{(2)}(A)$ is generated by

$$\mathcal{D}^{(2)}(A) = \langle\ x(\partial_x+\partial_y)^2,\ (x^2-xy)\partial_y^2,\ (x^2-xy)\partial_x\partial_y\ \rangle.$$

Holm's Method

In the case where $p = Q(\mathcal{A}) = x(x-y)$, we are going to apply Proposition 4.28 to obtain the generators of $\mathcal{D}^{(1)}(A)$ and $\mathcal{D}^{(2)}(A)$:

In the first case, we have that $m = 1$ and $r = 2$:

$$\mathcal{D}^{(1)}(A) = \langle\ (x-y)\partial_y,\ x(\partial_x + \partial_y)\ \rangle.$$

In the second case, as $m = 2$ and $r = 2$:

$$\mathcal{D}^{(2)}(A) = \langle\ (x-y)\partial_y^2,\ x(\partial_x + \partial_y)^2\ \rangle.$$

4.3 Comparison between the different computations

Our aim is to compare the different systems of generators of $\mathcal{D}^{(1)}(A)$ and $\mathcal{D}^{(2)}(A)$ previously obtained:

- Jacobian method:

$$\begin{aligned}\mathcal{D}^{(1)}(A) =& \langle\ (x-y)\partial_y,\ x(\partial_x + \partial_y)\ \rangle,\\ \mathcal{D}^{(2)}(A) =& \langle x\partial_x^2 + 2x\partial_x\partial_y + (2x-y)\partial_y^2, x^2\partial_x^2 + (-3x^2+xy)\partial_x\partial_y + (-3x^2+3xy-y^2)\partial_y^2, (x-y)\partial_y^2\rangle.\end{aligned}$$

- The method of intersecting modules:

$$\begin{aligned}\mathcal{D}^{(1)}(A) =& \langle x\partial_x + y\partial_y,\ (x-y)\partial_y\ \rangle,\\ \mathcal{D}^{(2)}(A) =& \langle\ x(\partial_x+\partial_y)^2,\ (x^2-xy)\partial_y^2,\ (x^2-xy)\partial_x\partial_y\ \rangle.\end{aligned}$$

- Holm's method:

$$\begin{aligned}\mathcal{D}^{(1)}(A) =& \langle\ (x-y)\partial_y,\ x(\partial_x + \partial_y)\ \rangle,\\ \mathcal{D}^{(2)}(A) =& \langle\ (x-y)\partial_y^2,\ x(\partial_x + \partial_y)^2\ \rangle.\end{aligned}$$

- One can obtain another system of generators using Macaulay2:

$$\begin{aligned}\mathcal{D}^{(1)}(A) =& \langle\ (x-y)\partial_y,\ x\partial_x + y\partial_y\ \rangle,\\ \mathcal{D}^{(2)}(A) =& \langle\ (x-y)\partial_y^2,\ x\partial_x^2 + 2x\partial_x\partial_y + y\partial_y^2\ \rangle.\end{aligned}$$

Seemingly there are discrepancies in the set of generators obtained in each method, but recall that

$$\mathcal{D}^{(m)}(A) \simeq \mathcal{D}^{(m)}(I)/I\mathcal{D}^{(m)}(R).$$

so these are classes of differential operators. It is quite difficult to compare them by hand and, to the best of our knowledge, this has not been implemented in computer algebra systems such as Macaulay2.

5. Conclusions

Over regular rings, the theory of D-modules has been an active area of research over the last fifty years since the works [Bernstein, 1971, Kashiwara, 1970]. It has motivated the development of many techniques and it has been applied to several other areas of mathematics such as Algebraic Geometry, Singularity Theory, Representation Theory and Commutative algebra.

A natural question is: what parts of the theory of D-modules extend to the case of singular rings? Answering this question requires tackling several issues. We mention some of them and some interesting approaches in the following.

For a non-regular ring S, describing the ring $D_{S|\mathbb{k}}$ of $\mathbb{k}$-linear differential operators is difficult, even in the particular case of hyperplane arrangements. For example, we saw in Section 4.3 that, even in a simple example, computing the generators of $\mathcal{D}^{(i)}(A)$ for small i may require some computational power. Few non-regular examples have been studied so far. The development of computational algebra systems such as Macaulay2 and Singular would be useful to grasp a better understanding of the area.

It is not known in what cases Bernstein's inequality holds. We have seen in Section 3.5 that linear simplicity is a fundamental tool to prove it. However, there are cases which are not linearly simple. For example the ring of differential operators on the cubic curve described in Example 3.14 does not contain elements of negative degree, therefore it is not linearly simple. So one can ask whether linear simplicity is related to the type of the singularity and, if so, what kind of singularities it holds on.

Back to the question on Bernstein's inequality, in the case of hyperplane arrangements we have seen that the $D_{A|\mathbb{k}}$-submodule Γ_A of A from Definition 3.31 is non trivial. Therefore, Bernstein's inequality does not hold for hyperplane arrangements. It may be interesting to investigate whether Bernstein's inequality holds if we consider a generalized version of rings of differential operators.

Finally, another approach to prove Bernstein's inequality is using characteristic varieties, which we have not defined, as in the reference [Castro Jiménez, 2010] on the regular case. Further work is needed to generalize this approach for singular rings.

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