MASTER'S DEGREE THESIS

Master of Science in Advanced Mathematics and Mathematical Engineering

Title: Asymptotics in the schema of simple varieties of trees

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Abstract

We study the functional equation $y(x) = xA(y(x))$ satisfied by the generating functions in the schema of simple varieties of trees. The radii of convergence r, R of y, A respectively satisfy $y(r) \leq R$. In the subcritical case ($y(r) < R$), one has $y_n \sim C \cdot r^{-n} n^{-3/2}$. In the critical case ($y(r) = R$), we approach the problem of determining the asymptotics of a_n when the information of $y(x)$ is well known. To that end, we give sufficient conditions that ensure that $A(u)$ can be extended to a delta domain around its dominant singularity R, which is needed to be able to apply the transfer theorem when determining the asymptotics. We do a similar analysis for the equation $y(x) = x + B(y(x))$, which appears in the additive schema of simple varieties of trees. The general framework we propose includes the three examples of extension to a delta domain of the functions $A(u)$ and $B(u)$ encountered in the literature so far.

Keywords

asymptotics of generating functions, delta domain, schema of simple varieties of trees, sharpness criterion, D-finite functions

MSC2020: 05A16

Contents

Introduction

The topic of this thesis was motivated from two papers in the literature [3] and [6] and a manuscript in preparation [2] where the supervisor of the thesis is involved.

The problem studied in Section 2 of this thesis occurs in the context of the functional equation

$$y(x) = xA(y(x)),$$

where $y(x), A(x)$ are the generating functions modelling the enumeration of trees and children in the schema of simple varieties of trees. Under suitable conditions, the radii of convergence r, R of $y(z), A(u)$, respectively, satisfy $y(r) \leq R$ (see Lemma 2.5). It is convenient to write the previous equation as

$$\psi(y(z)) = z, \text{ where } \psi(u) := \frac{u}{A(u)}, \tag{1}$$

as an inverse-function problem. Throughout the thesis, we review to which domains the functions $y(z)$ and $\psi(u)$ can be extended (analytically) so that Equation 1 holds.

In that context, when the strict inequality $y(r) < R$ occurs, one can prove that ψ' vanishes at $y(r)$ for the first time in the real positive axis (see Sharpness Criterion 2.7), which in turn allows one to determine the dominant singularity of $y(z)$. In that case, the asymptotics of the coefficients y_n of $y(z)$ can be easily determined (see Theorem 2.10) and one has

$$y_n \sim C\rho^{-n}n^{-\frac{3}{2}},$$

where ρ is the radius of convergence of $y(z)$ and C is a positive constant.

In this thesis, we are interested in the critical case $y(r) = R$, which occurs for instance if y_n does not admit the previous asymptotic expression. In our case of study, we know in detail the information of the singularities of the function $y(z)$ and its behavior around them, and we seek to determine the asymptotic behavior of the coefficients a_n of $A(u)$. In that process, in order to be able to use the transfer theorem (1.17), which relates the asymptotics of the coefficients of the generating function to behavior around the dominant singularity of the generating function, we need to prove the extension of $A(u)$ to a delta domain. This has been done with *ad hoc* arguments in two of the examples of the literature [3] [6]. We use the technique known as analytic continuation along paths. We provide a set of sufficient conditions that ensure that $A(u)$ can be extended to a delta domain around its dominant singularity (see Proposition 2.13). The two examples from the literature fit under this set of sufficient conditions (see section 2.4 for a detailed analysis).

Section 3 of this thesis follows similar lines to Section 2, but we study the additive version of the previous functional equation

$$y(z) = z + B(y(z)).$$

This type of functional equation appeared in the first paper with this type of problem that was published in the literature [3]. In Section 3, we replicate the analysis done in Section 2 to show that $B(u)$ can be extended to a delta domain in the critical case $y(r) = R$ when the information of $y(z)$ is well-known. We do so in a context general enough where the example of this type from the literature fits.

Section 1 of the thesis is a theoretical introduction to the main concepts used in the next two sections. Its contents are mainly based on the book [4] and are complemented with textbooks in complex analysis [5] [7].

1. Analytic combinatorics

In this chapter, we present the main theoretical results that will be used in Sections 2 and 3. Most of the results are on Analytic Combinatorics, for which the book [4] has been taken as a reference. We also present some general results in Complex Analysis that we will need later on. The results in this section are stated without a proof.

1.1 Results in Complex Analysis

We just present some concepts and results on Complex Analysis. A more solid background in Complex Analysis and the proofs of the results can be found in [5] or [7].

Definition 1.1 (Analytic continuation). Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open sets and let $f_i \colon \Omega_i \to \mathbb{C}$ be analytic functions (for $i = 1, 2$). If $\Omega_1 \cap \Omega_2 \neq \varnothing$, and $f_1(z) = f_2(z)$ for all $z \in \Omega_1 \cap \Omega_2$, f_2 is said to be an analytic continuation of f_1 to Ω_2.

Theorem 1.2 (Unicity of analytic continuation). *Given and open set $\Omega \subset \mathbb{C}$ and two analytic functions $f_1, f_2 \colon \Omega \subset \mathbb{C}$, if*

$$S := \{z \in \mathbb{C} \mid f_1(z) = f_2(z)\}$$

has an accumulation point, then $f_1(z) = f_2(z)$ for all $z \in \Omega$.

In particular, if there exists an analytic continuation f_2 of a function $f_1 \colon \Omega_1 \to \mathbb{C}$ to a domain Ω_2, then it is unique.

Theorem 1.3 (Analytic inversion). *Let $\psi(z)$ be analytic at y_0 with $\psi(y_0) = z_0$. If $\psi'(y_0) \neq 0$. Then, there exists a small neighborhood U_0 of z_0 and an analytic function $y(z)$ in U_0 such that $y(z_0) = y_0$ and $\psi(y(z)) = z$ in U_0.*

Definition 1.4. Let $\Omega \subset \mathbb{C}$ be an open set and $f \colon \Omega \to \mathbb{C}$ an analytic function. A point $z_0 \in \partial\Omega$ is said to be a singularity of f if for every neighborhood U of z_0, there is no analytic continuation of f to U.

Theorem 1.5 (Singular inversion). *Let $\psi(z)$ be analytic at y_0 with $\psi(y_0) = z_0$. If $\psi'(y_0) = 0$ and $\psi''(y_0) \neq 0$. Then, there exists a small neighborhood U_0 of z_0 such that for any angle $\theta \in [0, 2\pi)$, there exist two analytic functions y_1, y_2 in $U_0^{\setminus\theta} = \{z \in U_0 \mid \arg(z - z_0) \neq \theta\}$ such that $\psi(y_i(z)) = z$ in $U_0^{\setminus\theta}$, y_i have a singularity at z_0 and $\lim_{z\to z_0} y_i(z) = y_0$.*

1.2 Generating functions

Definition 1.6. Let $a_n \in \mathbb{C}$ for $n \in \mathbb{N}$ be a sequence of complex numbers. The associated generating function (GF for short) $A(x)$ is defined as

$$A(x) := \sum_{n\geq 0} a_n x^n,$$

understood as an element of $\mathbb{C}[[x]]$, the ring of formal power series over $\mathbb{C}$.

Theorem 1.7 (Lagrange Inversion Theorem). *Let $A(t) = \sum_{k\geq 0} a_k t^k \in \mathbb{C}[[t]]$ be a power series with $a_0 \neq 0$. Then, the equation $y(x) = xA(y(x))$ admits a unique solution in $\mathbb{C}[[x]]$ with coefficients given by:*

$$[x^n]\, y(x) = \frac{1}{n}\left[t^{n-1}\right] \phi(t)^n.$$

Definition 1.8. Given a generating function $A(x) := \sum_{n\geq 0} a_n x^n$, we define:

- The order of a GF $A(x) \not\equiv 0$ is

$$\text{ord}(A) := \min\{n \in \mathbb{N} \mid a_n \neq 0\}.$$

- The period of a GF $A(x)$ with at least two non-zero coefficients is

$$\text{Period}(A) := \gcd\{n - \text{ord}(A) \mid n \in \mathbb{N}, a_n \neq 0\}.$$

  A GF is said to be aperiodic if its period is 1.

- The radius of convergence of $A(x)$ at zero is

$$R = \sup\left\{r \in \mathbb{R} \mid \sum_{n\geq 0} a_n z^n \text{ converges for } |z| = r, z \in \mathbb{C}\right\}.$$

Lemma 1.9 (Daffodil's Lemma)**.** *Let $f(z) = \sum_{n\geq 0} f_n z^n$ with $f_n \in \mathbb{R}_{\geq 0}$ be absolutely convergent in the region $|z| \leq R$. If $f(z)$ is aperiodic, then*

$$|f(z)| < f(|z|)$$

for any $z \in \{z \in \mathbb{C} \mid |z| \leq R, z \notin \mathbb{R}_{\geq 0}\}$.

Proposition 1.10. *Given a GF $A(x)$ with radius of convergence R, let $D := \{z \in \mathbb{C} \mid |z| < R\}$. The function*

$$\begin{aligned} A\colon D &\to \mathbb{C} \\ z &\mapsto A(z) = \sum_{n\geq 0} a_n z^n \end{aligned}$$

is analytic.

Theorem 1.11 (Composition of generating functions)**.** *Let $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{n\geq 1} b_n x^n$ be power series over $\mathbb{C}$ such that $B(x)$ has null constant term. Assume that $A(u)$ is absolutely convergent for $|u| \leq R$, with $R > 0$. Let $r > 0$ be such that*

$$\sum_{n\geq 0} |b_n| r^n \leq R.$$

Consider the formal power series C obtained by composing (formally) A and B:

$$C(x) = A(B(x)) = \sum_{n\geq 0} a_n B^n(x) = \sum_{n\geq 0} c_n x^n,$$

where $c_n = \sum_{i=0}^{n} a_i [x^n] B^i(x)$. Then, C converges absolutely for $|z| \leq r$ and for such z,

$$C(z) = A(B(z)).$$

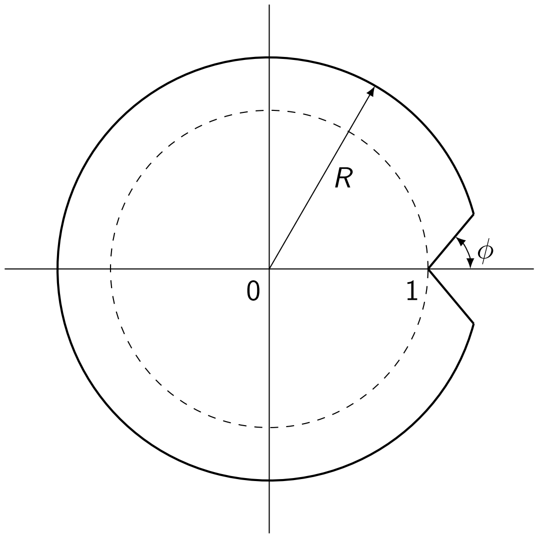

Figure 1: Drawing of a delta domain

1.3 Singularities and asymptotics of generating functions

Theorem 1.12 (Pringsheim's Theorem). *Let $f(z)$ be a complex function analytic at the origin, whose series expansion at the origin has non-negative real coefficients and has a finite radius of convergence R. Then, $z = R$ is a singularity of $f(z)$.*

Proposition 1.13 (Standard function scale). *Given $\alpha \in \mathbb{R}\backslash\mathbb{Z}_{\leq 0}$, the n-th coefficient of the series expansion at the origin of the complex function $f(z) = (1-z)^{-\alpha}$ admits an asymptotic expansion in descending powers of n:*

$$[z^n]\, f(z) = \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \sum_{k\geq 1} \frac{e_k(\alpha)}{n^k}\right)$$

where $e_k(\alpha)$ are polynomials of degree $2k$ in α.

Proposition 1.14. *Given $m \in \mathbb{N}$, the n-th coefficient of the series expansion at the origin of the complex function $f(z) = (1-z)^m \log\left(\frac{1}{1-z}\right)$ is given explicitly by:*

$$[z^n]\, f(z) = (-1)^m \frac{m!}{n(n-1)\cdots(n-m)} \sim (-1)^m \frac{m!}{n^{m+1}}.$$

Definition 1.15. Given $R \in \mathbb{R}_{>1}$ and $\phi \in \left(0, \frac{\pi}{2}\right)$, the open domain $\Delta(\phi, R)$ is defined as:

$$\Delta(\phi, R) = \{z \mid |z| < R, z \neq 1, |\arg(z-1)| > \phi\}$$

A domain is a delta domain (at 1) when it is $\Delta(\phi, R)$ for certain R, ϕ.

A function $f(z)$ is said to be delta-analytic if it is analytic in some delta domain.

See Figure 1 for a picture of a delta domain.

Definition 1.16 (Singularity of the logarithmic type)**.** Define the following set $\mathcal{F}$ of functions:

$$\mathcal{F} := \left\{ (1-z)^{-\alpha} \log\left(\frac{1}{1-z}\right)^{\beta}, \alpha, \beta \in \mathbb{C} \right\}.$$

Let $f(z)$ be a function analytic at 0 with a singularity at $z = \zeta$. Consider the auxiliary function $\widetilde{f}(z) := f(\zeta z)$. The singularity ζ of f is said to be of the logarithmic type if there exist functions $\sigma(z) = \sum_{i=1}^{k} c_i \sigma_i(z)$ and $\tau(z)$, with and $c_1, ..., c_k \in \mathbb{C}$ and $\sigma_k(z), \tau(z) \in \mathcal{F}$, such that near $z = 1$, $\widetilde{f}(z)$ has the following behavior:

$$\widetilde{f}(z) = \sigma(z) + O(\tau(z)) \quad \text{as } z \to 1 \text{ in } \Delta_0.$$

Theorem 1.17 (Singularity analysis)**.** *Let $f(z)$ be a function analytic at 0 with a singularity at $z = \zeta$ of the logarithmic type, with the same notation as in Definition 1.16.*

Then, the coefficients f_n of the series expansion of $f(z)$ at 0 satisfy:

$$f_n = \zeta^{-n} \sum_{i=1}^{k} c_k \sigma_{k,n} + O\left(\zeta^{-n} \tau_n\right)$$

where $\sigma_{k,n}$ and τ_n are the n-th coefficients of the series expansion at 0 of $\sigma_k(z)$, $\tau(z)$, respectively (obtained using Propositions 1.13 and 1.14).

We introduce the set of D-finite functions, which will appear later on in the examples from the literature.

Definition 1.18. A generating function $y(z)$ is said to be D-finite if there exists $k \in \mathbb{N}$ and polynomials $p_0(z), \cdots, p_k(z) \in \mathbb{C}[z]$ such that $y(z)$ is a solution to the following differential equation

$$p_k(z)\frac{\mathrm{d}^k f}{\mathrm{d}z^k}(z) + \cdots + p_1(z)\frac{\mathrm{d}f}{\mathrm{d}z}(z) + p_0(z)f(z) = 0. \tag{2}$$

Theorem 1.19 (Singularities of D-finite functions)**.** *The singularities of a D-finite generating function $y(z)$ satisfying Equation (2) are roots of $p_k(z)$.*

Remark 1.20. In the context of the previous theorem, the roots of $p_k(z)$ are not necessarily singularities of $f(z)$. A further analysis is needed to determine whether a root of $p_k(z)$ is actually a singularity or not. The details of this topic are outside the scope of this thesis (see [10]) for an extensive analysis). It is worth mentioning that under suitable conditions, the singularities of a D-finite function are of the logarithmic type (see Definition 1.16) and its coefficients are amenable to asymptotic analysis.

2. Schema of simple varieties of trees

This section aims to study the asymptotics of the coefficients of the formal power series $y(x)$ and $A(x)$ involved in the following functional equation:

$$y(x) = xA(y(x)) \tag{3}$$

This type of functional equation arises when studying symbolically tree-like structures, where $y(x)$ corresponds to the generating function counting the number of trees and $A(x)$ corresponds to the generating function counting the possible children that a given node can have. For instance, Equation (3) is satisfied by the ordinary generating function (OGF) of plane rooted unlabeled trees and the exponential generating function (EGF) of plane and non-plane rooted trees. Table 1 shows the explicit form of the above relation for each of these cases.

Type of rooted tree	Type of GF	Symbolic description	Functional equation
Plane unlabeled	OGF	$\mathcal{T} = \mathcal{Z} \times \mathrm{Seq}_N(\mathcal{T})$	$T(x) = x \cdot \sum_{n\in N} T(x)^n$
Plane labelled	EGF	$\mathcal{T} = \mathcal{Z} \times \mathrm{Seq}_N(\mathcal{T})$	$T(x) = x \cdot \sum_{n\in N} T(x)^n$
Non-plane labelled	EGF	$\mathcal{T} = \mathcal{Z} \times \mathrm{Set}_N(\mathcal{T})$	$T(x) = x \cdot \sum_{n\in N} \frac{1}{n!} T(x)^n$

Table 1: Generating functions for some types of rooted trees with allowed number of children in the set N.

For the moment, and whenever the variable x is used, we understand Equation (3) as an equality between formal power series (i.e. their coefficients are equal). We will deal with the analytic interpretation of the equation later on. We start by stating a couple of basic remarks regarding Equation (3).

Observation 2.1. Let $y(x) = \sum_{i\geq 0} y_i x^i$ and $A(x) = \sum_{i\geq 0} a_i x^i$ be a pair of generating functions satisfying Equation (3). Then,

- $y_0 = 0$, since the right-hand side of Equation (3) has order at least 1.
- $y_1 = a_0$.
- $a_0 = 0 \Leftrightarrow y(x) \equiv 0$: If $a_0 = 0$, then $y(x) = xy(x)\widetilde{A}(y(x))$, where $\widetilde{A}(x) = \frac{A(x)}{x}$, so either $y(x) = 0$ or $1 = x\widetilde{A}(y(x))$ and the latter expression is impossible because the right-hand side is either 0 or has order at least 1, so it must be $y(x) = 0$. Conversely, if $y(x) \equiv 0$, Equation (3) becomes $0 = za_0$, so $a_0 = 0$.
- $A(x) \equiv a_0 \Leftrightarrow y(x) \equiv a_0 x$, which is a direct consequence of the previous two items.

Observation 2.2. Let $y(x) = \sum_{i\geq 1} y_i x^i \not\equiv 0$ and $A(x) = \sum_{i\geq 1} a_i x^i$ be a pair of generating functions satisfying Equation (3) and such that $A(x) \not\equiv a_0$. Then, $\mathrm{Period}(A(x)) = \mathrm{Period}(y(x))$.

Proof. Taking into account Observation 2.1, denoting $p = \mathrm{Period}(A(x))$ and $q = \mathrm{Period}(y(x))$, we can write $A(x)$ and $y(x)$ as:

- $A(x) = a_0 + a_p x^p + a_{2p} x^{2p} + \cdots + a_{kp} x^{kp} + \cdots = \widetilde{A}(x^p)$, where $\widetilde{A}(x) = \sum_{k\geq 0} a_{kp} x^k$.
- $y(x) = x\left(y_1 + y_{q+1} x^q + y_{2q+1} x^{2q} + \cdots + y_{kq+1} x^{kq} + \cdots\right) = x\widetilde{y}(x^q)$, , where $\widetilde{y}(x) = \sum_{k\geq 0} y_{kq} x^k$.

By Lagrange Inversion Theorem (1.7), we have:

$$[x^n]y(x) = \frac{1}{n}[t^{n-1}]\left(A^n(t)\right) = \frac{1}{n}[t^{n-1}]\left(\widetilde{A}^n(t^p)\right),$$

which will be zero whenever $n-1 \not\equiv 0 \pmod{p}$. In particular, this means that $p \mid q$.

To prove the equality of the two periods, we start by rewriting Equation (3) as:

$$\widetilde{y}(x^q) = \sum_{k\geq 0} a_{kp} x^{kp} \left(\widetilde{y}(x^q)\right)^{kp} \tag{4}$$

Assume there exists some $k \in \mathbb{N}$ such that $a_{kp} \neq 0$ and $q \nmid kp$. Let k_0 be the smallest such k. Then in Equation (4) the (k_0p)-th coefficient in the right-hand side is $a_{k_0p} y_1^{k_0p} \neq 0$ (there are no contribution of smaller order terms because we chose k_0 to be the smallest natural satisfying the above condition and all contributions of smaller order terms go to powers of x which are multiples of q), whereas the (k_0p)-th coefficient in the left-hand side is 0 since $q \nmid k_0p$, a contradiction. This means that $a_i = 0$ whenever $q \nmid i$, so $q \mid p$ and thus we must have the equality $p = q$. □

2.1 Analytic considerations and sharpness criterion

The discussion along this chapter will be dealing with the question of when Equation (3) makes sense analytically. We start by introducing the basic notation that will be used throughout this chapter. The results discussed here are taken from [8].

Definition 2.3. A pair of generating functions $(y(z), A(u))$ satisfying Equation (3) which are analytic at 0 are said to belong to the schema of *smooth simple varieties of trees*.

Notation 2.4. In the smooth simple varieties of trees schema, we will denote by:

- r the radius of convergence of $y(z)$. The disk of convergence of $y(z)$ is denoted by

$$\Omega := \{z \in \mathbb{C} \mid |z| < r\}.$$

- R the radius of convergence of $A(u)$. The disk of convergence of $A(u)$ is denoted by

$$\Lambda := \{u \in \mathbb{C} \mid |u| < R\}.$$

Lemma 2.5. *Let $a_0 > 0$ be a real positive number and let $A(x) = a_0 + \sum_{i\geq 1} a_i x^i$ and $y(x) = \sum_{i\geq 1} y_i x^i$ be formal power series over $\mathbb{R}$ satisfying Equation (3). Let r, R denote the radii of convergence of y, A, respectively; and let Ω be the disk of convergence of y. Assume that $A(x)$ has non-negative coefficients. Then,*

(i) $y(z)$ has non-negative coefficients,

(ii) $y(r) \leq R$ (it may be that $y(r) = \infty$),

(iii) $y(z) = zA(y(z))$ holds in $\overline{\Omega} = \{z \in \mathbb{C} \mid |z| \leq r\}$.

Proof. (i) It is an immediate consequence of Lagrange Inversion Theorem (1.7) and the fact that A has non-negative coefficients.

(ii) If $r = 0$, the result is trivial. Assume that $r > 0$. First observe that if $z \in \mathbb{R}^+$, in Equation (3), the summations involved can be rearranged because all the coefficients are real and non-negative:

$$y(z) = \sum_{i \geq 1} y_i z^i = zA(y(z)) = z \sum_{i \geq 0} a_i \left(\sum_{j \geq 1} y_j z^j \right)^i = \sum_{i \geq 1} \left[\left(\sum_{j=1}^{i} c_{ij} \right) z^i \right],$$

where $c_{ij} = a_j [x^{i-1}] \left(\sum_{k \geq 1} y_k x^k \right)^j$. In particular, Equation (3) can be evaluated for $z \in [0, r)$ (since we have $y(z)$ in the left-hand side of the last equation). Observe now that, since $y_1 = a_0 > 0$ and y has non-negative coefficients, $y\colon [0, r) \to [0, y(r))$ is strictly increasing and continuous (and thus bijective). Therefore, if $u \in (0, y(r))$, there is some $z \in (0, r)$ such that $y(z) = u$ and $A(u) = A(y(z)) = \frac{y(z)}{z} < \infty$ since z is in the disk of convergence of $y(z)$. Therefore, the power series A is convergent in $[0, y(r))$, so $y(r) \leq R$ (since A has non-negative coefficients, by Pringsheim's Theorem (1.12), the dominant singularity is in the real positive axis). Note that the argument is also valid when $y(r) = \infty$.

(iii) If $r = 0$, the result is trivial. Assume that $r > 0$. Since $y(r) \leq R$, we can use Theorem 1.11 with $B(x) = y(x)$, so the expression $A(y(z))$ converges for all $|z| \leq r$ and the equality (3) holds for $z \in \overline{\Omega}$. □

Observe that whenever it makes sense to evaluate at $z \in \mathbb{C} \setminus \{0\}$, we may rewrite Equation (3) as:

$$\psi(y(z)) = z, \quad \text{where} \quad \psi(u) := \frac{u}{A(u)}.$$

This way, we reformulate the equation into an inverse function equation, which will be key to the arguments concerning asymptotics.

We also note here that the derivative of $\psi(u)$ is given by the following expression:

$$\psi'(u) = \frac{A(u) - uA'(u)}{A^2(u)}, \tag{5}$$

which will play an essential role in the rest of the chapter.

Lemma 2.6. *In the hypotheses of Lemma 2.5, the function $y\colon \Omega \to y(\Omega)$ is a biholomorphism with inverse function $\psi(u) := \frac{u}{A(u)}$ (i.e. $\psi(y(z)) = z$ holds for $z \in \Omega$). Moreover,*

- *If $A(x) \not\equiv a_0$, then $r < \infty$.*
- *If $y(r) < \infty$, then y extends continuously to $\partial\Omega$ and ψ extends continuously to $\partial(y(\Omega))$ and $\psi(y(z)) = z$ holds for $z \in \partial\Omega$.*

Proof. Let's first prove the injectivity of $y\colon \Omega \to y(\Omega)$. Observe that if $y(z) = 0$, then by Equation (3),

$$0 = y(z) = zA(y(z)) = zA(0) = za_0 \Leftrightarrow z = 0.$$

In particular, if $z \in \Omega \setminus \{0\}$, $A(y(z)) = \frac{y(z)}{z} \neq 0$ and this implies that $A(u)$ does not vanish on $y(\Omega) \setminus \{0\}$. If $y(z_1) = y(z_2) \neq 0$, then by Equation (3),

$$z_1 A(y(z_1)) = y(z_1) = y(z_2) = z_2 A(y(z_2)) = z_2 A(y(z_1)) \Rightarrow z_1 = z_2,$$

where we have used that $A(y(z_1)) \neq 0$ since $y(z_1) \neq 0$. This completes the proof of the injectivity of $y\colon \Omega \to y(\Omega)$. Since y is analytic in Ω (because it is its disk of convergence) and injective, its inverse function is also analytic and the first part of the statement follows.

- If $A(x) \not\equiv a_0$, assume on the contrary that y is entire. Then, by Lemma 2.5, $R = \infty$ and y maps bijectively $[0,\infty)$ to $[0,\infty)$. For $u \in [0,\infty)$, consider the function $N(u) = A(u) - zA'(u) = a_0 - \sum_{n\in\mathbb{N}}(n-1)a_n u^n$, the numerator of Equation (5). $N(0) = a_0 > 0$ and $\lim_{u\to\infty} N(u) = -\infty$ because $a_k > 0$ for some $k \geq 1$, so there exists some $\tau = y(\alpha) > 0$ (for some $\alpha > 0$) such that $N(\tau) = 0$. By Lemma 2.5, y is analytic in a neighborhood of α and ψ is its inverse function, whose derivative at τ is given by Equation (5), so $\psi'(\tau) = 0$, which contradicts the fact that ψ is analytically invertible at τ.

- If $y(r) < \infty$, by Weierstrass M-test, $y(z)$ converges on $\overline{\Omega}$, and this gives the continuous extension of y to $\partial\Omega$. Similarly, using Theorem 1.11, $A(y(z))$ also converges on $\overline{\Omega}$. By continuity, $zA(y(z)) - y(z) = 0$ still holds on $\partial\Omega$, so $A(u)$ does not vanish on $\partial(y(\Omega))$ (the same argument used in the proof of injectivity is valid here) and $\psi(u)$ is well-defined on $\partial(y(\Omega))$. Thus, $\psi(y(z)) = z$ holds on $\partial\Omega$.

□

Proposition 2.7 (Sharpness Criterion). *Let $a_0 > 0$ be a real positive number and let $A(x) = a_0 + \sum_{i\geq 1} a_i x^i$ and $y(x) = \sum_{i\geq 1} y_i x^i$ be formal power series over $\mathbb{R}^+$ satisfying Equation (3). Let r, R denote the radii of convergence of y, A, respectively. If $y(r) < R \leq \infty$, then $A(u) - uA'(u)$ vanishes at $u = y(r)$.*

Proof. The proof of this Proposition is a bit technical, and it can be helpful looking at the diagram in Figure 2, which depicts the regions of the complex plane that play a role in the proof.

First observe that since $y_1 = a_0 > 0$, we have $r < \infty$ since otherwise $y(r) = \infty$. Assume that $A(y(r)) - y(r)A'(y(r)) \neq 0$. By Lemma 2.6, the function $\psi(u) = \frac{u}{A(u)}$ is analytic and the inverse of y on $y(\Omega)$.

Since $y(r) < R$ and R is the radius of convergence of A, there exists a small open disk V_1 around $y(r)$ where A is analytic. Since $A(y(r)) = \frac{y(r)}{r} > 0$, we can shrink V_1 to a smaller open disk V_2 so that A does not vanish on V_2. Thus, $\psi = \frac{u}{A(u)}$ is also analytic in V_2 and Equation (5) holds there.

By Equation (5), the fact that $A(y(r)) - y(r)A'(y(r)) \neq 0$ means that $\psi'(y(r)) \neq 0$. By the Analytic Inversion Theorem (1.3), there is some open disk $V \subset V_2$ around $y(r)$ such that $\psi\big|_V\colon V \to \psi(V)$ is a biholomorphism. By injectivity of $\psi\big|_V$, it follows that y and $\psi\big|_V^{-1}$ agree on the open set $\psi(y(\Omega)\cap V)$. Thus, y is analytically continuable to the larger set $\Omega\cup\psi(V)$. Since $r \in \psi(V)$, this contradicts Pringsheim's Theorem (1.12), which states that since $y(z)$ has non-negative coefficients, its radius of convergence must be a singularity. □

2.2 Asymptotics in the smooth inverse-function schema

Definition 2.8. Let $(y(z), A(u))$ belong to the smooth simple varieties of trees schema with radii of convergence r, R, respectively. We will say that the pair of functions belongs to the *smooth inverse-function schema* if in addition they satisfy the following two conditions:

(H1) The function $A(u)$ satisfies

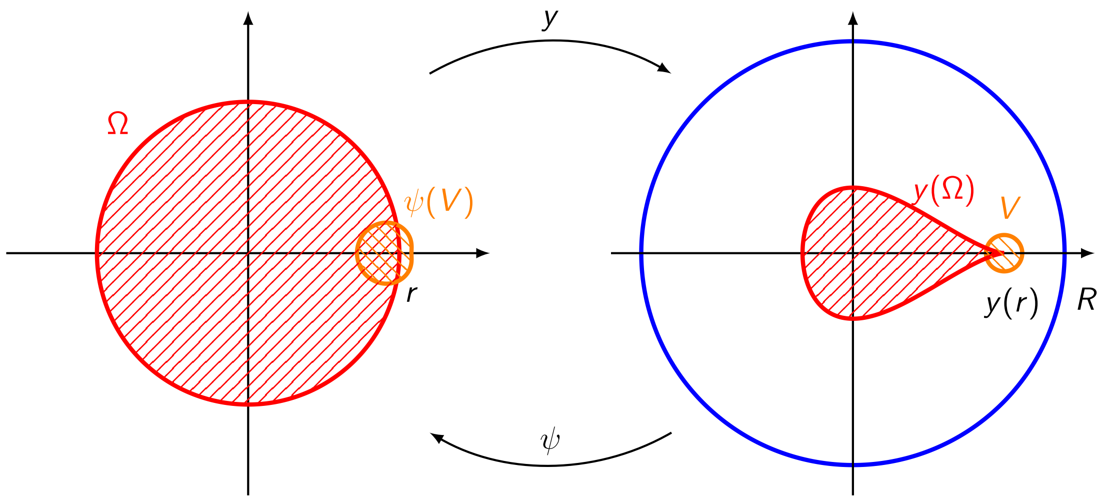

Figure 2: Diagram of the domains in the proof of Proposition 2.7

- $A(0) \neq 0$,
- $A(u) \not\equiv a_0 + a_1 u$,
- $[z^n]A(u) \geq 0$ for all $n \in \mathbb{N}$.

(H2) In the open disk of convergence $\Lambda = \{u \in \mathbb{C} \mid |u| < R\}$ of A at 0 exists a positive solution $0 < \tau < R$ to the characteristic equation:

$$A(u) - uA'(u) = 0. \tag{6}$$

If in addition the function $A(u)$ is aperiodic, the schema is said to be aperiodic.

Observation 2.9. Some comments on the conditions of the smooth inverse-function schema:

(i) In condition (H1), an easy check shows that the condition $A(u) \not\equiv a_0 + a_1 u$ just rules out the case in which $y(z) = \frac{a_0 z}{1-a_1 z} = a_0 \sum_{n\geq 0} a_1^n z^{n+1}$, whose asymptotics are trivial.

(ii) In condition (H2), if a positive real solution to Equation (6) exists, then it is unique since $A(u) - uA'(u) = a_0 - \sum_{n\in\mathbb{N}}(n-1)a_n u^n$ and the function $\sum_{n\in\mathbb{N}}(n-1)a_n u^n$ is strictly increasing in $(0, R)$ because $a_n \geq 0$ and $a_k > 0$ for some $k \geq 2$.

The asymptotics of the function y in the smooth inverse-function schema can be explicitly obtained using the following result.

Theorem 2.10. *Let $(y(z), A(u))$ belong to the smooth inverse-function schema in the aperiodic case*[1]*. Let $0 < \tau < R$ be the solution to the characteristic equation (6) and let $\rho := \frac{\tau}{A(\tau)}$. Then, ρ is the radius of convergence of y and one has the following asymptotic expansion:*

$$y_n = [z^n]y(z) \sim \sqrt{\frac{A(\tau)}{2A''(\tau)}} \frac{\rho^{-n}}{\sqrt{\pi n^3}} \left(1 + O\left(\frac{1}{n}\right)\right) \tag{7}$$

[1]The periodic case can be studied similarly. For a function $A(u)$ of period p, the same result holds adding a factor p to the asymptotic constant and the asymptotics is only valid for coefficients y_n with $n \equiv 1(\mathrm{mod}\,p)$

Proof. We start by proving that ρ is the radius of convergence of $y(z)$. By Observation 2.9, $y(z)$ has non-negative coefficients, so by Pringsheim's Theorem (1.12), the dominant singularity will be in the positive real axis (it will be the radius of convergence) and by aperiodicity of $A(u)$ (and hence of $y(z)$ by Observation 2.9), it will be the only dominant singularity. Consider the function $\psi(u) := \frac{u}{A(u)}$, the functional inverse of $y(z)$ in a neighborhood of $z = 0$. The derivative of this function is given by Equation (5).

Let r be the radius of convergence of $y(z)$. We now use the monotonicity of functions $y(z)$ and $A(u) - uA'(u)$ to prove that $y(r) = \tau$:

- If $y(r) < \tau$, then by monotonicity of the function $A(u) - uA'(u)$ in the positive real axis, $\psi'(y(r)) \neq 0$ and by the Analytic Inversion Theorem (1.3), y is analytic at r, a contradiction with the fact that r is a singularity of y.
- If $y(r) > \tau$, then by monoticity of $y(z)$ in the positive real axis, there exists $s < r$ such that $y(s) = \tau$, so $\psi'(y(s)) = 0$ and by the Singular Inversion Theorem, y is not analytic at s, a contradiction with the fact that $r > s$ is the radius of convergence of y.

Thus, $y(r) = \tau$. Moreover, since $\psi(y(z)) = z$ in $[0, r)$ and ψ is continuous at $y(r) = \tau < R$, we have:

$$r = \lim_{z \to r^-} z = \lim_{z \to r^-} \psi(y(z)) = \psi\left(\lim_{z \to r^-} y(z)\right) = \psi(y(r)) = \psi(\tau) = \frac{\tau}{A(\tau)}$$

Thus, ρ is the radius of convergence of y and this completes the proof of the first part.

Once we know the radius of convergence, we address the proof of the asymptotic expression of y_n. Since $\tau < R$ and R is the radius of convergence of $\psi(u)$, we can expand ψ around $u = \tau$ in the equation $\psi(y) = z$, which is equivalent to Equation (3):

$$z = \psi(\tau) + \psi'(\tau)(y - \tau) + \frac{\psi''(\tau)}{2!}(y - \tau)^2 + \frac{\psi'''(\tau)}{3!}(y - \tau)^3 + \ldots \tag{8}$$

We compute the first coefficients in the Taylor expansion:

$$\psi(\tau) = \rho,$$

$$\psi'(u) = \frac{A(u) - uA'(u)}{A^2(u)} \Rightarrow \psi'(\tau) = 0,$$

$$\psi''(u) = -\frac{A''(u)}{A^2(u)} - 2\frac{(A(u) - uA'(u))A'(u)}{A^3(u)} \Rightarrow \psi''(\tau) = -\frac{A''(\tau)}{A^2(\tau)},$$

where we have used the fact that $A(\tau) - \tau A'(\tau) = 0$. We may then rewrite Equation (8) as

$$\rho - z = \frac{A''(\tau)}{2A^2(\tau)}(y - \tau)^2 - \frac{\psi'''(\tau)}{3!}(y - \tau)^3 - \ldots$$

Taking square roots (locally) on both sides of the last equation, we have:

$$-\sqrt{\rho - z} = \sqrt{\frac{A''(\tau)}{2A^2(\tau)}}(y - \tau)\left(1 + c_1(y - \tau) + c_2(y - \tau)^2 + \cdots\right),$$

for certain coefficients c_i. The minus sign has been taken on the left-hand side since $y(z)$ tends increasingly to τ as $z \to \rho^-$. We can finally solve with respect to $(y-\tau)$:

$$(y-\tau) = -\sqrt{\frac{2A^2(\tau)}{A''(\tau)}}\sqrt{\rho - z}\left(1 + d_1\sqrt{\rho - z} + d_2(\rho - z) + d_3(\rho - z)^{\frac{3}{2}}\cdots\right),$$

for some coefficients d_i. This expression holds in a domain of the form $D_{\varepsilon,\delta} := \{z \in \mathbb{C} \mid |z-\rho| < \epsilon, \operatorname{Arg}(z-r) > \delta\}$ for some $\delta \in \left(0, \frac{\pi}{2}\right)$.

For any other point ζ such that $|\zeta| = \rho$, we have $|y(\zeta)| < y(r) = \tau$ (since y is aperiodic and by Pringsheim's Theorem 1.12). Note that $\psi'(y(\zeta)) \neq 0$ since otherwise, looking at expression (5) we would have:

$$A(y(\zeta)) - uA'(y(\zeta)) = 0 \Rightarrow a_0 = \left|\sum_{n\geq 1} na_n y(\zeta)^{n-1}\right| \leq \sum_{n\geq 1} na_n |y(\zeta)|^{n-1} < \sum_{n\geq 1} na_n \tau^{n-1} = a_0,$$

which is a contradiction. We then have:

$$0 \neq \psi'(y(\zeta)) = \lim_{z\to\zeta, z\in\Omega} \psi'(y(z)) = \lim_{z\to\zeta, z\in\Omega} \frac{1}{y'(z)}$$

Thus $\lim_{z\to\zeta} y'(z)$ exists and is finite and, by analytic inversion (theorem 1.3), y can be analytically continuated to a neighborhood of ζ. Taking this into account and using a compacity argument totally analogous to the one done later on at the proof of Proposition 2.13 in Step (9), $y(z)$ extends to a delta domain around its dominant singularity.

We can finally use Theorem 1.17 to show that the asymptotic expression in Equation (7) holds. □

Note that the previous Theorem is a particular case of a wider schema of functions implicitly defined which also have an asymptotic behavior of the type $Cr^{-n}n^{-3/2}$, where $C > 0$ is a constant and r the radius of convergence. See Theorem 2.19 in [1] for a more general result of this kind.

This result, together with Proposition 2.7, yields the following Corollary.

Corollary 2.11. *Let $a_0 > 0$ be a real positive number and let $A(x) = a_0 + \sum_{i\geq 1} a_i x^i$ and $y(x) = \sum_{i\geq 1} y_i x^i$ be formal power series over $\mathbb{R}^+$ satisfying Equation (3). Let r, R denote the radii of convergence of y, A, respectively. Then, exactly one of the following statements is true:*

(1) $A(u) - uA'(u)$ is non-vanishing for $z \in (0, R)$, in which case $R = y(r)$.

(2) $R > y(r) = \tau$, where τ is the unique solution to $A(\tau) - \tau A'(\tau) = 0$ on $(0, R)$ and $y_n = Cr^{-n}n^{-\frac{3}{2}}(1 + o(1))$ ad $n \to \infty$, for some constant $C > 0$.

In particular, if $y_n \not\sim Cr^{-n}n^{-\frac{3}{2}}$ for some constant $C > 0$, then we are in Case (1) and $R = y(r)$.

Remark 2.12. It is still possible however that in Case (1) of Corollary 2.11, $y_n \sim Cr^{-n}n^{-\frac{3}{2}}$ for some constant $C > 0$. Example 2 of Section 2.5. in [8], shows that this type of situation is possible. The details of the example are outside the scope of this thesis and are omitted here.

2.3 Extension to a delta domain of the inverse generating function

In this subsection, we are dealing with a pair of functions $A(u), y(z)$ that belong to the schema of smooth simple varieties of trees and in addition satisfy $y(r) = R$ (we are in Case (1) of Corollary 2.11). In contrast to the smooth inverse-function schema (2.2), here we know information of the function $y(z)$ and we seek to determine the asymptotic behavior of the coefficients of $A(u)$. The results discussed here are a generalization of the arguments developed in section 4.3. of [8].

In order to obtain the asymptotic behavior of $A(u)$ in this context, the general method is given by the following steps:

(1) Study the singularities of $y(z)$, determine its asymptotic behavior, and obtain a singular expansion of $y(z)$ around its dominant singularity r.

(2) Show that $\psi(u)$ (and thus $A(u)$) can be extended to a delta domain.

(3) Determine the singular expansion of $\psi(u)$ around its dominant singularity R by using a bootstrapping technique in Equation (3).

(4) Determine the singular expansion of $A(u)$ around its dominant singularity R from the singular expansion of $\psi(u)$.

(5) Determine the asymptotic behavior of a_n by using the singularity analysis theorem 1.17 on the singular expansion of A around R (since in Step (2) we proved that it can be extended to a delta domain).

The focus of this thesis is on Step (2). The following result gives sufficient conditions to guarantee it.

Proposition 2.13. *Let $a_0 > 0$ be a real positive number and let $A(x) = a_0 + \sum_{i\geq 1} a_i x^i$ and $y(x) = \sum_{i\geq 1} y_i x^i$ be formal power series over $\mathbb{R}^+$ satisfying Equation (3). Let r, R denote the radii of convergence and Ω, Λ the disks of convergence of y, A, respectively. Let $\widetilde{\Omega}$ be a maximal domain of analyticity of y. Assume that $y(r) = R$ and that $A(x)$ (equivalently $y(x)$) is aperiodic. Assume moreover the following set of conditions:*

(i) $\frac{y(r)}{r} \leq 2a_0$.

(ii) All the singularities of $y(z)$ are real and of the logarithmic type.

(iii) $y(z) = y(r) + c(z-r) + o(z-r)$ *as* $z \to r$ *in* $\widetilde{\Omega}$ *for some real strictly positive constant* $0 < c = y'(r)$.

(iv) y extends continuously to r and for every non-dominant singularity z_k of y, one of the following two conditions holds:

- *y extends continuously to z_k and* $\lim_{z\to z_k, z\in\widetilde{\Omega}} |y(z)| \neq y(r)$,
- $|z_k| \geq \frac{R}{2a_0 - A(R)}$.

Then, $A(u)$ extends to a delta domain Δ around its dominant singularity R. Moreover, in that domain, $y(\psi(u)) = u$ holds, where $\psi(u) = \frac{u}{A(u)}$.

Proof. The proof of this result is quite technical, and the diagram depicted in Figure 3 may help to understand which are the regions of the complex plane involved in the argument.

The proof of the result is divided into several steps:

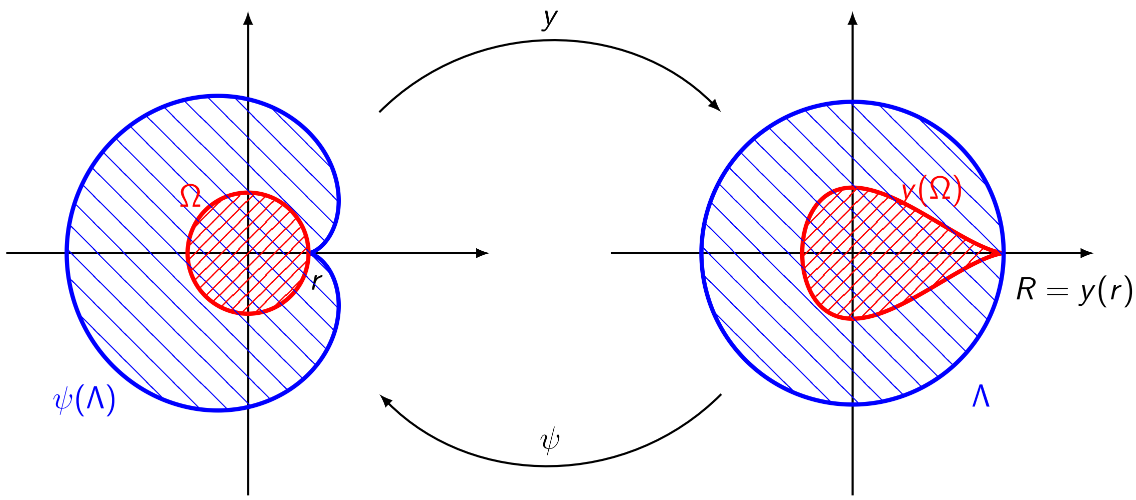

Figure 3: Diagram of the domains involved in the proof of Proposition 2.13

(1) $A(u) \neq 0$ for $|u| \leq R$.

First note that by Lemma 2.6, $A(y(z)) = \frac{y(z)}{z}$ holds at $\partial\Omega$, so in particular at $z = r$ we have:

$$A(R) = A(y(r)) = \frac{y(r)}{r} \leq 2a_0, \tag{9}$$

where we have used condition (i). The result then follows from the fact that R is the radius of convergence of $A(u)$ and the fact that if $A(u) = 0$ for some $|u| \leq R$ then $a_0 = -\sum_{i\geq 1} a_i u^i < \sum_{i\geq 1} a_i |u|^i \leq \sum_{i\geq 1} a_i R^i = A(R) - a_0 \leq a_0$, where we have used Equation (9) and the strict triangle inequality (which holds due to the non-periodicity of $A(u)$).

(2) ψ extends analytically to Λ and continuously to $\partial\Lambda$.

This is immediate from the expression $\psi(u) = \frac{u}{A(u)}$ and (1).

(3) ψ' does not vanish on Λ.

First observe that

$$\psi'(u) = \frac{A(u) - uA'(u)}{A^2(u)} = \frac{1}{A^2(u)} \left(a_0 + \sum_{n\geq 1} a_n(1-n)u^n \right). \tag{10}$$

Since we are in Case (1) from Corollary 2.11, $\psi'(u) \neq 0$ for $u \in [0, R)$. Since $\psi'(0) = 1$, ψ' is positive in the interval $[0, R)$. Now, if there is some u such that $|u| < R$ and $\psi'(u) = 0$, we have

$$1 = \left| \sum_{n\geq 1} a_n(n-1)u^n \right| < \sum_{n\geq 1} a_n(n-1)\,|u|^n = 1 - \psi'(|u|)A^2(|u|) < 1,$$

where the first strict inequality holds due to the non-periodicity of $A(u)$. Thus, ψ' does not vanish on Λ.

(4) ψ' extends continuously to $\partial\Lambda$ and is finite.

Differentiating the relation $y(z) = zA(y(z))$ with respect to z, one gets

$$A'(y(z)) = \frac{y'(z) - A(y(z))}{zy'(z)}.$$

Since $y'(r)$ exists and is non-zero and $y(r) = R$, $A'(R)$ exists and is finite. By Weierstrass M-test, $A'(u)$ also exists and is finite for $|z| = R$ and taking into account Equation (10), the same property holds for $\psi'(u)$.

(5) Given a path $\gamma \subset \Lambda$ from 0 to $\widetilde{\omega} \in \Lambda$, y can be analytically continued along $\psi \circ \gamma$, so $y = \psi^{-1}$ on $\psi(\Lambda)$.

The proof of the first part of the statement uses the technique of "analytic continuation along paths via local inversion".

Let $\gamma \subset \Lambda$ be a path from 0 to $\widetilde{\omega}$. By (3), $\psi'(\omega) \neq 0$ for any $\omega \in \gamma$. By the inverse function theorem, there is a neighborhood $U_\omega \subset \Gamma$ and an analytic homeomorphism $y_\omega \colon \psi(U_\omega) \to U_\omega$ such that $(y_\omega \circ \psi)\,|_{U_\omega} = \mathrm{Id}\,|_{U_\omega}$. The open sets U_ω cover the compact set γ, so a finite collection $U_0, \ldots, U_m$ of them also covers it, where $0 = \omega_0, \ldots, \omega_m = \widetilde{\omega}$ are taken in the order in which they are traversed (i.e. in such a way that $U_i \cap U_{i+1} \neq \varnothing$) . Starting at $\omega_0 = 0$, since $0 \in y(\Omega)$, $\psi(U_0) \cap \Omega$ is a non-empty open set. Moreover, y_0 and y agree on the open set $\psi(U_0 \cap y(\Omega)) \subset \Omega \cap \psi(U_0)$ since y is injective there (the inverse function theorem guarantees it since $y'(0) = 1 \neq 0$) and both y and y_0 are inverse functions of ψ in that domain (the inverse function of any injective function is unique). By analytic continuation, y and y_0 agree on the open set $\Omega \cap \psi(U_0)$. Similarly, y_1 and y_0 agree on the non-empty open set $\psi(U_0 \cap U_1) \subset \psi(U_0) \cap \psi(U_1)$ (they are both inverses of ψ there), so by analytic continuation $y_0 = y_1$ on $\psi(U_0) \cap \psi(U_1)$. In $\psi(U_0) \cap \psi(U_1)$, y_0 is an analytic continuation of y, so y_1 extends it to $\psi(U_1)$. We can inductively repeat those arguments to argue that y_k is an analytic continuation of y to $\psi(U_k)$ knowing that this is true for $0, 1, \ldots, k-1$.

We have produced an analytic continuation of y along the path $\psi \circ \gamma$, which yields an analytic continuation of y to the neighborhood $\psi(U_m)$ of $\psi(\widetilde{\omega})$ satisfying $y \circ \psi\,|_{U_m} = \mathrm{Id}\,|_{U_m}$.

Notice that it could happen that there are two points $\omega_1, \omega_2 \in \Lambda$ such that $\psi(\omega_1) = \psi(\omega_2)$, with different paths γ_1, γ_2 starting at the origin and reaching them, and different sequences of open sets $U_0, \ldots, U_n$ and $V_0, \ldots, V_m$. However, the analytic continuations y_1, y_2 cannot be different at $\psi(U_n \cap V_m)$ since $U_0 \cap V_0$ is an open non-empty set and the function in $\psi(U_0 \cap V_0)$ is fixed by y, and thus the principle of analytic continuation guarantees that $y_1 = y_2$ in $\psi(U_n \cap V_m)$.

(6) $\psi(\Lambda) \subset \widetilde{\Omega}$ and $y \circ \psi = \mathrm{id}\,|_\Lambda$.

Assume that some point in the rays of the real line excluded by the logarithmic singularities is equal to $\psi(u)$ for some $u \in \Gamma$. We construct the triangle $T : 0 \to u \to \overline{u} \to 0$, which is a closed path in Λ and we consider its image through ψ, which is symmetric with respect to the real axis due to the fact that ψ has real coefficients. We then retract the triangle T to the origin, which translates also in a retraction when applying ψ, but this retraction will necessarily go through the singularity in the real line, which yields a contradiction.

(7) ψ can be analytically continued near $z = R$ to a domain

$$D_{\varepsilon,\delta} := \{z \in \mathbb{C} \mid |z - r| < \epsilon, \mathrm{Arg}\,(z - r) > \delta\}$$

for some $\delta \in \left(0, \frac{\pi}{2}\right)$, so $y = \psi^{-1}$ on $D_{\varepsilon,\delta}$.

By condition (iii), $\lim_{z \to r, z \in \widetilde{\Omega}} y'(z) \neq 0$, so y is continuous and injective in a neighborhood in $\widetilde{\Omega}$ of r and thus it admits a local inverse there y^{-1}. Moreover, condition (iii) means that $y(z) - y(r) = c(z-r) + o(z-r)$. We now take a ray of the form $\gamma(t) = r + te^{i\frac{\pi}{4}}$ for $t \geq 0$ and condition (iii) implies that $y(\gamma(t)) - y(r) = cte^{i\frac{\pi}{4}} + o(t)$, so there is some small enough $T > 0$ such that for $0 \leq t \leq T$, $0 < \operatorname{Arg}(y(\gamma(t)) - y(r)) < \frac{\pi}{3}$. The same T is valid for the conjugate of γ due to the fact that y has real coefficients. In that case, we will have $0 > \operatorname{Arg}(y(\overline{\gamma}(t)) - y(r)) > -\frac{\pi}{3}$. This means that y^{-1} is defined in $D_{\epsilon,\frac{\pi}{3}}$ for some $\varepsilon > 0$. Moreover, $D_{\epsilon,\frac{\pi}{3}} \cap y(\Omega)$ is an open non-empty set, so, by analytic continuation, y^{-1} extends ψ to $D_{\varepsilon,\frac{\pi}{3}}$.

(8) Given $\omega \in \partial\Lambda \setminus \{R\}$, $\psi(\omega)$ is not a singularity of y.

Given $\omega \neq R$ such that $|\omega| = R = y(r)$, let's first see that $\psi(\omega) \neq r$, i.e. it cannot be the dominant singularity. By aperiodicity of A, we can use the strong triangle inequality

$$\frac{R}{|\psi(\omega)|} = \frac{|\omega|}{|\psi(\omega)|} = |A(\omega)| < A(|\omega|) = A(R) = \frac{R}{\psi(R)},$$

which yields $|\psi(\omega)| > \psi(R) = \psi(y(r)) = r$, where the last equality is due to the fact that $\psi \circ y = \mathrm{Id}$ in Ω and the fact that y extends continuously to r and ψ extends continuously to $\partial\Lambda$. This implies that $\psi(\omega) \neq r$.

Let's now see that $\psi(\omega) \neq z_k$ for any non-dominant singularity z_k of y. Assume that $\psi(\omega) = z_k$ and take a path γ in Λ ending at ω.

In the first case, if $\lim_{z \to z_k, z \in \widetilde{\Omega}} |y(z)| \neq y(r)$, we have:

$$\lim_{z \to z_k, z \in \psi \circ \gamma} |y(z)| = \lim_{u \to \omega, u \in \gamma} |y(\psi(u))| = \lim_{u \to \omega, u \in \gamma} |u| = |\omega| = y(r),$$

which is a contradiction.

In the second case, if $|z_k| > \frac{R}{2a_0 - A(R)}$, we have:

$$|z_k| = \lim_{z \to z_k, z \in \psi \circ \gamma} |z| = \lim_{z \to z_k, z \in \psi \circ \gamma} |\psi(y(z))| = \lim_{u \to \omega, u \in \gamma} |\psi(u)| = \frac{|\omega|}{|A(\omega)|} < \frac{R}{a_0 - \sum_{i \geq 1} a_i |\omega|^i}$$
$$= \frac{R}{2a_0 - A(R)} \leq |z_k|,$$

which is again a contradiction. In the previous equation, we have used the inequality $|A(\omega)| \geq a_0 - \left|\sum_{i \geq 1} a_i \omega^i\right| > a_0 - \sum_{i \geq 1} a_i |\omega|^i$ which follows from the non-negativity and aperiodicity of the coefficients a_i.

(9) Given $\omega \in \partial\Lambda \setminus \{R\}$, ψ can be analytically continued to a neighborhood of ω.

Given $\omega \in \partial\Lambda \setminus \{R\}$, take a path γ form 0 to ω contained in Λ, except for the last point ω. Then, $y(\psi(u)) = u$ for $u \in \gamma$, $u \neq \omega$; so $y'(\psi(u)) = \frac{1}{\psi'(u)}$ and we have

$$\lim_{u \to \omega, u \in \gamma} |y'(\psi(\omega))| = \lim_{u \to \omega, u \in \gamma} \frac{1}{\psi'(u)}$$

By (4), $\lim_{u\to\omega, u\in\gamma} \psi'(u)$ exists and is finite, so it cannot be that the previous equation is zero. This means that $0 \neq y'(\psi(\omega))$. Using this and the fact that y is analytic at $\psi(\omega)$ (by Step (8)), we can use the analytic inversion theorem (1.3) to conclude that y is locally invertible at $\psi(\omega)$, which extends ψ analytically to a neighborhood of ω.

(10) ψ (and thus also A) extends to a delta domain around Λ. Moreover, in that domain, $y(\psi(u)) = u$ holds.

We take first the domain $D_{\epsilon,\delta}$ from Step (7). For sufficiently small ϵ, the intersection of $D_{\epsilon,\delta}$ and $\partial\Lambda$ is the arc centered at the origin of radius R between angles $[-\phi, \phi]$, for some $\phi > 0$. This arc is a compact set, and around each point we can extend ψ analytically to a neighborhood of the point (by Step (9)). By compactness, we can find a finite number of such neighborhoods, which together with $D_{\epsilon,\delta}$ yield a delta domain where ψ is analytic.

□

2.4 Application to examples from the literature

We now present two examples from the literature where Proposition 2.13 is applicable. We will not focus on the combinatorial origin of the problems, but rather on the applicability of the previous result to the different examples.

Example 2.14 (Triangulations in Kuperberg's G_2 spider)**.** This example was originally introduced in Section 8.4. of [6]. It is in turn the motivation for Robert Shcerer's paper [8], where the results of this section are based on.

Let a_n be a sequence counting the number of triangulations of a regular n-gon such that each internal vertex has degree at least 6 (OEIS A060049), and let b_n be the dimension of vector subspace of invariants of n-th tensor power of 7-dimensional irreducible representation of G_2 (OEIS A059710), with $a_0 = b_0 = 1$. Their generating functions $A(x)$, $B(x)$ satisfy the following functional equation:

$$B(x) = A(xB(x)).$$

Setting $y(x) = xB(x)$ is exactly Equation (3). Moreover, the asymptotic behavior of b_n (equivalently y_n) is known (see Proposition 9 in [8]):

$$b_n \sim K\frac{7^n}{n^7},$$

for some constant $K > 0$. Since the asymptotic behavior of b_n is not of the form $Kr^{-n}n^{-\frac{3}{2}}$, this means that we are in Case (1) and thus $y(r) = R$, where $r = \frac{1}{7}$ is the radius of convergence of y and R is the radius of convergence of A.

A detailed analysis of the singularities of $y(z)$ is done in (1). It has two singularities, the dominant one at $z = \frac{1}{7}$ and a subdominant one at $z = -\frac{1}{2}$ and both are of the logarithmic type, meaning that y admits an analytic extension to $\widetilde{\Omega} = \mathbb{C} \setminus \left(\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{7}, +\infty\right)\right)$. Near the singularities, one has:

$$y\left(\frac{1}{7}\right) \approx 0.1466 \qquad y'\left(\frac{1}{7}\right) \approx 1.090 \qquad y\left(-\frac{1}{2}\right) \approx 0.65.$$

The previous values actually admit explicit rational expressions in π and $\sqrt{3}$, but the given approximations are enough to verify that conditions (i)-(iv) from Proposition 2.13 hold. This guarantees the extension to a delta domain of $A(u)$. A further analysis can be done to determine the asymptotics of a_n (see [4]).

Example 2.15 (Tree-rooted planar maps)**.** This example is taken from [2], a manuscript which is in preparation at the time of publishing this thesis.

In that context, $M(z,x)$ is the bivariate generating function counting the tree-rooted maps, with z marking edges and x marking vertices minus 1. It is explicitly given by:

$$M(z,x) = \sum_{n\geq 0}\sum_{k=0}^{n} \frac{(2n)!}{k!(k+1)!(n-k)!(n-k+1)!} z^n x^k$$

For each fixed value of x, the function $zM(z,x)$ is D-finite with dominant singularity

$$z_0(x) = \frac{1}{4(1+\sqrt{x})^2}.$$

When $x \neq 1$, it has one other subdominant singularity

$$z_1(x) = \frac{1}{4(1-\sqrt{x})^2}.$$

The asymptotic expansion of $M(z,x)$ near its dominant singularity for x sufficiently close to 1 is given by:

$$M(z,x) = M_0 + M_1 Z + M_2 Z^2 \log(Z) + O(Z^2), \tag{11}$$

where M_0, M_1, M_2 are explicit functions on x, whose expressions are omitted for the sake of simplicity and

$$Z = 1 - \frac{z}{z_0(x)}.$$

Moreover, a lengthy analysis based on an integral representation of $M(z,x)$ yields the following explicit expressions for $M(z_0(x),x)$ and $M'(z_0(x),x)$:

$$M(z_0(x),x) = \frac{4(1+\sqrt{x})^2}{\pi\sqrt{x}}\left(\frac{4}{3(x^{1/4}+x^{-1/4})} - (x^{1/4}+x^{-1/4}) + \frac{\pi\sqrt{x}}{2} + (x^{-1/2}-x^{1/2})\arctan(x^{1/4})\right)$$

$$M'(z_0(x),x) = -\frac{M(z_0(x),x)}{z_0(x)} + \frac{8(x^{1/4}+x^{-1/4})}{3\pi z_0(x)}$$

The bivariate generating function $N(z,x)$ counting 2-connected maps is related to $M(z,x)$ through the following functional equation:

$$M(z,x) = 1 + z(1+x)M(z,x)^2 + N(zM(z,x)^2, x) \tag{12}$$

The interest of the manuscript is studying the asymptotics of $N(z,x)$ for x sufficiently close to 1. For that end, one needs to justify the extension to a delta domain around its dominant singularity. Setting

$$y(z,x) := zM^2(z,x) \qquad A(u,x) := (1+(1+x)u+N(u,x))^2,$$

we may rewrite Equation (12) as

$$y(z,x) = zA(y(z,x),x),$$

which coincides with Equation (3).

Equation (11) yields the following asymptotic behavior of $y(z,x) = zM(z,x)^2$ near $z_0(x)$:

$$y(z,x) = z_0(x)M_0^2 + z_0(x)(2M_1 - M_0)M_0 Z + 2z_0(x)M_0M_2Z^2\log(Z) + O(Z^2).$$

In particular, this means that $z_0(z,x)$ is a singularity of $y(z,x) = zM(z,x)^2$ and thus it is also the radius of convergence (since the singularities of $zM(z,x)^2$ are also singularities of $M(z,x)$).

Using now Theorem 1.17, the asymptotic behavior of its n-th coefficient $y_n(x)$ is given by:

$$y_n(x) \sim 4z_0(x)M_0M_2z_0(x)^{-n}n^{-3}.$$

Since the asymptotic behavior of $y_n(x)$ is not of the form $K \cdot z_0(x)^{-n}n^{-\frac{3}{2}}$ (for some constant $K > 0$), this means that we are in Case (1) and thus $y(z_0(x),x) = R(x)$, where $R(x)$ is the radius of convergence of $A(u,x)$.

Next, from Equation (11) and taking into account that $[z^0]M(z,x) = 1$, it immediately follows that $[z^0]N(z,x) = 0$ and thus $a_0(x) = [z^0]A(z,x) = 1$. For $x = 1$, $M(z_0(1),1) = 8 - \frac{64}{3\pi}$, so

$$\frac{y(z_0(1),1)}{z_0(1)} = M(z_0(1),1)^2 = \left(8 - \frac{64}{3\pi}\right)^2 \approx 1.4626 < 2a_0 = 2$$

and condition (i) is verified. Since the expression of $\frac{y(z_0(x),x)}{z_0(x)} = M(z_0(x),x)^2$ is a continuous function in x (near $x = 1$), this inequality still holds in a sufficiently small neighborhood of $x = 1$ and condition (i) is verified.

Condition (ii) holds due to the behavior of singularities of D-finite functions. Condition (iii) holds for $x = 1$ since

$$M'(z_0(1),1) = \frac{1280}{3\pi} - 128 \approx 7.8122180$$

and

$$y'(z_0(1),1) = M(z_0(1),1)^2 + 2\frac{1}{16}M(z_0(1),1)M'(z_0(1),1) \approx 2.6436 > 0.$$

By continuity, for x sufficiently close to $x = 1$, condition (iii) still holds.

Finally, regarding condition (iv), if $x = 1$, $y(z,1)$ has only one singularity and there is nothing to check. If $x \neq 1$, but it is sufficiently close to $x = 1$, we check that the second condition is verified:

$$\frac{R(x)}{2a_0(x) - A(R(x))} = \frac{y(z_0(x))}{2 - \frac{y(z_0(x))}{z_0(x)}} \to \frac{y(z_0(1))}{2 - \frac{y(z_0())}{z_0(1)}} = \frac{z_0(1)M(z_0(1),1)^2}{2 - M(z_0(1),1)^2} \approx 0.17011 \quad \text{as } x \to 1,$$

whereas $\lim_{x\to 1} z_1(x) = \infty$, so again by continuity, the following equation

$$|z_1(x)| > \frac{R(x)}{2a_0(x) - A(R(x))}$$

holds for x sufficiently close to 1.

With all conditions of Proposition 2.13 verified, we can now guarantee the extension to a delta domain of the function $A(u,x) = (1+(1+x)u+N(u,x))^2$. Since $A(u,x)$ does not vanish in the delta domain (it is one of the intermediate steps of Proposition 2.13), its square root can also be extended to a delta domain, and since the term $1 + (1 + x)u$ is entire, $N(u, x)$ can also be extended to a delta domain, which was our initial goal. With that in hand, one can proceed to determine the asymptotic behavior of $N(u, x)$ near its dominant singularity using a bootstrapping method, but this is outside the scope of this thesis.

3. Schema of additive simple varieties of trees

This section aims to study the asymptotics of the coefficients of the formal power series $y(x)$ and $B(x)$ involved in the following functional equation:

$$y(x) = x + B(y(x)). \tag{13}$$

This Equation is similar to Equation (3), the only difference being that the term x is being added instead of multiplied. This equation describes a recurrence for simple varieties of trees enumerated by its number of leaves (instead of the total number of vertices).

In that context, $y(x)$ is the GF of a class of rooted plane trees, whose size is given by the number of non-root leaves (i.e. the total number of leaves minus one) and a single vertex tree has size 1, so that the equation makes sense. $B(x)$ is the GF of the possible children that a fixed node can have.

3.1 Analytic considerations and sharpness criterion

As in section 2, we are interested in studying the analytic properties of Equation (13).

Definition 3.1. A pair of generating functions $(y(z), B(u))$ satisfying Equation (13) which are analytic at 0 are said to belong to the schema of *additive smooth simple varieties of trees*.

Notation 3.2. In the additive smooth simple varieties of trees schema, we will denote by:

- r the radius of convergence of $y(z)$. The disk of convergence of $y(z)$ is denoted by

$$\Omega := \{z \in \mathbb{C} \mid |z| < r\}.$$

- R the radius of convergence of $B(u)$. The disk of convergence of $B(u)$ is denoted by

$$\Lambda := \{u \in \mathbb{C} \mid |u| < R\}.$$

The goal of this section is developing a theory that allows us to justify that the generating function in Example 3.14 which corresponds to $B(x)$ in Equation (13) can be extended to a delta domain around its disk of convergence. For that reason, it is not enough to consider just GF $B(x)$ which have non-negative coefficients, and we have to enlarge a bit the set of allowed GFs $B(x)$ so that Example 3.14 fits in our discussion. We introduce the class of pairs of generating functions $(y(z), B(u))$ that we will be studying throughout this section.

Definition 3.3. A pair of generating functions $(y(z), B(u))$ belonging to the additive smooth simple varieties of trees belong to the schema $\mathcal{S}$ if the following conditions hold:

(S1) $y(x) = \sum_{i\geq 1} y_i x^i$, where $y_i \in \mathbb{R}_{\geq 0}$ and $y_k > 0$ for some $k \in \mathbb{N}$.

(S2) $B(x) = C(x) - D(x)$, where:

- $D(x) = \sum_{i=1}^{N} d_i x^i$ for some $N \in \mathbb{N}$, where $d_i \in \mathbb{R}_{\geq 0}$.
- $C(x) = \sum_{i\geq 1} c_i x^i$, where $c_i \in \mathbb{R}_{\geq 0}$ and $c_k > 0$ for some $k \geq \max(2, N+1)$, where N is the degree of the polynomial $D(x)$.

(S3) $B''(\omega) > 0$ for all $\omega \in [0, R]$, where R is the radius of convergence of $B(u)$.

(S4) The functions $y(x)$ and $C(x)$ are aperiodic.

Remark 3.4. In the schema $\mathcal{S}$, note that since $D(u)$ is a polynomial and thus an entire function, the radius of convergence R of $C(u)$ and $B(u)$ is the same and also all their other singularities.

Observation 3.5. In the previous definition, we are considering generating functions $B(x)$ which have non-negative coefficients, except for maybe a finite set of coefficients which may be negative. Note that when $D(x) \equiv 0$, condition (S2) immediately implies condition (S3).

Lemma 3.6. *Let $(y(z), B(u))$ belong to the schema $\mathcal{S}$ (see Definition 3.3). Let r, R denote the radii of convergence of y, B, respectively; and let Ω be the disk of convergence of y. Then,*

(i) $y(r) \leq R$ (it may be that $y(r) = \infty$),

(ii) $y(z) = z + B(y(z))$ holds in $\overline{\Omega} = \{z \in \mathbb{C} \mid |z| \leq r\}$.

Proof. The proof follows very similar lines to the proof of Lemma 2.5.

(i) If $r = 0$, the result is trivial. Assume that $r > 0$. First observe that if $z \in \mathbb{R}^+$, in Equation (13), the summations involved (after the N-th term) can be rearranged because all the coefficients are real and non-negative:

$$y(z) = \sum_{i\geq 1} y_i z^i = z + B(y(z)) = z + \sum_{i\geq 1} b_i \left(\sum_{j\geq 1} y_j z^j\right)^i = z + \sum_{i=1}^{N} b_i y(z)^i + \sum_{i\geq N+1} \left[\left(\sum_{j=1}^{i} e_{ij}\right) z^i\right],$$

where $e_{ij} = c_j[x^i]\left(\sum_{k\geq 1} y_k x^k\right)^j$. In particular, Equation (13) can be evaluated for $z \in [0, r)$ (since we have $y(z)$ in the left-hand side of the last equation). Observe now that since $y_k > 0$ for some $k \geq 0$ and y has non-negative coefficients, $y\colon [0, r) \to [0, y(r))$ is strictly increasing and continuous (and thus bijective). Therefore, if $u \in (0, y(r))$, there is some $z \in (0, r)$ such that $y(z) = u$ and $B(u) = B(y(z)) = y(z) - z < \infty$ since z is in the disk of convergence of $y(z)$. Therefore, the power series B is convergent in $[0, y(r))$. Since $B(u) = C(u) - D(u)$ and $D(u)$ is a polynomial, the power series C is also convergent in $[0, y(r))$, so $y(r) \leq R$ (since C has non-negative coefficients, by Pringsheim's Theorem (1.12), the dominant singularity is in the real positive axis). Note that the argument is also valid when $y(r) = \infty$.

(ii) Once we have proven that $y(r) \leq R$, the same argument of Lemma 2.5 (iii) can be used here.

□

As in section 2, whenever it makes sense to evaluate at $z \in \mathbb{C} \setminus \{0\}$, we may rewrite Equation (13) as

$$\phi(y(z)) = z, \quad \text{where} \quad \phi(u) := u - B(u),$$

which is an inverse function equation.

Note that the derivative of $\phi(u)$ is given by

$$\phi'(u) = 1 - B'(u). \tag{14}$$

Lemma 3.7. *In the hypotheses of Lemma 3.6, the function* $y\colon \Omega \to y(\Omega)$ *is a biholomorphism with inverse function* $\phi(u) := u - B(u)$ *(i.e.* $\phi(y(z)) = z$ *holds for* $z \in \Omega$*). Moreover,*

- $r < \infty$.
- *If* $y(r) < \infty$*, then* y *extends continuously to* $\partial\Omega$ *and* ϕ *extends continuously to* $\partial(y(\Omega))$ *and* $\phi(y(z)) = z$ *holds for* $z \in \partial\Omega$*.*

Proof. The proof of this result follows analogous arguments to the proof of Lemma 2.6.

Let's first prove the injectivity of $y\colon \Omega \to y(\Omega)$. If $y(z_1) = y(z_2)$, then by Equation (13),

$$z_1 + B(y(z_1)) = y(z_1) = y(z_2) = z_2 + B(y(z_2)) = z_2 + B(y(z_1)) \Rightarrow z_1 = z_2.$$

This completes the proof of the injectivity of $y\colon \Omega \to y(\Omega)$. Since y is analytic in Ω (because it is its disk of convergence) and injective, its inverse function is also analytic and the first part of the statement follows.

- Assume on the contrary that y is entire. Then, by Lemma 3.6, $R = \infty$ and y maps bijectively $[0, \infty)$ to $[0, \infty)$. For $u \in [0, \infty)$, the function $\phi'(u)$ satisfies $\phi'(0) = 1$ and $\lim_{u\to\infty} \phi'(u) = -\infty$ because $b_k > 0$ for some $k \geq \max(2, N + 1)$, so there exists some $\tau = y(\beta) > 0$ (for some $\beta > 0$) such that $\phi'(\tau) = 0$. By Lemma 3.6, y is analytic in a neighborhood of β and ϕ is its inverse function, whose derivative at τ is null, which contradicts the fact that ϕ is analytically invertible at τ.
- If $y(r) < \infty$, by Weierstrass M-test, $y(z)$ converges on $\overline{\Omega}$, and this gives the continuous extension of y to $\partial\Omega$. Similarly, using Theorem 1.11, $C(y(z))$ and thus $B(y(z))$ also converge on $\overline{\Omega}$, so ϕ is well-defined on $\partial(y(\Omega))$. By continuity, $z + B(y(z)) - y(z) = 0$ still holds on $\partial\Omega$, so $\phi(y(z)) = z$ holds on $\partial\Omega$.

□

Proposition 3.8 (Sharpness Criterion)**.** *Let* $(y(z), B(u))$ *belong to the schema* $\mathcal{S}$ *(see Definition 3.3) and let* r, R *denote the radii of convergence of* y, B*, respectively. If* $y(r) < R \leq \infty$*, then* $\phi'(u)$ *vanishes at* $u = y(r)$*.*

Proof. With Lemma 3.7 in hand, the proof of this Proposition is totally analogous to the proof of Proposition 2.7 (it is even simpler here because we do not have to guarantee that a denominator does not vanish). □

3.2 Asymptotics in the smooth additive inverse-function schema

Definition 3.9. Let $(y(z), B(u))$ belong to the schema $\mathcal{S}$ (see Definition 3.3) and let R denote the radius of convergence of B. We say that $(y(z), B(u))$ belongs to the *Smooth additive inverse-function schema* if there exists $0 < \tau < R$ such that

$$\phi'(\tau) = 1 - B'(\tau) = 0. \tag{15}$$

Theorem 3.10 (Smooth additive inverse-function schema)**.** *Let* $(y(z), B(u))$ *belong to the smooth additive inverse-function schema. Let* $0 < \tau < R$ *be the solution to Equation (15). Define* $\rho := \tau - B(\tau)$*. Then,* ρ *is the radius of convergence of* y *and one has the following asymptotic expansion:*

$$y_n = [z^n]y(z) \sim \sqrt{\frac{1}{2B''(\tau)}} \frac{\rho^{-n}}{\sqrt{\pi n^3}} \left(1 + O\left(\frac{1}{n}\right)\right) \tag{16}$$

Proof. The proof is analogous to the proof of Theorem 2.10. In a totally analogous way, one can prove that $y(r) = \tau$ (we need to use the monotonicity of $1 - B'(u)$ in $[0, R]$, which immediately follows from condition (S3)). The same continuity argument developed there yields

$$r = \phi(\tau) = \tau - B(\tau).$$

We next obtain the asymptotic expansion of y_n. Since $\tau < R$ and R is the radius of convergence of $\phi(u)$, we can expand ϕ around $u = \tau$ in the equation $\phi(y) = z$, which is equivalent to Equation (13):

$$z = \phi(\tau) + \phi'(\tau)(y - \tau) + \frac{\phi''(\tau)}{2!}(y - \tau)^2 + \frac{\phi'''(\tau)}{3!}(y - \tau)^3 + ... \tag{17}$$

The first coefficients are $\phi(\tau) = \rho$, $\phi'(\tau) = 0$ and $\phi''(\tau) = -B''(\tau) \neq 0$ (condition (S3)).

The argument in the proof of Theorem 2.10 that shows that $y(z)$ can be extended to a delta domain can be used here too.

Finally, a totally analogous argument to the one in the proof of Theorem 2.10 (changing $\frac{A''(\tau)}{A^2(\tau)}$ by $B''(\tau)$) yields the asymptotic expression in Equation (7). □

Corollary 3.11. *Let $(y(z), B(u))$ belong to the schema $\mathcal{S}$ (see Definition 3.3). Let r, R denote the radii of convergence of y, B, respectively. Then, exactly one of the following statements is true:*

(1) $1 - B'(u)$ is non-vanishing for $u \in (0, R)$, in which case $R = y(r)$.

(2) $R > y(r) = \tau$, where τ is the unique solution to $1 - B'(\tau) = 0$ on $(0, R)$ and $y_n = Cr^{-n}n^{-\frac{3}{2}}(1 + o(1))$ ad $n \to \infty$, for some constant $C > 0$.

In particular, if $y_n \not\sim Cr^{-n}n^{-\frac{3}{2}}$ for some constant $C > 0$, then we are in Case (1) and $R = y(r)$.

3.3 Extension to a delta domain of the inverse generating function

We now address the study of the asymptotics of the coefficients of $B(x)$ in case (1) of Corollary 3.11 when the singularities of $y(z)$ and its expansion around singularities are known.

Proposition 3.12. *Let $(y(z), B(u))$ belong to the schema $\mathcal{S}$ (see Definition 3.3). Let r, R denote the radii of convergence and Ω, Λ the disks of convergence of y, B, respectively. Let $\widetilde{\Omega}$ be a maximal domain of analyticity of y. Assume that $y(r) = R$ and that $B(x)$ and the following set of conditions:*

(i) All the singularities of $y(z)$ are real and of the logarithmic type.

(ii) $y(z) = y(r) + c(z - r) + o(z - r)$ as $z \to r$ in $\widetilde{\Omega}$ for some real strictly positive constant $0 < c = y'(r)$.

(iii) $\frac{1}{y'(r)} \geq 2D'(y(r))$.

(iv) $|\phi(\omega)| > \phi(R) = r$ in $\partial\Lambda \setminus \{R\} = \{u \in \mathbb{C} \mid u \neq R, |u| = R\}$.

(v) For every non-dominant singularity z_k of y, one of the following two conditions holds:

- *y extends continuously to z_k and $\lim_{z \to z_k, z \in \widetilde{\Omega}} |y(z)| \neq y(r)$,*
- *$|z_k| \geq 2y(r) + 2D(y(r)) - r$.*

Then, $B(u)$ extends to a delta domain Δ around its dominant singularity R. Moreover, in that domain, $y(\phi(u)) = u$ holds, where $\phi(u) = u - B(u)$.

Proof. The proof of this result follows analogous arguments to the proof of Proposition 2.13. We will follow a similar step-by-step structure along the proof.

(1) ϕ extends analytically to Λ and continuously to $\partial\Lambda$.

This is immediate from the expression $\phi(u) = u - B(u)$.

(2) ϕ' does not vanish on Λ.

Since we are in Case (1) from Corollary 3.11, $\phi'(u) \neq 0$ for $u \in [0, R)$. Since $\phi'(0) = 1$, ϕ' is positive in the interval $[0, R)$. Now, if there is some u such that $|u| < R$ and $\phi'(u) = 0$, we have

$$1 = \left| \sum_{n \geq 1} n b_n u^{n-1} \right| \leq \sum_{n \geq 1} n\, |b_n|\, |u|^{n-1} \leq \sum_{n \geq 1} n c_n\, |u|^{n-1} + \sum_{n=1}^{N} n d_n\, |u|^{n-1} < C'(|u|) + D'(|u|), \quad (18)$$

where the strict inequality holds due to the non-periodicity of $C(u)$.

Now consider the auxiliary function $\alpha(u) := C'(u) + D'(u) = 1 - \phi'(u) + 2D'(u)$ for $u \in [0, y(r)]$. We have $\alpha'(u) = C''(u) + D''(u) > 0$ (since $c_k > 0$ for some $k \geq 2$ and C and D have non-negative coefficients), so it is strictly increasing, continuous and satisfies:

$$\alpha(0) = 0,$$

$$\alpha(y(r)) = 1 - \phi'(y(r)) + 2D'(y(r)) = 1 - \frac{1}{y'(r)} + 2D'(y(r)) \leq 1,$$

where we are using the fact that ϕ is the inverse of y to compute its derivative and condition (iii). This means that $\alpha(u) \leq 1$ for $u \in [0, y(r)]$

Going back to Equation (18), this yields the contradiction $1 < 1$. Thus, ϕ' does not vanish on Λ.

(3) ϕ' extends continuously to $\partial\Lambda$ and is finite.

Differentiating the relation $y(z) = z + B(y(z))$ with respect to z, one gets

$$B'(y(z)) = \frac{y'(z) - 1}{y'(z)}.$$

Since $y'(r)$ exists and is finite and $y(r) = R$, $B'(R)$ exists and is finite. By Weierstrass M-test, $B'(u)$ also exists and is finite for $|u| = R$ and taking into account Equation (14), the same property holds for $\phi'(u)$.

(4) Given a path $\gamma \subset \Lambda$ from 0 to $\widetilde{\omega} \in \Lambda$, y can be analytically continued along $\phi \circ \gamma$, so $y = \phi^{-1}$ on $\phi(\Lambda)$.

The proof of this step is totally analogous to that of item (5) of Proposition 2.13.

(5) $\phi(\Lambda) \subset \widetilde{\Omega}$ and $y \circ \phi = \mathrm{id}\,|_\Lambda$.

The proof of this step is totally analogous to that of item (6) of Proposition 2.13.

(6) ϕ can be analytically continued near $z = R$ to a domain

$$D_{\varepsilon,\delta} := \{z \in \mathbb{C} \mid |z - r| < \epsilon, \operatorname{Arg}(z - r) > \delta\}$$

for some $\delta \in \left(0, \frac{\pi}{2}\right)$, so $y = \phi^{-1}$ on $D_{\varepsilon,\delta}$.

The proof of this step is totally analogous to that of item (7) of Proposition 2.13.

(7) Given $\omega \in \partial\Lambda \setminus \{R\}$, $\phi(\omega)$ is not a singularity of y.

Given $\omega \neq R$ such that $|\omega| = R$, first observe that condition (iv) immediately implies that $\phi(\omega) \neq r$.

For a non-dominant singularity z_k of y, if the first condition is met (i.e. y extends continuously to z_k and $\lim_{z \to z_k, z \in \widetilde{\Omega}} |y(z)| \neq y(r)$), the proof is totally analogous to the proof of (8) of Proposition 2.13.

If the second condition is met (i.e. $|z_k| \geq 2y(r) + 2D(y(r)) - r$), for a given $\omega \neq R$ such that $|\omega| = R$, if $z_k = \phi(\omega)$ we have:

$$\begin{aligned}|z_k| = |\phi(\omega)| = |\omega - C(\omega) + D(\omega)| \leq |\omega| + |C(\omega)| + |D(\omega)| < R + C(R) + D(R) \\ = 2R + 2D(R) - \phi(R) = 2y(r) + 2D(y(r)) - r < |z_k|\,,\end{aligned}$$

which is a contradiction. The strict inequality in the previous equation holds because C is not periodic.

(8) Given $\omega \in \partial\Lambda \setminus \{R\}$, ϕ can be analytically continued to a neighborhood of ω.

The proof of this step is totally analogous to that of item (9) of Proposition 2.13.

(9) ϕ (and thus also B) extends to a delta domain around Λ. Moreover, in that domain, $y(\phi(u)) = u$ holds.

The proof of this step is totally analogous to that of item (10) of Proposition 2.13.

□

Observation 3.13. In Proposition 3.12, Conditions (iii) and (iv) are not necessary when $D(x) \equiv 0$, since they are already given by condition (ii).

Proof. The fact that condition (iii) follows from condition (ii) is immediate. Seeing that (iv) also follows from there is slightly more difficult.

Given $\omega \neq R$ such that $|\omega| = R$, let's see that $\phi(\omega) \neq r$, i.e. it cannot be the dominant singularity. We can use $|a| - |b| \leq |a - b|$ for any $a, b \in \mathbb{C}$ and the strong triangle inequality because B is aperiodic:

$$R - |\phi(\omega)| = |\omega| - |\phi(\omega)| \leq |\omega - \phi(\omega)| = |B(\omega)| < B(|\omega|) = R - \phi(R),$$

which yields $|\phi(\omega)| > \phi(R) = \phi(y(r)) = r$, where the last equality is due to the fact that $\phi \circ y = \mathrm{Id}$ in Ω and the fact that y extends continuously to r (which follows from (ii)) and ϕ extends continuously to $\partial\Lambda$. □

3.4 Application to an example from the literature

Example 3.14 (Prime 2-orientations)**.** This example is taken from the article [3], which is the first published result that fits in the setting studied in this thesis.

Let q_n be the number of 2-orientations among all quadrangulations with n inner faces (OEIS A001181). A quadrangulation is said to be prime if it has at least 3 inner faces and any of its 4-cycles is the boundary of a face. p_n counts the number of prime 2-orientations with n inner faces. Their generating functions satisfy the following equation (see Lemma 2 in [3]):

$$Q(x) = x + \frac{2Q^2(x)}{1+Q(x)} + P(Q(x)). \tag{19}$$

Introducing $\overline{P}(u) := \frac{2u^2}{1+u} + P(u)$, we have

$$Q(x) = x + \overline{P}(Q(x)),$$

the same functional Equation as in (13).

The asymptotics of q_n is given by:

$$q_n \sim Kn^{-4}8^n,$$

for some constant $K > 0$. Moreover, the singularities of $Q(z)$ can be studied since it is a D-finite function. Its only two singularities are $\left\{-1, \frac{1}{8}\right\}$.

The coefficients q_n admit the following explicit expression:

$$q_n = \frac{1}{\binom{n+1}{1}\binom{n+1}{2}} \sum_{r=0}^{n-1} \binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2}$$

With the explicit expression of the coefficients, we can obtain good enough approximations for $Q\left(\frac{1}{8}\right)$ and $Q'\left(\frac{1}{8}\right)$ (which converge since $q_n\left(\frac{1}{8}\right)^n \sim Kn^{-4}$ and $nq_n\left(\frac{1}{8}\right)^{n-1} \sim K'n^{-3}$ and their sums are convergent).

$$0.18 < Q\left(\frac{1}{8}\right) < 0.19 \qquad 2.4 < Q'\left(\frac{1}{8}\right) < 2.5.$$

The first coefficients of $P(u)$ are:

$$P(u) = 2u^5 + 0u^6 + 12u^7 + 24u^8 + \ldots$$

and all the next coefficients are greater or equal than 2. Thus,

$$\overline{P}(u) = \frac{2u^2}{1+u} + P(u) = 2u^2 - 2u^3 + 2u^4 + 0u^5 + 2u^6 + 10u^7 + 22u^8 + \ldots = C(u) - D(u),$$

where $D(u) := 2u^3$ and $C(u) := \frac{2u^2}{1+u} + P(u) + 2u^3$.

Let's check that condition (S3) of the schema $\mathcal{S}$ (see Definition 3.3) holds. Observe that $\overline{P}''(u) = 4 - 12u + 24u^2 + \beta(u)$, where $\beta(u)$ has real positive coefficients. Moreover, an easy check shows that $4 - 12u + 24u^2 > 0$ for all $u \in \mathbb{R}$. Therefore, $\overline{P}''(u) > 0$ for all $u \in \mathbb{R}_{>0}$ (in particular in $[0, R]$).

Since the pair of generating functions $(Q(x), \overline{P}(x))$ belong to the schema $\mathcal{S}$ and the asymptotic behavior of q_n does not have a factor $n^{-\frac{3}{2}}$ means that we are in case (1) of Corollary 3.11 and thus the radius of convergence of $\overline{P}$ is $R = Q\left(\frac{1}{8}\right)$.

We finally use Proposition 3.12. Conditions (i) and (ii) hold by the previous discussion of the singularities of $Q(z)$. Condition (iii) holds since

$$\frac{1}{Q'\left(\frac{1}{8}\right)} > \frac{1}{2.5} = 0.4 > 0.2166 = 6 \cdot 0.19^2 = 2D'(0.19) > 2D'(y(r)).$$

Let's now verify condition (iv). Given $\omega \neq R$ such that $|\omega| = R$, we have:

$$|\phi(\omega)| = \left|\omega - \frac{2\omega^2}{1+\omega} - P(\omega)\right| = \left|\frac{\omega(1-\omega)}{1+\omega} - P(\omega)\right| \geq \left|\frac{\omega(1-\omega)}{1+\omega}\right| - |P(\omega)| > \frac{|\omega|\left(1-|\omega|\right)}{1+|\omega|} - P(|\omega|)$$
$$= \phi(R) = \phi(y(r)) = r,$$

where the strict inequality is due to the aperiodicity of $P(x)$.

We finally verify the second possibility for condition (v) at the subdominant singularity $z_1 = -1$:

$$2y(r) + 2D(y(r)) - r \leq 2 \cdot 0.19 + 4 \cdot 0.19^3 - \frac{1}{8} = 0.268718 < 1 = |z_1|.$$

Thus, all conditions of Proposition 3.12 are met, and this guarantees the extension of $\overline{P}$ to a delta domain. Since $\overline{P}(u) := \frac{2u^2}{1+u} + P(u)$ and $\frac{2u^2}{1+u}$ has a single singularity, which is a simple pole at $u = -1$, $P(u)$ also has radius of convergence $R = Q\left(\frac{1}{8}\right) < 1$ and it can be extended to a delta domain around its dominant singularity. A further analysis of Equation (19) taking into account the singular expansion of $Q(z)$ around $r = \frac{1}{8}$ allows to determine the singular expansion of $P(u)$ around R and using the singularity analysis theorem (1.17), one can determine the asymptotics of p_n.

Final comments and future work

The main progress done in this thesis is giving a more general framework where the argument for extending $A(u)$ to a delta domain done in [6] works. We have also given a general framework where this argument works for the function $B(u)$ in the additive schema of simple varieties of trees.

The following list includes some limitations found in the process of writing this thesis, which could be further studied in the future.

- For the sake of simplicity, we have limited our results in sections 2 and 3 to aperiodic functions, but working with periodic functions should pose no major difficulty and I think one can get similar results in that case without much extra work.
- In Proposition 2.13, we have limited the singularities of $y(z)$ to be real. This is used only in Step (6). If we had an argument to ensure this property for a non-necessarily real singularity, we could extend the general result of Proposition 2.13 to a larger framework. The same applies to Proposition 3.12.
- When using the technique of analytic continuation along paths in Proposition 2.13 (or in Proposition 3.12), we are just using that $\psi'(u) \neq 0$ (or $\phi'(u) \neq 0$) for $u \in \Lambda$. From this, one may think it could be easy to extend the arguments to a more general framework of general inversion (i.e. for a larger set of inverse functions, not just $\psi(u) = \frac{u}{A(u)}$ and $\phi(u) = u - B(u)$). Although this may be possible, it seems hard to get a very general framework with the arguments discussed in this thesis, since many steps involve using the explicit expressions of ψ or ϕ (or at least using that the sign of their derivatives behaves monotonously). However, for other specific inverse functions which are similar to ψ or ϕ, it seems likely that very similar arguments would still work, and the ideas could be reused there.

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