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Per a la meva germana, na Irene.

Abstract

In this work we generalize the construction of p-adic anticyclotomic L-functions associated to an elliptic curve E/F and a quadratic extension K/F, by defining a measure $\mu_{\phi_\lambda^p}$ attached to K/F and an automorphic form. In the case of parallel 2, the automorphic form is associated with an elliptic curve E/F.

The first main result is a p-adic Gross-Zagier formula: if E has split multiplicative reduction at $\mathfrak{p}$ and $\mathfrak{p}$ does not split at K/F, we compute the first derivative of the p-adic L-function by relating it with the conjugate difference of a Darmon point twisted by a character ξ. The proof uses the reciprocity map provided by class field theory as a natural way to interpret conjugate differences of points in $E(K_\mathfrak{p})$ as elements in the augmentation ideal for the evaluation at the character ξ. This generalizes a result of Bertolini and Darmon. With a similar argument, after discovering the work of Fornea and Gehrmann on plectic points, we prove an exceptional zero formula which relates a higher order derivative of $\mu_{\phi_\lambda^S}$ with plectic points.

We find an interpolating measure $\mu_{\Phi_\lambda^p}$ for $\mu_{\phi_\lambda^p}$ attached to an interpolating Hida family Φ_λ^p for ϕ_λ^p. Here $\mu_{\Phi_\lambda^p}$ can be regarded as a two variable p-adic L-function, which now includes the weight as a variable. Then we define the Hida-Rankin p-adic L-function $L_p(\phi_\lambda^p, \xi, \underline{k})$ as the restriction of $\mu_{\Phi_\lambda^p}$ to the weight space. Finally, we prove a formula which relates the weight-leading term of $L_p(\phi_\lambda^p, \xi, \underline{k})$ with plectic points. In short, the leading term is an explicit constant times Euler factors times the logarithm of the trace of a plectic point. This formula is a generalization of a result of Longo, Kimball and Hu, which has been used to prove the rationality of a Darmon point under some hypotheses.

MSC2020 classification: 11F67, 11F75, 11G05, 11F70, 11G40.

Resum

En aquesta tesi generalitzem la construcció de funcions L p-àdiques anticiclotòmiques associades a una corba el·líptica E/F i una extensió quadràtica K/F, definint una mesura $\mu_{\phi_\lambda^p}$ associada a K/F i una forma automorfa. En el cas de pes paral·lel 2, la forma automorfa s'associa a una corba el·líptica E/F.

El primer resultat és una fórmula p-àdica de Gross-Zagier: si E té reducció multiplicativa *split* a $\mathfrak{p}$ i $\mathfrak{p}$ descomposa a K/F, calculem la primera derivada de la funció L p-àdica relacionant-la amb la diferència conjugada d'un punt de Darmon *twistat* per un caràcter ξ. La demostració utilitza l'aplicació de reciprocitat de la teoria de cossos com una manera natural d'interpretar les diferències conjugades de punts de $E(K_\mathfrak{p})$ com elements en l'ideal d'augmentació de l'avaluació en el caràcter ξ. Això generalitza un resultat de Bertolini i Darmon. Amb un argument semblant, després de descobrir el treball de Fornea i Gehrmann sobre els punts plèctics, demostrem una fórmula de zero excepcional que relaciona una derivada d'ordre superior de $\mu_{\phi_\lambda^S}$ amb punts plèctics.

Trobem una mesura d'interpolació $\mu_{\Phi_\lambda^p}$ per a $\mu_{\phi_\lambda^p}$ associada a la família Hida Φ_λ^p que passa per ϕ_λ^p. Aquí $\mu_{\Phi_\lambda^p}$ es pot considerar com una funció L de dues variables, que ara inclou el pes com a variable. Aleshores definim una funció L de Hida-Rankin p-àdica $L_p(\phi_\lambda^p, \xi, \underline{k})$ com la restricció de $\mu_{\Phi_\lambda^p}$ a l'espai de pesos. Finalment, demostrem una fórmula que relaciona el terme principal de $L_p(\phi_\lambda^p, \xi, \underline{k})$ respecte al pes amb punts plèctics. En resum, el terme principal és una constant explícita multiplicada per factors d'Euler i pel logaritme de la traça d'un punt plèctic. Aquesta fórmula és una generalització d'un resultat de Longo, Kimball i Hu, que s'ha utilitzat per demostrar la racionalitat d'un punt de Darmon sota certes hipòtesis.

MSC2020 classification: `11F67`, `11F75`, `11G05`, `11F70`, `11G40`.

Contents

List of symbols

F	the base field
d	the degree $[F : \mathbb{Q}]$
$\mathcal{O}_F$	the ring of integers of F
K	a quadratic extension of F
E	an elliptic curve over F
$\mathfrak{p}$	a prime ideal of F
$\varpi_\mathfrak{p}$	an uniformizer of $F_\mathfrak{p}$
$v_\mathfrak{p}$	the $\mathfrak{p}$-adic valuation
$\mathbb{1}_X$	the characteristic function of a set X
π	an automorphic representation
ρ	a model of π over $\overline{\mathbb{Q}}$ satisfying $\rho \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \simeq \pi^\infty = \pi \mid_{G(\mathbb{A}_F^\infty)}$
ϕ_λ^p	an automorphic modular symbol that generates π
$\mu_{\phi_\lambda^p}$	the measure attached to ϕ_λ^p
η	the fundamental class
$q_\mathfrak{p}$	a Tate uniformizer for $E(K_\mathfrak{p})$
$\underline{k}$	a weight vector i.e. an element of $\mathbb{Z}^d$
$\mathcal{G}_T$	the Galois group of the abelian extension of K associated with T
$\mathcal{G}_T^\mathfrak{p}$	the Galois group of the abelian extension of K attached to $T(\mathbb{A}_F)/T(F_\mathfrak{p})$
$\Lambda_\mathfrak{p}$	the $\mathfrak{p}$-Iwasawa algebra $\mathbb{Z}_p[[\mathcal{O}_\mathfrak{p}^\times]]$
Λ_F	the Iwasawa algebra $\mathbb{Z}_p[[\mathcal{O}_{F_p}^\times]]$
$\mathbf{k}_p$	the universal character of Λ_F
ρ_χ	the specialization morphism of an Iwasawa algebra at the character χ
$\mathbf{k}_\mathfrak{p}$	the universal character of $\Lambda_\mathfrak{p}$
$\mathbf{a}_\mathfrak{p}$	the element of $\Lambda_F^\times$ used to extend $\mathbf{k}_\mathfrak{p}$ to evaluate elements of $F_\mathfrak{p}^\times$
$\hat{\phi}_\lambda^p$	the overconvergent modular symbol attached to ϕ_λ^p
Φ_λ^p	a Hida family passing through ϕ_λ^p
$\mu_{\Phi_\lambda^p}$	the measure attached to Φ_λ^p

Chapter 1

Introduction

Some first examples

Historically, there are some reasons to believe that L-series or zeta functions link arithmetic with analysis, linking local and global behaviour. The Riemann zeta function $\zeta(s)$ is defined by the series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ which converge for $s \in \mathbb{C}$ with $\Re s > 1$. Riemann showed that $\zeta(s)$ extends to a meromorphic function of $\mathbb{C}$ and proved its functional equation. Euler gave another expression for $\zeta(s)$, its *Euler product*

$$\zeta(s) = \prod_{p} \left(1 - p^{-s}\right)^{-1}$$

where p runs over the primes of $\mathbb{Z}$. It follows from the fundamental theorem of arithmetic. Using the Euler product, Riemann linked the prime counting function $\pi(x)$ with the zeroes of $\zeta(s)$, and formulated the *Riemann hypothesis*, still open. Given a number field $F/\mathbb{Q}$, its Dedekind zeta function is defined similarly and also has an Euler product

$$\zeta_F(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_F} (\mathbb{N}\mathfrak{a})^{-s} = \prod_{\mathfrak{p}} \left(1 - (\mathbb{N}\mathfrak{p})^{-s}\right)^{-1}$$

where $\mathfrak{a}$ runs over the ideals of $\mathcal{O}_F$ and $\mathfrak{p}$ over the prime ideals. The *class number formula* links several invariants of F with the behaviour of $\zeta_F(s)$ at $s = 1$,

$$\lim_{s \to 1} (s-1)\zeta_F(s) = \frac{2^{r_\mathbb{R}} \cdot (2\pi)^{r_\mathbb{C}} \cdot \mathrm{Reg}_K \cdot h_K}{w_K \cdot \sqrt{|d_K|}} \tag{1.1}$$

where $r_\mathbb{R}$ is the number of real embeddings of F into $\mathbb{C}$, $r_\mathbb{R} + 2r_\mathbb{C} = [F : \mathbb{Q}]$, Reg_K is the regulator of K, w_K the number of roots of unity in K, h_K the class number of K and d_K its discriminant.

The conjectures of Beilinson and Bloch-Kato, out of the scope of this text, are one of the first general attempts to link the special values of L-functions with arithmetic, from which the BSD conjecture can be regarded as an special case.

1.1 The BSD conjecture

Given an elliptic curve E/F of conductor $\mathfrak{N} \subset \mathcal{O}_F$, we have a natural geometric group law, giving rise to its abelian group of F-rational points $E(F)$. Mordell proved $E(F)$ is finitely generated and thus has finite $\mathbb{Z}$-rank, this is the *algebraic rank* $r_{\mathrm{alg}}(E/F)$ of E. We can also form its Hasse-Weil L-function with the following Euler product

$$L(E/F, s) = \prod_{\mathfrak{p} \nmid \mathfrak{N}} \left(1 - a_\mathfrak{p} q_\mathfrak{p}^{-s} + q_\mathfrak{p}^{1-2s}\right)^{-1} \cdot \prod_{\mathfrak{p} \mid \mathfrak{N}} \left(1 - a_\mathfrak{p} q_\mathfrak{p}^{-s}\right) \tag{1.2}$$

where each $a_{\mathfrak{p}} = a_{\mathfrak{p},E}$ is given explicitly in terms the reduction type and the number of rational points of E over $F_{\mathfrak{p}}$. The product converges absolutely on $\Re(s) > 3/2$ because the $a_{\mathfrak{p}}$ have a controlled behaviour: they satisfy *Hasse's inequality*.

In few words, automorphic representations are generated by automorphic forms, which are a generalization of modular forms. We say E is *modular* if there exists an automorphic representation π for $\mathbf{PGL}_2/F$ of parallel weight 2 such that

$$L\left(\pi, s - \frac{1}{2}\right) \doteq L(E/F, s)$$

where $\doteq$ denotes that we ignore the archimedean factors. It is a classic result of Hecke that the L-series $L(f, s)$ associated to a modular form $f \in S_k(\Gamma_0(N))$ of level N has an analytic continuation to $\mathbb{C}$, and that its normalized L-function $\Lambda(f, s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f, s)$ satisfies the *functional equation*

$$\Lambda(f, s) = \varepsilon \cdot \Lambda(f, k - s) \tag{1.3}$$

Similar results (functional equation and analytic continuation) hold for automorphic representations. The existence of the automorphic representation associated with E is now known when $F = \mathbb{Q}$ by the *modularity theorem*, and also for most elliptic curves when F is a totally real field, thus in these cases $L(E/F, s)$ extends to an entire function.

However, in the 60s Birch and Swinnerton-Dyer formulated a conjecture based on computational evidence, a prediction about the behaviour of $L(E/\mathbb{Q}, s)$ at $s = 1$, yet there was no modularity, no Langlands programme to support whether the L-function is defined at $s = 1$. The *BSD conjecture* can be stated by means of an equality which involves several invariants of the elliptic curve as in (1.1), together with the claim that

$$r_{\mathrm{alg}}(E/F) = \mathrm{ord}_{s=1} L(E/F, s) \tag{1.4}$$

That is, the algebraic rank is a computable number equal to the *analytic rank*, the order of vanishing of $L(E/K, s)$ at $s = 1$. The *weak* version of BSD conjecture only claims equation 1.4 and the finiteness of the Tate-Shafarevich group, an invariant of E.

From (1.4) we observe that if the L-function vanishes then $E(K)$ should be infinite, and at least a K-rational point of infinite order should exist. Some of the progress in proving partial versions of (1.4) use a set of special points, called *Heegner* points, that emerge when K is a totally imaginary quadratic extension of a totally real field F and E is modular and defined over F. In fact, for now the cases of rank 0 and 1 are the most understood due to the results of Gross, Zagier and Kolyvagin and their use of Heegner points.

1.2 p-adic L-functions

An automorphic form has an associated L-function, which as we have seen above conjecturally holds a lot of arithmetic information, but is inaccessible by current methods. Much of the recent progress on this problem comes from proving results about *p-adic L-functions*, rather than working with the L-function directly. These objects, which are constructed in many different ways, in some sense interpolate the classic L-function. We will later focus on one possible automorphic construction of a p-adic L-function, by regarding the twists of the L-function as the integral of the character χ over a certain measure.

In this approach the measure μ determines the p-adic L-function, and to find its *k-th Taylor coefficient* amounts to compute the image of μ in I^k/I^{k+1} when it has *a zero of order k* (see [MT87]), where I is an ideal called the *augmentation ideal.* To illustrate this idea, consider the following example. In the polynomial ring $\mathbb{C}[X]$ we can consider the morphism $\mathbb{C}[X] \to \mathbb{C}$ given by $X \mapsto 0$, inducing the sequence

$$0 \longrightarrow I \longrightarrow \mathbb{C}[X] \longrightarrow \mathbb{C} \longrightarrow 0$$

where $I = \ker(X \mapsto 0)$. Then an element $f \in \mathbb{C}[X]$ has a zero of order k at $X = 0$ when $f \in I^k$ and the coefficient of X^k in f can be identified with its image in I^k/I^{k+1}.

One usual way to construct p-adic L-functions is through measures of $\mathbb{Z}_p$-extensions of the base field F. There is a unique $\mathbb{Z}_p$-extension of $\mathbb{Q}$, namely $\mathbb{Q}_{\infty,p}^{\Delta}$ where $\mathbb{Q}_{\infty,p} = \mathbb{Q}(\zeta_{p^\infty})$ is the field generated by the p^n roots of unity for all $n \geq 1$ and

$$\mathrm{Gal}(\mathbb{Q}_{\infty,p}/\mathbb{Q}) \simeq \mathbb{Z}_p^\times = \Delta \times \Sigma, \quad \Delta = \mathbb{Z}/(p-1)\mathbb{Z}, \quad \Sigma = \mathbb{Z}_p$$

For a general number field this does no longer hold (see [Sha19]). Apart from the cyclotomic extension, an imaginary quadratic field K has an abelian $\mathbb{Z}_p$-extension L of K which is *anticyclotomic*: $L/\mathbb{Q}$ is Galois and $\mathrm{Gal}(K/\mathbb{Q})$ acts on $\mathrm{Gal}(L/K)$ by -1. For now, there have been constructions of both cyclotomic and anticyclotomic p-adic measures.

Cyclotomic case

The Mazur and Swinnerton-Dyer p-adic L-function $L_p(f,s)$ is defined through a measure μ_p of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Given a cusp form $f \in S_2(N)$, it can be defined as the p-adic Mellin transform

$$L_p(f,s) = \int_{\mathbb{Z}_p^\times} \exp_p(s \log_p \gamma) d\mu_p(\gamma) : \mathbb{C}_p \to \mathbb{C}_p.$$

Since the p-adic exponential and logarithm are p-adic analytic, so is $L_p(f,s)$. Its interpolation property shows $L_p(f,s)$ is related with the classical L-function: given a finite character $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ one has

$$\int_{\mathbb{Z}_p^\times} \chi(\gamma) d\mu_p(\gamma) = \varepsilon_p(\chi, f) \cdot L(f, \chi, 1),$$

where $L(f,\chi,1)$ is the classical L-function twisted by this character and $\varepsilon_p(\chi, f)$ some Euler factor.

Later, Mazur, Tate and Teitelbaum formulated the p-adic version of the BSD conjecture. One consequence of this conjecture is the *exceptional zero* conjecture: $L_p(E,s) = L_p(f,s)$ has exactly one more extra zero at the critical point than $L(E,s)$, when f is attached to an elliptic curve E with split multiplicative reduction at p. In rank zero situations it is now a theorem by Greenberg and Stevens and it states that in this case

$$L_p'(E,1) = \mathcal{L}(E) \cdot \frac{L(E,1)}{\Omega_E^+}, \tag{1.5}$$

where $\mathcal{L}(E) = \log_p(q_E)/\mathrm{ord}_p(q_E)$ is the $\mathcal{L}$-invariant and Ω_E^+ the real period of E.

Note that since $L_p(f,0) = \int_{\mathbb{Z}_p^\times} d\mu_p$, when $L_p(f,s)$ has a zero of order r at $s = 0$ its Taylor coefficient can be found by computing the image of μ_p in I^r/I^{r+1} where I is the augmentation ideal, the kernel of integrating the constant function 1. More generally, if R is a ring the R-valued measures of a topological space $\mathcal{X}$ form a ring $\mathrm{Meas}(\mathcal{X}, R)$ and I fits into the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & \mathrm{Meas}(\mathcal{X}, R) & \longrightarrow & R & \longrightarrow & 0 \\ & & & & \mu & \longmapsto & \int_{\mathcal{X}} d\mu & & \end{array}$$

Anticyclotomic case

Let K be a quadratic imaginary field over $\mathbb{Q}$. In [BD96; BD98; BD99], Bertolini and Darmon constructed the anticyclotomic p-adic L-function $L_p(E/K)$ associated to an elliptic curve E/K with split multiplicative reduction at p and the anticyclotomic extension of K. They proved an analogous Greenberg-Stevens exceptional zero formula if K/F splits, by relating the derivative $L_p'(E/K)$ at the critical point with the classical L-function and the $\mathcal{L}$-invariant similarly as in (1.5). Instead of using Hida families as Greenberg and Stevens, the proof uses p-adic integration on Shimura curves. Also, they showed that when K/F is inert the derivative $L_p'(E/K)$ can be used to obtain a K-rational point in $E(K)$. In fact, the image $\Phi_{\mathrm{Tate}}(L_p'(E/K))$ by the Tate uniformization is a difference of conjugate Heegner points α_K;

$$\Phi_{\mathrm{Tate}}(L_p'(E/K)) = \alpha_K - \overline{\alpha_K}. \tag{1.6}$$

In the papers [Ber+02; BD07] these results were generalized for higher weights.

1.3 Heegner points

The classical way to construct Heegner points on an (modular) elliptic curve $E/\mathbb{Q}$ of conductor N is essentially to take the image of CM points of the modular curve $X_0(N)$ of level N using its modular parametrization φ. The modular curve $X_0(N)$ can be regarded as the solution of the moduli problem (i.e. the corresponding functor is representable) of classifying pairs (E', C') where E' is an elliptic curve and C' a cyclic subgroup of order N modulo isomorphism. If $\mathfrak{H} = \{z \in \mathbb{C} : \Im z > 0\}$ is the upper half plane and $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$ the extended upper half-plane, each $\tau \in \mathfrak{H}$ has an associated elliptic curve $E_\tau = \mathbb{C}/\langle 1, \tau \rangle$ over $\mathbb{C}$ with endomorphism ring $\mathcal{O}_\tau$. From this it can be shown that

$$X_0(N)(\mathbb{C}) \simeq \mathfrak{H}^*/\Gamma_0(N).$$

The ring $\mathcal{O}_\tau$ is either $\mathbb{Z}$ or an order of a quadratic imaginary field $K/\mathbb{Q}$. In the last case τ is said to be a *CM point*; the theory of *complex multiplication* shows there is a finite number of isomorphism classes of elliptic curves E_τ with $\mathcal{O}_\tau = \mathcal{O}$ for a fixed order $\mathcal{O}$ of a quadratic imaginary field K, and that their j-invariants must be algebraic integers that generate abelian extensions of K. This is a remarkable property; by evaluating transcendental functions on certain arguments we obtain values that generate class fields in a systematic way, an instance of a realization of *Kronecker's Jugendtraum*.

Elliptic curves have a uniformization over $\mathbb{C}$ because $E(\mathbb{C})$ can be regarded as a torus $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$. There is an explicit parametrization φ: if $\mathbb{C}/\Lambda \simeq E(\mathbb{C})$ (and c is Manin's constant) then φ is given by

$$\begin{array}{rcl} \varphi : X_0(N)(\mathbb{C}) \simeq \mathfrak{H}/\Gamma_0(N) & \longrightarrow & \mathbb{C}/\Lambda \\ \tau & \longmapsto & c\int_{i\infty}^{\tau} 2\pi i f(z)dz \end{array} \tag{1.7}$$

where $f = f_E$ is a weight 2 modular form attached to E by modularity. The image of a CM point lies in fact in the group of rational points $E(H)$, where H is the ring class field associated to $\mathcal{O}$. The construction can be carried out under certain conditions on K and $\mathcal{O}$, the *Heegner hypothesis*: the conductor of $\mathcal{O} \subset K$ is prime to N and the primes ℓ dividing N split in $K/\mathbb{Q}$. For instance if the conductor of $\mathcal{O}$ is 1 then the point belongs to $E(H_1)$ where H_1 is the Hilbert class field of K.

This gives a quite general construction of algebraic points on E. The link between the L-function of E and Heegner points was found by Gross and Zagier in the 80s. Their formula relates the Néron-Tate pairing with the first derivative of $L(E/K, s)$; if $P_K = \mathrm{Tr}_{H_1/K} P_1$ is the trace of a Heegner point of conductor 1 then

$$\langle P_K, P_K \rangle = L'(E/K, 1)$$

where the equality is up to some explicit non-zero factor. In particular, P_K is non torsion if and only if $L'(E/K, 1) \neq 0$. This connection was strengthened by Kolyvagin's results; he proved that if P_K is non torsion then the Mordeil-Weil group of K-rational points $E(K)$ has rank 1. Altogether their results show that if $E/\mathbb{Q}$ has analytic rank ≤ 1 then E satisfies the weak BSD conjecture.

If one wants to relax the Heegner hypothesis, one must parametrize the elliptic curve with a certain Shimura curve X_{N^+,N^-} instead, otherwise P_K cannot be constructed. That is one of the reasons for which we will work with algebraic groups associated to a quaternion algebra, that we define in the next section. Here N^+, N^- is an *admissible factorization* of N. The complex points of X_{N^+,N^-} can be identified with a quotient of the upper half plane $\mathfrak{H}$ by a discrete subgroup Γ_{N^+,N^-} of a quaternion algebra. The curve is also defined over $\mathbb{Q}$ as $X_0(N)$ and is the solution to another moduli problem. In this setting, Zhang generalized the Gross-Zagier formula for Heegner points obtained from the Shimura curves X_{N^+,N^-}.

p-adic Heegner points

To obtain similar results to those of Gross, Zagier and Kolyvagin in other settings, for example when K is not imaginary, one should first generalize Heegner points. Note however that the algebraicity of Heegner points was provided by the theory of complex multiplication, which is either absent or conjectural in the real setting; there are some intriguing results for special cases and conjectures in this direction though, see [DPV21].

One first step is to phrase the construction of Heegner points differently, by using *modular symbols*. Recall the modular parametrization $\varphi : X_0(N) \to E$ and denote by Δ the free abelian group of divisors of $\mathfrak{H}$. Then there is a natural map $i(f)$ attached to $f = f_E$ from the degree zero divisors Δ_0 on $\mathfrak{H}$ to $\mathbb{C}$, namely

$$\begin{aligned} i : S_2(N) &\longrightarrow \mathbf{Hom}_{\Gamma_0(N)}(\Delta_0, \mathbb{C}) \\ f &\longmapsto i(f) : \left(\tau' - \tau \mapsto \int_\tau^{\tau'} f(z)dz\right) \end{aligned}$$

Since $i(f)$ is $\Gamma_0(N)$-equivariant, we have $i(f) \in \mathbf{Hom}_{\Gamma_0(N)}(\Delta_0, \mathbb{C}) = H^0(\Gamma_0(N), \mathbf{Hom}(\Delta_0, \mathbb{C}))$ and to evaluate an any point of $\mathfrak{H}$ rather than only at Δ_0, we should try to lift $i(f)$ to $\mathbf{Hom}_{\Gamma_0(N)}(\Delta, \mathbb{C})$. Consider the short exact sequence associated to the degree map

$$0 \longrightarrow \Delta^0 \longrightarrow \Delta \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

The $\mathbf{Hom}(\bullet, \mathbb{C})$ functor is contravariant, and exact in this case because $\mathbb{C}$ is an injective $\mathbb{Z}$-module, and we obtain

$$0 \longrightarrow \mathbf{Hom}(\mathbb{Z}, \mathbb{C}) \simeq \mathbb{C} \longrightarrow \mathbf{Hom}(\Delta, \mathbb{C}) \longrightarrow \mathbf{Hom}(\Delta^0, \mathbb{C}) \longrightarrow 0$$

Taking the long exact sequence of $\Gamma_0(N)$-cohomology we obtain

$$\cdots \longrightarrow \mathbf{Hom}_{\Gamma_0(N)}(\Delta, \mathbb{C}) \longrightarrow \mathbf{Hom}_{\Gamma_0(N)}(\Delta^0, \mathbb{C}) \xrightarrow{\kappa} H^1(\Gamma_0(N), \mathbb{C}) \longrightarrow \cdots$$

The connecting morphism here is $\kappa(h) = \int_\tau^{\gamma\tau} h(z)dz$ and its cohomology class does not depend on the choice of τ. It turns out the image of $i(f)$ is a lattice $\Lambda_f \subset \mathbb{C}$ such that E is isogeneous to $\mathbb{C}/\Lambda_f$. Rewriting the long exact sequence above but now applying $\mathbf{Hom}_\Gamma(\bullet, \mathbb{C}/\Lambda_f)$ we obtain by construction $i(f) \in \ker c$, so there exists a lift I_f of $i(f)$ to Δ. Historically, an element in $\mathbf{Hom}_{\Gamma_0(N)}(\Delta^0, M) = H^0(\Gamma_0(N), \mathbf{Hom}(\Delta^0), M))$ or rather is restriction to the cuspidal degree zero divisors, was understood as a modular symbol. In this work, we will call modular symbol to any cohomology class in $H^r(\Gamma_0(N), \mathbf{Hom}(N, M))$ for any $\Gamma_0(N)$-modules N and M. In §2.8 we will see automorphic analogues of these cohomology classes.

This argument can be translated to the p-adic setting, by using the p-adic uniformizations of the elliptic curve and the modular curve. If $\mathfrak{p}$ is a prime of F of split multiplicative reduction for E then the *Tate uniformization* provides an isomorphism $\Phi_{\text{Tate}} : \mathbb{C}_p^\times/q^{\mathbb{Z}} \xrightarrow{\simeq} E(\mathbb{C}_p^\times)$ for some $q \in F_\mathfrak{p}^\times$ with $|q|_\mathfrak{p} < 1$. The $\mathbb{C}_p$-points of $X_0(N)$ can be identified as a quotient of the p-adic upper half plane $\mathfrak{H}_p = \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$

$$X_0(N)(\mathbb{C}_p) \simeq \mathfrak{H}_p/\Gamma$$

for some subgroup Γ of the units of a definite quaternion algebra $B/\mathbb{Q}$ split at p i.e. there is an isomorphism of algebras $M_2(\mathbb{Q}_p) \simeq B_p = B \otimes_\mathbb{Q} \mathbb{Q}_p$. It can be shown that $f = f_E$ has a naturally attached element ψ_f of $\mathbf{Hom}_\Gamma(\mathrm{St}(\mathbb{Q}_p), \mathbb{Z})$, where $\mathrm{St}(\mathbb{Q}_p)$ is the *Steinberg representation* [1] that we introduce in §2.4.1 and §4.1.1. Now we only need to use the *multiplicative integral* instead, to account for the multiplicative nature of $\mathbb{C}_p^\times/q^{\mathbb{Z}}$. As before, we first define the modular symbol for a degree zero divisor $\tau - \tau' \in \Delta_p^0$ of $\mathfrak{H}_p$:

$$⨏_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{x-\tau}{x-\tau'} d\mu_f(x) = \lim_{\mathcal{U}} \prod_{U \in \mathcal{U}} \left(\frac{x_U - \tau}{x_U - \tau'}\right)^{\psi_f(\mathbb{1}_U)} \in \mathbb{C}_p^\times$$

where the limit is taken along coverings $\mathcal{U}$ of $\mathbb{P}^1(\mathbb{Q}_p)$ ordered by refinement and $x_U \in U$. This defines a morphism $i : \mathbf{Hom}_\Gamma(\mathrm{St}(\mathbb{Q}_p), \mathbb{Z}) \to \mathbf{Hom}_\Gamma(\Delta_p^0, \mathbb{C}_p^\times/q^{\mathbb{Z}})$. Repeating the same diagram above with these changes we obtain

$$\begin{array}{ccccccc} & & & \mathbf{Hom}_\Gamma(\mathrm{St}(\mathbb{Q}_p), \mathbb{Z}) & & & \\ & & & \downarrow i & \searrow^{\kappa \circ i} & & \\ \cdots \longrightarrow & \mathbf{Hom}_\Gamma(\Delta_p, \mathbb{C}_p^\times/q^{\mathbb{Z}}) & \longrightarrow & \mathbf{Hom}_\Gamma(\Delta_p^0, \mathbb{C}_p^\times/q^{\mathbb{Z}}) & \xrightarrow{\kappa} & H^1(\Gamma, \mathbb{C}_p^\times/q^{\mathbb{Z}}) & \longrightarrow \cdots \end{array}$$

[1] By the Jacquet-Langlands correspondence, f_E has an attached quaternionic analytic modular form g_E. In turn, g_E has an attached *harmonic cocycle* c_f, a function of the edges of the Bruhat-Tits tree of $\mathrm{PSL}_2(\mathbb{Q}_p)$, which can be regarded as a $\mathbb{Z}$-valued measure of $\mathbb{P}^1(\mathbb{Q}_p)$. For a complete exposition see [Dar04].

By construction $i\psi_f \in \ker\kappa$, so there exists a lift I_f to arbitrary divisors Δ_p. $K^\times$ acts on $\mathfrak{H}_p$ by fractional linear transformations, and the points of $\mathfrak{H}_p$ fixed by $K = \mathbb{Q}(\alpha)$ are those fixed by α, because $\mathbb{Q}$ acts trivially. If the image of α in $M_2(\mathbb{Q})$ is $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then a point τ_K fixed by α corresponds to an eigenvector of γ because

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix} = (c\tau + d)\begin{pmatrix} \gamma\tau \\ 1 \end{pmatrix}$$

The characteristic polynomial of γ is the same as that of α, and so the eigenvalue $c\tau_K + d$ and τ_K lies in the quadratic extension K. If we assume without loss of generality that $\alpha = c\tau_K + d$, modulo isogenies we can regard $y_K = I_f(\tau_K) \in \mathbb{C}_p^\times/q^{\mathbb{Z}}$ as a point in $E(\mathbb{C}_p)$, and we say y_K is a *p-adic Heegner point* of E. In fact it lies in $K_p^\times/q^{\mathbb{Z}} \simeq E(K_p)$, and it coincides with the algebraic point constructed previously; y_K is the image of the Heegner point under the natural inclusion $E(H) \subseteq E(K_p)$.

The natural analogue of Heegner points in the case $F = \mathbb{Q}$ and K real quadratic are Darmon's *Stark-Heegner* points. Darmon took the analytical construction of Heegner points and translated it to a local setting, by using the analytic uniformization of $E(K_\nu)$ as we have just explained, and obtained points of $E(K_\nu)$ for some non archimedean place ν.

These elements have been progressively defined in other settings over the years by Dasgupta, Greenberg, Sengun, Masdeu, Guitart and Molina for an arbitrary quadratic extension K/F and an elliptic curve E over F. We will refer to the points constructed in [GMM17] as *Darmon points*. In the archimedean case the uniformization always exists, and in the non archimedean case one imposes that E has split multiplicative reduction at ν to use the Tate uniformization. Conjecturally, these should be global points defined over K^{ab} and satisfy a Shimura reciprocity law. Together with *plectic points*, which are yet another generalization, they will be constructed in §4.1 with a similar diagram chasing.

Plectic points

As we will see in chapter 6, Gross-Zagier formulas for Darmon points consistent with the rank 1 case of the BSD conjecture can be obtained, but until recently not much was known in the case of rank > 2. In this direction, M. Fornea and L. Gehrmann came in [FG21] with a novel and interesting construction which provides elements - *plectic points* - in certain completed tensor products of elliptic curves, and stated p-adic Gross-Zagier formulas for these elements. Their construction generalizes that of Darmon points in §4.1.3, and requires the use of tensor products in both the local representations at p, the local uniformizations $E(K_\mathfrak{p})$ of the elliptic curve for a subset S of primes $\mathfrak{p}$ of F above p, and the groups of divisors.

Hida theory

In few words, this theory provides a framework to make sense of *interpolating automorphic or modular forms continuously*. For introductions on this topic see [BD07; Maz12].

Introduction

Historically, one of the first examples of p-adically interpolated objects are the Eisenstein series. Bernoulli numbers B_k can be defined by the equality $\frac{X}{e^X-1} = \sum_{n\geq 0} B_n \frac{X^n}{n!}$. The *Kummer congruences* assert that under some conditions on k and ℓ

$$\left(1-p^{2k-1}\right)\frac{B_{2k}}{2k} \equiv \left(1-p^{2\ell-1}\right)\frac{B_{2\ell}}{2\ell} \mod p^{a+1}$$

Let $E_{2k}(q)$ be the weight $2k$ Eisenstein series, $E^*_{2k}(q) = E_{2k}(q) - p^{2k-1}E_{2k}(q^p)$ and define the *weight space* as

$$W = \mathbf{Hom}_{\mathrm{cont}}\left(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times\right) \simeq \mathbf{Hom}_{\mathbb{Z}_p-\mathrm{alg}}(\Lambda, \mathbb{Z}_p)$$

Here W is endowed the topology of uniform convergence and $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ is the *Iwasawa algebra* of $\mathbb{Z}_p^\times$. By regarding $h \in \mathbb{Z}$ as the map $z \mapsto z^h$ we can embed $\mathbb{Z} \subset W$, and for $h \in W$ write $z^h := h(z)$. Similarly, for $\lambda \in \Lambda$ we write $\lambda(h)$ for the image of λ through the corresponding specialization morphism $h \in \mathbf{Hom}_{\mathbb{Z}_p\text{-alg}}(\Lambda, \mathbb{Z}_p)$.

Serre noticed that the Fourier coefficients of $E^*_h(q)$, regarded now as a formal power series in q, are continuous $\mathbb{Z}_p$-valued functions when h varies in the *even* weights $h \in W$ i.e. those h with $(-1)^h = 1$. In particular, the constant term of $E_{2k}(q)$ is a rational multiple of B_{2k}. This led him to the notion of *p-adic modular forms*, formal power series

$$f = \sum_n a_n q^n \in \mathbb{Q}_p[[q]]$$

which are limits in the v_p-metric of rational modular forms $f_m = \sum_n a_{n,m} q^n$ of $\mathbf{SL}_2(\mathbb{Z})$. That is, $\lim_m \inf_n v_p(a_n - a_{n,m}) = 0$.

Using these objects, Serre gave an alternative proof of the Kummer congruences. Nevertheless, it can be shown that the spaces of p-adic modular forms are infinite dimensional Banach spaces where the Hecke operators are not compact in general, so one cannot apply the spectral theorem to obtain a basis of eigenforms as in the classical case. One first idea to account for these obstructions is to focus on the *ordinary* subspace instead, as in Hida's work.

Hida families

Suppose that p is an odd prime and $v_p(N) = 1$. An eigenform $f \in S_k(\Gamma_0(N))$ is *p-ordinary* when its Hecke eigenvalue λ_p by U_p is in $\mathbb{Z}_p^\times$. In particular, the U_p Hecke operator is invertible when restricted to the p-ordinary subspace. One also says that f has *slope m* when $v_p(\lambda_p) = m$.

Then we have the identification

$$W \simeq (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}_p \simeq \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$$

For each open $U \subset W$ we define $A(U)$ to be the set of *analytic functions* f on U i.e. if $V = U \cap \left(\{a\} \times \mathbb{Z}_p\right)$ and $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, $f|_V$ is a power series. A *Λ-adic form* is then a formal q-expansion $f_\infty = \sum_n a_n q^n$ for which there exists a neighborhood of k in W such that its *weight k specialization* $f_k = \sum_n a_n(k) q^n$ is a weight k, p-ordinary, normalized eigenform. This is a first example of a *Hida family*, a formal q-expansion with coefficients in a convenient Iwasawa algebra specializing to classical modular forms.

Iwasawa algebras come equipped with an *universal character* $\mathbf{k}$ that has an universal property. As an example, for the group $G = \mathbb{Z}_p^\times$ with Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, the universal property of Λ and $\mathbf{k}$ is that for any character $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p$ the *specialization morphism* ρ_χ fits in the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z}_p^\times & \xrightarrow{\mathbf{k}} & \Lambda \\ & \searrow^{\chi} & \downarrow{\scriptstyle \rho_\chi} \\ & & \mathbb{C}_p \end{array} \tag{1.8}$$

One of Hida's main results is that p-ordinary eigenforms are always in some Hida family.

Overconvergent modular forms and the eigencurve

Coleman, by rephrasing Katz's geometric definition of modular forms, showed that most *overconvergent modular forms* of finite slope are in a p-adic family, generalizing Hida's result.

Although in our approach we do not seek a geometric interpretation of Hida families but rather use them as interpolating objects, there is a nice geometric picture behind this; the *eigencurve* defined by Coleman and Mazur. It is a rigid analytic curve with still many unknown properties. This object was further generalized by Buzzard to *eigenvarieties*, which deal with overconvergent quaternionic automorphic forms over totally real fields.

1.4 Overview of the main results

In this work we generalize the construction of p-adic anticyclotomic L-functions associated to an elliptic curve E/F and a quadratic extension K/F, by defining in §5.1.1 a measure $\mu_{\phi_\lambda^p}$ attached to K/F and an automorphic form of even weight vector

$$\underline{k} + 2 = (k_1 + 2, \cdots, k_d + 2) \in 2\mathbb{Z}_{\geq 2}^d$$

where $d = [F : \mathbb{Q}]$. In the case of parallel 2, the automorphic form is associated with an elliptic curve E/F. Given a set of places S above p we will recall the construction of the Fornea-Gehrmann plectic point $P_\xi^S \in \bigotimes_{\mathfrak{p}\in S} E(K_\mathfrak{p})$ attached to a finite order character ξ. When $S = \{\mathfrak{p}\}$ is a single place $P_\xi^\mathfrak{p}$ is a Darmon point. In this parallel weight 2 setting, we can define a measure $\mu_{\phi_\lambda^S}$ depending on S that can be regarded as the restriction of $\mu_{\phi_\lambda^p}$. Write I_ξ for the augmentation ideal attached to the evaluation at ξ. The first main result is a p-adic Gross-Zagier formula, namely

Theorem 6.1.1. *Assume that E has split multiplicative reduction at $\mathfrak{p}$ and $\mathfrak{p}$ does not split at K/F. Then $\mu_{\phi_\lambda^\mathfrak{p}} \in I_\xi$. Moreover,*

$$\mu_{\phi_\lambda^\mathfrak{p}} \equiv [\mathcal{O}_+^\mathfrak{p} : \mathcal{O}_+]^{-1} \cdot \varphi \circ \mathrm{rec}_\mathfrak{p} \left(\overline{P}_\xi^\mathfrak{p} - P_\xi^\mathfrak{p} \right) \mod I_\xi^2$$

where $\mathcal{O}_+$ and $\mathcal{O}_+^\mathfrak{p}$ are the groups of totally positive units and $\mathfrak{p}$-units, respectively.

In this result, the morphism $\varphi \circ \mathrm{rec}_\mathfrak{p}$ provided by class field theory is a natural way to interpret conjugate differences of points in $E(K_\mathfrak{p})$ as elements in I_ξ. This relates the difference of conjugate (twisted by ξ) Darmon points $P_\xi^\mathfrak{p}$ with the first derivative of the p-adic L-function, which generalizes result (1.6) of Bertolini and Darmon. With a similar argument, after discovering the work of Fornea and Gehrmann in [FG21] on plectic points, we prove an exceptional zero formula which relates the r-th derivative of $\mu_{\phi_\lambda^S}$ with plectic points, namely

Theorem 6.2.1. *Write $S = S_+ \cup S_-$, where S_+ is the subset of places where E has multiplicative reduction, and write $S_+ = S_+^1 \cup S_+^2$, where S_+^1 is the subset where K/F splits. Then $\mu_{\phi_\lambda^S} \in I_\xi^r$ where $r = \#S_+$. Moreover,*

$$\mu_{\phi_\lambda^S} \equiv (-1)^s \cdot [\mathcal{O}_+^{S_+^2} : \mathcal{O}_+]^{-1} \cdot \epsilon_{S_-}(\pi_{S_-}, \xi_{S_-}) \cdot \varphi \circ \mathrm{rec}_{S_+}\left(q_{S_+^1} \otimes \bigotimes_{\mathfrak{p} \in S_+^2} (\sigma_\mathfrak{p} - 1) P_\xi^{S_+^2} \right) \mod I_\xi^{r+1}$$

where $\sigma_\mathfrak{p} - 1$ stands for the conjugate difference, $q_{S_+^1}$ is the product of Tate uniformizers at $\mathfrak{p} \in S_+^1$, $\mathcal{O}_+^{S_+^2}$ is the group of positive S_+^2-units and $\epsilon_{S_-}(\pi_{S_-}, \xi_{S_-})$ is an explicit Euler factor.

In §5.4 we will introduce the Hida families of automorphic modular symbols ϕ_λ^p. We find an interpolating measure $\mu_{\Phi_\lambda^p}$ for $\mu_{\phi_\lambda^p}$ attached to an interpolating family Φ_λ^p for ϕ_λ^p (see (5.18)). Here $\mu_{\Phi_\lambda^p}$ can be regarded as a two variable p-adic L-function, which now includes the weight as a variable. Then we define the Hida-Rankin p-adic L-function $L_p(\phi_\lambda^p, \xi, \underline{k})$ in (6.12) as the restriction of $\mu_{\Phi_\lambda^p}$ to the weight space. Finally, we prove a formula which relates the weight-leading term of $L_p(\phi_\lambda^p, \xi, \underline{k})$ with plectic points, namely

Theorem 6.3.2. *Let I be the augmentation ideal for the weight parallel 2 specialization, and S the set of primes $\mathfrak{p}$ above p where E has multiplicative reduction and $\mathfrak{p}$ does not split at K. Then $L_p(\phi_\lambda^p, \xi, \underline{k}) \in I^r$ where $r = \#S$. Moreover*

$$L_p(\phi_\lambda^p, \xi, \underline{k}) \equiv C \cdot \prod_{\mathfrak{p} \notin S} \epsilon_\mathfrak{p}(\pi_\mathfrak{p}, \xi_\mathfrak{p}) \cdot \ell_{\mathbf{a}_S} \circ \mathrm{Tr}(P_\xi^S) \pmod{I^{r+1} \otimes_\mathbb{Z} \bar{\mathbb{Q}}}$$

where C is an explicit constant, $\epsilon_\mathfrak{p}(\pi_\mathfrak{p}, \xi_\mathfrak{p})$ are explicit Euler factors, Tr is the natural trace and $\ell_{\mathbf{a}_S}$ is the product at $\mathfrak{p} \in S$ of the natural $\mathfrak{p}$-adic logarithms attached to E.

That is, the leading term is an explicit constant C times Euler factors times the logarithm of the trace of a plectic point. This result is a generalization of [LMH20, Theorem 5.1], which is used there to prove the *rationality* of a Darmon point under some hypotheses.

Setup and notation

For any field L we will write $\mathcal{O}_L$ for its ring of integers, and denote $\mathcal{O}_{\overline{\mathbb{Q}}}$ by $\overline{\mathbb{Z}}$. Let G, G' be groups with $G \subset G'$, M a G-representation over a field L, and ρ an irreducible G'-representation over L. We will write

$$M_\rho := \mathbf{Hom}_G\left(\rho\mid_G, M\right).$$

as representations over L.

For any ring R and an even number k, let $\mathcal{P}(k)_R$ be the R-module of homogeneous polynomials of degree k in two variables with coefficients in R, together with the following $\mathbf{GL}_2(R)$-action; given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$

$$(\gamma \cdot P)(x, y) := (\det \gamma)^{-k/2} \cdot P\left((x, y) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \tag{1.9}$$

Note that the factor $(\det\gamma)^{-k/2}$ makes the central action trivial.

Denote by $V(k)_R$ the dual space $\mathbf{Hom}_R(\mathcal{P}(k)_R, R)$. In the case $R = \mathbb{C}$ we set $V(k) := V(k)_{\mathbb{C}}$. Given a vector $\underline{k} = (k_i) \in (2\mathbb{N})^n$ we define

$$V\left(\underline{k}\right)_R := \bigotimes_{i=1}^{n} V(k_i)_R$$

Note that, if $n = d = [F : \mathbb{Q}]$ and $\underline{k} = (k_{\bar{\sigma}}) \in (2\mathbb{N})^d$ is indexed by the embeddings $\bar{\sigma} : K \hookrightarrow \mathbb{C}$ of F, then $V(\underline{k}) := V(\underline{k})_{\mathbb{C}}$ has a natural action of $G(F_\infty)$. The subspace $V(\underline{k})_{\bar{\mathbb{Q}}}$ is fixed by the action of the subgroup $G(F) \subseteq G(F_\infty)$.

Note that, if we fix $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, we can associate to a prime $\mathfrak{p}$ of F above p the set $\Sigma_{\mathfrak{p}}$ of embeddings $\bar{\sigma} : F \hookrightarrow \bar{\mathbb{Q}}$ such that $v_p(\bar{\sigma}(\mathfrak{p})) > 0$. Hence $\bar{\sigma} \in \Sigma_{\mathfrak{p}}$ can be seen as an embedding $\bar{\sigma} : F_{\mathfrak{p}} \hookrightarrow \overline{\mathbb{Q}}_p$. We have fixed a bijection between embeddings $\bar{\sigma} : F \hookrightarrow \mathbb{C}$ and $\bar{\sigma} : F_{\mathfrak{p}} \hookrightarrow \mathbb{C}_p$ for all $\mathfrak{p} \mid p$. Thus, for any $\underline{k} \in (2\mathbb{N})^d$ we have also a natural action of $\mathbf{PGL}_2(F_p)$ on $V(\underline{k})_{\overline{\mathbb{Q}}_p}$.

Let $\mathfrak{p}$ be any place above p, and assume that $G(F_{\mathfrak{p}}) = \mathbf{PGL}_2(F_{\mathfrak{p}})$. The fixed embedding

$$\iota : K_{\mathfrak{p}}^\times \hookrightarrow \mathbf{GL}_2(F_{\mathfrak{p}}) \tag{1.10}$$

provides two eigenvectors $v_1^{\mathfrak{p}}, v_2^{\mathfrak{p}} \in (\bar{F}_{\mathfrak{p}})^2$ satisfying

$$v_1^{\mathfrak{p}}\iota(\tilde{t}_{\mathfrak{p}}) = \lambda_{\tilde{t}_{\mathfrak{p}}} v_1^{\mathfrak{p}}, \quad v_2^{\mathfrak{p}}\iota(\tilde{t}_{\mathfrak{p}}) = \bar{\lambda}_{\tilde{t}_{\mathfrak{p}}} v_2^{\mathfrak{p}}, \qquad \lambda_{\tilde{t}_{\mathfrak{p}}}, \bar{\lambda}_{\tilde{t}_{\mathfrak{p}}} \in \bar{F}_{\mathfrak{p}}, \quad \tilde{t}_{\mathfrak{p}} \in K_{\mathfrak{p}}^\times. \tag{1.11}$$

The quotient $\lambda_{\tilde{t}_{\mathfrak{p}}}/\bar{\lambda}_{\tilde{t}_{\mathfrak{p}}}$ depends on the class $t_{\mathfrak{p}} \in K_{\mathfrak{p}}^\times/F_{\mathfrak{p}}^\times = T(F_{\mathfrak{p}})$ of $\tilde{t}_{\mathfrak{p}}$. By abuse of notation we will denote $\lambda_{\tilde{t}_{\mathfrak{p}}}/\bar{\lambda}_{\tilde{t}_{\mathfrak{p}}} \in \bar{F}_{\mathfrak{p}}$ also by $t_{\mathfrak{p}}$. Write $K_{\mathfrak{p}}^\sharp$ for the minimum extension of $F_{\mathfrak{p}}$ where all $\lambda_{\tilde{t}_{\mathfrak{p}}}$ lie. Hence $K_{\mathfrak{p}}^\sharp = F_{\mathfrak{p}}$ if $\mathfrak{p}$ splits and $K_{\mathfrak{p}}^\sharp = K_{\mathfrak{p}}$ otherwise. For any $\sigma \in \Sigma_{\mathfrak{p}}$, we will fix once for all $\tilde{\sigma} : K_{\mathfrak{p}}^\sharp \hookrightarrow \overline{\mathbb{Q}}_p$ such that $\tilde{\sigma}\mid_{F_{\mathfrak{p}}} = \sigma$. Also by abuse of notation, we will write

$$\sigma(t_{\mathfrak{p}}) := \tilde{\sigma}(t_{\mathfrak{p}}) \in \overline{\mathbb{Q}}_p, \qquad t_{\mathfrak{p}} \in T(F_{\mathfrak{p}}).$$

Chapter 2

Automorphic forms

2.1 The ring of adèles

Since automorphic forms are functions on adèlic points of an algebraic group, in our case $\mathbf{GL}_2$ or the group of units of a quaternion algebra, we introduce the necessary notation and define of the ring of the adèles. The adèles will also be used to describe Galois groups using class field theory.

Let F be a number field of degree d over $\mathbb{Q}$ and $\mathcal{O}_F$ its ring of integers. We will denote by either ∞ or Σ_F the set of *archimedean* or *infinite* places of F i.e. classes of embeddings $\sigma : F \longrightarrow \mathbb{C}$ of F into $\mathbb{C}$ modulo complex conjugation, the non trivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Furthermore, sometimes p will denote the set of places of F above a rational prime p. Note that the complex embeddings are paired up into one place, since either $\operatorname{im}\sigma \subset \mathbb{R}$ or not. Given a place v let F_v be the completion of F at v. The *non archimedean* or *finite* places are the classes associated to the prime ideals $\mathfrak{p} \subset \mathcal{O}_F$ i.e. those for which F_v is neither $\mathbb{R}$ nor $\mathbb{C}$. For a finite place $\mathfrak{p}$ denote by $v_\mathfrak{p} : \overline{F_\mathfrak{p}^\times} \longrightarrow \mathbb{Q}$ its associated valuation, $\omega_\mathfrak{p}$ a fixed uniformizer with $v_\mathfrak{p}(\varpi_\mathfrak{p}) = 1$ and $q_\mathfrak{p}$ the cardinality of the residue field $\mathcal{O}_{F,\mathfrak{p}}/\mathfrak{p}$. If r_F, s_F are the number of real and complex places then $r_F + 2s_F = d$. If a place $\sigma \in \infty$ is given by the class of an embedding $\bar{\sigma} : F \hookrightarrow \mathbb{C}$, we write $\bar{\sigma} \mid \sigma$.

For a set of places S write

$$F_S := \prod_{v \in S} F_v \text{ and } F^S := \prod_{v \notin S} F_v$$

The ring of *adèles* is defined as the following restricted product

$$\mathbb{A}_F = F_\infty \times \widehat{\prod}_\mathfrak{p} F_\mathfrak{p}$$

where the $\widehat{\cdot}$ imposes that elements have a finite number of $\mathfrak{p}$-components which are not in $\mathcal{O}_{F,\mathfrak{p}}$. The *diagonal embedding* provides a way to regard F as a subset of $\mathbb{A}_F$,

$$\begin{aligned} F &\hookrightarrow \mathbb{A}_F \\ \alpha &\longmapsto (\alpha, \alpha, \cdots) \end{aligned}$$

For a set S of places define $\mathbb{A}_F^S := \mathbb{A}_F \cap F^S$. Since $\mathbb{A}_F^\times$ is a locally compact Hausdorff topological group, we can use its associated *Haar measure* to integrate functions ϕ of the idèles $\mathbb{A}_F^\times$. The defining properties of this measure are its inner and outer regularity, invariance by left multiplication i.e.

$$\mu(g \cdot X) = \mu(X), \text{ for any } g \in \mathbb{A}_F^\times \text{ and Borel subset } X$$

and uniqueness up to a constant. In particular, we have

$$\int_{\mathbb{A}_K^\times} \phi(t) d^\times t = \int_{\mathbb{A}_K^\times} \phi(\gamma t) d^\times t, \quad \forall \gamma \in \mathbb{A}_K^\times$$

2.2 Some algebraic groups associated to quaternion algebras

Let K/F be a quadratic extension and $\Sigma_{\text{un}}(K/F)$ be the set of archimedean places σ of F that split at K. Then

$$\Sigma_{\text{un}}(K/F) = \Sigma_{\mathbb{R}}^{\mathbb{R}}(K/F) \sqcup \Sigma_{\mathbb{C}}^{\mathbb{C}}(K/F)$$

where $\Sigma_{\mathbb{R}}^{\mathbb{R}}(K/F)$ is the set of real places of F which remain real in K and $\Sigma_{\mathbb{C}}^{\mathbb{C}}(K/F)$ the set of complex places of F. We will denote by $u, r_{K/F,\mathbb{R}}$ and s_F the cardinality of these three sets so that $u = r_{K/F,\mathbb{R}} + s_F$. Analogously, we write $\Sigma_{\mathbb{R}}^{\mathbb{C}}(K/F)$ for $\Sigma_F \backslash \Sigma_{\text{un}}(K/F)$.

A *quaternion algebra* B/F is a central simple algebra of dimension 4 over F. Let B/F be a quaternion algebra for which there exists an embedding $K \hookrightarrow B$, that we fix now. Let Σ_B be the set of archimedean places ν of F for which the quaternion algebra B splits i.e. such that

$$B \otimes_F F_\nu \simeq M_2(F_\nu)$$

We can define G an algebraic group associated to $B^\times/F^\times$ as follows: G represents the functor that sends any $\mathcal{O}_F$-algebra R to

$$G(R) = (\mathcal{O}_B \otimes_{\mathcal{O}_F} R)^\times / R^\times,$$

where $\mathcal{O}_B$ is a maximal order in B that we fix once and for all. Similarly, we define the algebraic group T associated to $K^\times/F^\times$ by

$$T(R) = (\mathcal{O}_c \otimes_{\mathcal{O}_F} R)^\times / R^\times,$$

where $\mathcal{O}_c := \mathcal{O}_B \cap K$ is an order of conductor c in $\mathcal{O}_K$. Note that $T \subset G$. We denote by $G(F_\infty)_+$ and $T(F_\infty)_+$ the connected component of the identity of $G(F_\infty)$ and $T(F_\infty)$, respectively. We also define $T(F)_+ := T(F) \cap T(F_\infty)_+$ and $G(F)_+ := G(F) \cap G(F_\infty)_+$.

Given $\sigma \in \Sigma_F$ note that $T(F_\sigma) = (\mathcal{O}_c \otimes_{\mathcal{O}_F,\sigma} F_\sigma)^\times / F_\sigma^\times$ is either $\mathbb{C}^\times/\mathbb{R}^\times, \mathbb{R}^\times$ or $\mathbb{C}^\times$. Define $T(F_\sigma)_0$ to be the intersection of all the connected subgroups N of 1 in $T(F_\sigma)$ for which the quotient $T(F_\sigma)/N$ is compact. Note that $T(F_\sigma)/T(F_\sigma)_0 \simeq \varprojlim_N T/N$ is then compact. The set $T(F_\sigma)_0$ depends on the ramification type of σ, as is described in the following table

ramification type	$T(F_\sigma)$	$T(F_\sigma)_0$	$T(F_\sigma)/T(F_\sigma)_0$
$\Sigma_{\mathbb{R}}^{\mathbb{C}}(K/F)$	$\mathbb{C}^\times/\mathbb{R}^\times$	1	$\mathbb{C}^\times/\mathbb{R}^\times \simeq S^1$
$\Sigma_{\mathbb{R}}^{\mathbb{R}}(K/F)$	$\mathbb{R}^\times$	$\mathbb{R}_+$	± 1
$\Sigma_{\mathbb{C}}^{\mathbb{C}}(K/F)$	$\mathbb{C}^\times$	$\mathbb{R}_+$	S^1

If $\sigma \in \Sigma_{\text{un}}(K/F)$, then only the second and third of the ramification types on the table actually occur. For any set of infinite places Σ, we write $T(F_\Sigma)_0 = \prod_\sigma T(F_\sigma)_0$.

2.3 Results from Class Field Theory

Class field theory describes the abelian extensions of a local (e.g. a finite extension of $\mathbb{Q}_p$) or global fields (e.g. a number field) in terms of their arithmetic. Essentially, if F is a global field it provides a surjective morphism from the idèles of F to the maximal abelian extension of F. For a complete exposition on this topic see [Mil20].

Let w be a place of M above a place v of F. Recall that given a finite abelian extension M/F of number fields we can consider the Galois action of $\mathrm{Gal}(M/F)$ on the places w of M; if w is archimedean then the action $\sigma w := \sigma \circ w$ is composition and if $w = \mathfrak{p}$ is non archimedean then the action is $\sigma\mathfrak{p} = \sigma(\mathfrak{p})$. Then the *decomposition group* of w is defined as the Galois subgroup which fixes w

$$D(w) = \{\sigma \in \mathrm{Gal}(M/F) \text{ such that } \sigma w = w\}$$

Local class field theory shows that this group is isomorphic to the local Galois group associated to w i.e. $D(w) \simeq \mathrm{Gal}(M_w/F_v)$. The *local reciprocity law* induces an isomorphism $\rho_{L/K} : F_v/\mathrm{Nm}_{M_w/F_v}(M_w^\times) \to \mathrm{Gal}(M_w/F_v)$.

Then it turns out that one can glue the maps $\rho_{L/K}$ - they form an inverse system and thus we can take the projective limit - to construct a surjective group morphism to the absolute Galois group of the abelian closure of F, known as the *Artin map*

$$\rho : \mathbb{A}_F^\times/F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

This result allows one to regard Galois groups as a certain quotient of an adèlic group.

Note that given a Galois character $\chi : \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathbb{C}$ of F we can construct a character of $\mathbb{A}_F^\times$, since by the above it can be regarded as a character of $\mathbf{GL}_1(\mathbb{A}_F) = \mathbb{A}_F^\times$.

$$\widehat{\chi} : \mathbb{A}_F^\times/F^\times \xrightarrow{\rho} \mathrm{Gal}\left(F^{\mathrm{ab}}/F\right) \xrightarrow{\chi} \mathbb{C}$$

This is an automorphic character, which generates a one dimensional irreducible representation. They were first studied in depth by Tate, who essentially tackled the $\mathbf{GL}_1$ case of automorphic forms in his thesis by using Fourier analysis on the adèles. It can be seen as a reformulation of Hecke's work on the analytic continuation of the L-series associated to a *Grossencharakter* and its functional equation.

2.4 Automorphic forms and representations for $\mathbf{GL}_2$

The theory of classical modular forms can be generalized in several directions: automorphic forms, Katz and overconvergent modular forms, Hida families, etc. One possible motivation to do so was explained in §2.3.

We will work with automorphic representations of a quaternion algebra or $\mathbf{GL}_2$, and regard them as group cohomology classes in the formalism Michael Spiess introduced in his study of automorphic $\mathcal{L}$-invariants (see [Spi13]). This approach does not rely on a geometric picture, but rather uses group representation theory and allows one to change the base field freely.

We remark that automorphic representations are not "representations" in the classical sense, but rather an infinite (restricted) tensor product of local representations for each place as we recall now below. For a complete exposition on this topic see [Bum98].

Let k be a field. A *group representation* of G is a group morphism $\pi : G \to \mathrm{Aut}(V)$ where V is a k-vector space, and an *irreducible* representation has no proper invariant subspaces. An *intertwining map* $\alpha : V \to V'$ between two representations $(\pi, V), (\pi', V')$ is a k-linear map that commutes with the group action i.e. the following diagram commutes for every $g \in G$

$$\begin{array}{ccc} V & \xrightarrow{\pi(g)} & V \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ V' & \xrightarrow{\pi'(g)} & V' \end{array}$$

Usually, some conditions are imposed on π to exclude wild or pathological representations. These conditions often have the same name but they are defined in a different way, depending on whether the place is archimedean or non archimedean, although the underlying idea is the same as exposed in the summary below.

2.4.1 Local factors at the non archimedean places

In this case, the group G_ν is $\mathbf{PGL}_2(F_\nu)$ where F is a number field and ν a finite place of F. If $k = \mathbb{C}$, π is *smooth* if every $v \in V$ has open stabilizer, and a smooth representation (π, V) of G_ν is *admissible* if for every compact subgroup $K' \subset G_\nu$ the space of K'-fixed vectors $V^{K'}$ is finite dimensional. We will frequently use the *Iwasawa decomposition* of G_ν to do explicit calculations with elements of G_ν. One can consider two notable subgroups of G_ν; the maximal compact subgroup of G_ν, which is $K_\nu = \mathbf{PGL}_2(\mathcal{O}_{F_\nu})$ where $\mathcal{O}_{F_\nu}$ is the ring of integers of F_ν, and the *Borel subgroup* of G_ν, which is just the image in G_ν of the subgroup of upper triangular matrices of $\mathbf{GL}_2(F_\nu)$. A representation is *unramified* if $\dim V^{K_\nu} = 1$. Then the Iwasawa decomposition of G_ν is

$$G_\nu = B_\nu \cdot K_\nu \tag{2.1}$$

A construction that will appear frequently is the induced representation from a given character. Note there is a natural action of G_ν on functions $f : G_\nu \longrightarrow \mathbb{C}$, the *right regular action* given by

$$(g \cdot f)(x) := f(xg)$$

Given a character $\chi : F_\nu^\times \longrightarrow \mathbb{C}^\times$ define $\mathcal{B}(\chi)$ as the following set of maps $f : G_\nu \longrightarrow \mathbb{C}$ which are smooth under the right action

$$\mathcal{B}(\chi) := \mathrm{Ind}_{B_\nu}^{G_\nu}(\chi) = \left\{ f : G_\nu \longrightarrow \mathbb{C} \text{ such that } f\left(\begin{pmatrix} a & b \\ & d \end{pmatrix} g \right) = \chi(a)\chi^{-1}(d) \cdot f(g) \right\}$$

By equation (2.1) we have $B_\nu \backslash G_\nu = K_\nu$ so elements $f \in \mathcal{B}(\chi)$ are determined by their restriction to K_ν and χ. Thus using that open compact subgroups are of finite index it follows that $\mathcal{B}(\chi)$ an admissible representation; whether it is irreducible or not depends on the character χ. Then the irreducible admissible representations of G_ν are of four kinds, and three of them are particular instances, subspaces or quotients of $\mathcal{B}(\chi)$:

1. The one dimensional representation induced by $\chi' \circ \det$ where χ' is a character.
2. The *principal series* representation is an element in the isomorphism class of $\mathcal{B}(\chi)$ when it is irreducible; this happens when $\chi^2 \neq |\cdot|_\nu^2$ and $\chi^2 \neq 1$.
3. The *special* and *Steinberg* representations can be obtained in at least two ways from $\mathcal{B}(\chi)$:
   (a) If $\chi^2 = |\cdot|_\nu^2$ then for some character χ' one has $\chi = \chi' \cdot |\cdot|_\nu$. Then $\mathcal{B}(\chi)$ has a one dimensional quotient where G_ν acts by $\chi' \circ \det$ and an infinite dimensional *subrepresentation* $\sigma(\chi')$.

(b) If $\chi^2 = 1$ then $\mathcal{B}(\chi)$ has a one dimensional submodule where G_v acts by $\chi \circ \det$ and an infinite dimensional *quotient* $\sigma'(\chi)$.

The subrepresentation in the first case and the quotient the second are isomorphic i.e. $\sigma(\chi') \simeq \sigma'(\chi)$. These are the *special* representations and the *Steinberg* is the particular case in which $\chi = 1$; in that case $\chi \circ \det$ is trivial and we can construct a *model* for the Steinberg that "works" for any ring

$$\mathrm{St}_R := \left\{ f : G_v \longrightarrow R \text{ such that } f(bg) = f(g) \text{ for all } g \in G_v, b \in B_v \right\} / R$$

Note we are modding out by the subspace of constant functions, that we identify with R.
In general, any special representation is a twist of the Steinberg i.e. $\sigma(\chi') \simeq \mathrm{St} \otimes \chi'$.

4. The *supercuspidal* representation, that cannot be obtained by induction from a character as the previous ones.

2.4.2 Local factors at the archimedean places

In this case, G_σ is either $\mathbf{PGL}_2(\mathbb{R})$ or $\mathbf{PGL}_2(\mathbb{C})$ depending on whether the embedding $\sigma : F \longrightarrow \mathbb{C}$ is real or complex, and G_σ is a Lie group. Now the maximal compact subgroup $K_\sigma \subset G_\sigma$ is either is the image in $\mathbf{PGL}_2(\mathbb{R})$ of the orthogonal group $O_2(\mathbb{R})$ or the image of the unitary group $U_2(\mathbb{C})$ in $\mathbf{PGL}_2(\mathbb{C})$. The difference with respect to non archimedean places is that one needs to introduce the notion of $(\mathfrak{g}_\sigma, K_\sigma)$-module developed by Harish-Chandra, where $\mathfrak{g}_\sigma$ is the Lie algebra of G_σ. As a motivation for the notion of $(\mathfrak{g}, K)$-module, suppose $\pi : G_\sigma \longrightarrow \mathbf{GL}(V)$ is a continuous representation of G_σ, where V is a Hilbert space. An element $v \in V$ is C^1 if for all $X \in \mathfrak{g} = \mathrm{Lie}(G)$ the derivative with respect to X is defined i.e. if

$$\pi(X)v := \frac{d}{dt}\pi(\exp(tX)v)|_{t=0}$$

exists, where exp is the exponential map provided by the Lie theory of G_σ. The vector will be C^k if $\pi(X_1)\cdots\pi(X_k)v$ is defined for all $X_i \in \mathfrak{g}$, and *smooth* if it is C^k for every $k \geq 1$. Here X acts on v by the *infinitesimal action* i.e. we have defined an action $\mathfrak{g} \longrightarrow \mathbf{End}\, V^\infty$. Let V^∞ be the space of smooth vectors. Then in general $V^\infty \neq V$. This is because in the non archimedean case, the group is locally profinite and admissible representations when restricted to a maximal compact subgroup decompose as a direct sum of irreducible representations of K appearing with finite multiplicity. Now G_σ is neither compact nor locally profinite when σ is archimedean. In order to classify them we use a different notion of admissibility and the representation theory of compact groups. It can be roughly summed up as follows: one can always assume the continuous representations are unitary, finite dimensional when irreducible, and that every representation is completely reducible i.e. is the direct sum of irreducible K-subrepresentations.

A $(\mathfrak{g}_\sigma, K_\sigma)$-*module* is a complex vector space V with representations $\pi : \mathfrak{g} \longrightarrow \mathbf{End}(V)$ and $\pi : K \longrightarrow \mathbf{GL}(V)$ such that :

- *Compatibility*: The given representation $\pi : \mathfrak{g} \longrightarrow \mathbf{End}(V)$ agrees with the infinitesimal action of K on smooth vectors V^∞, and for $X \in \mathfrak{g}$ and $k \in K$ we have $\pi(k)\pi(X)\pi(k^{-1}) = \pi\left((\mathrm{Ad}\ k)X\right)$.
- *K-finiteness*: V is a direct sum of finite dimensional representations of K

In addition, it will be *admissible* if every isomorphism class of K-subrepresentation appears with finite multiplicity.

There is a complete classification of irreducible admissible $(\mathfrak{g}, K)$-modules in the case $G_\sigma = \mathbf{PGL}_2(\mathbb{R})$. Since $K = \mathrm{SO}_2(\mathbb{R}) \simeq \mathbb{R}/2\pi\mathbb{Z}$, $V = \bigoplus_k V[k]$ where $\left(\begin{smallmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{smallmatrix}\right)$ acts by the character e^{ikt} on each $V[k]$, which are the K-isotypic components of V. It turns out that one can find an explicit basis, usually denoted by R, L, H, Z, for $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$, so that $RV[k] \subset V[k+2]$ and $LV[k] \subset V[k-2]$ i.e. R, L raise or lower k respectively, and the $V[k]$ have dimension one at most. Similar results hold for $\mathbf{PGL}_2(\mathbb{C})$, see [Mol22, §3].

After some considerations one shows there are at most three possibilities for V. Each possibility is then constructed explicitly: the *principal series*, the *discrete series*, and the finite-dimensional irreducible representations of $\mathbf{PGL}_2(\mathbb{R})$. These last are a twist by a 1-dimensional character of the space of homogeneous polynomials in two variables, with the action described in (1.9).

2.4.3 Local factors at ramified places

The above accounts for almost all places, but a quaternion algebra B over F is ramified (i.e. $B_v \neq M_2(F_v)$) in an even finite number of places of F. In this case the classification is simple. There are only finite dimensional representations: for the non archimedean places one dimensional representations induced by a character composed with the reduced norm of B (similar to the non archimedean kind 1 of §2.4.1), and for the archimedean places homogeneous polynomials in two variables with the $\mathbf{GL}_2(\mathbb{C})$-action given by (1.9). Here $B_\sigma^\times$ acts through the morphism $B_\sigma^\times \hookrightarrow (B_\sigma \otimes_\mathbb{R} \mathbb{C})^\times = \mathbf{GL}_2(\mathbb{C})$.

2.4.4 Definition of automorphic representation

Let I be a set and V_i vector spaces indexed by $i \in I$. Choose $v_{0,i} \in V_i$ for almost every $i \in I$. Then the *restricted tensor product* of the V_i with respect to the $v_{0,i}$ is the space

$$\bigotimes_{i\in I}{}' V_i := \{\otimes_{i\in I} v_i \text{ such that } v_i = v_{0,i} \text{ for almost every } i\}$$

An *irreducible admissible $G(\mathbb{A}_F)$-module* is a restricted tensor product

$$\pi = \bigotimes_v{}' \pi_v$$

where v runs over the places of F such that

- (π_∞, V_∞) is an irreducible admissible $(\mathfrak{g}_\infty, K_\infty)$-module
- For a finite place v, (π_v, V_v) is an irreducible admissible representation of $G(F_v)$.
- For almost every v, π_v is unramified, and the vectors $\pi_{0,v}$ are in $V_v^{K_v}$.

An (irreducible) *automorphic representation* is an irreducible admissible $G(\mathbb{A}_F)$-module isomorphic to a subquotient of $L^2(G(F)\backslash G(\mathbb{A}_F))$, the space of square-integrable functions in $G(F)\backslash G(\mathbb{A}_F)$.

Classical modular forms as automorphic forms

Classical modular forms are defined as certain functions on the complex upper half plane $\mathfrak{H}$. Recall that $\mathfrak{H} = \{z \in \mathbb{C} \text{ with } \Im(z) > 0\}$ and that $\mathbf{GL}_2(\mathbb{R})_+$ acts on $\mathfrak{H}$ by fractional lineal transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}$$

In fact, it is its isometry group with respect to the Poincaré metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. As a semisimple Lie group, $\mathbf{GL}_2(\mathbb{R})_+$ has an Iwasawa decomposition $\mathbf{GL}_2(\mathbb{R})_+ = KAN$ where $K = \left\{\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)\right\}$ is the stabilizer of i, $A = \left\{\left(\begin{smallmatrix} r & \\ & r^{-1} \end{smallmatrix}\right) \text{ with } r > 0\right\}$ and $N = \left\{\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right) \text{ where } x \in \mathbb{R}\right\}$. Modular forms are defined usually by satisfying a relation by elements of the discrete subgroup $\mathbf{SL}_2(\mathbb{Z}) \subset \mathbf{GL}_2(\mathbb{R})_+$, which tessellates $\mathfrak{H}$ by ideal triangles, or $\Gamma_0(N) \subset \mathbf{SL}_2(\mathbb{Z})$, the congruence subgroup of upper triangular matrices modulo N.

Let χ be a Dirichlet character modulo N, a morphism $(\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$. It can be extended to $\mathbb{Z}$ by setting $\chi(n) = 0$ when $(n, N) \neq 1$, and it induces a character on $\Gamma_0(N)$ by evaluating upper left entries i.e. $\chi(\gamma) = \chi(a)$ where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. A *modular form of weight k of level N and nebentypus χ* is an holomorphic function $f : \mathfrak{H} \longrightarrow \mathbb{C}$ that is holomorphic at infinity (i.e. having a Fourier expansion

$$f(z) = \sum_{n\geq 0} a_n q^n$$

where $q = \exp(2\pi i z)$) satisfying

$$f(\gamma z) = \chi(\gamma) \cdot j(\gamma, z)^k f(z) \text{ for all } \gamma \in \Gamma_0(N)$$

where $j(\gamma, z) = cz + d$ if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Note that $j(\gamma, z)$ satisfies the cocycle relation

$$j(\gamma\gamma', z) = j(\gamma, \gamma' z) j(\gamma', z)$$

A modular form is *cuspidal* if $a_0 = 0$. The set of cuspidal modular forms of weight k and level N form a finite dimensional $\mathbb{C}$-vector space $S_k(N, \chi)$.

One can regard classical modular forms as automorphic forms, by translating some the above objects to the adèles. The character χ has an associated idèlic character ω_χ of $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times$. We can define $K_0(N)$ as the subgroup of $\mathbf{GL}_2(\hat{\mathbb{Z}})$ of elements which are upper triangular modulo $N\hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. Then by the strong approximation theorem we can write any element $g \in \mathbf{GL}_2(\mathbb{A}_\mathbb{Q})$ as $g = \gamma h_\infty k$, with $\gamma \in \mathbf{GL}_2(\mathbb{Q}), h_\infty \in \mathbf{GL}_2(\mathbb{R})_+, k \in K_0(N)$, and the *adelization* of an element $f \in S_k(N, \chi)$ can be defined as a function $\varphi_f : \mathbf{GL}_2(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{C}$ given by

$$\varphi_f(g) = j(h_\infty, i)^{-k} f(h_\infty \cdot i)\, \omega_\chi(k)$$

It can be checked that φ_f is an automorphic form of $\mathbf{GL}_2$ with central character ω_χ.

Later on we will work with the automorphic representation π associated to an elliptic curve E; the local representation π_v at the archimedean places is a discrete series of weight 2, and at the non archimedean places is the principal series or the (possibly twisted) Steinberg representation depending on whether E has good or multiplicative reduction at v. The supercuspidal representation corresponds to the case of additive reduction.

2.5 Automorphic forms as cohomology classes

Thus in few words, an automorphic form of G is a function $G(\mathbb{A}_F) \to \mathbb{C}$ which is $G(F)$-invariant and has *good properties*. More precisely, let $\mathcal{A}(\mathbb{C})$ be the set of functions

$$f : G(\mathbb{A}_F) \to \mathbb{C} \tag{2.2}$$

which are

- right-U_f-invariant for some open subgroup $U_f \subset G(\mathbb{A}_F^\infty)$
- C^∞ when restricted to $G(F_\infty)$
- right-Z_σ-finite and right-K_σ-finite for all $\sigma \in \Sigma_F$

where Z_σ is the center of the universal enveloping algebra of $G(F_\sigma)$ and K_σ is the maximal compact subgroup of $G(F_\sigma)$.

We can obtain an admissible representation of $G(\mathbb{A}_F)$ by letting $G(\mathbb{A}_F)$ act by right translation on $\mathcal{A}(\mathbb{C})$, and if $G(F)$ acts by left translation instead then automorphic forms are just $G(F)$-invariant elements of $\mathcal{A}(\mathbb{C})$.

If we only want to work with forms which generate a fixed local representation or a fixed $(\mathfrak{g}_\infty, K_\infty)$-module at infinity, we can proceed as follows. Recall that a $(\mathfrak{g}_\infty, K_\infty)$-module is a tensor product

$$\mathcal{V} = \bigotimes_\sigma \mathcal{V}_\sigma$$

where $\mathcal{V}_\sigma$ is a $(\mathfrak{g}_\sigma, K_\sigma)$-module or a finite-dimensional $G(F_\sigma)$-representation, depending on whether $\sigma \in \Sigma_B$ or $\sigma \in \Sigma_F - \Sigma_B$. Then given a $(\mathfrak{g}_\infty, K_\infty)$-module $\mathcal{V}$ we define

$$\mathcal{A}^\infty(\mathcal{V}, \mathbb{C}) := \mathbf{Hom}_{(\mathfrak{g}_\infty, K_\infty)}(\mathcal{V}, \mathcal{A}(\mathbb{C}))$$

This notion can be further generalized, to fix representations at several places or change the coefficients of the forms. Let $H \subset G(F)$ be a subgroup, R a topological ring, and S a finite set of places of F above p. The subgroup H will usually be $G(F), G(F)_+, T(F)$ or $T(F)_+$. For any $R[H]$-modules M, N let

$$\mathcal{A}^{S\cup\infty}(M,N) := \left\{ \phi : G(\mathbb{A}_F^{S\cup\infty}) \to \mathbf{Hom}_R(M,N), \quad \begin{array}{c} \text{there exists an open compact} \\ \text{subgroup } U \subset G(\mathbb{A}_F^{S\cup\infty}) \\ \text{with } \phi(\cdot U) = \phi(\cdot) \end{array} \right\}, \tag{2.3}$$

and let also $\mathcal{A}^{S\cup\infty}(N) := \mathcal{A}^{S\cup\infty}(R, N)$. Then $\mathcal{A}^{S\cup\infty}(M,N)$ has a natural action of H and $G(\mathbb{A}_F^{S\cup\infty})$, namely

$$(h\phi)(x) := h \cdot \phi(h^{-1}x), \qquad (y\phi)(x) := \phi(xy),$$

where $h \in H$ and $x, y \in G(\mathbb{A}_F^{S\cup\infty})$.

As an example, let $\underline{k} \in 2\mathbb{Z}^d$ be a weight vector and consider the $(\mathfrak{g}_\infty, K_\infty)$-module $D(\underline{k}) = \bigotimes_\sigma D_\sigma(k_\sigma)$ where $D_\sigma(k_\sigma)$ is the discrete series of weight k_σ or $V(k_\sigma - 2)$, depending on whether $\sigma \in \Sigma_B$ or $\sigma \in \Sigma_F - \Sigma_B$ (see §3 of [Mol22]). Then an automorphic form of weight $\underline{k}$ defines a $G(F)$-invariant element of $\mathcal{A}^\infty(D(\underline{k}), \mathbb{C})$ that maps a generator of $D(\underline{k})$ to the automorphic form

$$\Phi \in H^0(G(F), \mathcal{A}^\infty(D(\underline{k}), \mathbb{C})) \tag{2.4}$$

Conversely, any such Φ defines an automorphic form of weight $\underline{k}$. If V is a finite-dimensional $G(F_\infty)$-representation then by lemma 2.3 of [GMM17]

$$\mathcal{A}^\infty(V,\mathbb{C}) = \mathcal{A}^\infty(\mathcal{V},\mathbb{C})$$

where $\mathcal{V}$ is the $(\mathfrak{g}_\infty, K_\infty)$-module associated with V.

2.6 Eichler-Shimura for automorphic forms

Overview

The Eichler-Shimura morphism can be seen as a method to regard automorphic forms as cohomological elements or cohomological modular symbols. Its first version establishes a correspondence between Γ-modular forms and the first cohomology group of Γ, where Γ is an arithmetic subgroup of $\mathbf{SL}_2(\mathbb{Z})$. In other words, the sheaf of (holomorphic) differential forms on $X(\Gamma)$ when regarded as a Riemann surface is isomorphic to the singular cohomology of $Y(\Gamma)$, since $X(\Gamma)$ is the compactification of $Y(\Gamma) = \mathfrak{H}/\Gamma$.

The classical argument is geometric, based on a careful study of the canonical structure of the modular curve $X_0(N)$ as a moduli space, its Jacobian and the Hecke algebra of $S_2(N)$. Moreover,

$$M_2(\Gamma) \oplus S_2(\Gamma) \simeq H^1(\Gamma,\mathbb{C})$$

For higher weight there is a natural generalization

$$\begin{array}{ccc} S_k(N) & \longrightarrow & H^1(\Gamma, V(k)) \\ f & \longmapsto & \left(\gamma \mapsto \left(P \mapsto \int_{z_0}^{\gamma z_0} P(1,-z) f(z) dz\right)\right) \end{array}$$

since $V(0) = \mathbf{Hom}_{\mathbb{C}}(\mathcal{P}(0)_{\mathbb{C}},\mathbb{C}) \simeq \mathbb{C}$.

This classical vision can be generalized to the automorphic case using group cohomology, as is done in [Mol17a] without relying on a moduli or geometric description. If $F = \mathbb{Q}$ the discrete series of weight k fits into the exact sequences

$$0 \longrightarrow D(k) \longrightarrow I(k)^\pm \longrightarrow V(k-2)^\pm \longrightarrow 0$$

where $I(k)^\pm$ is an appropriate induced representation and $\pm$ is a choice of sign. By (2.4), an automorphic form can be regarded as a $G(F)$-invariant element of $\mathcal{A}^\infty(D(k),\mathbb{C})$ i.e.

$$H^0\left(G(F), \mathcal{A}^\infty(D(k),\mathbb{C})\right)$$

Thus, the connecting morphism from the long exact sequence of group cohomology gives an element of $H^1(G(F)_+, \mathcal{A}^\infty(V(k-2)))^\pm$. For arbitrary F something similar happens, the discrete series of weight $\underline{k} \in \mathbb{Z}^d$ are the kernel of a sequence,

$$0 \longrightarrow D(k) \longrightarrow I_1(\underline{k})^\lambda \longrightarrow I_2(\underline{k})^\lambda \longrightarrow \cdots \longrightarrow I_s(\underline{k})^\lambda \longrightarrow V(\underline{k}-2)^\lambda \longrightarrow 0$$

where $V(\underline{k}-2) = \bigotimes_\sigma V(k_\sigma - 2)$, $s = \#\Sigma_B$, $I_i(\underline{k})^\lambda$ are appropriate induced representations, and $\lambda : G(F)/G(F)_+ \longrightarrow \pm 1$ is a character. The Eichler-Shimura morphism is regarded as the composition of connecting morphism of the underlying short exact sequences.

$$\mathrm{ES}^\lambda : H^0(G(F), \mathcal{A}^\infty(D(\underline{k}),\mathbb{C})) \to H^s(G(F)_+, \mathcal{A}^\infty(V(\underline{k}-2)))^\lambda.$$

2.7 Waldspurger's formula in higher cohomology

We will first recall Waldspurger's formula in its classic form. It can be regarded equivalently as providing an expression for the leading term of the L-function of an elliptic curve or an explicit relation between a toric period and local data. Although it has several modern counterparts, as the proposed generalization in Gan-Gross-Prasad conjectures, we use the one presented in [Mol22] for higher cohomology in the next sections.

Classic results and motivation

In equation 1.3 of the introduction, ε is called *the sign* of the functional equation. Note that $\varepsilon = \pm 1$ since applying (1.3) twice gives $\varepsilon^2 = 1$. Suppose now $F = \mathbb{Q}$ and that K/F is an imaginary quadratic field. Let $E/\mathbb{Q}$ be a modular elliptic curve of conductor N.

By considering the base change of E to K we obtain another sign $\varepsilon_K = \varepsilon(E_K)$ of the corresponding functional equation. Under *general Heegner hypothesis* on the conductor $N = N^+N^-$, that N^+ is a product of primes split in K and N^- is a product of primes inert in K and squarefree, we have

$$\varepsilon\left(E_K\right) = -(-1)^{\#\{p|N^-\}}$$

and for each case there are two well-known formulas that express the first non-trivial coefficient of the L-series of E.

- If $\varepsilon_K = -1$, the even terms of the Taylor series are 0 thus the first non-trivial coefficient is the value of $L'(E_K, 1)$. By considering the Shimura curve $X(N^+, N^-)$ associated to the quaternion algebra that ramifies precisely at N^-, a product of an even number of primes inert in K, the Yuan-Zhang-Zhang generalization of the Gross-Zagier formula shows $L'(E_K, 1)$ is related to Néron-Tate height of a CM point in a Shimura curve,

$$L'\left(E_K, 1\right) = \frac{(f, f)}{|d_K|^{\frac{1}{2}}\left|\mathcal{O}_K^{\times}/\{\pm 1\}\right|^2} \cdot \frac{\left\langle \varphi y_K, \varphi y_K\right\rangle_{\mathrm{NT}}}{\deg \varphi}$$

  where $\varphi : X(N^+, N^-) \longrightarrow E$ is a modular parametrization and y_K is the $\mathrm{Gal}(H_K/K)$-trace of a CM point of $X(N^+, N^-)$. Recall that φy_K is the trace of a Heegner point

- If $\varepsilon_K = +1$, Waldspurger showed that $L(E_K, 1)$ is related very explicitly to CM points in a Shimura set, which in this case is associated to a quaternion algebra which ramifies precisely at N^- and ∞, as follows

$$L\left(E_K, 1\right) = \frac{(f, f)}{|d_K|^{\frac{1}{2}}\left|\mathcal{O}_K^{\times}/\{\pm 1\}\right|^2} \cdot \mathcal{P}^2$$

  where now $\mathcal{P}$ is *Waldspurger's toric period.*

Automorphic analogues

In the automorphic side, Waldspurger's formula relates certain periods with the critical value of an automorphic L-function. Recall our setup: a number field F, a quadratic extension K/F, a quaternion algebra B/F with $K \hookrightarrow B$ and an irreducible automorphic representation π on $G = B^\times/F^\times$. Given a character χ on $T(F) = K^\times/F^\times$, for any $\phi \in \pi$ we can consider the following toric period

$$\mathcal{P}(\phi, \chi) := \int_{T(\mathbb{A}_F)/T(F)} \chi(t)\phi(t)d^\times t$$

Note that since ϕ is a function of $G(\mathbb{A}_F)$ it can be restricted to the idèles of K, and that the integral does not depend on the choice of a fundamental domain since $\chi(t)\phi(t)$ is $K^\times$-invariant. Note that we have a linear functional

$$\mathcal{P} \in \mathbf{Hom}_{T(\mathbb{A}_F)}(\pi \otimes \chi, \mathbb{C}) = \prod_v{}' \mathbf{Hom}_{T(F_v)}(\pi_v \otimes \chi_v, \mathbb{C})$$

The following results of Tunnel and Saito provide key information on the local spaces $\mathbf{Hom}_{T(F_v)}(\pi_v \otimes \chi_v, \mathbb{C})$:

Theorem 2.7.1. *We have multiplicity one*

$$\dim_{\mathbb{C}} \mathbf{Hom}_{T(F_v)}(\pi_v \otimes \chi_v, \mathbb{C}) \leq 1$$

and if for any π_v^0 irreducible representation of $\boldsymbol{PGL}_2(F_v)$, π_v^1 is its Jacquet-Langlands transfer to $B_v^\times/F_v^\times$ then

$$\dim_{\mathbb{C}} \mathbf{Hom}_{T(F_v)}(\pi_v^0 \otimes \chi_v, \mathbb{C}) + \dim_{\mathbb{C}} \mathbf{Hom}_{T(F_v)}(\pi_v^1 \otimes \chi_v, \mathbb{C}) = 1$$

Suppose now χ is trivial. If π comes from an elliptic curve $E/\mathbb{Q}$ of conductor N then it can be shown that $\pi_v^1 = 0$ when $v \nmid N$, and if $v|N$ then $\dim_{\mathbb{C}} \mathbf{Hom}_{T(F_v)}(\pi_v^0 \otimes \chi_v, \mathbb{C}) = 1$ if $v|N^+$ and $\dim_{\mathbb{C}} \mathbf{Hom}_{T(F_v)}(\pi_v^1 \otimes \chi_v, \mathbb{C}) = 1$ if $v|N^-$. Together these imply that $\dim \mathbf{Hom}_T(\mathbb{A}_F)(\pi \otimes \chi, \mathbb{C}) = 1$. Waldspurger formula then provides explicit expression for the "ratio" of two elements in a one dimensional space: for an appropriate choice of the Haar measure it asserts

Theorem 2.7.2 (Waldspurger)**.** *For some explicit constant C,*

$$\mathcal{P}(\phi, \chi)^2 = C \cdot L(\pi, \chi, 1/2) \cdot \prod_v \alpha_v \left(\phi_v, \chi_v\right)$$

This relates $\mathcal{P}^2$ with a critical value of the classical L-function associated to the automorphic representation π twisted by χ and the local factors α_v,

$$\alpha_v(\phi_v, \chi_v) := \int_{T(F_v)} \langle \pi_v(h)\phi_v, \phi_v \rangle \chi_v(h)dh \tag{2.5}$$

which are 1 for almost every v. Note that in some cases, depending on whether the central character of π and χ coincide, we may obtain a trivial equality $0 = 0$. Again when $\chi = 1$ and $F = \mathbb{Q}$, the value $L(\pi, \chi, 1/2)$ coincides with $L(E_K, 1)$. In fact

Corollary 2.7.3.

$$\mathcal{P}(\phi, \chi) \neq 0 \text{ if and only if } \mathbf{Hom}_{T(\mathbb{A}_F)}(\pi \otimes \chi) \neq 0 \text{ and } L(\pi \otimes \chi, 1/2) \neq 0$$

2.8 Automorphic cohomology classes

Let E/F be a modular elliptic curve. Hence, attached to E, we have an automorphic form for $\mathbf{PGL}_2/F$ of parallel weight 2. Let us assume that such form admits a Jacquet-Langlands lift to G, and denote by π the corresponding automorphic representation. Let $s := \#\Sigma_B$. As shown in [Mol17b] and overviewed in §2.6, once fixed a character $\lambda : G(F)/G(F)_+ \to \pm 1$, the image through the Eichler-Shimura isomorphism of such Jacquet-Langlands lift provides a cohomology class in $H^s(G(F)_+, \mathcal{A}^\infty(\mathbb{C}))^\lambda$, where the super-index λ stands for the subspace where the natural action of $G(F)/G(F)_+$ on the cohomology groups is given by the character λ. Moreover, since the coefficient ring of the automorphic forms is $\mathbb{Z}$, such a class is the extension of scalars of a class

$$\phi_\lambda \in H^s(G(F)_+, \mathcal{A}^\infty(\mathbb{Z}))^\lambda.$$

Indeed, $H^s(G(F)_+, \mathcal{A}^\infty(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{C} = H^s(G(F)_+, \mathcal{A}^\infty(\mathbb{C}))$ (see for example [Spi14, Proposition 4.6]). For general automorphic forms of arbitrary even weight $(\underline{k}+2) \in (2\mathbb{N})^d$, the Eichler-Shimura morphism provides a class

$$\phi_\lambda \in H^s(G(F)_+, \mathcal{A}^\infty(V(\underline{k})_{\overline{\mathbb{Q}}}))^\lambda.$$

The $G(\mathbb{A}_F^\infty)$-representation ρ over $\overline{\mathbb{Q}}$ generated by ϕ_λ satisfies $\rho \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \simeq \pi^\infty := \pi \mid_{G(\mathbb{A}_F^\infty)}$. This implies that ϕ_λ defines an element

$$\Phi_\lambda \in H^s(G(F)_+, \mathcal{A}^\infty(V(\underline{k})))^\lambda_{\pi^\infty}.$$

For any set S of places above p, write $V_S := \rho \mid_{G(F_S)}$, with $V_S = \bigotimes_{\mathfrak{p}\in S} V_{\mathfrak{p}}$, and for any ring R we denote by $V_S^R = \bigotimes V_{\mathfrak{p}}^R$ the R-module generated by ϕ_λ. By [GMM17, Remark 2.1] we have that

$$\Phi_\lambda \in H^s(G(F)_+, \mathcal{A}^\infty(V(\underline{k})))^\lambda_{\pi^\infty} = H^s(G(F)_+, \mathcal{A}^{S\cup\infty}(V_S, V(\underline{k})))^\lambda_{\pi^\infty}. \tag{2.6}$$

For any $x^S \in \rho^S := \rho \mid_{G(\mathbb{A}_F^{S\cup\infty})}$, the image $\Phi_\lambda(x^S)$ defines an element

$$\phi_\lambda^S \in H^s(G(F)_+, \mathcal{A}^{S\cup\infty}(V_S, V(\underline{k})_{\overline{\mathbb{Q}}}))^\lambda.$$

We will usually treat ϕ_λ^S as an element of the cohomology group $H^s(G(F)_+, \mathcal{A}^{S\cup\infty}(V_S, V(\underline{k})))^\lambda$ since

$$H^s(G(F)_+, \mathcal{A}^{S\cup\infty}(V_S^{\overline{\mathbb{Z}}}, V(\underline{k})_{\overline{\mathbb{Q}}})) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} = H^s(G(F)_+, \mathcal{A}^{S\cup\infty}(V_S, V(\underline{k}))),$$

again by [Spi14, Proposition 4.6]. The classes ϕ_λ^S are essential in our construction of anti-cyclotomic p-adic L-functions, and plectic points.

If ϕ^λ is associated with an elliptic curve as above and $x^S \in \rho^S$ is now an element of $\mathbb{Z}$-module generated by the translations of ϕ_λ, we can think ϕ_λ^S as an element

$$\phi_\lambda^S \in H^s(G(F)_+, \mathcal{A}^{S\cup\infty}(V_S^{\mathbb{Z}}, \mathbb{Z}))^\lambda,$$

since the coefficient ring of π is $\mathbb{Z}$. Similarly, we will sometimes regard ρ as a $\mathbb{Q}$-representation.

Chapter 3

The fundamental class

3.1 Fundamental classes of tori

In this section we define certain fundamental classes associated with the torus T. Their definition is a generalization to the one defined in [GMM17]; in the case of Darmon points only one v-uniformization is needed, but for plectic points we will need to account for several places $\mathfrak{p}$ above one fixed prime p.

3.1.1 The fundamental class η of the torus

Let $U = \prod_{\mathfrak{q}} T(\mathcal{O}_{F_{\mathfrak{q}}})$ and denote by $\mathcal{O} := T(F) \cap U$ the group of relative units. Similarly, we define $\mathcal{O}_+ := \mathcal{O} \cap T(F_\infty)_+$ to be the group of totally positive relative units. By an straightforward argument using Dirichlet Units Theorem

$$\mathrm{rank}_{\mathbb{Z}}\mathcal{O} = \mathrm{rank}_{\mathbb{Z}}\mathcal{O}_+ = (2r_{K/F,\mathbb{R}} + r_{K/F,\mathbb{C}} + 2s_F - 1) - (r_F + s_F - 1) = r_{K/F,\mathbb{R}} + s_F = u.$$

We also define the class group

$$\mathrm{Cl}(T)_+ := T(\mathbb{A}_F)/(T(F) \cdot U \cdot T(F_\infty)_+) \simeq T(\mathbb{A}_F^\infty)/(T(F)_+ \cdot U)\,.$$

Note that the subgroup

$$T(F_{\Sigma_{\mathrm{un}}(K/F)})_0 = \mathbb{R}_+^u = (\mathbb{R}_+)^{\#\Sigma_{\mathbb{R}}^{\mathbb{R}}(K/F)} \times (\mathbb{R}_+)^{\#\Sigma_{\mathbb{C}}^{\mathbb{C}}(K/F)} \subset T(F_{\Sigma_{\mathrm{un}}(K/F)}) = (\mathbb{R}^\times)^{\#\Sigma_{\mathbb{R}}^{\mathbb{R}}(K/F)} \times (\mathbb{C}^\times)^{\#\Sigma_{\mathbb{C}}^{\mathbb{C}}(K/F)}$$

is isomorphic to $\mathbb{R}^u$ by means of the homomorphism $T(F_\infty)_+ \longrightarrow \mathbb{R}^u$ given by $z \mapsto (\log|\sigma z|)_{\sigma \in \Sigma_{\mathrm{un}}(K/F)}$. Moreover, under this isomorphism the image of $\mathcal{O}_+$ is a $\mathbb{Z}$-lattice $\Lambda \subset \mathbb{R}^u$, as in the proof of Dirichlet's Unit Theorem.

We can identify Λ with its preimage in $T(F_{\Sigma_{\mathrm{un}}(K/F)})_0$. Then $T(F_{\Sigma_{\mathrm{un}}(K/F)})_0/\Lambda$ is a u-dimensional real torus. The *fundamental class* ξ is a generator of $H_u(T(F_{\Sigma_{\mathrm{un}}(K/F)})_0/\Lambda, \mathbb{Z}) \simeq \mathbb{Z}$. We can give a better description of ξ: let $M := T(F_{\Sigma_{\mathrm{un}}(K/F)})_0 \simeq \mathbb{R}^u$. The de Rham complex $\Omega_M^\bullet$ is a resolution for $\mathbb{R}$. This implies that we have an edge morphism of the corresponding spectral sequence

$$e : H^0(\Lambda, \Omega_M^u) \to H^u(\Lambda, \mathbb{R}).$$

We identify any $c \in H_u(M/\Lambda, \mathbb{Z})$ with $c \in H_u(\Lambda, \mathbb{Z})$ by means of the relation

$$\int_c \omega = e(\omega) \cap c, \text{ where } \omega \in H^0(\Lambda, \Omega_M^u) = \Omega_{M/\Lambda}^u.$$

We can think of $\xi \in H_u(\Lambda, \mathbb{Z})$ as such that:

$$e(\omega) \cap \xi = \int_{T(F_\infty)/\Lambda} \omega, \text{ for all } \omega \in H^0(\Lambda, \Omega_M^u). \tag{3.1}$$

Note that

$$T(F_\infty)_+ = \mathbb{T} \times T(F_{\Sigma_{\rm un}(K/F)})_0, \qquad \mathbb{T} := (S^1)^{\#\Sigma_{\mathbb{R}}^{\mathbb{C}}(K/F)+\#\Sigma_{\mathbb{C}}^{\mathbb{C}}(K/F)}. \tag{3.2}$$

Let $\mathcal{O}_{+,\mathbb{T}} = \mathcal{O}_+ \cap \mathbb{T}$. Since $\mathcal{O}_+$ is discrete and $\mathbb{T}$ is compact, $\mathcal{O}_{+,\mathbb{T}}$ is finite.

Lemma 3.1.1. *We have that $\mathcal{O}_+ \simeq \Lambda \times \mathcal{O}_{+,\mathbb{T}}$. In particular,*

$$T(F_\infty)_+/\mathcal{O}_+ \simeq T(F_{\Sigma_{\rm un}(K/F)})_0/\Lambda \times \mathbb{T}/\mathcal{O}_{+,\mathbb{T}} \tag{3.3}$$

Proof. The image of the morphism $\pi|_{\mathcal{O}_+} : \mathcal{O}_+ \hookrightarrow T(F_\infty)_+ \xrightarrow{\pi} T(F_{\Sigma_{\rm un}(K/F)})_0$ is Λ by definition. Since $\mathbb{T} = \ker \pi$, $\ker \pi|_{\mathcal{O}_+} = \mathbb{T} \cap \mathcal{O}_+ = \mathcal{O}_{+,\mathbb{T}}$ and we deduce the following exact sequence

$$0 \longrightarrow \mathcal{O}_{+,\mathbb{T}} \longrightarrow \mathcal{O}_+ \xrightarrow{\pi|_{\mathcal{O}_+}} \Lambda \longrightarrow 0.$$

Since Λ is a free $\mathbb{Z}$-module, such sequence splits and the result follows. □

The group $\mathrm{Cl}(T)_+$ fits in the following exact sequence

$$0 \longrightarrow T(F)_+/\mathcal{O}_+ \to T(\mathbb{A}_F^\infty)/U \xrightarrow{p} \mathrm{Cl}(T)_+ \longrightarrow 0.$$

We fix preimages $\bar{t}_i \in T(\mathbb{A}_F^\infty)$ for every element $t_i \in \mathrm{Cl}(T)_+$ and we consider the compact set

$$\mathcal{F} = \bigcup_i \bar{t}_i U \subset T(\mathbb{A}_F^\infty)$$

It is compact because $\mathrm{Cl}(T)_+$ is finite and U compact.

Lemma 3.1.2. *For any $t \in T(\mathbb{A}_F)$ there exists a unique $\tau_t \in T(F)/\mathcal{O}_+$ such that ${\tau_t}^{-1} t \in T(F_\infty)_+ \times \mathcal{F}$.*

Proof. Since $T(F_\infty)/T(F_\infty)_+ = T(F)/T(F)_+$, given $t = (t_\infty, t^\infty) \in T(\mathbb{A}_F)$ there exists $\gamma \in T(F)$ such that $\gamma t_\infty \in T(F_\infty)_+$. On the other hand, $p(\gamma t^\infty U) = t_i$ for some i. By the definition of $\mathrm{Cl}(T)_+$, there exists $\tau \in T(F)_+$ such that $\tau\gamma t^\infty U = \bar{t}_i U$. Hence

$$\tau\gamma t = (\tau\gamma t_\infty, \tau\gamma t^\infty) \in T(F_\infty)_+ \times \bar{t}_i U \subset T(F_\infty)_+ \times \mathcal{F}.$$

By considering the image of $\tau^{-1}\gamma^{-1}$ in $T(F)/\mathcal{O}_+$, we deduce the existence of τ_t.

For the unicity, suppose there exist $\tau, \tau' \in T(F)$ with

$$(\tau t_\infty, \tau t^\infty) = \tau t \in T(F_\infty)_+ \times \mathcal{F} \ni \tau' t = (\tau' t_\infty, \tau' t^\infty)$$

Then $\tau^{-1}\tau' = (\tau t_\infty)^{-1}(\tau' t_\infty) \in T(F_\infty)_+$. On the other hand, $\tau t^\infty = \bar{t}_i u$ for some $\bar{t}_i$ and $u \in U$, and $\tau' t^\infty = \bar{t}_j u'$ for some $\bar{t}_j$ and $u' \in U$. So $\tau^{-1}\tau' = (\bar{t}_i)^{-1}\bar{t}_j u^{-1} u'$. This implies $t_i = p(\bar{t}_i U) = p(\bar{t}_j U) = t_j$. By the construction of $\mathcal{F}$, we must have $\bar{t}_i = \bar{t}_j$. Then $\tau^{-1}\tau' \in U \cap T(F) \cap T(F_\infty)_+ = \mathcal{O}_+$. □

The set of continuous functions $C(\mathcal{F}, \mathbb{Z})$ has a natural action of $\mathcal{O}_+$ given by translation. The characteristic function $\mathbb{1}_{\mathcal{F}}$ is $\mathcal{O}_+$-invariant. Consider the cap product

$$\eta = \mathbb{1}_{\mathcal{F}} \cap \xi \in H_u(\mathcal{O}_+, C(\mathcal{F}, \mathbb{Z})), \tag{3.4}$$

where $\mathbb{1}_{\mathcal{F}} \in H^0(\mathcal{O}_+, C(\mathcal{F}, \mathbb{Z}))$ is the indicator function and $\xi \in H_u(\mathcal{O}_+, \mathbb{Z})$ is the image of ξ through the corestriction morphism.

For any ring R and any R-module M, write $C_c^0(T(\mathbb{A}_F), M)$ for the set of locally constant M-valued functions of $T(\mathbb{A}_F)$ that are compactly supported when restricted to $T(\mathbb{A}_F^\infty)$. If M is endowed with a natural $T(F)$-action, then we have a natural action of $T(F)$ on $C_c^0(T(\mathbb{A}_F), M)$.

Lemma 3.1.3. *There is an isomorphism of $T(F)$-modules*

$$\mathrm{Ind}_{\mathcal{O}_+}^{T(F)}(C(\mathcal{F}, \mathbb{Z})) \simeq C_c^0(T(\mathbb{A}_F), \mathbb{Z}).$$

Proof. Since $\mathcal{F} \subset T(\mathbb{A}_F)$ is compact there is an $\mathcal{O}_+$-equivariant embedding

$$\begin{aligned} \iota : C(\mathcal{F}, \mathbb{Z}) &\hookrightarrow C_c^0(T(\mathbb{A}_F), \mathbb{Z}) \\ \phi &\longmapsto (\iota\phi)(t_\infty, t^\infty) = \mathbb{1}_{T(F_\infty)_+}(t_\infty) \cdot \phi(t^\infty) \cdot \mathbb{1}_{\mathcal{F}}(t^\infty). \end{aligned}$$

By definition, an element $\Phi \in \mathrm{Ind}_{\mathcal{O}_+}^{T(F)}(C(\mathcal{F}, \mathbb{Z}))$ is a function $\Phi : T(F) \to C(\mathcal{F}, \mathbb{Z})$ finitely supported in $T(F)/\mathcal{O}_+$, and satisfying the compatibility condition $\Phi(t\lambda) = \lambda^{-1} \cdot \Phi(t)$ for all $\lambda \in \mathcal{O}_+$.

We define the morphism

$$\begin{aligned} \varphi : \mathrm{Ind}_{\mathcal{O}_+}^{T(F)}(C(\mathcal{F}, \mathbb{Z})) &\longrightarrow C_c^0(T(\mathbb{A}_F), \mathbb{Z}) \\ \Phi &\longmapsto \varphi(\Phi) = \textstyle\sum_{t \in T(F)/\mathcal{O}_+} t \cdot \iota(\Phi(t)). \end{aligned}$$

The sum is finite because Φ is finitely supported, and from the $\mathcal{O}_+$-equivariance of ι and the $\mathcal{O}_+$-compatibility it follows that it is well-defined and $T(F)$-equivariant.

By lemma 3.1.2, for all $x \in T(\mathbb{A}_F)$ there exists a unique $\tau_x \in T(F)/\mathcal{O}_+$ such that ${\tau_x}^{-1} x \in T(F_\infty)_+ \times \mathcal{F}$. Hence

$$\varphi(\Phi)(x) = \iota(\Phi(\tau_x))({\tau_x}^{-1} x) \tag{3.5}$$

Then φ is bijective because

- It is injective: Suppose $\varphi(\Phi) = 0$. Since $\tau_x = \tau_{xy}$ for all $y \in T(F_\infty)_+ \times \mathcal{F}$, by (3.5)

  $$\varphi(\phi)(xy) = \iota(\Phi(\tau_{xy}))({\tau_{xy}}^{-1} xy) = \iota(\Phi(\tau_x))({\tau_x}^{-1} xy) = 0$$

  Thus $\iota(\Phi(\tau_x)) = 0$ is identically zero for all $x \in T(\mathbb{A}_F)$. Then $\Phi(t) = 0$ for all $t \in T(F)$ because ι is injective and $\tau_t = t$ for all $t \in T(F)$.
- It is surjective: Let $\phi \in C_c^\varnothing(T(\mathbb{A}_F), \mathbb{Z})$. By lemma 3.1.2

  $$T(\mathbb{A}_F) = \bigsqcup_{t \in T(F)/\mathcal{O}_+} t(T(F_\infty)_+ \times \mathcal{F})$$

  Let

  $$\Phi(t)(x^\infty) := \phi(t, tx^\infty) = (t^{-1} \cdot \phi)(1, x^\infty) \in C(\mathcal{F}, \mathbb{Z})$$

  where $t \in T(F)$. There are finitely many $t(T(F_\infty)_+ \times \mathcal{F})$ in the support of ϕ, by lemma 3.1.2 and because ϕ has compact support. Thus $\Phi(t) = 0$ except for finitely many t. Note ϕ is $T(F_\infty)_+$-invariant since it is $\mathbb{Z}$-valued and $T(F_\infty)_0$-invariant. For $x \in T(\mathbb{A}_F)$

$$\varphi(\Phi)(x) = \sum_{t\in T(F)/O_+} t\cdot \iota(\Phi(t))(x) = \sum_{t\in T(F)/O_+} \mathbb{1}_{T(F_\infty)_+}(t^{-1}x_\infty)\cdot \Phi(t)(t^{-1}x^\infty)\cdot \mathbb{1}_{\mathcal{F}}(t^{-1}x^\infty)$$
$$= \sum_{t\in T(F)/O_+} \phi(t,x^\infty)\cdot \mathbb{1}_{t(T(F_\infty)_+\times\mathcal{F})}(x) = \phi(x)$$

Since φ is bijective the result follows. □

Thus, by Shapiro's lemma one may regard

$$\eta \in H_u(T(F), C_c^0(T(\mathbb{A}_F),\mathbb{Z})).$$

The S-fundamental classes η^S

In 3.1.1 we have defined a fundamental class η. As shown in [Mol22], by means of η we can compute certain periods related with certain critical values of classical L-functions. Nevertheless in [GMM17] different fundamental classes $\eta^{\mathfrak{p}}$ are defined in order to construct Darmon points. We devote this section to slightly generalize the definition of $\eta^{\mathfrak{p}}$ in [GMM17] and we will relate η and $\eta^{\mathfrak{p}}$ in the next section.

Let S be a set of places $\mathfrak{p}$ of F above p. Write $S := S^1 \cup S^2$, being S^1 the set of places $\mathfrak{p}$ in S where T splits, $T(F_{\mathfrak{p}}) = F_{\mathfrak{p}}^\times$, and S^2 the set of places where T does not split. We consider:

$$\mathcal{F}^S = \textstyle\bigsqcup_i \bar{s}_i U^S, \qquad U^S = T(\hat{O}_F^S), \qquad \hat{O}_F^S = \prod_{\mathfrak{q}\notin S} O_{F_{\mathfrak{q}}}, \tag{3.6}$$

where $\bar{s}_i \in T(\mathbb{A}_F^{S\cup\infty})$ are representatives of the elements of $\mathrm{Cl}(T)_+^S$ and

$$\mathrm{Cl}(T)_+^S = T(\mathbb{A}_F^{S\cup\infty})/\left(U^S T(F)_+\right). \tag{3.7}$$

Let us consider also the set of totally positive relative S-units $O_+^S := U^S \cap T(F)_+$. Note that we have an exact sequence

$$0 \longrightarrow T(F_S)/O_+^S T(O_{F,S}) \longrightarrow \mathrm{Cl}(T)_+ \longrightarrow \mathrm{Cl}(T)_+^S \longrightarrow 0 \tag{3.8}$$

where $O_{F,S} := \prod_{\mathfrak{p}\in S} O_{F_{\mathfrak{p}}}$.

It is clear that $\mathbb{Z}$-rank of the quotient $T(F_S)/T(O_{F,S})$ is $r = \#S^1$. Moreover, we have the natural exact sequence

$$0 \longrightarrow O_+ \longrightarrow O_+^S \longrightarrow T(F_S)/T(O_{F,S}) \tag{3.9}$$

Note that (3.9) and (3.8) imply that we have an exact sequence

$$0 \longrightarrow O_+ \longrightarrow O_+^S \longrightarrow T(F_S)/T(O_{F,S}) \longrightarrow \mathrm{Cl}(T)_+ \longrightarrow \mathrm{Cl}(T)_+^S \longrightarrow 0 \tag{3.10}$$

This implies that the $\mathbb{Z}$-rank of O_+^S is $r+u$, since $\mathrm{Cl}(T)_+$ is finite. Write $H^S := O_+^S/O_+$. The free part Λ_H^S of H^S provides a fundamental class $c^S \in H_r(H^S,\mathbb{Z})$. Indeed, the map

$$(v_{\mathfrak{p}})_{\mathfrak{p}\in S^1} : H^S \longrightarrow \mathbb{R}^r$$

given by the p-adic valuations identifies Λ_H^S as a lattice in $\mathbb{R}^r$, hence we can proceed as in the previous section to define c^S. We can consider the image $\xi^S \in H_{u+r}(\mathcal{O}_+^S, \mathbb{Z})$ of c^S through the composition

$$c^S \in H_r(H^S, \mathbb{Z}) \xrightarrow{1 \mapsto \xi} H_r(H^S, H_u(\mathcal{O}_+, \mathbb{Z})) \longrightarrow H_{u+r}(\mathcal{O}_+^S, \mathbb{Z}),$$

where $\xi \in H_u(\mathcal{O}_+, \mathbb{Z})$ is the fundamental class defined previously, and the last arrow is the edge morphism of the Lyndon–Hochschild–Serre spectral sequence $H_p(H^S, H_q(\mathcal{O}_+, \mathbb{Z})) \Rightarrow H_{p+q}(\mathcal{O}_+^S, \mathbb{Z})$. We have, similarly as in Lemma 3.1.3,

$$C_c^0(T(\mathbb{A}_F^{S\cup\infty}), \mathbb{Z}) = \operatorname{Ind}_{\mathcal{O}_+^S}^{T(F)_+} C(\mathcal{F}^S, \mathbb{Z}),$$

where $C_c^0(T(\mathbb{A}_F^{S\cup\infty}), M)$ is the set of M-valued locally constant and compactly supported functions of $T(\mathbb{A}_F^{S\cup\infty})$. Thus, the cap product $\mathbb{1}_{\mathcal{F}^S} \cap \xi^S$ provides an element

$$\eta^S := \mathbb{1}_{\mathcal{F}^S} \cap \xi^S \in H_{u+r}(\mathcal{O}_+^S, C(\mathcal{F}^S, \mathbb{Z})) \overset{\text{(a)}}{=} H_{u+r}(T(F)_+, C_c^0(T(\mathbb{A}_F^{S\cup\infty}), \mathbb{Z})), \tag{3.11}$$

where (a) follows from Shapiro's Lemma.

Relation between fundamental classes

Since $T(F_\infty)/T(F_\infty)_+ \simeq T(F)/T(F)_+$, there is a $T(F)$-equivariant isomorphism

$$\varphi : C_c^0(T(\mathbb{A}_F), M) \longrightarrow \operatorname{Ind}_{T(F)_+}^{T(F)} C_c^0(T(\mathbb{A}_F^\infty), M) := C_c^0(T(\mathbb{A}_F^\infty), M) \otimes_{R[T(F)_+]} R[T(F)] \tag{3.12}$$

given by $\varphi(f) := \sum_{\bar{t} \in T(F)/T(F)_+} \left((t^{-1} \cdot f)|_{T(\mathbb{A}^\infty)} \otimes t\right)$. Moreover, for any ring R

$$C_c^0(T(\mathbb{A}_F^\infty), R) \simeq C_c^0(T(\mathbb{A}_F^{S\cup\infty}), R) \otimes_R C_c^0(T(F_S), R) \simeq C_c^0(T(\mathbb{A}_F^{S\cup\infty}), C_c^0(T(F_S), R)), \tag{3.13}$$

where $C_c^0(T(F_S), R)$ is the set of R-valued locally constant and compactly supported functions on $T(F_S)$, seen as $T(F)$-module by means of the diagonal embedding $T(F) \hookrightarrow T(F_S)$. Putting these two identities together we obtain

$$C_c^0(T(\mathbb{A}_F), \mathbb{Z}) = \operatorname{Ind}_{T(F)_+}^{T(F)} \left(C_c^0(T(\mathbb{A}_F^{S\cup\infty}), \mathbb{Z}) \otimes_{\mathbb{Z}} C_c^0(T(F_S), \mathbb{Z})\right).$$

The following result relates the previously defined fundamental classes (see also [BG18, Lemma 1.4]):

Lemma 3.1.4. *We have that*

$$\#(H_{\mathrm{tor}}^S) \cdot \eta = \eta^S \cap \bigcup_{\mathfrak{p} \in S} \operatorname{res}_{T(F)_+}^{T(F_\mathfrak{p})} z_\mathfrak{p}$$

in $H_u\left(T(F)_+, C_c^0(T(\mathbb{A}_F^{S\cup\infty}), \mathbb{Z}) \otimes_{\mathbb{Z}} C_c^0(T(F_S), \mathbb{Z})\right) = H_u(T(F), C_c^0(T(\mathbb{A}_F), \mathbb{Z}))$*, where* H_{tor}^S *for the torsion subgroup of* H^S*,*

$$z_\mathfrak{p} = \mathbb{1}_{T(F_\mathfrak{p})} \in H^0(T(F_\mathfrak{p}), C_c^0(T(F_\mathfrak{p}), \mathbb{Z})), \qquad \text{if } \mathfrak{p} \in S^2,$$

and $z_\mathfrak{p} \in H^1(T(F_\mathfrak{p}), C_c^0(T(F_S), \mathbb{Z}))$ *is the class associated with the exact sequence*

$$0 \longrightarrow C_c^0(T(F_\mathfrak{p}), \mathbb{Z})) \longrightarrow C_c^0(F_\mathfrak{p}, \mathbb{Z})) \xrightarrow{f \mapsto f(0)} \mathbb{Z} \longrightarrow 0,$$

if $\mathfrak{p} \in S^1$*.*

Proof. Note that on the one hand that, if $\mathfrak{p} \in S^1$, the class $z_\mathfrak{p}$ has representative $z_\mathfrak{p}(t) = \mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}} - t\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}$. Hence $z_\mathfrak{p}(t)$ is characterized to be the function such that $\sum_{n\in\mathbb{Z}} t^n z_\mathfrak{p}(t) = \mathbb{1}_{T(F_\mathfrak{p})}$, for all $t \in T(F_\mathfrak{p})$ with $v_\mathfrak{p}(t) > 0$. Thus, $\left(\bigotimes_{\mathfrak{p}\in S} z_\mathfrak{p}\right)(c^S)$ is characterized so that

$$\sum_{\gamma\in\Lambda_H^S} \gamma \left(\bigotimes_{\mathfrak{p}\in S} z_\mathfrak{p}\right)(c^S) = \mathbb{1}_{T(F_S)}.$$

On the other hand, the exact sequence (3.8) implies

$$0 \longrightarrow H^S \longrightarrow T(F_S)/T(\mathcal{O}_{F,S}) \longrightarrow W \longrightarrow 0$$

where $W := \ker\left(\mathrm{Cl}(T)_+ \to \mathrm{Cl}(T)_+^S\right)$, and note that

$$\mathcal{F} = \bigsqcup_{\bar{s}_i\in\mathrm{Cl}(T)_+} s_i U = \bigsqcup_{\bar{t}_j\in W} \bigsqcup_{\bar{s}_k\in\mathrm{Cl}(T)_+^S} s_k U^S \times t_j T(\mathcal{O}_{F,S}) = \bigsqcup_{\bar{t}_j\in W} \mathcal{F}^S \times t_j T(\mathcal{O}_{F,S}).$$

Hence

$$\sum_{\beta\in\Lambda_H^S} \beta \sum_{\bar{\gamma}\in H_{\mathrm{tor}}^S} \gamma\cdot\mathbb{1}_{\mathcal{F}} = \sum_{\bar{\alpha}\in H^S} \alpha\cdot\left(\sum_{\bar{t}_j\in T(F_\mathfrak{p})/T(\mathcal{O}_{F,S})\mathcal{O}_+^S} \mathbb{1}_{\mathcal{F}^S}\otimes t_j\mathbb{1}_{T(\mathcal{O}_{F,S})}\right) = \mathbb{1}_{\mathcal{F}^S}\otimes\mathbb{1}_{T(F_S)},$$

and so $\sum_{\bar{\gamma}\in H_{\mathrm{tor}}^S} \gamma\cdot\mathbb{1}_{\mathcal{F}} = \mathbb{1}_{\mathcal{F}^S}\otimes\left(\bigotimes_{\mathfrak{p}\in S} z_\mathfrak{p}\right)(c^S)$ implying that

$$\eta^S \cap \bigcap_{\mathfrak{p}\in S} \mathrm{res}_{T(F)_+}^{T(F_\mathfrak{p})} z_\mathfrak{p} = \left(\mathbb{1}_{\mathcal{F}^S} \cap \bigcap_{\mathfrak{p}\in S} \mathrm{res}_{T(F)_+}^{T(F_\mathfrak{p})} z_\mathfrak{p}\right) \cap \xi^S = \left(\sum_{\bar{\gamma}\in H_{\mathrm{tor}}^S} \gamma\mathbb{1}_{\mathcal{F}}\right) \cap \xi$$
$$= \#(H_{\mathrm{tor}}^S)\cdot(\mathbb{1}_{\mathcal{F}}\cap\xi) = \#(H_{\mathrm{tor}}^S)\cdot\eta,$$

and the result follows. □

For $S = \{\mathfrak{p}\} = S^2$, we recover the definition $\eta^\mathfrak{p}$ given in [GMM17], and the previous lemma relates it with η.

3.2 Pairings

In order to relate the p-arithmetic cohomology groups defined in §2.8 and the homology groups defined in §3.1, we will define certain pairings that will allow us to perform cap products. For this purpose, we assume for the rest of this work the following hypothesis:

Hypothesis 3.2.1. *Assume that $\Sigma_B = \Sigma_{\mathrm{un}}(K/F)$. Hence, in particular, $u = s$ and $G(F)/G(F)_+ = T(F)/T(F)_+$.*

As above, let S be a set of primes $\mathfrak{p}$ above p. For any $T(F)$-modules M and N, let us consider the $T(F)_+$-equivariant pairing

$$\begin{aligned} \langle\cdot,\cdot\rangle_+ : C_c^0(T(\mathbb{A}_F^{S\cup\infty}), M) \times \mathcal{A}^{S\cup\infty}(M,N) &\longrightarrow N, \\ (f,\phi) &\longmapsto \langle f,\phi\rangle_+ := \int_{T(\mathbb{A}_F^{S\cup\infty})} \phi(t)(f(t))d^\times t, \end{aligned} \tag{3.14}$$

where $d^\times t$ is the corresponding Haar measure. This implies that, once fixed a character $\lambda : G(F)/G(F)_+ = T(F)/T(F)_+ \to \pm 1$, it induces a well-defined $T(F)$-equivariant pairing

$$\begin{aligned} \langle \cdot , \cdot \rangle : \ & \mathrm{Ind}_{T(F)_+}^{T(F)} C_c^0(T(\mathbb{A}_F^{S\cup\infty}), M) \times \mathcal{A}^{S\cup\infty}(M,N)(\lambda) \longrightarrow N, \\ \left\langle \textstyle\sum_{\bar{t}\in T(F)/T(F)_+} f_t \otimes t, \phi \right\rangle & := [T(F):T(F)_+]^{-1} \textstyle\sum_{\bar{t}\in T(F)/T(F)_+} t \cdot \langle f_t, t^{-1}\phi\rangle_+, \end{aligned} \tag{3.15}$$

where $\mathcal{A}^{S\cup\infty}(M,N)(\lambda)$ is the twist of the $T(F)$-representation $\mathcal{A}^{p\cup\infty}(M,N)$ by the character λ.

Assume that $M = C_c^0(T(F_S), R) \otimes_R V$, for a finite rank R-module V,

$$\mathrm{Ind}_{T(F)_+}^{T(F)} C_c^0(T(\mathbb{A}_F^{S\cup\infty}), M) = C_c^0(T(\mathbb{A}_F), R) \otimes_R V.$$

This implies that (3.15) provides a final $T(F)$-equivariant pairing

$$\begin{aligned} \langle \cdot | \cdot \rangle : C_c^0(T(\mathbb{A}_F), R) \otimes_R V \times \mathcal{A}^{S\cup\infty}(C_c^0(T(F_S), R) \otimes_R V, N)(\lambda) \longrightarrow N, \\ \langle f_S \otimes f^S \otimes v | \phi \rangle = [T(F):T(F)_+]^{-1} \textstyle\sum_{\bar{x}\in T(F)/T(F)_+} \lambda(x)^{-1} \int_{T(\mathbb{A}^{S\cup\infty})} f^S(x,t)\cdot \phi(t)(f_S \otimes v) d^\times t. \end{aligned} \tag{3.16}$$

All the pairings above induce cap products in H-(co)homology by their H-equivariance. Now denote by f_λ the projection of f to the subspace

$$C_c^0(T(\mathbb{A}_F), R)_\lambda := \left\{ f \in C_c^0(T(\mathbb{A}_F), R) \text{ with } f|_{T(F_\infty)} = \lambda \right\}.$$

One easily computes that for $v \in V$, $f \in C_c^0(T(\mathbb{A}_F), R)$ and $\phi \in \mathcal{A}^{S\cup\infty}(C_c^0(T(F_S), R), N)(\lambda)$

$$\langle f \otimes v | \phi \rangle = \langle f_\lambda \otimes v | \phi \rangle = \langle f_\lambda \mid_{T(\mathbb{A}^\infty)} \otimes v, \phi \rangle_+.$$

Since we can identify $H^u(G(F)_+, \bullet)^\lambda \simeq H^u(G(F), \bullet(\lambda))$, we deduce that for all $f \otimes v \in H_u(T(F), C_c^0(T(\mathbb{A}_F), R) \otimes_R V)$ and $\phi \in H^u(T(F)_+, \mathcal{A}^{S\cup\infty}(C_c^0(T(F_S), R) \otimes_R V, N))^\lambda$,

$$(f \otimes v) \cap \phi = (f_\lambda \otimes v) \cap \phi = (f_\lambda \mid_{T(\mathbb{A}_F^\infty)} \otimes v) \cap \mathrm{res}_{T(F)_+}^{T(F)} \phi \in N, \tag{3.17}$$

where $\mathrm{res}_{T(F)_+}^{T(F)}$ is the restriction morphism and the cap products are the induced by (3.14),(3.16), respectively.

Chapter 4

Darmon and plectic points

4.1 Construction of Darmon points

In this section we expose the construction of non-archimedean Darmon points given in [GMM17]. Throughout this section we will assume that $S = \{\mathfrak{p}\}$, $T(F_\mathfrak{p})$ is non-split and $V_S^{\mathbb{Z}} = \mathrm{St}_{\mathbb{Z}}(F_\mathfrak{p})$. We will write

$$\phi_\lambda^\mathfrak{p} := \phi_\lambda^S \in H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\mathrm{St}_{\mathbb{Z}}(F_\mathfrak{p}), \mathbb{Z}))^\lambda,$$

the class associated with the elliptic curve E/F where $u = \Sigma_T = \Sigma_B$. Since E has split multiplicative reduction at $\mathfrak{p}$, it admits a p-adic Tate's uniformization at $\mathfrak{p}$.

4.1.1 Extensions of the Steinberg representation

Let $\mathfrak{H}_\mathfrak{p} = K_\mathfrak{p} \setminus F_\mathfrak{p}$ be the $\mathfrak{p}$-adic upper half plane and

$$\mathrm{X}_\mathfrak{p} := \begin{cases} \mathfrak{H}_\mathfrak{p}, & \text{if } T \text{ does not split at } \mathfrak{p}, \\ \mathbb{P}^1(F_\mathfrak{p}), & \text{if splits at } \mathfrak{p}. \end{cases}$$

In any case, $\mathrm{X}_\mathfrak{p}$ is endowed with a natural action of $G(F_\mathfrak{p})$ given by fractional linear transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * x = \frac{ax+b}{cx+d} \tag{4.1}$$

Let $\tau_\mathfrak{p}, \bar\tau_\mathfrak{p} \in \mathrm{X}_\mathfrak{p}$ be the two fixed points by $\iota(T(F_\mathfrak{p}))$, where ι is the embedding fixed in (1.10). We will assume that $\iota(K_\mathfrak{p}^\times) \cap P = F_\mathfrak{p}^\times$, hence in the split case $\tau_\mathfrak{p}, \bar\tau_\mathfrak{p} \neq \infty$ (see also hypothesis 5.1.5). In fact, $\tau_\mathfrak{p}$ and $\bar\tau_\mathfrak{p}$ correspond to the two simultaneous eigenvectors of all matrices $\iota(t)$ because of the following identity.

$$\iota(t)\begin{pmatrix} \tau_\mathfrak{p} \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tau_\mathfrak{p} \\ 1 \end{pmatrix} = (c\tau_\mathfrak{p} + d)\begin{pmatrix} \iota(t)\cdot\tau_\mathfrak{p} \\ 1 \end{pmatrix} = \lambda_t \begin{pmatrix} \tau_\mathfrak{p} \\ 1 \end{pmatrix}, \qquad \iota(t)\begin{pmatrix} \bar\tau_\mathfrak{p} \\ 1 \end{pmatrix} = \bar\lambda_t \begin{pmatrix} \bar\tau_\mathfrak{p} \\ 1 \end{pmatrix}. \tag{4.2}$$

being $t \mapsto t$ and $t \mapsto \bar t$ the two embeddings $K_\mathfrak{p} \hookrightarrow \mathbb{C}_p$.

Assume that the local representation $\pi_\mathfrak{p}$ is Steinberg. Thus $\pi_\mathfrak{p}$ is the quotient of the induced representation $\mathrm{Ind}_P^G 1$ modulo the constant functions. Let $C^0(\mathbb{P}^1(F_\mathfrak{p}), \mathbb{Z})$ be the set of locally constant $\mathbb{Z}$-valued functions of the projective line with the action (4.1). Since $G(F_\mathfrak{p})/P \simeq \mathbb{P}^1(F_\mathfrak{p})$, we can give a simple description of $V_\mathfrak{p}^{\mathbb{Z}} = \mathrm{St}_{\mathbb{Z}}(F_\mathfrak{p})$,

$$\mathrm{St}_{\mathbb{Z}}(F_\mathfrak{p}) = C^0(\mathbb{P}^1(F_\mathfrak{p}), \mathbb{Z})/\mathbb{Z}.$$

Denote by $\mathrm{Cov}(X)$ the poset of open coverings of a topological space X ordered by refinement. Let

$$\Delta_\mathfrak{p} := \mathrm{Div}(\mathfrak{H}_\mathfrak{p}) \text{ and } \Delta_\mathfrak{p}^0 := \mathrm{Div}^0(\mathfrak{H}_\mathfrak{p}) \subset \Delta_\mathfrak{p}$$

the set of divisors and degree zero divisors. The *multiplicative integral* i is defined by

$$\begin{aligned} i:\mathbf{Hom}(\mathrm{St}_{\mathbb{Z}}(F_{\mathfrak{p}}),\mathbb{Z}) &\longrightarrow \mathbf{Hom}(\Delta_{\mathfrak{p}}^0,K_{\mathfrak{p}}^\times) \\ \psi &\longmapsto \left(z_2-z_1\mapsto \times\!\!\!\!\!\!\int_{\mathbb{P}^1(F_{\mathfrak{p}})}\tfrac{x-z_2}{x-z_1}d\mu_\psi(x)\right) \end{aligned} \tag{4.3}$$

where

$$\times\!\!\!\!\!\!\int_{\mathbb{P}^1(F_{\mathfrak{p}})}\frac{x-z_2}{x-z_1}d\mu_\psi(x)=\lim_{\mathcal{U}\in\mathrm{Cov}(\mathbb{P}^1(F_{\mathfrak{p}}))}\prod_{U\in\mathcal{U}}\left(\frac{x_U-z_2}{x_U-z_1}\right)^{\psi(\mathbb{1}_U)}$$

Here each $x_U\in U$ and $\mathbb{1}_U$ is the characteristic function of U.

In [GMM17] the multiplicative integral is described alternatively: for any topological group M, write $\mathrm{St}_M:=C(\mathbb{P}^1(F_{\mathfrak{p}}),M)/M$. Then we have a natural morphism

$$\varphi_{\mathrm{unv}}:\mathbf{Hom}(\mathrm{St}_{\mathbb{Z}}(F_{\mathfrak{p}}),\mathbb{Z})\longrightarrow\mathbf{Hom}(\mathrm{St}_{F_{\mathfrak{p}}^\times},F_{\mathfrak{p}}^\times),\quad \varphi_{\mathrm{unv}}(\psi)(f):=\lim_{\mathcal{U}\in\mathrm{Cov}(\mathbb{P}^1(F_{\mathfrak{p}}))}\prod_{U\in\mathcal{U}}f(x_U)^{\psi(\mathbb{1}_U)} \tag{4.4}$$

For any multiplicative top. group M and any continuous character $\ell:F_{\mathfrak{p}}^\times\to M$ let

$$\mathcal{E}(\ell):=\left\{(\phi,y)\in C(\mathbf{GL}_2(F_{\mathfrak{p}}),M)\times\mathbb{Z}:\phi\left(\begin{pmatrix}s&x\\&t\end{pmatrix}g\right)=\ell(t)^y\cdot\phi(g)\right\}/(M,0). \tag{4.5}$$

Then the *universal extension* of $\mathrm{St}_{F_{\mathfrak{p}}^\times}$ is $\mathcal{E}_{F_{\mathfrak{p}}^\times}=\mathcal{E}(\mathrm{id})$ in the sense that, for any ℓ as above, the morphism $(y,\phi)\mapsto(y,\ell\phi)$ provides a morphism $\ell:\mathcal{E}_{F_{\mathfrak{p}}^\times}\to\mathcal{E}(\ell)$. If we define

$$\mathrm{ev}(z):=(\phi_z,1),\qquad \phi_z\begin{pmatrix}a&b\\c&d\end{pmatrix}:=cz+d\in K_{\mathfrak{p}}^\times. \tag{4.6}$$

then we have a natural morphism $\mathrm{ev}:\Delta_{\mathfrak{p}}\to\mathcal{E}_{K_{\mathfrak{p}}^\times}$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta_{\mathfrak{p}}^0 & \longrightarrow & \Delta_{\mathfrak{p}} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0\\ & & \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & \mathrm{St}_{K_{\mathfrak{p}}^\times} & \xrightarrow{\iota_{\mathrm{St}}} & \mathcal{E}_{K_{\mathfrak{p}}^\times} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array} \tag{4.7}$$

The morphism (4.3) is then the composition $\mathrm{ev}^*\circ\varphi_{\mathrm{unv}}$, where ev^* is the associated pull-back.

4.1.2 p-adic uniformization

The morphism in (4.3) induces a $G(F)_+$-equivariant morphism

$$i:\mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\mathrm{St}_{\mathbb{Z}}(F_{\mathfrak{p}}),\mathbb{Z})\to\mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_{\mathfrak{p}}^0,K_{\mathfrak{p}}^\times)$$

which induces morphisms in $G(F)_+$-cohomology. The degree map short exact sequence $0\to\Delta_{\mathfrak{p}}^0\to\Delta_{\mathfrak{p}}\to\mathbb{Z}\to 0$ and its associated long exact sequence in $G(F)_+$-cohomology, provides a commutative diagram

$$\begin{array}{ccccc} & & H^u(G(F)_+,\mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\mathrm{St}_{\mathbb{Z}}(F_{\mathfrak{p}}),\mathbb{Z}))^\lambda & & \\ & & \downarrow{\scriptstyle i} & \searrow{\scriptstyle c} & \\ \cdots\to H^u(G(F)_+,\mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_{\mathfrak{p}},K_{\mathfrak{p}}^\times))^\lambda & \longrightarrow & H^u(G(F)_+,\mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_{\mathfrak{p}}^0,K_{\mathfrak{p}}^\times))^\lambda & \longrightarrow & H^{u+1}(G(F)_+,\mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(K_{\mathfrak{p}}^\times))^\lambda\to\cdots \end{array}$$

One can show that there exists $q_\mathfrak{p} \in F_\mathfrak{p}^\times$ such that $c\phi_\lambda^\mathfrak{p} \in H^{u+1}(G(F)_+, \mathcal{A}^{\mathfrak{p},\infty}(q_\mathfrak{p}^\mathbb{Z}))$. An isogeny $K_\mathfrak{p}^\times/q_\mathfrak{p}^\mathbb{Z} \sim E(K_\mathfrak{p})$ is expected (see [GMM17, Conjecture 3.8]): this has been verified in many situations, see [GR21] and [Spi21]. Recall that the map $\rho^{\{\mathfrak{p}\}} \ni x^\mathfrak{p} \mapsto \phi_\lambda^\mathfrak{p}$ defines an element

$$\Phi_\lambda \in H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\mathrm{St}_\mathbb{Z}(F_\mathfrak{p}), \mathbb{Q}))_\rho^\lambda. \tag{4.8}$$

Hence, if we write $E_\mathbb{Q}(K_\mathfrak{p}) := E(K_\mathfrak{p}) \otimes \mathbb{Q}$, we can consider instead the diagram

$$\begin{array}{ccccc} & & H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\mathrm{St}_\mathbb{Z}(F_\mathfrak{p}), \mathbb{Q}))_\rho^\lambda & \overset{\overline{c}}{\searrow} & \\ & & \downarrow i & & \\ \cdots \to H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_\mathfrak{p}, E_\mathbb{Q}(K_\mathfrak{p})))_\rho^\lambda & \longrightarrow & H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_\mathfrak{p}^0, E_\mathbb{Q}(K_\mathfrak{p})))_\rho^\lambda & \longrightarrow & H^{u+1}(G(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(E_\mathbb{Q}(K_\mathfrak{p})))_\rho^\lambda \to \cdots \end{array}$$

obtained reducing modulo $q_\mathfrak{p}^\mathbb{Z}$, and by definition $\Phi_\lambda \in \ker \overline{c}$. This implies that there exists an element

$$\Psi_\lambda \in H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_\mathfrak{p}, E_\mathbb{Q}(K_\mathfrak{p})))_\rho^\lambda \tag{4.9}$$

which maps to $i\Phi_\lambda$. It is unique by [GMM17, Lemma 3.6]. By [Spi13]

$$H^S(G(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_\mathfrak{p}, E_\mathbb{Q}(K_\mathfrak{p})))^\lambda = H^S(G(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_\mathfrak{p}, E(K_\mathfrak{p})))^\lambda \otimes_\mathbb{Z} \mathbb{Q}.$$

so we can write

$$\psi_\lambda^\mathfrak{p} \in H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_\mathfrak{p}, E(K_\mathfrak{p})))^\lambda,$$

for the image of $x^\mathfrak{p} \in \rho^{\{\mathfrak{p}\}}$ once we get rid of denominators.

4.1.3 Darmon points

Let $\tau_\mathfrak{p} \in \mathfrak{H}_\mathfrak{p}$ be the point fixed by $T(F)$ as in §4.1.1. Consider the following morphism of $G(F)_+$-modules

$$\begin{aligned} \cdot|_{\tau_\mathfrak{p}} : \mathbb{Z}[G(F)_+/T(F)_+] &\longrightarrow \Delta_\mathfrak{p} \\ n \cdot gT(F)_+ &\longmapsto n \cdot g(\tau_\mathfrak{p}) \end{aligned} \tag{4.10}$$

It is well-defined because $\tau_\mathfrak{p}$ is $T(F)$-invariant. Then composing by $\cdot|_{\tau_\mathfrak{p}}$ induces a $G(F)_+$-equivariant morphism

$$\cdot|_{\tau_\mathfrak{p}} : \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_\mathfrak{p}, E(K_\mathfrak{p})) \longrightarrow \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\mathbb{Z}[G(F)_+/T(F)_+], E(K_\mathfrak{p}))$$

which in turn induces morphisms in the $G(F)_+$-cohomology. By [GMM17, Lemma 4.1]

$$\mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\mathbb{Z}[G(F)_+/T(F)_+], E(K_\mathfrak{p})) \simeq \mathrm{coInd}_{T(F)_+}^{G(F)_+}\left(\mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(E(K_\mathfrak{p}))\right).$$

Hence we obtain an element

$$\psi_\lambda^\mathfrak{p}|_{\tau_\mathfrak{p}} \in H^u(T(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(E(K_\mathfrak{p})))^\lambda,$$

by Shapiro's lemma. Here we regard λ as a character of $T(F)/T(F)_+$ since we have an isomorphism $G(F)/G(F)_+ \simeq T(F)/T(F)_+$ induced by the natural embedding $T \hookrightarrow G$ and the determinant.

Let $\mathcal{G}_T^\mathfrak{p}$ be the Galois group of the abelian extension of K attached to $T(\mathbb{A}_F)/T(F_\mathfrak{p})$. Note $T(F_\mathfrak{p})$ is compact because $\mathfrak{p}$ does not split in K. Then the Artin map factors through the following map

$$\mathbb{A}_K^\times/K^\times \longrightarrow T(F_\infty)/T(F_\infty)_+ \times T(\mathbb{A}_F^\infty)/T(F_\mathfrak{p})T(F)_+ = T(F)/T(F)_+ \times T(\mathbb{A}_F^{\{\mathfrak{p}\}\cup\infty})/T(F)_+. \tag{4.11}$$

and hence we have a decomposition into λ-eigenspaces

$$C(\mathcal{G}_T^{\mathfrak{p}}, \overline{\mathbb{Q}}) = \bigoplus_{\lambda : T(F)/T(F)_+ \to \{\pm 1\}} H^0\left(T(F)_+, C^0(T(\mathbb{A}_F^{\{\mathfrak{p}\}\cup\infty}), \overline{\mathbb{Q}})\right). \tag{4.12}$$

If $\xi \in C(\mathcal{G}_T^{\mathfrak{p}}, \overline{\mathbb{Z}})$ is a locally constant character with $\xi\mid_{T(F_\infty)} = \lambda$ then we can consider its λ-component $\xi_\lambda \in H^0\left(T(F)_+, C^0(T(\mathbb{A}_F^{\{\mathfrak{p}\}\cup\infty}, \overline{\mathbb{Z}}))\right)$. We can define a twisted *Darmon point*

$$P_\xi^{\mathfrak{p}} = \left(\eta^{\mathfrak{p}} \cap \xi_\lambda\right) \cap \psi_\lambda^{\mathfrak{p}}|_{\tau_{\mathfrak{p}}} \in E(K_{\mathfrak{p}}) \otimes_{\mathbb{Z}} \overline{\mathbb{Z}},$$

where the $\cap$ are with respect to $\langle \cdot, \cdot \rangle_+$ of (3.14).

These points are conjectured to be defined over abelian extensions of K (see [GMM17, Conjecture 4.3]). Moreover, in the special case F is totally real and K is totally imaginary, this construction fits with the p-adic construction of classical Heegner points. For a more detailed description of these facts see §5 in [GMM17].

4.2 Fornea-Gehrmann plectic points

A plectic point will be defined as an element of the (completed) tensor product $\otimes_{\mathfrak{p}} E(K_{\mathfrak{p}})$, and they owe their name to the plectic conjectures made by Nekovár and Scholl, see [NS16].

In the case of Darmon points one had to assume that the local representation at p is the special representation, which corresponds to the case of split multiplicative reduction at $\mathfrak{p}$, while in the plectic approach of Fornea and Gehrmann we can allow the non split reduction case. This is done by choosing an appropriate uniformization, as in (4.15).

Moreover, they are conjectured to be non-zero in rank $r \leq [F : \mathbb{Q}]$ situations, hence they open the door to important progress towards the understanding of the Birch and Swinnerton-Dyer conjecture in rank $r \geq 2$ situations, although we are restricted to the case $r \leq [F : \mathbb{Q}]$. In this direction, they prove in [FG21, Theorem 5.13] the analogous of Theorem 6.1.1 with Darmon points replaced by plectic points. Our aim in the following sections is to generalize their result adapting our proof of Theorem 6.1.1.

Let $\phi_\lambda^S \in H^u(G(F)_+, \mathcal{A}^{S\cup\infty}(V_S^{\mathbb{Z}}, \mathbb{Z}))^\lambda$ be the modular symbol associated with an elliptic curve E/F as in §2.8, and S a set of finite places above p such that T does not split at any $\mathfrak{p} \in S$. We will assume that

$$G(F_S) = \mathbf{PGL}_2(F_S) \text{ and } V_S^{\mathbb{Z}} = \bigotimes_{\mathfrak{p}\in S} \mathrm{St}_{\mathbb{Z}}(F_{\mathfrak{p}})(\varepsilon_{\mathfrak{p}}), \tag{4.13}$$

where $\varepsilon_{\mathfrak{p}} : G(F_{\mathfrak{p}}) \longrightarrow \pm 1$ is given by $g \mapsto (\pm 1)^{v_{\mathfrak{p}} \det g}$.

4.2.1 S-adic uniformization

This section is analogous to §4.1.2 but for several primes, which requires an induction step. For all $\mathfrak{p} \in S$, the $G(F_{\mathfrak{p}})$-invariant morphism (4.3) induces the $G(F_{\mathfrak{p}})$-invariant morphism

$$i_{\mathfrak{p}} : \mathbf{Hom}(\mathrm{St}_{\mathbb{Z}}(F_{\mathfrak{p}})(\varepsilon_{\mathfrak{p}}), \mathbb{Z}) \longrightarrow \mathbf{Hom}(\Delta_{\mathfrak{p}}^0, K_{\mathfrak{p}}^\times)(\varepsilon_{\mathfrak{p}}), \tag{4.14}$$

where $(\varepsilon_\mathfrak{p})$ denotes again the twist by the character $\varepsilon_\mathfrak{p}$. The restriction $\varepsilon_\mathfrak{p} : T(F_\mathfrak{p}) \longrightarrow \pm 1$ is non trivial only if $\varepsilon_\mathfrak{p} \neq 1$ and T ramifies at $\mathfrak{p}$. Moreover, if $H_{\varepsilon_\mathfrak{p}}/K_\mathfrak{p}$ is the extension cut out by $\varepsilon_\mathfrak{p}$, by [Sil94, Corollary 5.4]

$$E(K_\mathfrak{p}) = \left\{ u \in H_{\varepsilon_\mathfrak{p}}^\times / q_\mathfrak{p}^\mathbb{Z} : \; u^{-\varepsilon_\mathfrak{p}(\tau)} u^\tau \in q_\mathfrak{p}^\mathbb{Z} \right\}, \qquad 1 \neq \tau \in \mathrm{Gal}(H_{\varepsilon_\mathfrak{p}}/K_\mathfrak{p}). \tag{4.15}$$

Hence, we no longer have a Tate uniformization $K_\mathfrak{p}^\times / q_\mathfrak{p}^\mathbb{Z} \sim E(K_\mathfrak{p})$ but an isomorphism

$$K_\mathfrak{p}^\times / q_\mathfrak{p}^\mathbb{Z} \sim E(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}} := \{ P \in E(H_{\varepsilon_\mathfrak{p}});\; P^\tau = \varepsilon_\mathfrak{p}(\tau) \cdot P \}.$$

Thus, applying the same construction given in §4.1.2 one obtains

$$\Psi_\lambda \in H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_\mathfrak{p}, E_\mathbb{Q}(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}})(\varepsilon_\mathfrak{p}))_\rho^\lambda,$$

as in (4.9), where $\varepsilon_\mathfrak{p}$ is now seen as a character of $G(F)$ by means of the composition $G(F) \hookrightarrow G(F_\mathfrak{p}) \xrightarrow{\varepsilon_\mathfrak{p}} \pm 1$.

If we consider another $\mathfrak{q} \in S$, we can apply again [GMM17, Remark 2.1] to obtain

$$\Psi_\lambda \in H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_\mathfrak{p}, E(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}})(\varepsilon_\mathfrak{p}) \otimes \mathbb{Q})_\rho^\lambda$$

$$= H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\mathrm{St}_\mathbb{Z}(F_\mathfrak{q})(\varepsilon_\mathfrak{q}) \otimes \Delta_\mathfrak{p}, E(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}})(\varepsilon_\mathfrak{p}) \otimes \mathbb{Q})_\rho^\lambda.$$

After evaluating at certain $x^{\mathfrak{p}\mathfrak{q}} \in \rho^{\{\mathfrak{p},\mathfrak{q}\}}$ and clearing denominators, we obtain

$$\psi_\lambda^\mathfrak{p} \in H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\mathrm{St}_\mathbb{Z}(F_\mathfrak{q})(\varepsilon_\mathfrak{q}) \otimes \Delta_\mathfrak{p}, E(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}})(\varepsilon_\mathfrak{p}))^\lambda.$$

From the definition given in (4.3), we deduce that (4.14) provides a $G(F)$-equivariant morphism

$$i_\mathfrak{p} : \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\mathrm{St}_\mathbb{Z}(F_\mathfrak{q})(\varepsilon_\mathfrak{q}) \otimes \Delta_\mathfrak{p}, E(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}})(\varepsilon_\mathfrak{p})(\lambda) \longrightarrow \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\Delta_\mathfrak{q}^0 \otimes \Delta_\mathfrak{p}, \hat{K}_\mathfrak{q}^\times \otimes_{\mathbb{Z}_p} \hat{E}(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}})(\varepsilon_{\mathfrak{p},\mathfrak{q}})(\lambda),$$

where $\widehat{(\cdot)}$ stands for the p-adic completion of the torsion free part and $\varepsilon_{\mathfrak{p},\mathfrak{q}} := \varepsilon_\mathfrak{p} \cdot \varepsilon_\mathfrak{q}$. Thus we can consider $i_\mathfrak{p}\psi_\lambda^\mathfrak{p} \in H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\Delta_\mathfrak{q}^0 \otimes \Delta_\mathfrak{p}, \hat{K}_\mathfrak{q}^\times \otimes_{\mathbb{Z}_p} \hat{E}(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}})(\varepsilon_{\mathfrak{p},\mathfrak{q}}))^\lambda$. Let $M := \hat{K}_\mathfrak{q}^\times \otimes_{\mathbb{Z}_p} \hat{E}(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}}(\varepsilon_{\mathfrak{p},\mathfrak{q}})$ and consider the commutative diagram given by the degree long exact sequences

$$\begin{array}{ccccc}
H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\Delta_\mathfrak{q} \otimes \Delta_\mathfrak{p}, M))^\lambda & \xrightarrow{\varphi_2} & H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\Delta_\mathfrak{q}^0 \otimes \Delta_\mathfrak{p}, M))^\lambda & \xrightarrow{c_1} & H^{u+1}(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\Delta_\mathfrak{p}, M))^\lambda \\
\downarrow & & \downarrow \varphi_1 & & \downarrow \\
H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\Delta_\mathfrak{q} \otimes \Delta_\mathfrak{p}^0, M))^\lambda & \longrightarrow & H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\Delta_\mathfrak{q}^0 \otimes \Delta_\mathfrak{p}^0, M))^\lambda & \xrightarrow{c_2} & H^{u+1}(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\Delta_\mathfrak{p}^0, M))^\lambda \\
\downarrow & & \downarrow & & \downarrow \\
H^{u+1}(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\Delta_\mathfrak{q}, M))^\lambda & \longrightarrow & H^{u+1}(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\Delta_\mathfrak{q}^0, M))^\lambda & \longrightarrow & H^{u+2}(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(M))^\lambda
\end{array}$$

Clearly $i_\mathfrak{p}\psi_\lambda^\mathfrak{p}$ is a preimage through φ_1 of $i_{\mathfrak{p},\mathfrak{q}}\phi_\lambda^{\mathfrak{p},\mathfrak{q}} \in H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\Delta_\mathfrak{q}^0 \otimes \Delta_\mathfrak{p}^0, \hat{K}_\mathfrak{q}^\times \otimes_{\mathbb{Z}_p} \hat{K}_\mathfrak{p}^\times))^\lambda$ modulo $\hat{K}_\mathfrak{q}^\times \otimes q_\mathfrak{p}^{\mathbb{Z}_p}$, where $i_{\mathfrak{p},\mathfrak{q}} = i_\mathfrak{p} \circ i_\mathfrak{q}$ and

$$\phi_\lambda^{\mathfrak{p},\mathfrak{q}} = \Phi_\lambda(x^{\mathfrak{p}\mathfrak{q}}) \in H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\mathrm{St}_\mathbb{Z}(F_\mathfrak{p})(\varepsilon_\mathfrak{p}) \otimes_\mathbb{Z} \mathrm{St}_\mathbb{Z}(F_\mathfrak{q})(\varepsilon_\mathfrak{q}), \mathbb{Z}))^\lambda$$

once we have the identification

$$\Phi_\lambda \in H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\mathrm{St}_{\mathbb{Z}}(F_\mathfrak{p})(\varepsilon_\mathfrak{p}), \mathbb{Q}))_\rho^\lambda = H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\mathrm{St}_{\mathbb{Z}}(F_\mathfrak{p})(\varepsilon_\mathfrak{p})\otimes_{\mathbb{Z}}\mathrm{St}_{\mathbb{Z}}(F_\mathfrak{q})(\varepsilon_\mathfrak{q}), \mathbb{Q}))_\rho^\lambda$$

of (4.8). Conjecture [GMM17, Conjecture 3.8] implies that $c_2 i_{\mathfrak{p},\mathfrak{q}}\phi_\lambda^{\mathfrak{p},\mathfrak{q}}$ lies in the line $q_\mathfrak{q}^{\mathbb{Z}_p}\otimes_{\mathbb{Z}_p}\hat{E}(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}}(\varepsilon_{\mathfrak{p},\mathfrak{q}})\subseteq M$, hence without loss of generality, we can assume that $c_1 i_\mathfrak{p}\psi_\lambda^\mathfrak{p}$ lies in $q_\mathfrak{q}^{\mathbb{Z}_p}\otimes_{\mathbb{Z}_p}\hat{E}(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}}(\varepsilon_{\mathfrak{p},\mathfrak{q}})$. The preimage of $i_\mathfrak{p}\psi_\lambda^\mathfrak{p}$ through φ_2 modulo $q_\mathfrak{q}^{\mathbb{Z}_p}\otimes_{\mathbb{Z}_p}\hat{E}(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}}(\varepsilon_{\mathfrak{p},\mathfrak{q}})$ provides, after taking ρ-isotypical components, a unique

$$\Psi_\lambda \in H^u(G(F)_+, \mathcal{A}^{\{\mathfrak{p},\mathfrak{q}\}\cup\infty}(\Delta_\mathfrak{p}\otimes\Delta_\mathfrak{q}, \hat{E}(K_\mathfrak{q})_{\varepsilon_\mathfrak{q}}\otimes_{\mathbb{Z}_p}\hat{E}(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)(\varepsilon_{\mathfrak{p},\mathfrak{q}}))_\rho^\lambda.$$

Repeating this construction for all places in S, one obtains

$$\Psi_\lambda \in H^u(G(F)_+, \mathcal{A}^{S\cup\infty}(\Delta_S, \hat{E}(K_S)_{\varepsilon_S}\otimes_{\mathbb{Z}_p}\mathbb{Q}_p)(\varepsilon_S))_\rho^\lambda,$$

Here $\Delta_S := \bigotimes_{\mathfrak{p}\in S}\Delta_\mathfrak{p}$ where the tensor product is taken over $\mathbb{Z}$, $\hat{E}(K_S)_{\varepsilon_S} := \bigotimes_{\mathfrak{p}\in S}\hat{E}(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}}$ where the tensor product is taken over $\mathbb{Z}_p$, and $\varepsilon_S := \prod_{\mathfrak{p}\in S}\varepsilon_\mathfrak{p}$. After getting rid of denominators, the image of $x^S\in\rho^S$ corresponding to ϕ_λ^S provides a class

$$\psi_\lambda^S \in H^u(G(F)_+, \mathcal{A}^{S\cup\infty}(\Delta_S, \hat{E}(K_S)_{\varepsilon_S})(\varepsilon_S))^\lambda.$$

By construction, the restriction of ψ_λ^S at $\Delta_S^0 := \bigotimes_{\mathfrak{p}\in S}\Delta_\mathfrak{p}^0$ is the projection modulo

$$\sum_{\mathfrak{p}\in S}\left(q_\mathfrak{p}^{\mathbb{Z}_p}\otimes\bigotimes_{\mathfrak{q}\in S\setminus\{\mathfrak{p}\}}\hat{K}_\mathfrak{q}^\times\right)$$

of

$$i_S\phi_\lambda^S \in H^u(G(F)_+, \mathcal{A}^{S\cup\infty}(\Delta_S^0, \hat{K}_S^\times)(\varepsilon_S))^\lambda, \qquad K_S^\times := \bigotimes_{\mathfrak{p}\in S}\hat{K}_\mathfrak{p}^\times,$$

where $i_S = \prod_{\mathfrak{p}\in S} i_\mathfrak{p}$ is the morphism induced by (4.14) and the tensor product is taken over $\mathbb{Z}_p$.

4.2.2 Plectic points

For all $\mathfrak{p}\in S$, let $\tau_\mathfrak{p}\in\mathfrak{H}_\mathfrak{p}$ be the point fixed by $T(F)$ as in §4.1.3. The morphism of $G(F)_+$-modules

$$\mathbb{Z}[G(F)_+/T(F)_+] \longrightarrow \Delta_S; \qquad n\cdot gT(F)_+ \longmapsto n\left(\bigotimes_{\mathfrak{p}\in S} g\tau_\mathfrak{p}\right). \tag{4.16}$$

is well-defined and it induces a $G(F)_+$-equivariant morphism

$$\mathcal{A}^{S\cup\infty}(\Delta_S, \hat{E}(K_S)_{\varepsilon_S})(\varepsilon_S) \longrightarrow \mathcal{A}^{S\cup\infty}(\mathbb{Z}[G(F)_+/T(F)_+], \hat{E}(K_S)_{\varepsilon_S})(\varepsilon_S), \qquad \psi\mapsto\psi\mid_{(\tau_\mathfrak{p})_\mathfrak{p}}.$$

Since we have (see [GMM17, Lemma 4.1])

$$\mathcal{A}^{S\cup\infty}(\mathbb{Z}[G(F)_+/T(F)_+], \hat{E}(K_S)_{\varepsilon_S})(\varepsilon_S) \simeq \mathrm{coInd}_{T(F)_+}^{G(F)_+}\left(\mathcal{A}^{S\cup\infty}(\hat{E}(K_S)_{\varepsilon_S})(\varepsilon_S)\right),$$

we obtain by Shapiro's lemma

$$\psi_\lambda^S \mid_{(\tau_\mathfrak{p})_\mathfrak{p}} \in H^u(T(F)_+, \mathcal{A}^{S\cup\infty}(\hat{E}(K_S)_{\varepsilon_S})(\varepsilon_S))^\lambda.$$

Consider the subspace of functions

$$C(\mathcal{G}_T, \overline{\mathbb{Q}})^{\varepsilon_S} := \left\{ f \in C(\mathcal{G}_T, \overline{\mathbb{Q}}),\ \rho^* f \mid_{T(F_S)} = \varepsilon_S \right\},$$

where $\varepsilon_S : T(F_S) \to \pm 1$ is now the product of the local $\varepsilon_\mathfrak{p}$. This is consistent with the construction given in §4.1.3, since if $S = \{\mathfrak{p}\}$ and $\varepsilon_\mathfrak{p} = 1$ then $C(\mathcal{G}_T, \overline{\mathbb{Q}})^{\varepsilon_S} = C(\mathcal{G}_T^\mathfrak{p}, \overline{\mathbb{Q}})$. In this situation, the Artin map provides a decomposition:

$$C(\mathcal{G}_T, \overline{\mathbb{Q}})^{\varepsilon_S} = \bigoplus_{\lambda: T(F)/T(F)_+ \to \{\pm 1\}} H^0\left(T(F)_+, C^0(T(\mathbb{A}_F^{S\cup\infty}), \overline{\mathbb{Q}})(\varepsilon_S) \right). \quad (4.17)$$

Given a locally constant character $\xi \in C(\mathcal{G}_T, \overline{\mathbb{Z}})^{\varepsilon_S}$ we can consider

$$\xi_\lambda \in H^0\left(T(F)_+, C^0(T(\mathbb{A}_F^{S\cup\infty}, \overline{\mathbb{Z}})(\varepsilon_S)) \right)$$

its λ-component, and define the *twisted plectic point*

$$P_\xi^S = \left(\eta^S \cap \xi_\lambda \right) \cap \psi_\lambda^S \mid_{(\tau_\mathfrak{p})_\mathfrak{p}} \in \hat{E}(K_S)_{\varepsilon_S} \otimes_\mathbb{Z} \overline{\mathbb{Z}},$$

where again the cap product is with respect to the pairing $\langle \cdot, \cdot \rangle_+$ of (3.14) twisted by ϵ_S.

4.2.3 Conjectures

As with Darmon points in [GMM17, Conjecture 4.3], it is conjectured that plectic points come from certain rational points of the elliptic curve E. The corresponding conjectures that can be found in [FG21, §1.4].

Let $\xi \in C(\mathcal{G}_T, \overline{\mathbb{Q}})^{\varepsilon_S}$ be a locally constant character such that $\xi \mid_{T(F_\infty)} = \lambda$ (namely, $\xi_\lambda \neq 0$). Let H_ξ/K the abelian extension cut out by ξ. Since $\rho^*\xi \mid_{T(F_S)} = \varepsilon_S$, we can embed $H_\xi \subset H_{\varepsilon_S}$. Moreover, if we consider

$$E(K)_\xi = \{P \in E(H_\xi) \otimes_\mathbb{Z} \bar{\mathbb{Z}};\quad P^\sigma = \xi(\sigma) \cdot P\},$$

clearly $E(K)_\xi \subset E(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}}$. Fornea and Gehrmann consider the natural morphism

$$\imath_\mathfrak{p} : E(K)_\xi \hookrightarrow E(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}} \otimes_\mathbb{Z} \bar{\mathbb{Z}} \longrightarrow \hat{E}(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}} \otimes_\mathbb{Z} \bar{\mathbb{Z}},$$

and, if we write $r := \#S$, we define

$$\det : \bigwedge^r E(K)_\xi \longmapsto \hat{E}(K_S)_{\varepsilon_S};\qquad \det(P_1 \wedge \cdots \wedge P_r) := \det \begin{pmatrix} \imath_{\mathfrak{p}_1}(P_1) & \cdots & \imath_{\mathfrak{p}_r}(P_1) \\ & \cdots & \\ \imath_{\mathfrak{p}_1}(P_r) & \cdots & \imath_{\mathfrak{p}_r}(P_r) \end{pmatrix}.$$

Recall that our construction of P_ξ^S depends on the choice of a vector $x^S = \bigotimes'_{v \notin S\cup\infty} x_v^S \in \rho^S$ corresponding to the modular symbol ϕ_λ^S. The following conjecture is analogous to those of [FG21, §1.4]. Let $\xi_v : T(F_v) \to \mathbb{C}^\times$ be the local characters corresponding to ξ via class field theory.

Conjecture 4.2.1. *We have that* $P_\xi^S = 0$ *unless* $\prod_{v \notin S \cup \infty} \alpha_v(x_v^S, \xi_v) \neq 0$*, where* $\alpha_v \in \mathbf{Hom}_{T(F_v)}(\pi_v \otimes \xi_v, \mathbb{C})$ *is the morphism of* (2.5). *In this case:*

- Algebraicity and reciprocity law*: There exists* $R_\xi \in \bigwedge^r E(K)_\xi$ *such that*

$$\det R_\xi = P_\xi^S.$$

- Connection with BSD*: Assume that* $\mathrm{rank}(E(K)_\xi) \geq r$*. If* $P_\xi^S \neq 0$*, we have that*

$$\mathrm{rank}(E(K)_\xi) = r$$

Chapter 5

p-adic L-functions

5.1 Anticyclotomic p-adic L-functions

In this chapter we will define the anticyclotomic p-adic L-functions associated with p, T and the automorphic cohomology class

$$\phi_\lambda^p \in H^u(G(F)_+, \mathcal{A}^{\infty,p}(V_p, V(\underline{k}))$$

for an even weight $\underline{k} \in 2\mathbb{N}^d$ (here we choose $S = p$).

5.1.1 Defining the distribution

Let $C_{\underline{k}}(T(F_p), \overline{\mathbb{Q}}_p)$ be the space of $\overline{\mathbb{Q}}_p$-valued locally polynomial functions of $T(F_p)$ of degree less than $\underline{k}$. These correspond to the set of functions $f : T(F_p) \to \overline{\mathbb{Q}}_p$ such that in an small neighbourhood U of $t_p = (t_\mathfrak{p})_{\mathfrak{p}|p} \in T(F_p)$

$$f(s_p) = \sum_{|m_\sigma| \leq \frac{k_\sigma}{2}} a_{\underline{m}}(U) \prod_{\mathfrak{p}|p} \prod_{\sigma \in \Sigma_\mathfrak{p}} \sigma(s_\mathfrak{p})^{m_\sigma}$$

where $\underline{m} = (m_\sigma) \in \mathbb{Z}^d$, $a_{\underline{m}}(U) \in \overline{\mathbb{Q}}_p$ and $s_p \in U$. If we denote by C^0 the set of locally constant functions, there is an isomorphism

$$\imath : C^0(T(F_p), \overline{\mathbb{Q}}) \otimes_{\bar{\mathbb{Q}}} \mathcal{P}(\underline{k})_{\bar{\mathbb{Q}}} \otimes \overline{\mathbb{Q}}_p \longrightarrow C_{\underline{k}}(T(F_p), \overline{\mathbb{Q}}_p) \tag{5.1}$$

defined by

$$h \otimes \bigotimes_\sigma P_\sigma \longmapsto \left(t_p = (t_\mathfrak{p})_{\mathfrak{p}|p} \mapsto h(t_p) \prod_{\mathfrak{p}|p} \prod_{\sigma \in \Sigma_\mathfrak{p}} P_\sigma \left(\tilde{\sigma} \left(v_1^\mathfrak{p} + t_\mathfrak{p} v_2^\mathfrak{p} \right) \right) \sigma(t_\mathfrak{p})^{\frac{-k_\sigma}{2}} \right).$$

It is $T(F_p)$-equivariant, indeed for any $x_p = (x_\mathfrak{p}) \in T(F_p)$,

$$\begin{aligned} x_p \cdot \imath \left(h \otimes \bigotimes_\sigma P_\sigma \right)(t_p) &= \imath \left(h \otimes \bigotimes_\sigma P_\sigma \right)(x_p^{-1} t_p) \\ &= h(x_p^{-1} t_p) \prod_{\mathfrak{p}|p} \prod_{\sigma \in \Sigma_\mathfrak{p}} P_\sigma \left(\tilde{\sigma} \left(v_1^\mathfrak{p} + \frac{\bar{\lambda}_{\tilde{x}_\mathfrak{p}}}{\lambda_{\tilde{x}_\mathfrak{p}}} t_\mathfrak{p} v_2^\mathfrak{p} \right) \right) \tilde{\sigma} \left(\frac{\bar{\lambda}_{\tilde{x}_\mathfrak{p}}}{\lambda_{\tilde{x}_\mathfrak{p}}} t_\mathfrak{p} \right)^{\frac{-k_\sigma}{2}} \end{aligned}$$

$$= h(x_p^{-1}t_p)\prod_{\mathfrak{p}|p}\prod_{\sigma\in\Sigma_\mathfrak{p}} P_\sigma\left(\tilde\sigma\left(\lambda_{\tilde x_\mathfrak{p}}v_1^\mathfrak{p}+t_\mathfrak{p}\bar\lambda_{\tilde x_\mathfrak{p}}v_2^\mathfrak{p}\right)\right)\tilde\sigma\left(\lambda_{\tilde x_\mathfrak{p}}\bar\lambda_{\tilde x_\mathfrak{p}}t_\mathfrak{p}\right)^{\frac{-k_\sigma}{2}}$$

$$= (x_p\cdot h)(t_p)\prod_{\mathfrak{p}|p}\prod_{\sigma\in\Sigma_\mathfrak{p}} P_\sigma\left(\tilde\sigma\left(v_1^\mathfrak{p}\iota(\tilde x_\mathfrak{p})+t_\mathfrak{p}v_2^\mathfrak{p}\iota(\tilde x_\mathfrak{p})\right)\right)\sigma\left(\det(\iota(\tilde x_\mathfrak{p}))t_\mathfrak{p}\right)^{\frac{-k_\sigma}{2}}$$

$$= (x_p\cdot h)(t_p)\prod_{\mathfrak{p}|p}\prod_{\sigma\in\Sigma_\mathfrak{p}}((\sigma\iota(x_\mathfrak{p})\cdot P_\sigma)\left(\tilde\sigma\left(v_1^\mathfrak{p}+t_\mathfrak{p}v_2^\mathfrak{p}\right)\right)\sigma\left(t_\mathfrak{p}\right)^{\frac{-k_\sigma}{2}}$$

$$= \imath\left((x_p\cdot h)\otimes\left(x_p\cdot\left(\bigotimes_\sigma P_\sigma\right)\right)\right)(t_p).$$

Let $\mathcal{G}_T$ be the Galois group of the abelian extension of K associated with T. By class field theory, there is a continuous morphism

$$\rho:\left(T(F_\infty)/T(F_\infty)_+\times T(\mathbb{A}_F^\infty)\right)/T(F)\to\mathcal{G}_T. \tag{5.2}$$

Let us consider the subset of $C(T(\mathbb{A}_F),\overline{\mathbb{Q}}_p)$:

$$C_{\underline{k}}(T(\mathbb{A}_F),\overline{\mathbb{Q}}_p):=\{f:T(\mathbb{A}_F^p)\to C_{\underline{k}}(T(F_p),\overline{\mathbb{Q}}_p),\ \text{locally constant}\},$$

and write also $C_{\underline{k}}(\mathcal{G}_T,\overline{\mathbb{Q}}_p)$ for the subspace of continuous functions $f:\mathcal{G}_T\to\overline{\mathbb{Q}}_p$ such that $\rho^*(f)\in C_{\underline{k}}(T(\mathbb{A}_F),\overline{\mathbb{Q}}_p)$. The pullback of ρ together with the cap product by η give the following morphism

$$\delta:C_{\underline{k}}(\mathcal{G}_T,\overline{\mathbb{Q}}_p)\xrightarrow{\rho^*}H^0(T(F),C_{\underline{k}}(T(\mathbb{A}_F),\overline{\mathbb{Q}}_p))\xrightarrow{\cap\eta}H_u(T(F),C_{\underline{k},c}(T(\mathbb{A}_F),\overline{\mathbb{Q}}_p)). \tag{5.3}$$

where $C_{\underline{k},c}(T(\mathbb{A}_F),\overline{\mathbb{Q}}_p))$ is the subspace of functions in $C_{\underline{k}}(T(\mathbb{A}_F),\overline{\mathbb{Q}}_p))$ which are compactly supported when restricted to $T(\mathbb{A}_F^\infty)$. Note that $\imath$ of (5.1) provides an isomorphism

$$C_{\underline{k},c}(T(\mathbb{A}_F^\infty),\overline{\mathbb{Q}}_p))=C_c^0(T(\mathbb{A}_F^\infty),\overline{\mathbb{Q}})\otimes_{\overline{\mathbb{Q}}}\mathcal{P}(\underline{k})_{\overline{\mathbb{Q}}}\otimes\overline{\mathbb{Q}}_p. \tag{5.4}$$

In order to define the distribution we need to construct a $T(F_p)$-equivariant morphism:

$$\delta_p=(\delta_\mathfrak{p})_{\mathfrak{p}|p}:C_c^0(T(F_p),\overline{\mathbb{Q}})\longrightarrow V_p,\qquad \delta_\mathfrak{p}:C_c^0(T(F_\mathfrak{p}),\overline{\mathbb{Q}})\longrightarrow V_\mathfrak{p}. \tag{5.5}$$

Given such a δ_p we can directly define the distribution associated with

$$\phi_\lambda^p\in H^u\left(G(F)_+,\mathcal{A}^{p\cup\infty}(V_p,V(\underline{k})_{\overline{\mathbb{Q}}})\right)^\lambda$$

as follows:

$$\int_{\mathcal{G}_T}gd\mu_{\phi_\lambda^p}:=\delta(g)\cap\delta_p^*\phi_\lambda^p,\qquad\text{for all } g\in C_{\underline{k}}(\mathcal{G}_T,\overline{\mathbb{Q}}_p) \tag{5.6}$$

where the cap product is induced by the pairing (3.16) and

$$\delta_p^*:\mathcal{A}^{p\cup\infty}(V_p,V(\underline{k})_{\overline{\mathbb{Q}}})\longrightarrow\mathcal{A}^{p\cup\infty}\left(C_c^0(T(F_p),\overline{\mathbb{Q}})\otimes_{\overline{\mathbb{Q}}}\mathcal{P}(\underline{k})_{\overline{\mathbb{Q}}},\overline{\mathbb{Q}}_p\right)$$

is the corresponding $T(F)$-equivariant pullback.

5.1.2 Admissibility

A continuous function $f \in C(T(F_p), \mathbb{C}_p)$ is *locally analytic* if, for any $x_p = (x_\mathfrak{p})_{\mathfrak{p}|p} \in T(F_p)$ there exists a neighbourhood U of t_p and $a_{\underline{n}}(U) \in \overline{\mathbb{Q}}_p$ such that

$$f\mid_U (s_p) = \sum_{\underline{n}\in\mathbb{N}^d} a_{\underline{n}}(U) \prod_{\mathfrak{p}|p} \prod_{\sigma\in\Sigma_\mathfrak{p}} (\sigma(s_\mathfrak{p}) - \sigma(x_\mathfrak{p}))^{n_\sigma}, \qquad \text{for all } s_p = (s_\mathfrak{p})_{\mathfrak{p}|p} \in U \subset T(F_p).$$

Write $C_{\rm an}(T(F_p), \mathbb{C}_p)$ for the subspace of locally analytic functions. Note that $C_{\underline{k}}(T(F_p), \mathbb{C}_p) \subset C_{\rm an}(T(F_p), \mathbb{C}_p)$. Indeed, it is clear that $\sigma(t_\mathfrak{p})^{m_\sigma} \in C^{\rm an}(T(F_p), \mathbb{C}_p)$ when m_σ is positive, but when $m_\sigma < 0$,

$$\sigma(t_\mathfrak{p})^{m_\sigma} = \sum_{i\geq 0} \binom{m_\sigma}{i} \sigma(x_\mathfrak{p})^{m_\sigma - i} (\sigma(t_\mathfrak{p}) - \sigma(x_\mathfrak{p}))^i \in C_{\rm an}(T(F_p), \mathbb{C}_p).$$

Similarly as above, we consider

$$C_{\rm an}(T(\mathbb{A}_F), \mathbb{C}_p) := \{f : T(\mathbb{A}_F^p) \to C_{\rm an}(T(F_p), \mathbb{C}_p), \text{ locally constant}\} \subset C(T(\mathbb{A}_F), \mathbb{C}_p),$$

and $C_{\rm an}(\mathcal{G}_T, \mathbb{C}_p) := (\rho^*)^{-1} C_{\rm an}(T(\mathbb{A}_F), \mathbb{C}_p)$.

In the previous section we have constructed a locally polynomial distribution

$$\mu_{\phi_\lambda^p} \in {\rm Dist}_{\underline{k}}(\mathcal{G}_T, \overline{\mathbb{Q}}_p) := \mathbf{Hom}(C_{\underline{k}}(\mathcal{G}_T, \overline{\mathbb{Q}}_p), \overline{\mathbb{Q}}_p).$$

In this section we aim to extend it to a locally analytic distribution

$$\mu_{\phi_\lambda^p} \in {\rm Dist}_{\rm an}(\mathcal{G}_T, \mathbb{C}_p) := \mathbf{Hom}_{\rm cnt}(C_{\rm an}(\mathcal{G}_T, \mathbb{C}_p), \mathbb{C}_p).$$

Write also

$$\begin{aligned} {\rm Dist}_{\underline{k}}(T(F_p), \overline{\mathbb{Q}}_p) &:= \mathbf{Hom}(C_{\underline{k},c}(T(F_p), \overline{\mathbb{Q}}_p), \overline{\mathbb{Q}}_p), \\ {\rm Dist}_{\rm an}(T(F_p), \mathbb{C}_p) &:= \mathbf{Hom}_{\rm cnt}(C_{{\rm an},c}(T(F_p), \mathbb{C}_p), \mathbb{C}_p). \end{aligned}$$

Consider the open compact subsets

$$U_\mathfrak{p}(a,n) := a\left(\left(\mathcal{O}_{F,p} + \varpi_\mathfrak{p}^{n - v_\mathfrak{p}(a)} \mathcal{O}_{K,p}\right)^\times / \mathcal{O}_{F,p}^\times\right) \subset K_p^\times / F_p^\times = T(F_p),$$

for any $a \in T(F_\mathfrak{p})$, where a is regarded as an element $a = \lambda_a / \overline{\lambda}_a \in \overline{F}_\mathfrak{p}$ as in (1.11). Note that for all $\mathfrak{p} \mid p$, the $U_\mathfrak{p}(a,n)$ generate a basis for the topology of $T(F_p)$. Moreover, it can be described as

$$U_\mathfrak{p}(a,n) = \{t \in T(F_\mathfrak{p});\ \varpi_\mathfrak{p}^n \mid (t-a)\}.$$

We will assume that $n \in \mathbb{N}$ if $T(F_\mathfrak{p})$ does not ramify. If $T(F_\mathfrak{p})$ ramifies we will choose by convenience that $n \in \frac{1}{2} + \mathbb{N}$, understanding that $\varpi_\mathfrak{p}^{\frac{1}{2}}$ is the uniformizer of $\mathfrak{p}^{\frac{1}{2}}$, the unique prime ideal of $K_\mathfrak{p}$ above $\mathfrak{p}$.

Definition 5.1.1. *For any $\mathfrak{p} \mid p$, a locally polynomial distribution $\mu \in {\rm Dist}_{\underline{k}}(T(F_p), \overline{\mathbb{Q}}_p)$ is* $h_\mathfrak{p}$-admissible *at $\mathfrak{p}$ if for every $a \in T(F_\mathfrak{p})$ there exists a fixed constant $A_\mathfrak{p} \in \mathbb{C}_p$ only depending on $\mathfrak{p}$, T and a neighbourhood of a such that*

$$\int_{U_\mathfrak{p}(a,n)} g d\mu \in A_\mathfrak{p} p^{-nh_\mathfrak{p}} \mathcal{O}_{\mathbb{C}_p},$$

for any $n \in \mathbb{N}$ and $g \in C_{\underline{k}}(T(F_p), \mathcal{O}_{\overline{\mathbb{Q}}_p})$.

Proposition 5.1.2. *Let $e_\mathfrak{p}$ be the ramification index of $\mathfrak{p}$. If $e_\mathfrak{p}h_\mathfrak{p} < \min\{k_\sigma+1,\ \sigma\in\Sigma_\mathfrak{p}\}$, a distribution $\mu\in\mathrm{Dist}_{\underline{k}}(T(F_p),\overline{\mathbb{Q}}_p)$ that is $h_\mathfrak{p}$-admissible at every $\mathfrak{p}\mid p$ can be extended to a unique locally analytic measure in $\mathrm{Dist}_{\mathrm{an}}(T(F_p),\mathbb{C}_p)$ such that*

$$\int_{U_\mathfrak{p}(a,n)} g\,d\mu \in A_\mathfrak{p}\cdot p^{-nh_\mathfrak{p}}\mathcal{O}_{\mathbb{C}_p},$$

for any $g\in C_{\mathrm{an}}(T(F_p),\mathcal{O}_{\mathbb{C}_p})$ which is analytic in $U_\mathfrak{p}(a,n)$.

Proof. Any locally analytic function is topologically generated by functions of the form

$$P_{\underline{m}}^{a,N}(x) := \mathbb{1}_{U_\mathfrak{p}(a,N)}(x)\cdot\left(\frac{x_\mathfrak{p}-a}{\varpi_\mathfrak{p}^N}\right)^{\underline{m}} x_\mathfrak{p}^{-\frac{\underline{k}}{2}},\quad a\in T(F_\mathfrak{p}),$$

where $\underline{m}\in\mathbb{N}^{\Sigma_\mathfrak{p}}$ and

$$\left(\frac{x_\mathfrak{p}-a}{\varpi_\mathfrak{p}^N}\right)^{\underline{m}} x_\mathfrak{p}^{-\frac{\underline{k}}{2}} := \prod_{\sigma\in\Sigma_\mathfrak{p}}\sigma\left(\frac{x_\mathfrak{p}-a}{\varpi_\mathfrak{p}^N}\right)^{m_\sigma}\sigma(x_\mathfrak{p})^{-\frac{k_\sigma}{2}}.$$

By definition, we have the values $\mu(P_{\underline{m}}^{a,N})$ for every $m_\sigma\le k_\sigma$. If there exist $\tau\in\Sigma_\mathfrak{p}$ such that $m_\tau > e_\mathfrak{p}h_\mathfrak{p}$, we define $\mu(P_{\underline{m}}^{a,N}) = \lim_{n\to\infty} a_n$, where

$$a_n = \sum_{\substack{b \bmod \varpi_\mathfrak{p}^n\\ b\equiv a \bmod \varpi_\mathfrak{p}^N}}\ \sum_{j_\sigma\le e_\mathfrak{p}h_\mathfrak{p}}\left(\frac{b-a}{\varpi_\mathfrak{p}^N}\right)^{\underline{m}-\underline{j}}\binom{\underline{m}}{\underline{j}}\varpi_\mathfrak{p}^{\underline{j}(n-N)}\mu(P_{\underline{j}}^{b,n}),$$

and

$$\binom{\underline{m}}{\underline{j}} = \prod_{\sigma\in\Sigma_\mathfrak{p}}\binom{m_\sigma}{j_\sigma},\qquad \varpi_\mathfrak{p}^{\underline{j}(n-N)} := \prod_{\sigma\in\Sigma_\mathfrak{p}}\sigma(\varpi_\mathfrak{p})^{j_\sigma(n-N)}$$

The definition agrees with μ when there exist $\tau\in\Sigma_\mathfrak{p}$ such that $e_\mathfrak{p}h_\mathfrak{p} < m_\tau \le k_\tau$ because $\varpi_\mathfrak{p}^{\underline{j}(n-N)}\mu(P_{\underline{j}}^{b,n}) \xrightarrow{n} 0$ when $j_\tau > e_\mathfrak{p}h_\mathfrak{p}$.

It can be checked similarly as in [Mol21, Proposition 2] that the sequence a_n is Cauchy, hence the limit exists. It is clear by the definition that $\mu(P_{\underline{m}}^{a,N})\in A_\mathfrak{p}\cdot p^{-Nh_\mathfrak{p}}\mathcal{O}_{\mathbb{C}_p}$ for all $\underline{m}$ and N. Hence, it extends to a locally analytic measure as described. □

Let $M_T = \begin{pmatrix} v_1^\mathfrak{p}\\ v_2^\mathfrak{p}\end{pmatrix}\in\mathbf{GL}_2(\overline{\mathbb{Q}}_p)$ be the matrix given by the eigenvectors $v_i^\mathfrak{p}$. Recall that $M_T\in\mathbf{GL}_2(K_\mathfrak{p}^\sharp)$ acts also on $\mathcal{P}(\underline{k})_{\overline{\mathbb{Q}}_p}$ and $V(\underline{k})_{\overline{\mathbb{Q}}_p}$ since we have fixed embeddings $\tilde\sigma: K_\mathfrak{p}^\sharp\hookrightarrow\overline{\mathbb{Q}}_p$ extending each σ. In the non-split situation, we will fix $v_1^\mathfrak{p} = (1,-\bar\tau_\mathfrak{p})$ and $v_2^\mathfrak{p} = (-1,\tau_\mathfrak{p})$, where $\tau_\mathfrak{p},\bar\tau_\mathfrak{p}\in K_\mathfrak{p}^\times$ are the points fixed by the action of $\iota(K_\mathfrak{p}^\times)$ on $\mathbb{P}^1(K_\mathfrak{p})$ given by linear fractional transformations. Note that, $M_T^{-1}\cdot t_\mathfrak{p}\in\mathbb{P}^1(F_\mathfrak{p})$, for all $t_\mathfrak{p}\in T(F_\mathfrak{p})$. Indeed,

$$M_T{}^{-1}\cdot t_\mathfrak{p} = \frac{1}{\tau_\mathfrak{p}-\bar\tau_\mathfrak{p}}\begin{pmatrix}\tau_\mathfrak{p} & \bar\tau_\mathfrak{p}\\ 1 & 1\end{pmatrix}\cdot t_\mathfrak{p} = \frac{\tau_\mathfrak{p}t_\mathfrak{p}+\bar\tau_\mathfrak{p}}{t_\mathfrak{p}+1}.$$

Since $\overline{t_{\mathfrak{p}}} = t_{\mathfrak{p}}^{-1}$ where $\overline{(\cdot)}$ stands for the non-trivial automorphism of $\mathrm{Gal}(K_{\mathfrak{p}}/F_{\mathfrak{p}})$, we obtain

$$\overline{M_T{}^{-1} \cdot t_{\mathfrak{p}}} = \frac{\bar{\tau}_{\mathfrak{p}} t_{\mathfrak{p}}{}^{-1} + \tau_{\mathfrak{p}}}{t_{\mathfrak{p}}{}^{-1} + 1} = M_T{}^{-1} \cdot t_{\mathfrak{p}}.$$

which is in $\mathbb{P}^1(F_{\mathfrak{p}})$ as claimed.

Write $v_p : \mathbb{C}_p \to \mathbb{Q}$ for the p-adic valuation satisfying $v_p(p) = 1$.

Proposition 5.1.3. *Assume that $G(F_{\mathfrak{p}}) = \mathbf{PGL}_2(F_{\mathfrak{p}})$ for all $\mathfrak{p} \mid p$ and there exists $\alpha_{\mathfrak{p}} \in \mathbb{C}_p^{\times}$ such that the $T(F_p)$-equivariant morphism $\delta_p = (\delta_{\mathfrak{p}})_{\mathfrak{p}|p}$ of (5.5) satisfies for n big enough*

$$\begin{cases} \begin{pmatrix} 1 & a \\ & \varpi_{\mathfrak{p}}^n \end{pmatrix} M_T \delta_{\mathfrak{p}}(\mathbb{1}_{U_{\mathfrak{p}}(a,n)}) = \frac{1}{\alpha_{\mathfrak{p}}^n} c(a,n) V, & T(F_{\mathfrak{p}}) \text{ splits}, \\ \begin{pmatrix} \varpi_{\mathfrak{p}}^{\beta} & s\varpi_{\mathfrak{p}}^{\beta} \\ & \varpi_{\mathfrak{p}}^{n+m-\beta} \end{pmatrix} \delta_{\mathfrak{p}}(\mathbb{1}_{U_{\mathfrak{p}}(a,n)}) = \frac{1}{\alpha_{\mathfrak{p}}^n} c(a,n) V, & T(F_{\mathfrak{p}}) \text{ is non-split, } s = -M_T{}^{-1}(-a) \in \mathcal{O}_{F_{\mathfrak{p}}}, \\ \begin{pmatrix} s^{-1}\varpi_{\mathfrak{p}}^{\beta} & \varpi_{\mathfrak{p}}^{\beta} \\ -\varpi_{\mathfrak{p}}^{n+m-\beta} & \end{pmatrix} \delta_{\mathfrak{p}}(\mathbb{1}_{U_{\mathfrak{p}}(a,n)}) = \frac{1}{\alpha_{\mathfrak{p}}^n} c(a,n) V, & T(F_{\mathfrak{p}}) \text{ is non-split, } s^{-1} = \left(-M_T{}^{-1}(-a)\right)^{-1} \in \mathfrak{p}, \end{cases} \tag{5.7}$$

where m is the $\mathfrak{p}$-valuation of $\tau_{\mathfrak{p}} - \bar{\tau}_{\mathfrak{p}}$, $\beta = \lfloor v_{\mathfrak{p}}(1-a) \rfloor$, if $s \in \mathcal{O}_{F_{\mathfrak{p}}}$, and $\beta = \lfloor v_{\mathfrak{p}}(a\tau_{\mathfrak{p}} - \bar{\tau}_{\mathfrak{p}}) \rfloor$, if $s^{-1} \in \mathfrak{p}$. Moreover, $V \in V_{\mathfrak{p}}$ do not depend neither a nor n, and $c(a,n)$ has p-adic valuation only depending on a neighbourhood of a. If

$$e_{\mathfrak{p}} v_p(\alpha_{\mathfrak{p}}^*) < \min\{k_\sigma + 1,\ \sigma \in \Sigma_{\mathfrak{p}}\}, \qquad \text{where} \quad \alpha^* = \alpha_{\mathfrak{p}} \varpi_{\mathfrak{p}}^{\frac{k}{2}},$$

then the distribution $\mu_{\phi_\lambda^p}$ extends to a unique locally analytic p-adic distribution.

Remark 5.1.4. *Note that*

$$\tau_{\mathfrak{p}} - \bar{\tau}_{\mathfrak{p}} = (s + \tau_{\mathfrak{p}})(1-a) = (1 + \tau_{\mathfrak{p}} s^{-1})(a\tau_{\mathfrak{p}} - \bar{\tau}_{\mathfrak{p}}).$$

This implies that $|\beta|$ is bounded. Indeed, it is easy to show that $\beta \leq \max\{m, m - v_{\mathfrak{p}}(\tau_{\mathfrak{p}})\}$.

Proof. Let $C_p = (C_{\mathfrak{p}})_{\mathfrak{p}} \subset G(\mathcal{O}_{F_p})$ be finite index subgroup such that $V \in V_{\mathfrak{p}}^{C_{\mathfrak{p}}}$ for all $\mathfrak{p} \mid p$. We have a well defined $G(F_p)$-equivariant morphism

$$\theta_V : \mathbf{Hom}\left(V_p, V(\underline{k})_{\overline{\mathbb{Q}}}\right) \longrightarrow \mathrm{coInd}_{C_p}^{G(F_p)}(V(\underline{k})_{\overline{\mathbb{Q}}})$$

where $\theta_V(\varphi)(g)(P) := \varphi(gV)(gP)$ and

$$\mathrm{coInd}_{C_p}^{G(F_p)}(V(\underline{k})_{\overline{\mathbb{Q}}}) = \left\{f : G(F_p) \to V(\underline{k})_{\overline{\mathbb{Q}}};\ f(gc) = c^{-1}f(g),\ c \in C_p\right\}.$$

Thus, we have a $G(F)_+$-equivariant morphism

$$\theta_V : \mathcal{A}^{p\cup\infty}(V_p, V(\underline{k})_{\bar{\mathbb{Q}}_p}))^{\lambda} \longrightarrow \mathcal{A}^{p\cup\infty}\left(\mathrm{coInd}_{C_p}^{G(F_p)}(V(\underline{k})_{\overline{\mathbb{Q}}_p})\right) \simeq \mathcal{A}^{\infty}\left(V(\underline{k})_{\overline{\mathbb{Q}}_p}\right)^{C_p}.$$

By [Spi14, Proposition 4.6] we have that

$$H^u\left(G(F)_+, \mathcal{A}^{\infty}(V(\underline{k})_{\overline{\mathbb{Q}}_p})\right)^{\lambda, C_p} \simeq H^u\left(G(F)_+, \mathcal{A}^{\infty}(V(\underline{k})_{\overline{\mathbb{Z}}_p})\right)^{\lambda, C_p} \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p$$

hence up to a constant we can assume that

$$\theta_V\phi_\lambda^p \in H^u\left(G(F)_+, \mathcal{A}^\infty(V(\underline{k})_{\overline{\mathbb{Z}}_p})\right)^{\lambda,C_p} = H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}\left(\mathrm{coInd}_{C_p}^{G(F_p)}(V(\underline{k})_{\overline{\mathbb{Z}}_p})\right)\right)^\lambda$$

($V(\underline{k})_{\overline{\mathbb{Z}}_p}$ admits an action of $C_p \subseteq G(\mathcal{O}_{F_p})$).

On the other hand, let $\varphi \in \mathbf{Hom}\left(V_p, V(\underline{k})_{\overline{\mathbb{Q}}_p}\right)$ such that $\theta_V(\varphi) \in \mathrm{coInd}_{C_p}^{G(F_p)}(V(\underline{k})_{\overline{\mathbb{Z}}_p})$. We compute for all $\underline{m} \in \mathbb{N}^{\Sigma_\mathfrak{p}}$ with $m_\sigma \leq k_\sigma$,

$$\int_{U_\mathfrak{p}(a,n)}\left(\frac{u_\mathfrak{p}-a}{\varpi_\mathfrak{p}^n}\right)^{\underline{m}} u_\mathfrak{p}^{-\frac{\underline{k}}{2}} d\delta_p^*\varphi(u_\mathfrak{p}) = \varphi(\delta_\mathfrak{p}\mathbb{1}_{U_\mathfrak{p}(a,n)})\left(M_T^{-1}\bigotimes_{\sigma\in\Sigma_\mathfrak{p}}\left(\frac{y_\sigma-\sigma(a)x_\sigma}{\sigma(\varpi_\mathfrak{p})^n}\right)^{m_\sigma} x_\sigma^{k_\sigma-m_\sigma}\right)\cdot\det(M_T)^{-\frac{\underline{k}}{2}}$$

$$= \varphi(\delta_\mathfrak{p}\mathbb{1}_{U_\mathfrak{p}(a,n)})\left(M_T^{-1}\begin{pmatrix}1 & a\\ & \varpi_\mathfrak{p}^n\end{pmatrix}^{-1}\bigotimes_{\sigma\in\Sigma_\mathfrak{p}} y_\sigma^{m_\sigma}x_\sigma^{k_\sigma-m_\sigma}\right)\cdot\det(M_T)^{-\frac{\underline{k}}{2}}\cdot\varpi_\mathfrak{p}^{-n\frac{\underline{k}}{2}}$$

If $T(F_\mathfrak{p})$ splits, we have $\gamma_{a,n} = \begin{pmatrix}1 & a\\ & \varpi_\mathfrak{p}^n\end{pmatrix}M_T \in \mathbf{GL}_2(F_\mathfrak{p})$, hence

$$\int_{U_\mathfrak{p}(a,n)}\left(\frac{u_\mathfrak{p}-a}{\varpi_\mathfrak{p}^n}\right)^{\underline{m}} u_\mathfrak{p}^{-\frac{\underline{k}}{2}} d\delta_p^*\varphi(u_\mathfrak{p}) = \frac{c(a,n)}{(\alpha_\mathfrak{p}^*)^n\cdot\det(M_T)^{\frac{\underline{k}}{2}}}\cdot\varphi\left({\gamma_{a,n}}^{-1}V\right)\left({\gamma_{a,n}}^{-1}\bigotimes_{\sigma\in\Sigma_\mathfrak{p}} y_\sigma^{m_\sigma}x_\sigma^{k_\sigma-m_\sigma}\right)$$

$$= \frac{c(a,n)}{(\alpha_\mathfrak{p}^*)^n\cdot\det(M_T)^{\frac{\underline{k}}{2}}}\cdot\theta_V\left(\varphi\right)\left({\gamma_{a,n}}^{-1}\right)\left(\bigotimes_{\sigma\in\Sigma_\mathfrak{p}} y_\sigma^{m_\sigma}x_\sigma^{k_\sigma-m_\sigma}\right) \in \frac{A_\mathfrak{p}}{p^{nv_p(\alpha_\mathfrak{p}^*)}}\mathcal{O}_{\mathbb{C}_p},$$

where $A_\mathfrak{p} = \frac{c(a,n)}{\det(M_T)^{\frac{\underline{k}}{2}}}$. Since all such $\left(\frac{u_\mathfrak{p}-a}{\varpi_\mathfrak{p}^n}\right)^{\underline{m}} u_\mathfrak{p}^{-\frac{\underline{k}}{2}}\mathbb{1}_{U_\mathfrak{p}(a,n)}(u_\mathfrak{p})$ generate the space $C_{\underline{k}}(T(F_p), \mathcal{O}_{\overline{\mathbb{Q}}_p})$, we obtain that $\delta_p^*\varphi$ is $v_p(\alpha_\mathfrak{p}^*)$-admissible for all $\mathfrak{p} \mid p$.

If $T(F_\mathfrak{p})$ does not split, we write $s = -M_T^{-1}\cdot(-a) = \frac{a\tau_\mathfrak{p}-\bar\tau_\mathfrak{p}}{1-a}$. Assume that $n > \beta$, then we have decompositions

$$\begin{pmatrix}1 & a\\ & \varpi_\mathfrak{p}^n\end{pmatrix}M_T = A(a,n)\gamma_{s,n},$$

where

$$\begin{cases}\gamma_{s,n} := \begin{pmatrix}\varpi_\mathfrak{p}^\beta & s\varpi_\mathfrak{p}^\beta\\ & \varpi_\mathfrak{p}^{n+m-\beta}\end{pmatrix} \in \mathbf{GL}_2(F_\mathfrak{p}), & A(a,n) := \begin{pmatrix}\frac{1-a}{\varpi_\mathfrak{p}^\beta} & \\ -\varpi_\mathfrak{p}^{n-\beta} & \frac{\tau_\mathfrak{p}-\bar\tau_\mathfrak{p}}{1-a}\varpi_\mathfrak{p}^{\beta-m}\end{pmatrix}, \quad \text{if } s\in\mathcal{O}_{F_\mathfrak{p}},\\ \gamma_{s,n} := \begin{pmatrix}s^{-1}\varpi_\mathfrak{p}^\beta & \varpi_\mathfrak{p}^\beta\\ -\varpi_\mathfrak{p}^{n+m-\beta} & \end{pmatrix} \in \mathbf{GL}_2(F_\mathfrak{p}), & A(a,n) := \begin{pmatrix}\frac{a\tau_\mathfrak{p}-\bar\tau_\mathfrak{p}}{\varpi_\mathfrak{p}^\beta} & \\ -\varpi_\mathfrak{p}^{n-\beta} & \frac{\tau_\mathfrak{p}-\bar\tau_\mathfrak{p}}{a\tau_\mathfrak{p}-\bar\tau_\mathfrak{p}}\varpi_\mathfrak{p}^{\beta-m}\end{pmatrix}, \quad \text{if } s^{-1}\in\mathfrak{p}.\end{cases}$$

The matrices $A(a,n)^{-1}$ have bounded denominators. Indeed, the determinant is in $\mathcal{O}_{K_\mathfrak{p}}^\times$ by the definition of m, and the coefficients have valuation not less than $\epsilon/2$, where $\epsilon = 0$, if $T(F_\mathfrak{p})$ inert, and $\epsilon = 1$, if $T(F_\mathfrak{p})$ ramifies. We obtain that

$$\int_{U_\mathfrak{p}(a,n)}\left(\frac{u_\mathfrak{p}-a}{\varpi_\mathfrak{p}^n}\right)^{\underline{m}} u_\mathfrak{p}^{-\frac{\underline{k}}{2}} d\delta_p^*\varphi(u_\mathfrak{p}) = \frac{c(a,n)}{(\alpha_\mathfrak{p}^*)^n\cdot\det(M_T)^{\frac{\underline{k}}{2}}}\cdot\varphi\left({\gamma_{s,n}}^{-1}V\right)\left({\gamma_{s,n}}^{-1}A(a,n)^{-1}\bigotimes_{\sigma\in\Sigma_\mathfrak{p}} y_\sigma^{m_\sigma}x_\sigma^{k_\sigma-m_\sigma}\right)$$

$$= \frac{c(a,n)}{(\alpha_{\mathfrak{p}}^*)^n \cdot \det(M_T)^{\frac{k}{2}}} \theta_V(\varphi)\left(\gamma_{a,n}{}^{-1}\right)\left(A(a,n)^{-1} \bigotimes_{\sigma\in\Sigma_{\mathfrak{p}}} y_\sigma^{m_\sigma} x_\sigma^{k_\sigma - m_\sigma}\right) \in \frac{A_{\mathfrak{p}}}{p^{n v_p(\alpha_{\mathfrak{p}}^*)}} \mathcal{O}_{\mathbb{C}_p},$$

where $A_{\mathfrak{p}}$ are similarly as above with an extra control of the bounded denominators of $A(a,n)^{-1} \bigotimes_{\sigma\in\Sigma_{\mathfrak{p}}} y_\sigma^{m_\sigma} x_\sigma^{k_\sigma-m_\sigma}$. The same argument as above shows that the distribution is $v_p(\alpha_{\mathfrak{p}}^*)$-admissible for all $\mathfrak{p} \mid p$ in this setting.

In summary, since $\theta_V \phi_\lambda^p \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}\left(\mathrm{coInd}_{C_p}^{G(F_p)}(V(\underline{k})_{\overline{\mathbb{Z}}_p})\right)\right)^\lambda$, we deduce that

$$\delta_p^* \phi_\lambda^p \in H^u\left(T(F)_+, \mathcal{A}^{p\cup\infty}\left(\mathrm{Dist}_{\underline{k}}(T(F_p), \overline{\mathbb{Q}}_p)^{h_p}\right)\right)^\lambda, \qquad h_p = (v_p(\alpha_{\mathfrak{p}}^*)),$$

where superindex h_p means $v_p(\alpha_{\mathfrak{p}}^*)$-admissible for all $\mathfrak{p} \mid p$. The result follows by Proposition 5.1.2. □

5.1.3 The morphism δ_p

Assume that $G(F_p) = \mathbf{PGL}_2(F_p)$. As seen in the previous §, we want to construct a morphism

$$\delta_p = (\delta_{\mathfrak{p}})_{\mathfrak{p}|p} : C_c^0(T(F_p), \overline{\mathbb{Q}}) \longrightarrow V_p, \qquad \delta_{\mathfrak{p}} : C_c^0(T(F_{\mathfrak{p}}), \overline{\mathbb{Q}}) \longrightarrow V_{\mathfrak{p}}.$$

satisfying relation (5.7).

Let us fix a place $\mathfrak{p} \mid p$, and let $\pi_{\mathfrak{p}}$ be the local representation. From now on we will do the following assumptions:

Hypothesis 5.1.5. *Let P be the subgroup of upper triangular matrices, then we assume that $P \cap \imath(K_{\mathfrak{p}}^\times) = F_{\mathfrak{p}}^\times$. Moreover we will assume that $\pi_{\mathfrak{p}}$ is either principal series $\mathcal{B}(\hat{\chi}_{\mathfrak{p}})$ or Steinberg $\sigma'(\hat{\chi}_{\mathfrak{p}})$ (see §2.4.1).*

By the previous assumption, $V_{\mathfrak{p}}$ is a quotient of

$$\mathrm{Ind}_P^G(\hat{\chi}_{\mathfrak{p}})^0 = \left\{ f \in \mathbf{GL}_2(F_{\mathfrak{p}}) \to \overline{\mathbb{Q}}, \text{ locally constant } f\left(\begin{pmatrix} x_1 & y \\ & x_2 \end{pmatrix} g\right) = \hat{\chi}_{\mathfrak{p}}\left(\frac{x_1}{x_2}\right) \cdot f(g) \right\},$$

for a locally constant character $\hat{\chi}_{\mathfrak{p}}$. We construct

$$\begin{aligned} \delta_{\mathfrak{p}} : C_c^0(T(F_{\mathfrak{p}}), \overline{\mathbb{Q}}) &\longrightarrow \mathrm{Ind}_P^G(\hat{\chi}_{\mathfrak{p}})^0 \\ f &\longmapsto \delta_{\mathfrak{p}}(f)(g) := \begin{cases} \hat{\chi}_{\mathfrak{p}}\left(\frac{x_1}{x_2}\right) \cdot f(t^{-1}), & g = \left(\begin{smallmatrix} x_1 & y \\ & x_2 \end{smallmatrix}\right) \imath(t) \in P\imath(K_{\mathfrak{p}}^\times) \\ 0, & g \notin P\imath(K_{\mathfrak{p}}^\times) \end{cases} \end{aligned} \tag{5.8}$$

where $\imath$ is the embedding fixed in (1.10). It is clearly $T(F_{\mathfrak{p}})$-equivariant. Moreover, it induces the wanted $T(F_{\mathfrak{p}})$-equivariant morphism

$$\delta_{\mathfrak{p}} : C_c^0(T(F_{\mathfrak{p}}), \overline{\mathbb{Q}}) \longrightarrow V_{\mathfrak{p}}.$$

The following result is a generalization of [BM21, Lemma 5.2] for more general induced representations:

Lemma 5.1.6. *The morphism $\delta_\mathfrak{p}$ is given by*

$$\delta_\mathfrak{p} : C_c(T(F_\mathfrak{p}), \overline{\mathbb{Q}}) \longrightarrow \mathrm{Ind}_P^G(\hat{\chi}_\mathfrak{p})^0; \qquad \delta_\mathfrak{p}(f)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \hat{\chi}_\mathfrak{p}\left(\frac{ad-bc}{(c\tau_\mathfrak{p}+d)(c\overline{\tau}_\mathfrak{p}+d)}\right) \cdot f\left(\frac{c\overline{\tau}_\mathfrak{p}+d}{c\tau_\mathfrak{p}+d}\right).$$

In particular, if $T(F_\mathfrak{p})$ does not split then $\delta_\mathfrak{p}$ is bijective.

Proof. From the relations

$$\iota(t)\begin{pmatrix} \tau_\mathfrak{p} \\ 1 \end{pmatrix} = \lambda_t \begin{pmatrix} \tau_\mathfrak{p} \\ 1 \end{pmatrix}, \qquad \iota(t)\begin{pmatrix} \overline{\tau}_\mathfrak{p} \\ 1 \end{pmatrix} = \bar{\lambda}_t \begin{pmatrix} \overline{\tau}_\mathfrak{p} \\ 1 \end{pmatrix};$$

we deduce

$$\iota(t) = \frac{1}{\tau_\mathfrak{p} - \overline{\tau}_\mathfrak{p}} \begin{pmatrix} \lambda_t\tau_\mathfrak{p} - \overline{\lambda}_t\overline{\tau}_\mathfrak{p} & \tau_\mathfrak{p}\overline{\tau}_\mathfrak{p}(\overline{\lambda}_t - \lambda_t) \\ \lambda_t - \overline{\lambda}_t & \overline{\lambda}_t\tau_\mathfrak{p} - \lambda_t\overline{\tau}_\mathfrak{p} \end{pmatrix}.$$

Hence, if we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x_1 & y \\ & x_2 \end{pmatrix}\iota(t) = \frac{\overline{\lambda}_t}{\tau_\mathfrak{p} - \overline{\tau}_\mathfrak{p}} \begin{pmatrix} x_1 & y \\ & x_2 \end{pmatrix} \begin{pmatrix} t\tau_\mathfrak{p} - \overline{\tau}_\mathfrak{p} & \tau_\mathfrak{p}\overline{\tau}_\mathfrak{p}(1-t) \\ t-1 & \tau_\mathfrak{p} - t\overline{\tau}_\mathfrak{p} \end{pmatrix}$$

where $t = \lambda_t/\overline{\lambda}_t \in T(F_\mathfrak{p})$, we obtain the identities

$$-\frac{d}{c} = \frac{t^{-1}\tau_\mathfrak{p} - \overline{\tau}_\mathfrak{p}}{t^{-1}-1}, \qquad ad-bc = x_1x_2\lambda_t\overline{\lambda}_t, \qquad t^{-1} = \frac{\overline{\tau}_\mathfrak{p} + \frac{d}{c}}{\tau_\mathfrak{p} + \frac{d}{c}} = \frac{c\overline{\tau}_\mathfrak{p}+d}{c\tau_\mathfrak{p}+d}, \qquad c = \frac{x_2(t-1)\overline{\lambda}_t}{\tau_\mathfrak{p} - \overline{\tau}_\mathfrak{p}}.$$

Thus, we obtain

$$\begin{aligned} \delta_\mathfrak{p}(f)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \hat{\chi}_\mathfrak{p}\left(\frac{(ad-bc)}{c^2}\frac{(t-1)^2}{t(\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p})^2}\right) \cdot f\left(\frac{c\overline{\tau}_\mathfrak{p}+d}{c\tau_\mathfrak{p}+d}\right) \\ &= \hat{\chi}_\mathfrak{p}\left(\frac{(ad-bc)}{c^2}\frac{\left(\frac{c\tau_\mathfrak{p}+d}{c\overline{\tau}_\mathfrak{p}+d}-1\right)\left(1-\frac{c\overline{\tau}_\mathfrak{p}+d}{c\tau_\mathfrak{p}+d}\right)}{(\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p})^2}\right) \cdot f\left(\frac{c\overline{\tau}_\mathfrak{p}+d}{c\tau_\mathfrak{p}+d}\right), \end{aligned}$$

and the result follows. □

Let us check that it satisfies the relations (5.7): We aim to compute for n big enough:

$$\gamma_{a,n}\delta_\mathfrak{p}\left(\mathbb{1}_{U_\mathfrak{p}(a,n)}\right), \qquad \begin{cases} \gamma_{a,n} = \begin{pmatrix} 1 & a \\ & \varpi_\mathfrak{p}^n \end{pmatrix} M_T, & \text{if } T(F_\mathfrak{p}) \text{ splits}, \\ \gamma_{a,n} = \begin{pmatrix} \varpi_\mathfrak{p}^\beta & s\varpi_\mathfrak{p}^\beta \\ & \varpi_\mathfrak{p}^{n+m-\beta} \end{pmatrix}, & \text{if } T(F_\mathfrak{p}) \text{ is non-split and } s = -M_T(-a) \in \mathcal{O}_{F_\mathfrak{p}}, \\ \gamma_{a,n} := \begin{pmatrix} s^{-1}\varpi_\mathfrak{p}^\beta & \varpi_\mathfrak{p}^\beta \\ -\varpi_\mathfrak{p}^{n+m-\beta} & \end{pmatrix}, & \text{if } T(F_\mathfrak{p}) \text{ is non-split and } s = -M_T(-a) \notin \mathcal{O}_{F_\mathfrak{p}}. \end{cases}$$

Split case

Assuming that $T(F_\mathfrak{p})$ splits, we can choose $M_T = \left(\begin{smallmatrix} 1 & -\overline{\tau}_\mathfrak{p} \\ -1 & \tau_\mathfrak{p} \end{smallmatrix}\right)$. Thus, we have

$$\gamma_{a,n}\delta_\mathfrak{p}\left(\mathbb{1}_{U_\mathfrak{p}(a,n)}\right)(\iota(t){M_T}^{-1}) = \delta_\mathfrak{p}(\mathbb{1}_{U_\mathfrak{p}(a,n)})\left(\frac{1}{(\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p})^2}\left(\begin{smallmatrix} \lambda_t\tau_\mathfrak{p}-\overline{\lambda}_t\overline{\tau}_\mathfrak{p} & \tau_\mathfrak{p}\overline{\tau}_\mathfrak{p}(\overline{\lambda}_t-\lambda_t) \\ \lambda_t-\overline{\lambda}_t & \overline{\lambda}_t\tau_\mathfrak{p}-\lambda_t\overline{\tau}_\mathfrak{p} \end{smallmatrix}\right)\left(\begin{smallmatrix} \tau_\mathfrak{p} & \overline{\tau}_\mathfrak{p} \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & a \\ & \varpi_\mathfrak{p}^n \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -\overline{\tau}_\mathfrak{p} \\ -1 & \tau_\mathfrak{p} \end{smallmatrix}\right)\right)$$

$$\begin{aligned}
&= \delta_\mathfrak{p}(\mathbb{1}_{U_\mathfrak{p}(a,n)})\left(\frac{\lambda_t}{(\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p})}\begin{pmatrix} \tau_\mathfrak{p}(1-a)-\varpi_\mathfrak{p}^n t^{-1}\overline{\tau}_\mathfrak{p} & a\tau_\mathfrak{p}^2+\tau_\mathfrak{p}\overline{\tau}_\mathfrak{p}(\varpi_\mathfrak{p}^n t^{-1}-1) \\ 1-a-\varpi_\mathfrak{p}^n t^{-1} & a\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p}+\varpi_\mathfrak{p}^n t^{-1}\tau_\mathfrak{p} \end{pmatrix}\right) \\
&= \hat{\chi}_\mathfrak{p}\left(\frac{\varpi_\mathfrak{p}^n}{t(a+\varpi_\mathfrak{p}^n t^{-1})}\right)\cdot\mathbb{1}_{U_\mathfrak{p}(a,n)}\left(a+\varpi_\mathfrak{p}^n t^{-1}\right) = \frac{\hat{\chi}_\mathfrak{p}(\varpi_\mathfrak{p})^n}{\hat{\chi}_\mathfrak{p}(a)}\cdot\hat{\chi}_\mathfrak{p}(t^{-1})\cdot\mathbb{1}_{\mathcal{O}_{F_\mathfrak{p}}}(t^{-1}),
\end{aligned}$$

if n is bigger then the conductor of $\chi_\mathfrak{p}$. We obtain that $\delta_\mathfrak{p}$ satisfies (5.7) with $\alpha_\mathfrak{p} = \hat{\chi}_\mathfrak{p}(\varpi_\mathfrak{p})^{-1}$ and $V = M_T V_0$ where

$$V_0(\iota(t)) = \hat{\chi}_\mathfrak{p}(t^{-1})\cdot\mathbb{1}_{\mathcal{O}_{F_\mathfrak{p}}}(t^{-1}).$$

Non-split case with $s = -M_T(-a) \in \mathcal{O}_{F_\mathfrak{p}}$

Write the bijective map

$$\varphi : T(F_\mathfrak{p}) \longrightarrow \mathbb{P}^1(F_\mathfrak{p}); \qquad t \mapsto \iota(t)\cdot\infty = \frac{t\tau_\mathfrak{p}-\bar{\tau}_\mathfrak{p}}{t-1}.$$

We compute for $n-\beta$ bigger than the conductor of χ

$$\left(\begin{smallmatrix} \varpi_\mathfrak{p}^\beta & s\varpi_\mathfrak{p}^\beta \\ & \varpi_\mathfrak{p}^{n+m-\beta} \end{smallmatrix}\right)\delta_\mathfrak{p}\left(\mathbb{1}_{U_\mathfrak{p}(a,n)}\right)(\iota(t)) = \delta_\mathfrak{p}\left(\mathbb{1}_{U_\mathfrak{p}(a,n)}\right)\left(\frac{1}{\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p}}\left(\begin{smallmatrix} \lambda_t\tau_\mathfrak{p}-\overline{\lambda}_t\overline{\tau}_\mathfrak{p} & \tau_\mathfrak{p}\overline{\tau}_\mathfrak{p}(\overline{\lambda}_t-\lambda_t) \\ \lambda_t-\overline{\lambda}_t & \overline{\lambda}_t\tau_\mathfrak{p}-\lambda_t\overline{\tau}_\mathfrak{p} \end{smallmatrix}\right)\left(\begin{smallmatrix} \varpi_\mathfrak{p}^\beta & s\varpi_\mathfrak{p}^\beta \\ & \varpi_\mathfrak{p}^{n+m-\beta} \end{smallmatrix}\right)\right)$$

$$\begin{aligned}
&= \delta_\mathfrak{p}\left(\mathbb{1}_{U_\mathfrak{p}(a,n)}\right)\left(\frac{\overline{\lambda}_t(t-1)}{\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p}}\left(\begin{smallmatrix} \varphi(t) & -\tau_\mathfrak{p}\overline{\tau}_\mathfrak{p} \\ 1 & -\varphi(t^{-1}) \end{smallmatrix}\right)\left(\begin{smallmatrix} \varpi_\mathfrak{p}^\beta & s\varpi_\mathfrak{p}^\beta \\ & \varpi_\mathfrak{p}^{n+m-\beta} \end{smallmatrix}\right)\right) \\
&= \hat{\chi}_\mathfrak{p}\left(\frac{\varpi_\mathfrak{p}^{n+m-2\beta}\frac{(1-a)^2}{a}\frac{t}{(t-1)^2}}{\mathrm{N}_{K/F}\left(1-\frac{\varphi(t^{-1})\varpi_\mathfrak{p}^{m+n-2\beta}}{(1-a)^{-1}(\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p})}\right)}\right)\cdot\mathbb{1}_{U_\mathfrak{p}(a,n)}\left(a+\varpi_\mathfrak{p}^n\frac{\varphi(t^{-1})(1-a)^2\varpi_\mathfrak{p}^{-2\beta}}{\varphi(t^{-1})\varpi_\mathfrak{p}^{n-2\beta}(1-a)+\frac{\bar{\tau}_\mathfrak{p}-\tau_\mathfrak{p}}{\varpi_\mathfrak{p}^m}}\right) \\
&= \hat{\chi}_\mathfrak{p}\left(\varpi_\mathfrak{p}^{n+m-2\beta}\frac{(1-a)^2}{a}\frac{t}{(t-1)^2}\right)\cdot\mathbb{1}_{\mathcal{O}_{K_\mathfrak{p}}}\left(\frac{(1-a)\varpi_\mathfrak{p}^{-\beta}}{\varpi_\mathfrak{p}^{n-\beta}-\frac{s+\tau_\mathfrak{p}}{\varpi_\mathfrak{p}^{m-\beta}}\varphi(t^{-1})^{-1}}\right) \\
&= \hat{\chi}_\mathfrak{p}(\varpi_\mathfrak{p})^{n+m}\cdot\hat{\chi}_\mathfrak{p}\left(\frac{(a-1)^2}{\varpi_\mathfrak{p}^{2\beta}a}\right)\cdot\hat{\chi}_\mathfrak{p}\left(\frac{t}{(t-1)^2}\right)\cdot\mathbb{1}_{\varphi^{-1}(\mathfrak{p}^{-\epsilon})}\left(t^{-1}\right),
\end{aligned}$$

where $\epsilon = 0$ if $T(F_\mathfrak{p})$ is inert and $\epsilon = 1$ if it ramifies. Since $v_\mathfrak{p}\left(\frac{(a-1)^2}{\varpi_\mathfrak{p}^{2\beta}a}\right) = \epsilon$, we obtain that $\delta_\mathfrak{p}$ satisfies (5.7) with $\alpha_\mathfrak{p} = \hat{\chi}_\mathfrak{p}(\varpi_\mathfrak{p})^{-1}$ and

$$V(\iota(t)) = \hat{\chi}_\mathfrak{p}\left(\frac{t}{(t-1)^2}\right)\cdot\mathbb{1}_{\varphi^{-1}(\mathfrak{p}^{-\epsilon})}\left(t^{-1}\right).$$

Non-split case with $s=-M_T(-a)\notin\mathcal{O}_{F_\mathfrak{p}}$

We compute for $\max\{n-\beta,n-\beta+v_\mathfrak{p}(\tau)\}$ bigger than the conductor of $\chi_\mathfrak{p}$

$$\left(\begin{smallmatrix} s^{-1}\varpi_\mathfrak{p}^\beta & \varpi_\mathfrak{p}^\beta \\ -\varpi_\mathfrak{p}^{n+m-\beta} & \end{smallmatrix}\right)\delta_\mathfrak{p}\left(\mathbb{1}_{U_\mathfrak{p}(a,n)}\right)(\imath(t))=\delta_\mathfrak{p}\left(\mathbb{1}_{U_\mathfrak{p}(a,n)}\right)\left(\frac{\overline{\lambda}_t(t-1)}{\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p}}\left(\begin{smallmatrix}\varphi(t) & -\tau_\mathfrak{p}\overline{\tau}_\mathfrak{p}\\ 1 & -\varphi(t^{-1})\end{smallmatrix}\right)\left(\begin{smallmatrix} s^{-1}\varpi_\mathfrak{p}^\beta & \varpi_\mathfrak{p}^\beta \\ -\varpi_\mathfrak{p}^{n+m-\beta} & \end{smallmatrix}\right)\right)$$

$$\begin{aligned}
&= \hat{\chi}_\mathfrak{p}\left(\frac{\varpi_\mathfrak{p}^{n+m-2\beta}\frac{(a\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p})^2}{a}\frac{t}{(t-1)^2}}{\mathrm{N}_{K/F}\left(1-\frac{\varphi(t^{-1})\tau_\mathfrak{p}\varpi_\mathfrak{p}^{m+n-2\beta}}{(a\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p})^{-1}(\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p})}\right)}\right)\cdot\mathbb{1}_{U_\mathfrak{p}(a,n)}\left(a+\varpi_\mathfrak{p}^n\frac{(\overline{\tau}_\mathfrak{p}-a\tau_\mathfrak{p})\varpi_\mathfrak{p}^{-\beta}\varphi(t^{-1})}{\varpi_\mathfrak{p}^{n-\beta}\tau_\mathfrak{p}\varphi(t^{-1})+\frac{1+\tau_\mathfrak{p}s^{-1}}{\varpi_\mathfrak{p}^{m-\beta}}}\right)\\
&= \hat{\chi}_\mathfrak{p}\left(\varpi_\mathfrak{p}^{n+m-2\beta}\frac{(a\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p})^2}{a}\frac{t}{(t-1)^2}\right)\cdot\mathbb{1}_{\mathcal{O}_{K_\mathfrak{p}}}\left(\frac{(\overline{\tau}_\mathfrak{p}-a\tau_\mathfrak{p})\varpi_\mathfrak{p}^{-\beta}}{\varpi_\mathfrak{p}^{n-\beta}\tau_\mathfrak{p}+\frac{1+\tau_\mathfrak{p}s^{-1}}{\varpi_\mathfrak{p}^{m-\beta}}\varphi(t^{-1})^{-1}}\right)\\
&= \hat{\chi}_\mathfrak{p}(\varpi_\mathfrak{p})^{n+m}\cdot\hat{\chi}_\mathfrak{p}\left(\frac{(a\tau_\mathfrak{p}-\overline{\tau}_\mathfrak{p})^2}{\varpi_\mathfrak{p}^{2\beta}a}\right)\cdot\hat{\chi}_\mathfrak{p}\left(\frac{t}{(t-1)^2}\right)\cdot\mathbb{1}_{\varphi^{-1}(\mathfrak{p}^{-\epsilon})}\left(t^{-1}\right).
\end{aligned}$$

Again since $v_\mathfrak{p}\left(\frac{(\tau_\mathfrak{p}a-\overline{\tau}_\mathfrak{p})^2}{\varpi_\mathfrak{p}^{2\beta}s}\right)=\epsilon$, we obtain that $\delta_\mathfrak{p}$ satisfies (5.7) with $\alpha_\mathfrak{p}$ and V as above.

Remark 5.1.7. *By Proposition 5.1.3, using the above δ_p the p-adic distribution extends to a locally analytic measure if $e_\mathfrak{p}v_p(\alpha_\mathfrak{p}^*)<\min_{\sigma\in\Sigma_\mathfrak{p}}(k_\sigma+1)$ where $\alpha_\mathfrak{p}^*=\hat{\chi}_\mathfrak{p}(\varpi_\mathfrak{p})^{-1}\varpi_\mathfrak{p}^{\frac{k}{2}}$. If this is the case we say that ϕ_λ^p has* non-critical slope.

5.1.4 Local integrals

Given the morphism $\delta_\mathfrak{p}$ defined in §5.1.3, we aim to calculate in this section the following integrals

$$\int_{T(F_\mathfrak{p})}\xi_\mathfrak{p}(t)\langle t\delta_\mathfrak{p}(1_H),\delta_\mathfrak{p}(1_H)\rangle d^\times t, \tag{5.9}$$

where $\xi_\mathfrak{p}$ is a locally constant character, $H\subset T(\mathcal{O}_{F_\mathfrak{p}})$ is an open and compact subgroup small enough so that $\xi_\mathfrak{p}$ is H-invariant, $d^\times$ is a Haar measure of $T(F_\mathfrak{p})$, and $\langle\cdot,\cdot\rangle$ is the natural $G(F_\mathfrak{p})$-equivariant pairing on $V_\mathfrak{p}$.

Since $\pi_\mathfrak{p}$ is unitary, by [Bum98, Proposition 4.6.11] the character $\hat{\chi}_\mathfrak{p}|\cdot|^{-\frac{1}{2}}$ is either unitary or real. In our setting, the second case corresponds to the Steinberg representations, namely $\hat{\chi}_\mathfrak{p}=\pm1$.

If $\hat{\chi}_\mathfrak{p}|\cdot|^{-\frac{1}{2}}$ is unitary, by [Bum98, Proposition 4.5.5] and [Bla19, Corollary 8.2] we have a $G(F_\mathfrak{p})$-invariant pairing

$$\langle\cdot,\cdot\rangle:\mathrm{Ind}_P^G(\hat{\chi}_\mathfrak{p})^0\times\mathrm{Ind}_P^G(\hat{\chi}_\mathfrak{p})^0\longrightarrow\mathbb{C};\qquad \langle f_1,f_2\rangle_{\chi_\mathfrak{p}}=\int_{T(F_\mathfrak{p})}f_1(\imath(\tau))\cdot\overline{f_2(\imath(\tau))}d^\times\tau.$$

We deduce

$$\langle t\delta_\mathfrak{p}(1_H),\delta_\mathfrak{p}(1_H)\rangle=\int_{T(F_\mathfrak{p})}\delta_\mathfrak{p}(1_{tH})(\imath(\tau))\cdot\overline{\delta_\mathfrak{p}(1_H)(\imath(\tau))}d^\times\tau$$

$$= \int_{T(F_\mathfrak{p})} 1_{tH}(\tau^{-1}) \cdot 1_H(\tau^{-1}) d^\times\tau = \mathrm{vol}(H) \cdot \mathbb{1}_H(t).$$

Thus we obtain

$$\int_{T(F_\mathfrak{p})} \xi_\mathfrak{p}(t)\langle t\delta_\mathfrak{p}(1_H), \delta_\mathfrak{p}(1_H)\rangle d^\times t = \mathrm{vol}(H)\int_H \xi_\mathfrak{p}(t) d^\times t = \mathrm{vol}(H)^2.$$

If $\hat{\chi}_\mathfrak{p} = \pm 1$ the computation is much more complicated but it can be found in [Bla19, §3.5]. Up to a constant depending on T, we have

$$\int_{T(F_\mathfrak{p})} \xi_\mathfrak{p}(t)\langle t\delta_\mathfrak{p}(1_H), \delta_\mathfrak{p}(1_H)\rangle d^\times t = \begin{cases} \mathrm{vol}(H)^2 \cdot L(\frac{1}{2}, \pi_\mathfrak{p}, \xi_\mathfrak{p}) \cdot L(-\frac{1}{2}, \pi_\mathfrak{p}, \xi_\mathfrak{p})^{-1}, & \mathrm{cond}(\xi_\mathfrak{p}) = 0, \\ \mathrm{vol}(H)^2 \cdot q^{n_{\xi_\mathfrak{p}}}, & \mathrm{cond}(\xi_\mathfrak{p}) = n_{\xi_\mathfrak{p}}, \end{cases}$$

where, if we write $\alpha_\mathfrak{p} = \hat{\chi}_\mathfrak{p}(\varpi_\mathfrak{p}) = \pm 1$

$$L(s, \pi_\mathfrak{p}, \xi_\mathfrak{p}) = \begin{cases} (1 - \alpha_\mathfrak{p}\xi_\mathfrak{p}(\varpi_\mathfrak{p})q^{-s-\frac{1}{2}})^{-1}(1 - \alpha_\mathfrak{p}\xi_\mathfrak{p}(\varpi_\mathfrak{p})^{-1}q^{-s-\frac{1}{2}})^{-1}, & T(F_\mathfrak{p}) \text{ splits}, \\ (1 - q^{-1-2s})^{-1}, & T(F_\mathfrak{p}) \text{ inert}, \\ (1 - \alpha_\mathfrak{p}\xi_\mathfrak{p}(\varpi_\mathfrak{p}^{\frac{1}{2}})q^{-s-\frac{1}{2}})^{-1}, & T(F_\mathfrak{p}) \text{ ramifies}. \end{cases} \tag{5.10}$$

Recall that in the ramified case $\varpi_\mathfrak{p}^{\frac{1}{2}}$ denotes the uniformizer of $K_\mathfrak{p}$.

Let $J \in G(F)$ such that $J \cdot \imath(t) = \imath(\bar{t}) \cdot J$ for all $t \in T(F)$. We write $J = \begin{pmatrix} J_1 & x \\ & J_2 \end{pmatrix} \cdot \imath(t_J)$ for some $t_J \in T(F_\mathfrak{p})$ and $J_i \in F_\mathfrak{p}^\times$. We compute

$$J\delta_\mathfrak{p}(f)(\imath(t)) = \delta_\mathfrak{p}(f)(\imath(t)\cdot J) = \delta_\mathfrak{p}(f)(J\cdot\imath(\bar{t})) = \hat{\chi}_\mathfrak{p}\left(\frac{J_1}{J_2}\right)\cdot\delta_\mathfrak{p}(f)(\imath(t_J\cdot t^{-1})) = \hat{\chi}_\mathfrak{p}\left(\frac{J_1}{J_2}\right)\cdot\delta_\mathfrak{p}(f^*)(\imath(t)),$$

where $f^*(t) = f(t^{-1}t_J)$. Since $J^2 \in F^\times$, we deduce that $\hat{\chi}_\mathfrak{p}\left(\frac{J_1}{J_2}\right) = \pm 1$. We aim to compute as well the integral

$$\int_{T(F_\mathfrak{p})} \xi_\mathfrak{p}(t)\langle t\delta_\mathfrak{p}(1_H), J\delta_\mathfrak{p}(1_H)\rangle d^\times t,$$

Using the above computation

$$\begin{aligned} \int_{T(F_\mathfrak{p})} \xi_\mathfrak{p}(t)\langle t\delta_\mathfrak{p}(1_H), J\delta_\mathfrak{p}(1_H)\rangle d^\times t &= \pm\int_{T(F_\mathfrak{p})} \xi_\mathfrak{p}(t)\langle t\delta_\mathfrak{p}(1_H), \delta_\mathfrak{p}(1_{t_J H})\rangle d^\times t \\ &= \pm\int_{T(F_\mathfrak{p})} \xi_\mathfrak{p}(t)\langle t\delta_\mathfrak{p}(1_H), t_J\delta_\mathfrak{p}(1_H)\rangle d^\times t \\ &= \pm\xi_\mathfrak{p}(t_J)\cdot\int_{T(F_\mathfrak{p})} \xi_\mathfrak{p}(t)\langle t\delta_\mathfrak{p}(1_H), \delta_\mathfrak{p}(1_H)\rangle d^\times t. \end{aligned}$$

Thus, we obtain

$$\int_{T(F_\mathfrak{p})} \xi_\mathfrak{p}(t)\langle t\delta_\mathfrak{p}(1_H), J\delta_\mathfrak{p}(1_H)\rangle d^\times t = \begin{cases} \pm\xi_\mathfrak{p}(t_J)\cdot\mathrm{vol}(H)^2 \cdot L(\frac{1}{2}, \pi_\mathfrak{p}, \xi_\mathfrak{p}) \cdot L(-\frac{1}{2}, \pi_\mathfrak{p}, \xi_\mathfrak{p})^{-1}, & \mathrm{cond}(\xi_\mathfrak{p}) = 0, \\ \pm\xi_\mathfrak{p}(t_J)\cdot\mathrm{vol}(H)^2 \cdot q^{n_{\xi_\mathfrak{p}}}, & \mathrm{cond}(\xi_\mathfrak{p}) = n_{\xi_\mathfrak{p}}. \end{cases}$$

5.1.5 Interpolation properties

As we have previously emphasized, we have to think of $\mu_{\phi_\lambda^p}$ as a generalization of Bertolini-Darmon anticyclotomic p-adic L-function. Hence it needs to have a link with the classical L-function, namely, an interpolation property.

Definition 5.1.8. *Let $\xi \in C_{\underline{k}}(\mathcal{G}_T, \mathbb{C}_p)$ be a locally polynomial character. Thus, in a neighbourhood U of 1 in $T(F_\mathfrak{p})$*

$$\rho^*\xi\mid_U(t_p) = \prod_{\mathfrak{p}\mid p}\prod_{\sigma\in\Sigma_\mathfrak{p}}\sigma(t_\mathfrak{p})^{m_\sigma}, \qquad \underline{m} = (m_\sigma); \quad -\frac{\underline{k}}{2} \leq \underline{m} \leq \frac{\underline{k}}{2}.$$

We define the archimedean avatar of ξ:

$$\tilde{\xi} : T(\mathbb{A}_F)/T(F) \longrightarrow \mathbb{C}^\times; \qquad \tilde{\xi}(t) = \rho^*\xi(t)\cdot\prod_{\mathfrak{p}\mid p}\prod_{\sigma\in\Sigma_\mathfrak{p}}\sigma(t_\mathfrak{p})^{-m_\sigma}\cdot\prod_{\tau\in\infty}\prod_{\sigma\mid\tau}\sigma(t_\tau)^{m_\sigma},$$

once identified the set of embeddings $\sigma : F \hookrightarrow \mathbb{C}$ with $\bigcup_\mathfrak{p}\Sigma_\mathfrak{p}$.

We will write $\rho^*\xi = \prod_v \xi_v$ and $\xi^p = \prod_{v\nmid p}\xi_v$. The following results provides the interpolation property of the distribution $\mu_{\phi_\lambda^p}$:

Theorem 5.1.9. *Given a locally polynomial character $\xi \in C_{\underline{k}}(\mathcal{G}_T, \mathbb{C}_p)$, we have that*

$$\int_{\mathcal{G}_T}\xi d\mu_{\phi_\lambda^p} = \begin{cases} K(x^p,\xi^p)\cdot\binom{\underline{k}}{\underline{k}-\underline{m}}^{-1}\cdot C(\underline{k})\cdot\epsilon_p(\pi_p,\xi_p)\cdot L(1/2,\pi,\tilde{\xi})^{\frac{1}{2}}, & \rho^*\xi\mid_{T(F_\infty)} = \lambda,\\ 0, & \rho^*\xi\mid_{T(F_\infty)} \neq \lambda,\end{cases}$$

where

$$\binom{\underline{k}}{\underline{k}-\underline{m}} = \prod_\sigma\binom{k_\sigma}{k_\sigma - m_\sigma}; \qquad C(\underline{k}) = \prod_{\tau\in\infty}\frac{\left(\sum_{\sigma\mid\tau}(k_\sigma+1)\right)!}{\prod_{\sigma\mid\tau}k_\sigma!},$$

$K(x^p,\xi^p)$ is an explicit constant depending on ξ^p and the image of ϕ_λ^p in $\pi^{p\cup\infty}$,

$$\epsilon_p(\pi_p,\xi_p) = \prod_{\mathfrak{p}\mid p}\varepsilon_\mathfrak{p}(\pi_\mathfrak{p},\xi_\mathfrak{p}); \quad \epsilon_\mathfrak{p}(\pi_\mathfrak{p},\xi_\mathfrak{p})^2 = \begin{cases} L(1/2,\pi_\mathfrak{p},\xi_\mathfrak{p})^{-1}, & \pi_\mathfrak{p}\neq \mathrm{St}_\mathbb{C}(F_\mathfrak{p})(\pm),\\ L(-1/2,\pi_\mathfrak{p},\xi_\mathfrak{p})^{-1}, & \pi_\mathfrak{p}\simeq \mathrm{St}_\mathbb{C}(F_\mathfrak{p})(\pm),\ \mathrm{cond}\xi_\mathfrak{p} = 0,\\ q^{n_x}L(1/2,\pi_\mathfrak{p},\xi_\mathfrak{p})^{-1}, & \pi_\mathfrak{p}\simeq \mathrm{St}_\mathbb{C}(F_\mathfrak{p})(\pm),\ \mathrm{cond}\xi_\mathfrak{p} = n_\xi,\end{cases}$$

and $\mathrm{St}_\mathbb{C}(F_\mathfrak{p})(\pm)$ denotes the Steinberg representation twisted by the character $g \mapsto (\pm 1)^{v_\mathfrak{p}\det(g)}$.

Proof. Let us consider the $T(F)$-equivariant morphism

$$\varphi : (C^0(T(\mathbb{A}_F),\mathbb{C})\otimes\mathcal{P}(\underline{k}))\otimes\mathcal{A}^\infty(V(\underline{k}))(\lambda) \longrightarrow C^0(T(\mathbb{A}_F),\mathbb{C});$$
$$\varphi((f\otimes P)\otimes\phi)(z,t) := f(z,t)\cdot\lambda(z)^{-1}\cdot\phi(t)(P),$$

for all $z \in T(F_\infty)$ and $t \in T(\mathbb{A}_F^\infty)$, and the natural pairing $\langle\cdot,\cdot\rangle_T : C^0(T(\mathbb{A}_F),\mathbb{C}) \times C_c^0(T(\mathbb{A}_F),\mathbb{C}) \to \mathbb{C}$ given by the Haar measure

$$\langle f_1, f_2\rangle_T := \frac{1}{n}\sum_{x\in CC}\int_{T(\mathbb{A}_F^\infty)} f_1(x,t)\cdot f_2(x,t)d^\times t, \qquad n := [T(F):T(F)_+], \quad CC := T(F_\infty)/T(F_\infty)_+.$$

For any $f\in C^0(T(\mathbb{A}_F),\mathbb{C})$ and $f_2\in C_c^0(T(\mathbb{A}_F),\mathbb{C})$, write $f\cdot f_2=f_p\otimes f^p$ and let $H\subseteq T(F_p)$ be a small enough open compact subgroup so that f_p is H-invariant, namely, $f_p=\sum_{t_p\in T(F_p)/H}f_p(t_p)1_{t_pH}$. If we consider the $G(F)$-equivariant morphism $\theta:\mathcal{A}^{p\cup\infty}(V_p,V(\underline{k}))(\lambda)\to\mathcal{A}^{\infty}(V(\underline{k}))(\lambda)$ defined by

$$\theta(\phi)(g_p,g^p):=\phi(g^p)(g_p\delta_p(1_H));\qquad g_p\in G(F_p),\ g^p\in G(\mathbb{A}_F^{p\cup\infty}),$$

we compute using the concrete description of $\langle\cdot|\cdot\rangle$ given in (3.16):

$$\langle\varphi(f\otimes P,\theta\phi),f_2\rangle_T=\frac{1}{n}\sum_{z\in CC}\int_{T(\mathbb{A}_F^\infty)}f(z,t)\cdot\lambda(z)^{-1}\cdot f_2(z,t)\cdot\theta\phi(t)(P)\,d^\times t$$

$$\begin{aligned}
&= \frac{1}{n}\sum_{z\in CC}\lambda(z)^{-1}\cdot\int_{T(\mathbb{A}_F^{p\cup\infty})}\int_{T(F_p)}f^p(z,t^p)\cdot f_p(t_p)\cdot\phi(t^p)(\delta_p(1_{t_pH}))(P)\,d^\times t^p d^\times t_p\\
&= \frac{\mathrm{vol}(H)}{n}\sum_{z\in CC}\lambda(z)^{-1}\cdot\int_{T(\mathbb{A}_F^{p\cup\infty})}f^p(z,t^p)\cdot\delta_p^*\phi(t^p)(f_p)(P)d^\times t^p\\
&= \mathrm{vol}(H)\cdot\langle f\cdot f_2\otimes P|\delta_p^*\phi\rangle,
\end{aligned}$$

for all $\phi\in\mathcal{A}^{p\cup\infty}(V_p,\mathbb{C}),\mathbb{C})(\lambda)$. Hence, we obtain by definition

$$\int_{\mathcal{G}_T}\xi d\mu_{\phi_\lambda^p}=(\rho^*\xi\cap\eta)\cap\delta_p^*\phi_\lambda^p=\frac{1}{\mathrm{vol}(H)}(\tilde{\xi}\cup\theta\phi_\lambda^p)\cap\eta,$$

where the cap and cup products in the third identity correspond to the pairings $\langle\cdot,\cdot\rangle_T$ and φ. In [Mol22, Theorem 4.25] an expression for $(\tilde{\xi}\cup\theta\phi_\lambda^p)\cap\eta$ is obtained in terms of the classical L-function:

$$(\tilde{\xi}\cup\theta\phi_\lambda^p)\cap\eta=\begin{cases}K\cdot\binom{\underline{k}}{\underline{k}-\underline{m}}^{-1}\cdot C(\underline{k})\cdot L(1/2,\pi,\tilde{\xi})^{\frac{1}{2}}\cdot\prod_{\nu\nmid\infty}\alpha(x_\nu)^{\frac{1}{2}}, & \rho^*\xi\mid_{T(F_\infty)}=\lambda,\\ 0, & \rho^*\xi\mid_{T(F_\infty)}\neq\lambda,\end{cases}\tag{5.11}$$

where $K\in\mathbb{C}$ is a non-zero explicit constant only depending on T and π, $x_v\in\pi_v$ is the image of $\theta\phi_\lambda^p$ in the corresponding local representation, and

$$\alpha(x_\nu)=\frac{L(1,\psi_\nu)}{\zeta_\nu(1)\cdot L(1/2,\pi_\nu,\xi_\nu)}\int_{T(F_\nu)}\xi_\nu(t)\langle\pi_\nu(t)x_\nu,\pi_\nu(J_\nu)x_\nu\rangle_\nu d^\times t,\tag{5.12}$$

are usual local factors appearing in classical Waldspurger formulas (see [Wal85] and [Hsi14] for more details).

The product $K(x^p,\xi^p)=K\cdot\prod_{\nu\notin p\cup\infty}\alpha(x_\nu^p)^{\frac{1}{2}}$ is a constant only depending on x^p and ξ^p. We can apply the formulas obtained in §5.1.4 to compute that, up to a constant factor depending on $T(F_\mathfrak{p})$,

$$\frac{\alpha(\delta_\mathfrak{p}(1_{H_\mathfrak{p}}))}{\mathrm{vol}(H_\mathfrak{p})^2}=\begin{cases}L(1/2,\pi_\mathfrak{p},\xi_\mathfrak{p})^{-1}, & \pi_\mathfrak{p}\neq\mathrm{St}_\mathbb{C}(F_\mathfrak{p})(\pm),\\ L(-1/2,\pi_\mathfrak{p},\xi_\mathfrak{p})^{-1}, & \pi_\mathfrak{p}\simeq\mathrm{St}_\mathbb{C}(F_\mathfrak{p})(\pm),\ \mathrm{cond}\chi_\mathfrak{p}=0,\\ q^{n_\chi}L(1/2,\pi_\mathfrak{p},\xi_\mathfrak{p})^{-1}, & \pi_\mathfrak{p}\simeq\mathrm{St}_\mathbb{C}(F_\mathfrak{p})(\pm),\ \mathrm{cond}\chi_\mathfrak{p}=n_\chi,\end{cases}$$

and the result follows. □

Remark 5.1.10. *If $\pi_\mathfrak{p}\simeq\mathrm{St}_\mathbb{C}(F_\mathfrak{p})(\pm)$ and $\xi_\mathfrak{p}$ is unramified, we observe in (5.10) that if $\xi_\mathfrak{p}(\varpi_\mathfrak{p})=\alpha_\mathfrak{p}=\pm1$ we obtain that $L(-1/2,\pi_\mathfrak{p},\xi_\mathfrak{p})^{-1}=0$. This phenomena is known as* **exceptional zero** *and the aim of the following chapters is to relate this exceptional zeroes with points in the extended Mordell-Weil group.*

5.2 p-adic L-functions attached to modular elliptic curves

We have constructed an admissible distribution $\mu_{\phi_\lambda^p}$ of $\mathcal{G}_T$ attached to an automorphic modular symbol ϕ_λ^p. In case of parallel weight 2, we can also consider a modular symbol

$$\phi_\lambda^S \in H^u(G(F)_+, \mathcal{A}^{\infty\cup S}(V_S, \overline{\mathbb{Q}}))^\lambda,$$

where S is any non-empty set of places $\mathfrak{p}$ above p. Hence, the same formalism as above applies to construct an admissible distribution associated with ϕ_λ^S:

$$\int_{\mathcal{G}_T} g\mu_{\phi_\lambda^S} = \delta(g) \cap \delta_S^* \phi_\lambda^S, \quad g \in C^0(\mathcal{G}_T, \overline{\mathbb{Q}}_p)$$

where

$$\delta : C^0(\mathcal{G}_T, \overline{\mathbb{Q}}_p) \xrightarrow{\rho^*} H^0(T(F), C^0(T(\mathbb{A}_F), \overline{\mathbb{Q}}_p)) \xrightarrow{\cap\eta} H_u(T(F), C_c^0(T(\mathbb{A}_F), \overline{\mathbb{Q}}_p)),$$

and

$$\delta_S : C_c^0(T(F_S), \overline{\mathbb{Q}}) \to V_S; \quad \delta_S = \prod_{\mathfrak{p}\in S} \delta_{\mathfrak{p}}.$$

We can also study the admissibility in this setting, obtaining analogously as in Remark 5.1.7, that if $e_{\mathfrak{p}} v_p(\alpha_{\mathfrak{p}}) = e_{\mathfrak{p}} v_p(\hat{\chi}_{\mathfrak{p}}(\varpi_{\mathfrak{p}})^{-1}) < 1$ where $V_{\mathfrak{p}}$ is a quotient of $\mathrm{Ind}_P^G \hat{\chi}_{\mathfrak{p}}$, then $\mu_{\phi_\lambda^S}$ extends to a distribution that is locally analytic at places $\mathfrak{p} \in S$.

5.3 Overconvergent modular symbols

In this section we extend non-critical modular symbols to overconvergent modular symbols. We will describe our admissible distributions in terms of the corresponding overconvergent modular symbols.

5.3.1 Distributions

Let us consider

$$\mathcal{L}_{\mathfrak{p}} := \left\{(x,y) \in \mathcal{O}_{F_{\mathfrak{p}}} \times \mathcal{O}_{F_{\mathfrak{p}}};\ (x,y) \notin \mathfrak{p}\times\mathfrak{p}\right\}, \qquad \mathcal{L}_p := \prod_{\mathfrak{p}} \mathcal{L}_{\mathfrak{p}}.$$

For any complete $\mathbb{Z}_p$-algebra R, and any continuous character $\chi_p : \mathcal{O}_{F_p}^\times \to R^\times$, let us consider the space of homogeneous functions

$$C_{\chi_p}(\mathcal{L}_p, R) = \left\{f : \mathcal{L}_p \to R, \text{ continuous s.t. } f(ax, ay) = \chi_p(a)\cdot f(x,y), \text{ for } a \in \mathcal{O}_{F_p}^\times\right\}.$$

We write $\mathbb{D}_{\chi_p}(R)$ for the R-dual space of $C_{\chi_p}(\mathcal{L}_p, R)$, namely,

$$\mathbb{D}_\chi(R) := \mathbf{Hom}(C_{\chi_p}(\mathcal{L}_p, R), R).$$

For any continuous extension $\hat{\chi}_p : F_p^\times \to R^\times$ of χ_p, we can consider the induced representation

$$\mathrm{Ind}_P^G(\hat{\chi}_p) = \left\{f : G(F_p) \to R, \text{ continuous }, f\left(\begin{pmatrix} x_1 & y \\ & x_2\end{pmatrix} g\right) = \hat{\chi}_p\left(\frac{x_1}{x_2}\right)\cdot f(g)\right\},$$

A choice of the extension $\hat{\chi}_p$ provides a natural $G(F_p)$-action on $C_{\chi_p^{-2}}(\mathcal{L}_p, R)$ since we have an isomorphism

$$\varphi_p : C_{\chi_p^{-2}}(\mathcal{L}_p, R) \to \mathrm{Ind}_P^G(\hat{\chi}_p) \tag{5.13}$$

given by

$$\varphi_p(f)\left(\begin{pmatrix} x_1 & y \\ & x_2 \end{pmatrix} k\right) = \hat{\chi}_p\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot f(c,d) \cdot \chi_p(\det(k)), \qquad k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}_2(\mathcal{O}_{F_p}).$$

This provides a well defined $G(F_p)$-action on $\mathbb{D}_{\chi_p^{-2}}(R)$ depending on the extension $\hat{\chi}_p$.

Definition 5.3.1. *To provide an extension $\hat{\chi}_p : F_p^\times \to R^\times$ it suffices to choose a tuple $\underline{\alpha}^* = (\alpha_\mathfrak{p}^*)_{\mathfrak{p}|p}$, where $\alpha_\mathfrak{p}^* \in R^\times$. Indeed, the extension depends on a choice $\alpha_\mathfrak{p}^* = \hat{\chi}_p(\varpi_\mathfrak{p})^{-1} \in R^\times$, for the fixed uniformizers $\varpi_\mathfrak{p}$. We will denote by $\mathbb{D}_{\chi_p^{-2}}(R)_{\underline{\alpha}^*}$ the space $\mathbb{D}_{\chi_p^{-2}}(R)$ with the action of $G(F)$ provided by the corresponding extension.*

Remark 5.3.2. *Given the extension $\hat{\chi}_p : F_p^\times \to R^\times$, we can directly describe the action of $g \in G(F_p)$ on $f \in C_{\chi_p^{-2}}(\mathcal{L}_p, R)$ compatible with φ_p^*. Indeed, for $(c,d) \in \mathcal{L}_p$,*

$$(gf)(c,d) = \hat{\chi}_p(x)^{-2} \cdot \hat{\chi}_p(\det g) \cdot f(x^{-1}(c,d)g), \tag{5.14}$$

where $x \in F_p^\times$ is such that $x^{-1}(c,d)g \in \mathcal{L}_p$.

Assume that $R \subseteq \mathbb{C}_p$ and χ_p is locally analytic. We can define the subspace $C^{\mathrm{an}}_{\chi_p^{-2}}(\mathcal{L}_p, R)$ of locally analytic functions. If we also assume that $\chi_p(a) = a^{-\frac{\underline{k}}{2}}\chi_p^0(a)$, for some $\underline{k} \in 2\mathbb{N}^d$ and some locally constant character χ_p^0, then we can consider the subspace $C^{\underline{k}}_{\chi_p^{-2}}(\mathcal{L}_p, R)$ of locally polynomial functions of homogeneous degree $\underline{k}$. If $\chi_p = \chi_p^0$ then the subspace is $C^0_{\chi_p^{-2}}(\mathcal{L}_p, R)$ the set of locally constant functions. For any extension $\hat{\chi}_p$ as above, we define

$$\mathrm{Ind}_P^G(\hat{\chi}_p)^* := \varphi_p\left(C^*_{\chi_p^{-2}}(\mathcal{L}_p, R)\right), \qquad \text{where} \quad * = 0, \mathrm{an}, \underline{k}.$$

We also write $\mathbb{D}^*_{\chi_p^{-2}}(R)$ for the dual space of $C^*_{\chi_p^{-2}}(\mathcal{L}_p, R)$, where $* = \underline{k}, 0$, and write $\mathbb{D}^{\mathrm{an}}_{\chi_p^{-2}}(R)$ for the continuous dual of $C^{\mathrm{an}}_{\chi_p^{-2}}(\mathcal{L}_p, R)$. If we write $\hat{\chi}_p(a) = a^{-\frac{\underline{k}}{2}}\hat{\chi}_p^0(a)$ for some locally constant character $\hat{\chi}_p^0$, we have $G(F_p)$-equivariant isomorphisms

$$\kappa : \mathrm{Ind}_P^G(\hat{\chi}_p^0)^0 \otimes_R \mathcal{P}(\underline{k})_R \xrightarrow{\simeq} \mathrm{Ind}_P^G(\hat{\chi}_p)^{\underline{k}},$$

where $\kappa\left(f \otimes P\right)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = f\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(c,d) \cdot (ad - bc)^{-\frac{\underline{k}}{2}}$, and

$$\kappa^* : \mathbb{D}^{\underline{k}}_{\chi_p^{-2}}(R)_{\underline{\alpha}^*} \xrightarrow{\simeq} \mathbf{Hom}\left(\mathrm{Ind}_P^G(\hat{\chi}_p^0)^0, V(\underline{k})_R\right),$$

where $\underline{\alpha}^* = (\alpha_\mathfrak{p}^*)_\mathfrak{p}$ with $\alpha_\mathfrak{p}^* = \hat{\chi}_p^0(\varpi_\mathfrak{p})^{-1}\varpi_\mathfrak{p}^{\frac{\underline{k}}{2}}$.

5.3.2 Admissibility

As above, we fix a locally polynomial character $\chi_p = (\chi_\mathfrak{p})_\mathfrak{p} : \mathcal{O}_{F_p}^\times \to \mathbb{C}_p$ where $\chi_p(a) = a^{-\frac{k}{2}}\chi_p^0(a)$, for some $\underline{k} \in 2\mathbb{N}^d$ and some locally constant character $\chi_p^0 = (\chi_\mathfrak{p}^0)_\mathfrak{p}$. For any $b \in \mathcal{O}_{F_\mathfrak{p}}$, $\underline{m} \in \mathbb{N}^d$, and $n \in \mathbb{N}$, let us consider the homogeneous locally analytic functions $f_{b,n}^{\underline{m}}, g_{b,n}^{\underline{m}} \in C_{\chi_\mathfrak{p}^{-2}}^{\mathrm{an}}(\mathcal{L}_\mathfrak{p}, \mathcal{O}_{\mathbb{C}_p})$:

$$f_{b,n}^{\underline{m}}(x,y) = \left(\frac{y - bx}{\varpi_\mathfrak{p}^n}\right)^{\underline{m}} x^{\underline{k}-\underline{m}} \cdot \chi_\mathfrak{p}^0(x)^{-2} \cdot \mathbb{1}_{U_\mathfrak{p}(b,n)}(x,y),$$

$$g_{b,n}^{\underline{m}}(x,y) = \left(\frac{x - by}{\varpi_\mathfrak{p}^n}\right)^{\underline{k}-\underline{m}} y^{\underline{m}} \cdot \chi_\mathfrak{p}^0(y)^{-2} \cdot \mathbb{1}_{V_\mathfrak{p}(b,n)}(x,y),$$

where

$$U_\mathfrak{p}(b,n) = \{(x,y) \in \mathcal{L}_\mathfrak{p};\ x \in \mathcal{O}_{F_\mathfrak{p}}^\times,\ yx^{-1} \equiv b \mod \varpi_\mathfrak{p}^n\},$$
$$V_\mathfrak{p}(b,n) = \{(x,y) \in \mathcal{L}_\mathfrak{p};\ y \in \mathcal{O}_{F_\mathfrak{p}}^\times,\ xy^{-1} \equiv b \mod \varpi_\mathfrak{p}^n\}.$$

It is clear that if $\underline{m} \le \underline{k}$ the functions $f_{b,n}^{\underline{m}}$ and $g_{b,n}^{\underline{m}}$ form a basis of $C_{\chi_\mathfrak{p}^{-2}}^{\underline{k}}(\mathcal{L}_\mathfrak{p}, \mathbb{C}_p)$. Moreover, any locally analytic function in $C_{\chi_\mathfrak{p}^{-2}}^{\mathrm{an}}(\mathcal{L}_\mathfrak{p}, \mathbb{C}_p)$ with support in $U_\mathfrak{p}(b,n)$ (resp. $V_\mathfrak{p}(b,n)$) can be written as a series $\sum_{\underline{m}} a_{\underline{m}} f_{b,n}^{\underline{m}}$ (resp. $\sum_{\underline{m}} a_{\underline{m}} g_{b,n}^{\underline{m}}$), where the coefficients $a_{\underline{m}}$ tend to 0.

Definition 5.3.3. *A distribution $\mu \in \mathbb{D}_{\chi_p^{-2}}^{\underline{k}}(\mathbb{C}_p)$ is $h_\mathfrak{p}$-admissible at $\mathfrak{p}$ if for every $b \in \mathcal{O}_{F_\mathfrak{p}}$ there exists a fixed constant $A_\mathfrak{p} \in \mathbb{C}_p$ only depending on $\mathfrak{p}$ such that*

$$\int_{U_\mathfrak{p}(b,n)} f d\mu \in A_\mathfrak{p} p^{-nh_\mathfrak{p}} \mathcal{O}_{\mathbb{C}_p}, \qquad \int_{V_\mathfrak{p}(b,n)} f d\mu \in A_\mathfrak{p} p^{-nh_\mathfrak{p}} \mathcal{O}_{\mathbb{C}_p},$$

for any $n \in \mathbb{N}$ and $f \in C_{\chi_\mathfrak{p}^{-2}}^{\underline{k}}(\mathcal{L}_\mathfrak{p}, \mathcal{O}_{\mathbb{C}_p})$.

We choose $\underline{\alpha}^* = (\alpha_\mathfrak{p}^*)_\mathfrak{p}$, with $\alpha_\mathfrak{p}^* \in \mathbb{C}_p^\times$, and we consider the $G(F_p)$-representation $\mathbb{D}_{\chi_p^{-2}}^{\underline{k}}(\mathbb{C}_p)_{\underline{\alpha}^*}$.

Lemma 5.3.4. *Let $h_p = (h_\mathfrak{p})_\mathfrak{p}$ where $h_\mathfrak{p} = v_p(\alpha_\mathfrak{p}^*)$. We write $\mathbb{D}_{\chi_p^{-2}}^{\underline{k}}(\mathbb{C}_p)_{\underline{\alpha}^*}^{h_p} \subseteq \mathbb{D}_{\chi_p^{-2}}^{\underline{k}}(\mathbb{C}_p)_{\underline{\alpha}^*}$ for the subspace of $h_\mathfrak{p}$-admissible distributions at every $\mathfrak{p} \mid p$. Then $\mathbb{D}_{\chi_p^{-2}}^{\underline{k}}(\mathbb{C}_p)_{\underline{\alpha}^*}^{h_p}$ is $G(F_p)$-invariant.*

Proof. By (5.14) the action of $G(\mathcal{O}_{F_\mathfrak{p}})$ on $C_{\chi_\mathfrak{p}^{-2}}^{\underline{k}}(\mathcal{L}_\mathfrak{p}, \mathbb{C}_p)$ fixes $C_{\chi_\mathfrak{p}^{-2}}^{\underline{k}}(\mathcal{L}_\mathfrak{p}, \mathcal{O}_{\mathbb{C}_p})$ (notice that $\chi_p(\mathcal{O}_{F_\mathfrak{p}}) \in \mathcal{O}_{\mathbb{C}_p}^\times$). Moreover,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathbb{1}_{U(\beta,n)} = \begin{cases} \mathbb{1}_{U\left(\frac{a\beta-b}{d-\beta c},n\right)}, & d - \beta c \notin \mathfrak{p} \\ \mathbb{1}_{V\left(\frac{d-c\beta}{\beta a-b},n\right)}, & a\beta - b \notin \mathfrak{p} \end{cases}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathcal{O}_{F_\mathfrak{p}}).$$

Together, this implies that $G(\mathcal{O}_{F_p}) \cdot \mathbb{D}_{\chi_p^{-2}}^{\underline{k}}(\mathbb{C}_p)_{\underline{\alpha}^*}^{h_p} \subseteq \mathbb{D}_{\chi_p^{-2}}^{\underline{k}}(\mathbb{C}_p)_{\underline{\alpha}^*}^{h_p}$.

Again by equation (5.14),

$$\begin{pmatrix} \varpi_{\mathfrak{p}} & \\ & 1 \end{pmatrix}^m F(x,y) = \begin{cases} \hat{\chi}_p(\varpi_{\mathfrak{p}})^{-m} \cdot F(x, \varpi_{\mathfrak{p}}^{-m} y); & v_{\mathfrak{p}}(y) = r \geq m, \\ \hat{\chi}_p(\varpi_{\mathfrak{p}})^{m-2r} \cdot F(\varpi_{\mathfrak{p}}^{m-r} x, \varpi_{\mathfrak{p}}^{-r} y); & v_{\mathfrak{p}}(y) = r < m. \end{cases}$$

Thus, if we write

$$I_U = \begin{pmatrix} \varpi_{\mathfrak{p}} & \\ & 1 \end{pmatrix}^m \left(f \cdot \mathbb{1}_{U(\beta,n)} \right)(x,y), \qquad I_V = \begin{pmatrix} \varpi_{\mathfrak{p}} & \\ & 1 \end{pmatrix}^m \left(f \cdot \mathbb{1}_{V(\beta,n)} \right)(x,y)$$

we obtain for every $f \in C^{\underline{k}}_{\chi_{\mathfrak{p}}^{-2}}(\mathcal{L}_{\mathfrak{p}}, \mathcal{O}_{\mathbb{C}_p})$,

$$\begin{aligned} I_U &= (\alpha_{\mathfrak{p}}^*)^m \cdot f(x, \varpi_{\mathfrak{p}}^{-m} y) \cdot \mathbb{1}_{U(\beta\varpi_{\mathfrak{p}}^m, n+m)}(x,y) \\ I_V &= \begin{cases} (\alpha_{\mathfrak{p}}^*)^{m-2v_{\mathfrak{p}}(\beta)} \cdot f(\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\beta)} x, \varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\beta)-m} y) \cdot \mathbb{1}_{U(\beta^{-1}\varpi_{\mathfrak{p}}^m, n+m-2v_{\mathfrak{p}}(\beta))}(x,y); & v_{\mathfrak{p}}(\beta) \leq m, \\ (\alpha_{\mathfrak{p}}^*)^{-m} \cdot f(\varpi_{\mathfrak{p}}^m x, y) \cdot \mathbb{1}_{V(\beta\varpi_{\mathfrak{p}}^{-m}, n-m)}(x,y); & v_{\mathfrak{p}}(\beta) > m. \end{cases} \end{aligned}$$

Hence if we write $r = v_{\mathfrak{p}}(\beta)$ and

$$J_U = \int_{U_{\mathfrak{p}}(\beta,n)} f d \begin{pmatrix} \varpi_{\mathfrak{p}} & \\ & 1 \end{pmatrix}^{-m} \mu, \qquad J_V = \int_{V_{\mathfrak{p}}(\beta,n)} f d \begin{pmatrix} \varpi_{\mathfrak{p}} & \\ & 1 \end{pmatrix}^{-m} \mu$$

then for any $\mu \in \mathbb{D}^{\underline{k}}_{\chi_p^{-2}}(\mathbb{C}_p)^{h_p}_{\underline{\alpha}^*}$,

$$\begin{aligned} J_U &= (\alpha_{\mathfrak{p}}^*)^m \left(\int_{U(\beta\varpi_{\mathfrak{p}}^m, n+m)} f(x, \varpi_{\mathfrak{p}}^{-m} y) d\mu \right) \in (\alpha_{\mathfrak{p}}^*)^m A_{\mathfrak{p}} p^{-(n+m)h_{\mathfrak{p}}} \mathcal{O}_{\mathbb{C}_p} = A_{\mathfrak{p}} p^{-nh_{\mathfrak{p}}} \mathcal{O}_{\mathbb{C}_p}, \\ J_V &= \begin{cases} \dfrac{\left(\int_{U(\beta^{-1}\varpi_{\mathfrak{p}}^m, n+m-2r)} f(\varpi_{\mathfrak{p}}^r x, \varpi_{\mathfrak{p}}^{r-m} y) d\mu \right)}{(\alpha_{\mathfrak{p}}^*)^{2r-m}} \in (\alpha_{\mathfrak{p}}^*)^{m-2r} A_{\mathfrak{p}} p^{-(n+m-2r)h_{\mathfrak{p}}} \mathcal{O}_{\mathbb{C}_p} = A_{\mathfrak{p}} p^{-nh_{\mathfrak{p}}} \mathcal{O}_{\mathbb{C}_p}; & r \leq m, \\ \dfrac{\left(\int_{V(\beta\varpi_{\mathfrak{p}}^{-m}, n-m)} f(\varpi_{\mathfrak{p}}^m x, y) d\mu \right)}{(\alpha_{\mathfrak{p}}^*)^m} \in (\alpha_{\mathfrak{p}}^*)^{-m} A_{\mathfrak{p}} p^{-(n-m)h_{\mathfrak{p}}} \mathcal{O}_{\mathbb{C}_p} = A_{\mathfrak{p}} p^{-nh_{\mathfrak{p}}} \mathcal{O}_{\mathbb{C}_p}; & r > m. \end{cases} \end{aligned}$$

We conclude that $\begin{pmatrix} \varpi_{\mathfrak{p}} & \\ & 1 \end{pmatrix}^{-m} \cdot \mathbb{D}^{\underline{k}}_{\chi_p^{-2}}(\mathbb{C}_p)^{h_p}_{\underline{\alpha}^*} \subseteq \mathbb{D}^{\underline{k}}_{\chi_p^{-2}}(\mathbb{C}_p)^{h_p}_{\underline{\alpha}^*}$. By Cartan decomposition

$$G(F_{\mathfrak{p}}) = \bigsqcup_{m \in \mathbb{N}} G(\mathcal{O}_{F_{\mathfrak{p}}}) \begin{pmatrix} \varpi_{\mathfrak{p}} & \\ & 1 \end{pmatrix}^{-m} G(\mathcal{O}_{F_{\mathfrak{p}}}).$$

Hence, the result follows. □

Lemma 5.3.5. *If we assume that* $e_{\mathfrak{p}} h_{\mathfrak{p}} < \min\{k_\sigma + 1, \ \sigma \in \Sigma_{\mathfrak{p}}\}$ *for all* $\mathfrak{p} \mid p$*, then any* $\mu \in \mathbb{D}^{\underline{k}}_{\chi_p^{-2}}(\mathbb{C}_p)^{h_p}_{\underline{\alpha}^*}$ *lifts to a unique locally analytic distribution* $\mu \in \mathbb{D}^{\mathrm{an}}_{\chi_p^{-2}}(\mathbb{C}_p)_{\underline{\alpha}^*}$.

Proof. The proof is completely analogous to that of Proposition 5.1.2. By definition, the values $\mu(f^{\underline{m}}_{a,N})$ and $\mu(g^{\underline{m}}_{a,N})$ are given for every $\underline{m} \leq \underline{k}$. We proceed to define $\mu(f^{\underline{m}}_{a,N})$ when there exist $\tau \in \Sigma_{\mathfrak{p}}$ such that $m_\tau > e_{\mathfrak{p}} h_{\mathfrak{p}}$, and the rest of values $\mu(g^{\underline{m}}_{a,N})$ can be defined analogously: we write $\mu(f^{\underline{m}}_{a,N}) = \lim_{n\to\infty} a_n$, where

$$a_n = \sum_{\substack{b \bmod \varpi_{\mathfrak{p}}^n \\ b \equiv a \bmod \varpi_{\mathfrak{p}}^N}} \sum_{j_\sigma \leq e_{\mathfrak{p}} h_{\mathfrak{p}}} \left(\frac{b-a}{\varpi_{\mathfrak{p}}^N} \right)^{\underline{m}-\underline{j}} \binom{\underline{m}}{\underline{j}} \varpi_{\mathfrak{p}}^{\underline{j}(n-N)} \mu(f^{\underline{j}}_{b,n}).$$

The definition agrees with μ when there exist $\tau \in \Sigma_\mathfrak{p}$ such that $e_\mathfrak{p} h_\mathfrak{p} < m_\tau \leq k_\tau$ because $\varpi_\mathfrak{p}^{\underline{j}(n-N)} \mu(f_{b,n}^{\underline{j}}) \xrightarrow{n} 0$ when $j_\tau > e_\mathfrak{p} h_\mathfrak{p}$. The usual computations show that the sequence a_n is Cauchy, hence the limit exists. Since the functions $f_{a,N}^{\underline{m}}$ and $g_{a,N}^{\underline{m}}$ topologically generate $C_{\chi_\mathfrak{p}^{-2}}^{\mathrm{an}}(\mathcal{L}_\mathfrak{p}, \mathbb{C}_p)$, this defines a locally analytic measure extending the locally polynomial distribution μ. □

5.3.3 Lifting the modular symbol

Assume that $\pi_\mathfrak{p}$ is principal series or special for all $\mathfrak{p} \mid p$. Thus, we have a projection

$$r : \mathrm{Ind}_P^G(\hat{\chi}_p^0)^0 \longrightarrow V_p^{\mathbb{C}_p} = \bigotimes_{\mathfrak{p}|p} V_\mathfrak{p}^{\mathbb{C}_p}, \qquad \hat{\chi}_p^0 = \prod_{\mathfrak{p}|p} \hat{\chi}_\mathfrak{p}^0,$$

as $G(F_p)$-representations for some locally constant character $\hat{\chi}_p^0 : F_p^\times \longrightarrow \mathbb{C}_p^\times$. Write χ_p^0 for the restriction of $\hat{\chi}_p^0$ to $\mathcal{O}_{F_p}^\times$. Moreover, we consider the locally polynomial character $\hat{\chi}_p = z^{-\frac{\underline{k}}{2}} \hat{\chi}_p^0 : F_p^\times \longrightarrow \mathbb{C}_p^\times$ and χ_p its restriction to $\mathcal{O}_{F_p}^\times$.

Under the above assumptions, the modular symbol

$$\phi_\lambda^p \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(V_p, V(\underline{k})_{\mathbb{C}_p})\right)^\lambda$$

satisfies

$$r^*\phi_\lambda^p \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}\left(\mathrm{Ind}_P^G(\hat{\chi}_p^0)^0, V(\underline{k})_{\mathbb{C}_p}\right)\right)^\lambda \overset{\kappa^*}{\simeq} H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}\left(\mathbb{D}_{\chi_p^{-2}}^{\underline{k}}(\mathbb{C}_p)_{\underline{\alpha}^*}\right)\right)^\lambda$$

where $\underline{\alpha}^* = (\alpha_\mathfrak{p}^*)_\mathfrak{p}$ with $\alpha_\mathfrak{p}^* = \hat{\chi}_\mathfrak{p}^0(\varpi_\mathfrak{p})^{-1}\varpi_\mathfrak{p}^{\frac{\underline{k}}{2}} = \hat{\chi}_\mathfrak{p}(\varpi_\mathfrak{p})^{-1}$.

Proposition 5.3.6. *Assume that for all $\mathfrak{p} \mid p$ we have $e_\mathfrak{p} \cdot v_p(\alpha_\mathfrak{p}^*) < \min\{k_\sigma + 1,\ \sigma \in \Sigma_\mathfrak{p}\}$, then any cohomology class $\phi_\lambda^p \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(V_p, V(\underline{k})_{\mathbb{C}_p})\right)^\lambda$ extends to a unique*

$$\hat{\phi}_\lambda^p \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathbb{D}_{\chi_p^{-2}}^{\mathrm{an}}(\mathbb{C}_p)_{\underline{\alpha}^*})\right)^\lambda.$$

Namely,

$$\kappa^*\hat{\phi}_\lambda^p = r^*\phi_\lambda^p \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathrm{Ind}_P^G(\hat{\chi}_p^0)^0, V(\underline{k})_{\mathbb{C}_p})\right)^\lambda.$$

Proof. We write $\chi_p^0 = \prod_\mathfrak{p} \chi_\mathfrak{p}^0$. If $V = \chi_\mathfrak{p}^0(x)^{-2} \cdot \mathbb{1}_{\mathcal{O}_{F_\mathfrak{p}}^\times \times \mathcal{O}_{F_\mathfrak{p}}}(x, y) \in C_{(\chi_\mathfrak{p}^0)^{-2}}^0(\mathcal{L}_\mathfrak{p}, \mathbb{C}_p)$, by equation (5.14)

$$\hat{\chi}_\mathfrak{p}(\varpi_\mathfrak{p})^{-n} \cdot f_{b,n}^{\underline{m}} = \begin{pmatrix} 1 & b \\ & \varpi_\mathfrak{p}^n \end{pmatrix}^{-1} \left(V \cdot y^{\underline{m}} x^{\underline{k}-\underline{m}}\right), \qquad \hat{\chi}_\mathfrak{p}(\varpi_\mathfrak{p})^{-n} \cdot g_{b,n}^{\underline{m}} = \begin{pmatrix} b & 1 \\ \varpi_\mathfrak{p}^n & \end{pmatrix}^{-1} \left(V \cdot y^{\underline{m}} x^{\underline{k}-\underline{m}}\right).$$

As in the proof of Proposition 5.1.3, let us consider the $G(F_p)$-equivariant morphism

$$\theta_V : \mathbb{D}_{\chi_p^{-2}}^{\underline{k}}(\mathbb{C}_p)_{\underline{\alpha}^*} \simeq \mathbf{Hom}\left(\mathrm{Ind}_P^G(\hat{\chi}_p^0)^0, V(\underline{k})_{\mathbb{C}_p}\right) \longrightarrow \mathrm{coInd}_{C_p}^{G(F_p)}(V(\underline{k})_{\mathbb{C}_p}),$$
$$\theta_V(\varphi)(g)(P) := \varphi(gV)(gP),$$

where $C_p \subseteq G(\mathcal{O}_{F_p})$ is a small enough subgroup of classes of matrices upper triangular modulo p. We compute, for any $\mu \in \mathbb{D}^{\underline{k}}_{\chi_p^{-2}}(\mathbb{C}_p)_{\underline{\alpha}^*}$ such that $\theta_V(\mu) \in \mathrm{coInd}_{C_p}^{G(F_p)}(V(\underline{k})_{\mathcal{O}_{\mathbb{C}_p}})$,

$$\begin{aligned}
\int_{U_\mathfrak{p}(b,n)} f^{\underline{m}}_{b,n} d\mu &= \left(\frac{1}{\alpha^*_\mathfrak{p}}\right)^n \int_{\mathcal{L}_\mathfrak{p}} \begin{pmatrix} 1 & b \\ & \varpi_\mathfrak{p}^n \end{pmatrix}^{-1} \left(V \cdot y^{\underline{m}} x^{\underline{k}-\underline{m}}\right) d\mu \\
&= \left(\frac{1}{\alpha^*_\mathfrak{p}}\right)^n \theta_V(\mu)\left(\begin{pmatrix} 1 & b \\ & \varpi_\mathfrak{p}^n \end{pmatrix}^{-1}\right)(y^{\underline{m}} x^{\underline{k}-\underline{m}}) \in p^{-nh_\mathfrak{p}} \mathcal{O}_{\mathbb{C}_p}; \\
\int_{V_\mathfrak{p}(b,n)} g^{\underline{m}}_{b,n} d\mu &= \left(\frac{1}{\alpha^*_\mathfrak{p}}\right)^n \int_{\mathcal{L}_\mathfrak{p}} \begin{pmatrix} b & 1 \\ \varpi_\mathfrak{p}^n & \end{pmatrix}^{-1} \left(V \cdot y^{\underline{m}} x^{\underline{k}-\underline{m}}\right) d\mu \\
&= \left(\frac{1}{\alpha^*_\mathfrak{p}}\right)^n \theta_V(\mu)\left(\begin{pmatrix} b & 1 \\ \varpi_\mathfrak{p}^n & \end{pmatrix}^{-1}\right)(y^{\underline{m}} x^{\underline{k}-\underline{m}}) \in p^{-nh_\mathfrak{p}} \mathcal{O}_{\mathbb{C}_p},
\end{aligned}$$

where $h_\mathfrak{p} = v_p(\alpha^*_\mathfrak{p})$. Since $f^{\underline{m}}_{b,n}$ and $g^{\underline{m}}_{b,n}$ generate the functions in $C_{\chi_\mathfrak{p}^{-2}}(\mathcal{L}_\mathfrak{p}, \mathcal{O}_{\mathbb{C}_p})$ with support in $U_\mathfrak{p}(b,n)$ and $V_\mathfrak{p}(b,n)$ respectively, we deduce $\mu \in \mathbb{D}^{\underline{k}}_{\chi_p^{-2}}(\mathbb{C}_p)^{h_p}_{\underline{\alpha}^*}$, with $h_p = (h_\mathfrak{p})_\mathfrak{p}$.

By [Spi14, Proposition 4.6] we have that

$$H^u\left(G(F)_+, \mathcal{A}^\infty(V(\underline{k})_{\mathbb{C}_p})\right)^{\lambda, C_p} \simeq H^u\left(G(F)_+, \mathcal{A}^\infty(V(\underline{k})_{\mathcal{O}_{\mathbb{C}_p}})\right)^{\lambda, C_p} \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p,$$

hence up to a constant we can assume that

$$\theta_V r^* \phi^p_\lambda \in H^u\left(G(F)_+, \mathcal{A}^\infty(V(\underline{k})_{\mathcal{O}_{\mathbb{C}_p}})\right)^{\lambda, C_p} = H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}\left(\mathrm{coInd}_{C_p}^{G(F_p)}(V(\underline{k})_{\mathcal{O}_{\mathbb{C}_p}})\right)\right)^\lambda$$

By the above computation, this implies that

$$r^* \phi^p_\lambda \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}\left(\mathbb{D}^{\underline{k}}_{\chi_p^{-2}}(R)^{h_p}_{\underline{\alpha}^*}\right)\right)^\lambda.$$

Hence, the result follows from Lemma 5.3.5. □

5.3.4 Relation with p-adic L-functions

Attached to the modular symbol ϕ^p_λ, we have a unique overconvergent cohomology class

$$\hat{\phi}^p_\lambda \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathbb{D}^{\mathrm{an}}_{\chi_p^{-2}}(\mathbb{C}_p)_{\underline{\alpha}^*})\right)^\lambda$$

On the other side, we have a fundamental class

$$\eta \in H_u(T(F), C_c(T(\mathbb{A}_F), \mathbb{Z}))$$

The formula used to define $\delta_\mathfrak{p}$ in (5.8) extends to

$$\delta_p : C_{\mathrm{an},c}(T(F_p), \mathbb{C}_p) \longrightarrow \mathrm{Ind}_P^G(\hat{\chi}_p)^{\mathrm{an}} \overset{\varphi_p^*}{\simeq} C^{\mathrm{an}}_{\chi_p^{-2}}(\mathcal{L}_p, \mathbb{C}_p).$$

Hence, for any $f \in C_{\mathrm{an}}(\mathcal{G}_T, \mathbb{C}_p)$, we can define the cap-product with respect to the pairings of §3.2

$$((\rho^* f) \cap \eta) \cap \delta_p^*(\hat{\phi}_\lambda^p),$$

where

$$\rho^* f \in H^0(T(F), C_{\mathrm{an}}(T(\mathbb{A}_F), \mathbb{C}_p)),$$

$$\delta_p^*(\hat{\phi}_\lambda^p) \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathrm{Dist}_{\mathrm{an}}(T(F_p), \mathbb{C}_p))\right)^\lambda,$$

$$\rho^* f \cap \eta \in H_u(T(F), C_{\mathrm{an},c}(T(\mathbb{A}_F), \mathbb{C}_p)) = H_u\left(T(F), \mathrm{Ind}_{T(F)_+}^{T(F)} C_c^0\left(T(\mathbb{A}_F^{p\cup\infty}), C_{\mathrm{an},c}(T(F_p), \mathbb{C}_p)\right)\right).$$

The following result follows directly from the definitions:

Theorem 5.3.7. *Assume that for all $\mathfrak{p} \mid p$ we have*

$$e_\mathfrak{p} \cdot v_p(\alpha_\mathfrak{p}^*) = e_\mathfrak{p} \cdot v_p(\hat{\chi}_\mathfrak{p}(\varpi_\mathfrak{p})^{-1}) < \min\{k_\sigma + 1,\ \sigma \in \Sigma_\mathfrak{p}\},$$

and let $\hat{\phi}_\lambda^p \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathbb{D}^{\mathrm{an}}_{z^{\underline{k}}\chi_p^{-2}}(\mathbb{C}_p))\right)^\lambda$ be the extension of ϕ_λ^p provided by Proposition 5.3.6. Then we have that

$$\int_{\mathcal{G}_T} f\, d\mu_{\phi_\lambda^p} = ((\rho^* f) \cap \eta) \cap \delta_p^*(\hat{\phi}_\lambda^p),$$

for any locally analytic function $f \in C_{\mathrm{an}}(\mathcal{G}_T, \mathbb{C}_p)$.

5.4 Hida families

From now on we will assume that the modular symbol $\phi_\lambda^p \in H^u(G(F)_+, \mathcal{A}^{p\cup\infty}(V_p, V(\underline{k}))^\lambda$ is *ordinary*, namely, the attached distribution is 0-admissible. By Proposition 5.1.3 this is to state that the valuations of $\alpha_\mathfrak{p}^* = \hat{\chi}_\mathfrak{p}(\varpi_\mathfrak{p})^{-1} = \varpi_\mathfrak{p}^{\underline{k}/2}\hat{\chi}_p^0(\varpi_\mathfrak{p})^{-1}$ are zero for all $\mathfrak{p} \mid p$. We are convinced that our work can be generalized to the finite slope situation by working with locally analytic distributions. From now on we will work with continuous functions and measures, namely bounded distributions, instead of locally analytic functions and admissible distributions.

Let Λ_F be the Iwasawa algebra associated with $\mathcal{O}_{F_p}^\times$, and let

$$\mathbf{k}_p : \mathcal{O}_{F_p}^\times \longrightarrow \Lambda_F^\times,$$

be the universal character. Recall that $\mathbf{k}_p$ is characterized by the following property: For any complete $\mathbb{Z}_p$-algebra R, and any continuous character $\chi_p : \mathcal{O}_{F_p}^\times \to R^\times$, there exists a morphism $\rho_{\chi_p} : \Lambda_F \to R$ such that $\chi_p = \rho_{\chi_p} \circ \mathbf{k}_p$.

Let $\underline{\mathbf{a}} = (\mathbf{a}_\mathfrak{p})_\mathfrak{p}$, where $\mathbf{a}_\mathfrak{p} \in \Lambda_F^\times$. Note that $\mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)_{\underline{\mathbf{a}}}$ satisfies that, for any such a pair (R, χ_p), we have a natural $G(F_p)$-equivariant morphism

$$\mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)_{\underline{\mathbf{a}}} \otimes_{\rho_{\chi_p}} R \longrightarrow \mathbb{D}_{\chi_p^{-2}}(R)_{\underline{\alpha}^*}, \tag{5.15}$$

where $\underline{\alpha}^* = \rho_{\chi_p}(\underline{\mathbf{a}})$.

5.4.1 Lifting the form to a family

In this ordinary setting, Proposition 5.3.6 states that the modular symbol ϕ_λ^p lifts to an overconvergent modular symbol $\hat\phi_\lambda^p \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathbb{D}_{\chi_p^{-2}}(O_{\mathbb{C}_p})_{\underline{\alpha}^*})\right)^\lambda$. Assume that there exists $\underline{\mathbf{a}} = (\mathbf{a}_\mathfrak{p})_\mathfrak{p}$, with $\mathbf{a}_\mathfrak{p} \in \Lambda_F^\times$ such that $\rho_{\chi_p}(\mathbf{a}_\mathfrak{p}) = \alpha_\mathfrak{p}^*$. This defines an extension $\hat{\mathbf{k}}_p$ of the universal character $\mathbf{k}_p$. Hence, the specialization map ρ_{χ_p} provides a morphism

$$\begin{array}{ccc} H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)_{\underline{\mathbf{a}}})\right)^\lambda & \longrightarrow & H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)_{\underline{\mathbf{a}}})\right)^\lambda \otimes_{\rho_{\chi_p}} O_{\mathbb{C}_p} \\ & \searrow^{\rho_{\chi_p}} & \downarrow \\ & & H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathbb{D}_{\chi_p^{-2}}(O_{\mathbb{C}_p})_{\underline{\alpha}^*}\right)^\lambda . \end{array} \tag{5.16}$$

In this section we will discuss the existence of both $\underline{\mathbf{a}}$ that specializes $\underline{\alpha}^*$ and a class $\Phi_\lambda^p \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)_{\underline{\mathbf{a}}})\right)^\lambda$ lifting the overconvergent modular symbol $\hat\phi_\lambda^p$.

Local systems and group cohomology

Given a compact subgroup $C \subseteq G(\mathbb{A}_F^\infty)$, we can construct the locally symmetric space

$$Y_C := G(F)_+\backslash G(F_\infty)_+ \times G(\mathbb{A}_F^\infty)/C_\infty C,$$

where C_∞ is the maximal compact subgroup of $G(F_\infty)_+$. When F is totally real and C is small enough, Y_C is in correspondence with the set of complex points of a Shimura variety.

Let $C_p := C \cap G(F_p)$. Given a C_p-module V, we can define the local system

$$\mathcal{V} := G(F)_+\backslash \left(G(F_\infty)_+ \times G(\mathbb{A}_F^\infty) \times V\right)/C_\infty C \longrightarrow Y_C,$$

where the left $G(F)_+$-action and right $C_\infty C$-action on $G(\mathbb{A}_F) \times V$ is given by

$$\gamma(g_\infty, g^\infty, v)(c_\infty, c) = (\gamma g_\infty c_\infty, \gamma g^\infty c, c_p{}^{-1}v),$$

being $c_p \in C_p$ the p-component of $c \in C$.

A locally constant section of the local system $\mathcal{V}$ amounts to a function $s : G(\mathbb{A}_F^\infty) \to V$ such that $s(g^\infty) = c_p s(\gamma g^\infty c)$, for all $\gamma \in G(F)_+$ and $c \in C$. Indeed, such a function provides the well defined section

$$Y_C \longrightarrow \mathcal{V}; \qquad g \longmapsto (g_\infty, g^\infty, s(g^\infty)).$$

Let us consider the coinduced representation

$$\mathrm{coInd}_{C_p}^{G(F_p)} V = \{f : G(F_p) \to V; \quad f(g_p c_p) = c_p{}^{-1} f(g_p), \quad g_p \in G(F_p),\ c_p \in C_p\},$$

with $G(F_p)$-action $(h_p f)(g_p) = f(h_p{}^{-1} g_p)$, $h_p \in G(F_p)$. Thus, to provide such an s is equivalent to provide an element

$$\hat s \in H^0\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathrm{coInd}_{C_p}^{G(F_p)} V)\right)^{C^p}, \qquad \hat s(g^p)(g_p) := s(g_p, g^p),$$

where $C^p := C \cap G(\mathbb{A}_F^{p\cup\infty})$. Hence, we can identify the cohomology of the sheaf of local sections of $\mathcal{V}$ with the C^p-invariant subgroup of the group cohomology of $\mathcal{A}^{p\cup\infty}(\mathrm{coInd}_{C_p}^{G(F_p)} V)$, namely,

$$H^k(Y_C, \mathcal{V}) = H^k\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathrm{coInd}_{C_p}^{G(F_p)} V)\right)^{C^p}. \tag{5.17}$$

Families and the eigencurve

Given a $\mathbb{Z}_p$-algebra R, write $C(\mathcal{O}_{F_p},R)$ and $D(\mathcal{O}_{F_p},R)$ for the set of R-valued continuous functions and its continuous dual, respectively. Write $\mathcal{I}_p\subset G(F_p)$ the usual Iwahori subgroup

$$\mathcal{I}_p=\prod_{\mathfrak{p}\mid p}\mathcal{I}_{\mathfrak{p}},\qquad \mathcal{I}_{\mathfrak{p}}:=\mathbf{PGL}_2(\mathcal{O}_{F_{\mathfrak{p}}})\cap\begin{pmatrix}\mathcal{O}_{F_{\mathfrak{p}}}^\times & \mathcal{O}_{F_{\mathfrak{p}}}\\ \varpi_{\mathfrak{p}}\mathcal{O}_{F_{\mathfrak{p}}} & \mathcal{O}_{F_{\mathfrak{p}}}^\times\end{pmatrix}/\mathcal{O}_{F_p}^\times.$$

Given a continuous character $\chi_p:\mathcal{O}_{F_p}^\times\to R^\times$, one can define a $\mathcal{I}_p$-action on $C(\mathcal{O}_{F_p},R)$ by means of the formula

$$(i_pf)(x)=f\left(\frac{b+dx}{a+cx}\right)\cdot\chi_p(\det i_p)\cdot\chi_p^{-2}(a+cx),\qquad i_p=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathcal{I}_p,\quad f\in C(\mathcal{O}_{F_p},R).$$

This provides the usual action on $D(\mathcal{O}_{F_p},R)$ given by $(i_p\mu)(f)=\mu(i_p{}^{-1}f)$. We write $D_{\chi_p}(\mathcal{O}_{F_p},R)$ for the space endowed with the above action of $\mathcal{I}_p$. Such an action can be extended to the semigroup $\Sigma_p{}^{-1}$ of inverses of

$$\Sigma_p:=\prod_{\mathfrak{p}\mid p}\Sigma_{\mathfrak{p}},\qquad \Sigma_{\mathfrak{p}}:=\mathbf{PGL}_2(\mathcal{O}_{F_{\mathfrak{p}}})\cap\begin{pmatrix}\mathcal{O}_{F_{\mathfrak{p}}}^\times & \mathcal{O}_{F_{\mathfrak{p}}}\\ \varpi_{\mathfrak{p}}\mathcal{O}_{F_{\mathfrak{p}}} & \mathcal{O}_{F_{\mathfrak{p}}}\end{pmatrix}/\mathcal{O}_{F_p}^\times.$$

Indeed, if we write $\langle\alpha\rangle=\frac{\alpha}{\prod_{\mathfrak{p}\mid p}\varpi_{\mathfrak{p}}^{v_{\mathfrak{p}}(\alpha)}}\in\mathcal{O}_p^\times$ for any $\alpha\in F_p^\times$, the action is defined by

$$(g_pf)(x)=f\left(\frac{b+dx}{a+cx}\right)\cdot\chi_p\left(\langle\det g_p\rangle\right)\cdot\chi_p^{-2}(a+cx),\qquad g_p=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Sigma_p.$$

Hence it makes sense to consider the action of the matrices $\left(\begin{smallmatrix}\varpi_{\mathfrak{p}}&i\\&1\end{smallmatrix}\right)$, $i\in\mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}$, defining Hecke operators $U_{\mathfrak{p}}$.

Remark 5.4.1. *We have a morphism*

$$C(\mathcal{O}_{F_p},R)\longrightarrow C_{\chi_p^{-2}}(\mathcal{L}_p,R);\qquad f\longmapsto\hat f(x,y)=\chi_p(x)^{-2}\cdot f\left(\frac{y}{x}\right)\cdot\mathbb{1}_{\mathcal{O}_{F_p}^\times\times\mathcal{O}_{F_p}}(x,y),$$

satisfying for all $k=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in G(\mathcal{O}_{F_p})$

$$(k\hat f)(x,y)=\chi_p(\det k)\cdot\hat f((x,y)k)=f\left(\frac{bx+dy}{ax+cy}\right)\cdot\chi_p(\det k)\cdot\chi_p^{-2}(ax+cy)\cdot\mathbb{1}_{(\mathcal{O}_{F_p}^\times\times\mathcal{O}_{F_p})k^{-1}}(x,y).$$

In particular it is $\mathcal{I}_p$-equivariant with respect to the above action. More generally, given $g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Sigma_p$

$$\begin{aligned}g^{-1}\widehat{gf}(x,y)&=\hat\chi_p(\alpha)^{-2}\cdot\hat\chi_p{}^{-1}(\det g)\cdot\widehat{gf}(\alpha^{-1}(x,y)g^{-1})\\&=\hat\chi_p(\alpha)^{-2}\cdot\hat\chi_p{}^{-1}(\det g)\cdot\widehat{gf}\left(\frac{1}{\alpha\det(g)}(dx-cy,ay-bx)\right),\end{aligned}$$

by (5.14), where $\alpha\in F_p^\times$ is such that $\alpha^{-1}(x,y)g^{-1}\in\mathcal{L}_p$. Since $a\in\mathcal{O}_{F_p}^\times$ and $c\in\prod_{\mathfrak{p}}\varpi_{\mathfrak{p}}$, a necessary condition for $(dx-cy,ay-bx)$ being in $F_p^\times(\mathcal{O}_{F_p}^\times\times\mathcal{O}_{F_p})$ is $x\in\mathcal{O}_{F_p}^\times$. Since the support of $\widehat{gf}$ is precisely $\mathcal{O}_{F_p}^\times\times\mathcal{O}_{F_p}$, we conclude that $x\in\mathcal{O}_{F_p}^\times$ and

$$v_{\mathfrak{p}}\left(\frac{y}{x}-\frac{b}{a}\right)=v_{\mathfrak{p}}(ay-bx)\geq v_{\mathfrak{p}}(dx-cy)=v_{\mathfrak{p}}\left(d-c\frac{y}{x}\right)=v_{\mathfrak{p}}\left(\frac{\det(g)}{a}+c\left(\frac{b}{a}-\frac{y}{x}\right)\right).$$

Thus, $\frac{y}{x}\in\frac{b}{a}+\det(g)\mathcal{O}_{F_p}$ *and* $\alpha=1$*. Hence, if we write* $U(g):=\frac{b}{a}+\det(g)\mathcal{O}_{F_p}$*, we compute,*

$$\begin{aligned} g^{-1}\widehat{gf}(x,y) &= \hat{\chi}_p{}^{-1}(\det g)\cdot\chi_p^{-2}\left(\frac{dx-cy}{\det(g)}\right)\cdot(gf)\left(\frac{ay-bx}{dx-cy}\right)\cdot\mathbb{1}_{U(g)}\left(\frac{y}{x}\right)\\ &= f\left(\frac{y}{x}\right)\cdot\frac{\chi_p\left(\langle\det g\rangle\right)}{\hat{\chi}_p(\det g)}\cdot\hat{\chi}_p^{-2}(x)\cdot\mathbb{1}_{U(g)}\left(\frac{y}{x}\right)=\frac{\chi_p\left(\langle\det g\rangle\right)}{\hat{\chi}_p(\det g)}\cdot\widehat{f\cdot\mathbb{1}_{U(g)}}(x,y).\end{aligned}$$

The classical strategy to construct the eigenvariety is to consider finite slope subspaces for the action of $U_\mathfrak{p}$ of the locally analytic analogues of cohomology spaces

$$H^k(Y_C,\mathcal{D}_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F)),$$

where $\mathcal{D}_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F)$ is the local system associated with $D_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F)$ and $C\subset G(\mathbb{A}_F^{p\cup\infty})$ is a compact open subgroup such that $C_p=\mathcal{I}_p$. Since our setting is ordinary, it is enough for us to consider $D_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F)$. A connected component of the eigenvariety passing through π provides a system of eigenvalues for the Hecke operators in Λ_F.

Once we have a system of eigenvalues, in particular eigenvalues $\mathbf{a}_\mathfrak{p}$ for the $U_\mathfrak{p}$-operators, a different and challenging problem is to provide eigenvectors living in the space of cohomology $H^u(Y_C,\mathcal{D}_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F))^\lambda$. If F is totally real, then one can make use of the étaleness of the eigenvariety to prove the existence of such families in middle degree $k=u$ (see [BDJ21, Theorem 2.14] for the case $G=\mathbf{PGL}_2$, and notice that techniques of [BDJ21, §2.5] can be extended to general G). For arbitrary number fields F the situation is more complicated (see [GR21] or [BW21]). Since these questions are beyond the scope of this work, we will directly assume the existence of a family $\Phi_\lambda^p\in H^u(Y_C,\mathcal{D}_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F))^{\lambda,U_\mathfrak{p}=\mathbf{a}_\mathfrak{p}}$ and we will address the reader to the previous references for details in each concrete situation. By Equation (5.17)

$$H^u(Y_C,\mathcal{D}_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F))^{\lambda,U_\mathfrak{p}=\mathbf{a}_\mathfrak{p}}=H^u\left(G(F)_+,\mathcal{A}^{p\cup\infty}(\mathrm{coInd}_{\mathcal{I}_p}^{G(F_p)}D_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F)^{U_\mathfrak{p}=\mathbf{a}_\mathfrak{p}})\right)^{\lambda,C^p},$$

where the action of $U_\mathfrak{p}$ on $\phi\in\mathrm{Ind}_{\mathcal{I}_p}^{G(F_p)}D_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F)$ is given by

$$U_\mathfrak{p}\phi(g_p)=\sum_{i\in\mathcal{O}_{F_\mathfrak{p}}/\mathfrak{p}}\begin{pmatrix}1&i\\&\varpi_\mathfrak{p}\end{pmatrix}^{-1}\phi\left(g_p\begin{pmatrix}1&i\\&\varpi_\mathfrak{p}\end{pmatrix}^{-1}\right).$$

Remark 5.4.2. *Given* $a\in\mathcal{O}_{F_\mathfrak{p}}/\mathfrak{p}^n$ *write* $g_a=\left(\begin{smallmatrix}1&a\\&\varpi_\mathfrak{p}^n\end{smallmatrix}\right)$*. For any*

$$\phi\in\mathrm{coInd}_{\mathcal{I}_p}^{G(F_p)}D_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F)^{U_\mathfrak{p}=\mathbf{a}_\mathfrak{p}})$$

it is clear that

$$\int_{i+\mathfrak{p}\mathcal{O}_{F_p}}d\phi(g)=\frac{1}{\alpha_\mathfrak{p}}\int_{\mathcal{O}_{F_p}}d\phi(gg_i{}^{-1}),$$

by the definition of $U_\mathfrak{p}$*. Applying this fact inductively, we deduce*

$$\int_{a+\mathfrak{p}^n\mathcal{O}_{F_p}}d\phi(g)=\frac{1}{\alpha_\mathfrak{p}^n}\int_{\mathcal{O}_{F_p}}d\phi(gg_a{}^{-1}),$$

for all $a\in\mathcal{O}_{F_\mathfrak{p}}/\mathfrak{p}^n$*. This implies that the integrals* $\int_{\mathcal{O}_{F_p}}d\phi(g)$ *characterize the element* ϕ*.*

Proposition 5.4.3. *We have a $G(F_p)$-equivariant isomorphism.*

$$\mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)_{\underline{\mathbf{a}}} \xrightarrow{\simeq} \mathrm{coInd}_{\mathcal{I}_p}^{G(F_p)} D_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F)^{U_{\mathfrak{p}}=\mathbf{a}_{\mathfrak{p}}},$$

where $\mathbf{a} = (\mathbf{a}_{\mathfrak{p}})_{\mathfrak{p}}$.

Proof. By Remark 5.4.1, we have a $\mathcal{I}_p$-equivariant morphism

$$\mathrm{res}: \mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)_{\underline{\mathbf{a}}} \longrightarrow D_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F); \qquad \int_{\mathcal{O}_{F_p}} f(z)d(\mathrm{res}\mu)(z) := \int_{\mathcal{O}_{F_p}^\times \times \mathcal{O}_{F_p}} \hat{f}(x,y)d\mu(x,y).$$

Hence it provides a well defined $G(F_p)$-equivariant morphism

$$\varphi: \mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)_{\underline{\mathbf{a}}} \longrightarrow \mathrm{coInd}_{\mathcal{I}_p}^{G(F_p)} D_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F), \qquad \varphi(\mu)(g_p) := \mathrm{res}({g_p}^{-1}\mu).$$

Let us check that the image lies in the subspace where $U_{\mathfrak{p}}$ acts like $\mathbf{a}_{\mathfrak{p}}$: If we write $g_i = \left(\begin{smallmatrix} 1 & i \\ & \varpi_{\mathfrak{p}} \end{smallmatrix}\right)$, then by Remark 5.4.1,

$$\begin{aligned}
\int_{\mathcal{O}_{F_p}} f(z)dU_{\mathfrak{p}}(\varphi(\mu))(g_p)(z) &= \sum_{i\in\mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}} \int_{\mathcal{O}_{F_p}} f d\left(g_i^{-1}\varphi(\mu)\left(g_p g_i^{-1}\right)\right) \\
&= \sum_{i\in\mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}} \int_{\mathcal{O}_{F_p}} \widehat{g_i f} d\left(g_i {g_p}^{-1}\mu\right) \\
&= \sum_{i\in\mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}} \int_{\mathcal{O}_{F_p}} {g_i}^{-1}\widehat{g_i f} d\left({g_p}^{-1}\mu\right) \\
&= \hat{\mathbf{k}}(\varpi_{\mathfrak{p}})^{-1} \cdot \sum_{i\in\mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}} \int_{\mathcal{O}_{F_p}} \widehat{f \cdot \mathbb{1}_{i+\mathfrak{p}}} d\left({g_p}^{-1}\mu\right) \\
&= \mathbf{a}_{\mathfrak{p}} \cdot \int_{\mathcal{O}_{F_p}} \hat{f} d\left({g_p}^{-1}\mu\right) = \mathbf{a}_{\mathfrak{p}} \cdot \int_{\mathcal{O}_{F_p}} f(z)d\varphi(\mu)(g_p)(z).
\end{aligned}$$

Thus $U_{\mathfrak{p}}(\varphi(\mu)) = \mathbf{a}_{\mathfrak{p}} \cdot \varphi(\mu)$. This implies that we have a well defined $G(F)$-equivariant morphism

$$\varphi: \mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)_{\underline{\mathbf{a}}} \longrightarrow \mathrm{coInd}_{\mathcal{I}_p}^{G(F_p)} D_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F)^{U_{\mathfrak{p}}=\mathbf{a}_{\mathfrak{p}}}.$$

It is clearly injective since the vanishing of $\varphi(\mu)$ in particular implies that the distribution μ vanishes when restricted to $(\mathcal{O}_{F_p}^\times \times \mathcal{O}_{F_p})G(\mathcal{O}_{F_p}) = \mathcal{L}_p$.

Let us consider the function $f_0 \in C_{\mathbf{k}_p^{-2}}(\mathcal{L}_p,\Lambda_F)$, defined by $f_0(x,y) = \mathbf{k}_p^{-2}(x)\cdot \mathbb{1}_{\mathcal{O}_{F_p}^\times}(x)$. Note that $\widehat{\mathbb{1}_{\mathcal{O}_{F_p}}} = f_0$. Then it is clear that the translates $g_p f_0$, where $g_p \in G(F_p)$, topologically generate $C_{\mathbf{k}_p^{-2}}(\mathcal{L}_p,\Lambda_F)$. Thus, we can define

$$\psi: \mathrm{coInd}_{\mathcal{I}_p}^{G(F_p)} D_{\mathbf{k}_p}(\mathcal{O}_{F_p},\Lambda_F)^{U_{\mathfrak{p}}=\mathbf{a}_{\mathfrak{p}}} \longrightarrow \mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)_{\underline{\mathbf{a}}}, \qquad \int_{\mathcal{L}_p}(g_p f_0)d\psi(\phi) = \int_{\mathcal{O}_{F_p}} d\phi(g_p).$$

It is easy to check that the morphism ψ is $G(F_p)$-equivariant, indeed, for $h_p \in G(F_p)$

$$\int_{\mathcal{L}_p}(g_p f_0)d\psi(h_p\phi) = \int_{\mathcal{O}_{F_p}} d\phi({h_p}^{-1}g_p) = \int_{\mathcal{L}_p}({h_p}^{-1}g_p f_0)d\psi(\phi) = \int_{\mathcal{L}_p}(g_p f_0)d\left(h_p\psi(\phi)\right).$$

We also have to check that it is well defined, namely, given any linear relation $\sum_i c_i h_i f_0 = \sum_j h_j f_0$ we have that

$$\sum_i c_i \int_{\mathcal{L}_p} h_i f_0 d\psi(\phi) = \sum_j c_j \int_{\mathcal{L}_p} h_j f_0 d\psi(\phi).$$

But such relations are $G(F_p)$-generated by

$$f_0 = \frac{1}{\alpha_{\mathfrak{p}}} \sum_{i \in \mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}} \begin{pmatrix} \varpi_{\mathfrak{p}} & i \\ & 1 \end{pmatrix} f_0 = \frac{1}{\alpha_{\mathfrak{p}}} \sum_{i \in \mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}} g_i^{-1} f_0,$$

Hence, by Remark 5.4.2, the morphism ψ is well defined since

$$\int_{\mathcal{L}_p} f_0 d\psi(\phi) = \int_{\mathcal{O}_{F_p}} d\phi(1) = \sum_{i \in \mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}} \int_{i+\mathfrak{p}\mathcal{O}_p} d\phi(1)$$

$$= \frac{1}{\alpha_{\mathfrak{p}}} \sum_{i \in \mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}} \int_{\mathcal{O}_{F_p}} d\phi(g_i^{-1}) = \frac{1}{\alpha_{\mathfrak{p}}} \sum_{i \in \mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}} \int_{\mathcal{L}_p} (g_i^{-1} f_0) d\psi(\phi).$$

We compute

$$\int_{\mathcal{L}_p} (g_p f_0) d\left(\psi \circ \varphi(\mu)\right) = \int_{\mathcal{O}_{F_p}} d\varphi(\mu)(g_p) = \int_{\mathcal{O}_{F_p}^\times \times \mathcal{O}_{F_p}} \widehat{\mathbb{1}_{\mathcal{O}_{F_p}}} d\left(g_p{}^{-1}\mu\right) = \int_{\mathcal{L}_p} (g_p f_0) d\mu,$$

hence $\psi \circ \varphi(\mu) = \mu$. Moreover, for all $g_p \in G(F_p)$,

$$\begin{aligned} \int_{a+\mathfrak{p}^n\mathcal{O}_{F_p}} d\varphi \circ \psi(\phi)(g_p) &= \frac{1}{\alpha_{\mathfrak{p}}^n} \int_{\mathcal{O}_{F_p}} d\varphi \circ \psi(\phi)(g_p g_a{}^{-1}) = \frac{1}{\alpha_{\mathfrak{p}}^n} \int_{\mathcal{L}_p} \widehat{\mathbb{1}_{\mathcal{O}_{F_p}}} d(g_a g_p{}^{-1}\psi(\phi)) \\ &= \frac{1}{\alpha_{\mathfrak{p}}^n} \int_{\mathcal{L}_p} (g_p g_a{}^{-1} f_0) d\psi(\phi) = \frac{1}{\alpha_{\mathfrak{p}}^n} \int_{\mathcal{O}_{F_p}} d\phi(g_p g_a{}^{-1}) \\ &= \int_{a+\mathfrak{p}^n\mathcal{O}_{F_p}} d\phi(g_p), \end{aligned}$$

by Remark 5.4.2. Hence $\varphi \circ \psi(\phi) = \phi$ and the result follows.

□

The above result together with equation (5.17) implies that we have an isomorphism

$$H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathbb{D}_{\mathbf{k}_p-2}(\Lambda_F)_{\underline{\mathbf{a}}})\right)^\lambda \xrightarrow{\simeq} H^u(Y_C, \mathcal{D}_{\mathbf{k}_p}(\mathcal{O}_{F_p}, \Lambda_F))^{\lambda, U_{\mathfrak{p}}=\mathbf{a}_{\mathfrak{p}}}.$$

Thus, the assumption on the existence of a classical family $\Phi_\lambda^p \in H^u(Y_C, \mathcal{D}_{\mathbf{k}_p}(\mathcal{O}_{F_p}, \Lambda_F))^{\lambda, U_{\mathfrak{p}}=\mathbf{a}_{\mathfrak{p}}}$ ensures the existence of the lift

$$\Phi_\lambda^p \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathbb{D}_{\mathbf{k}_p-2}(\Lambda_F)_{\underline{\mathbf{a}}})\right)^\lambda \tag{5.18}$$

5.4.2 $\mathfrak{p}$-Iwasawa algebras

Let $\Lambda_\mathfrak{p} := \mathbb{Z}_p[[\mathcal{O}_{F_\mathfrak{p}}^\times]]$ the $\mathfrak{p}$-Iwasawa algebra with universal character $\mathbf{k}_\mathfrak{p}$. For any continuous character $\chi_\mathfrak{p} : \mathcal{O}_{F_\mathfrak{p}}^\times \to R^\times$, we consider the space of homogeneous measures

$$\mathbb{D}_{\chi_\mathfrak{p}}(R) = \mathbf{Hom}_{\mathrm{cnt}}(C_{\chi_\mathfrak{p}}(\mathcal{L}_\mathfrak{p}, R), R), \qquad \mathbb{D}_{\mathbf{k}_\mathfrak{p}}(\Lambda_\mathfrak{p})^0 = \{\mu \in \mathbb{D}_{\mathbf{k}_\mathfrak{p}}(\Lambda_\mathfrak{p}) : \ (\rho_1\mu)(1) = 0\},$$

where the specializations $\rho_{\chi_\mathfrak{p}} : \mathbb{D}_{\mathbf{k}_\mathfrak{p}}(\Lambda_\mathfrak{p}) \to \mathbb{D}_{\chi_\mathfrak{p}}(R)$ are defined analogously as in (5.15). Recall that any extension $\hat{\mathbf{k}}_\mathfrak{p} : F_\mathfrak{p}^\times \to \Lambda_\mathfrak{p}^\times$ of the universal character provides an isomorphism $C_{\mathbf{k}_\mathfrak{p}^{-2}}(\mathcal{L}_\mathfrak{p}, \Lambda_\mathfrak{p}) \overset{\varphi_\mathfrak{p}}{\simeq} \mathrm{Ind}_P^G(\hat{\mathbf{k}}_\mathfrak{p})$ as in (5.13), and this provides an action of $G(F_\mathfrak{p})$ on $\mathbb{D}_{\mathbf{k}_\mathfrak{p}^{-2}}(\Lambda_\mathfrak{p})$. If we write $\mathbf{a}_\mathfrak{p} = \hat{\mathbf{k}}_\mathfrak{p}(\varpi_\mathfrak{p})^{-1}$ as usual, then we denote the space with such an action by $\mathbb{D}_{\mathbf{k}_\mathfrak{p}^{-2}}(\Lambda_\mathfrak{p})_{\mathbf{a}_\mathfrak{p}}$.

Lemma 5.4.4. *If $\rho_1(\mathbf{a}_\mathfrak{p}) = 1$, then the subspace $\mathbb{D}_{\mathbf{k}_\mathfrak{p}^{-2}}(\Lambda_\mathfrak{p})^0$ is $G(F_\mathfrak{p})$-invariant. We will denote the subspace with the corresponding action by $\mathbb{D}_{\mathbf{k}_\mathfrak{p}^{-2}}(\Lambda_\mathfrak{p})^0_{\mathbf{a}_\mathfrak{p}}$.*

Proof. Given $\mu \in \mathbb{D}_{\mathbf{k}_\mathfrak{p}}(\Lambda_\mathfrak{p})^0$ we have to check that $g\mu \in \mathbb{D}_{\mathbf{k}_\mathfrak{p}}(\Lambda_\mathfrak{p})^0$ for all $g \in G(F_\mathfrak{p})$. By $G(F_\mathfrak{p})$-equivariance

$$(\rho_1 g\mu)(1) = (g\rho_1\mu)(1) = (\rho_1\mu)(g^{-1}1).$$

But if $\rho_1(\mathbf{a}_\mathfrak{p}) = 1$ we have $\rho_1 C_{\mathbf{k}_\mathfrak{p}}(\mathcal{L}_\mathfrak{p}, \Lambda_\mathfrak{p}) \overset{\varphi_\mathfrak{p}}{\simeq} \mathrm{Ind}_P^G(1)$, and $\varphi_\mathfrak{p}(1)$ is $G(F_\mathfrak{p})$-invariant. Hence $g^{-1}1 = 1$ and the result follows. □

Throughout the rest of this section we will assume that $\rho_1(\mathbf{a}_\mathfrak{p}) = 1$, hence we are in the setting of the above Lemma. Write $I_\mathfrak{p}$ for the augmentation ideal

$$I_\mathfrak{p} := \ker\left(\Lambda_\mathfrak{p} \xrightarrow{\rho_1} \mathbb{Z}_p\right). \tag{5.19}$$

There is a natural isomorphism

$$\mathcal{O}_{F_\mathfrak{p}}^\times \otimes_\mathbb{Z} \mathbb{Z}_p =: \hat{\mathcal{O}}_{F_\mathfrak{p}}^\times \xrightarrow{\simeq} I_\mathfrak{p}/I_\mathfrak{p}^2; \qquad \alpha \mapsto (\mathbf{k}_\mathfrak{p}(\alpha) - 1) + I_\mathfrak{p}^2.$$

Since $\rho_1(\mathbf{a}_\mathfrak{p}) = 1$, the above isomorphism can be extended to a character

$$\ell_{\mathbf{a}_\mathfrak{p}} : K_\mathfrak{p}^\times \longrightarrow I_\mathfrak{p}/I_\mathfrak{p}^2; \qquad \alpha \mapsto (\hat{\mathbf{k}}_\mathfrak{p}(\alpha) - 1) + I_\mathfrak{p}^2.$$

Remark 5.4.5. *For any $y \in \mathbb{Z}$, $f \in C_{\mathbf{k}_\mathfrak{p}^y}(\mathcal{L}_\mathfrak{p}, \mathcal{O}_{F_\mathfrak{p}}^\times) \subset C_{\mathbf{k}_\mathfrak{p}^y}(\mathcal{L}_\mathfrak{p}, \Lambda_\mathfrak{p})$ and any $\mu \in \mathbb{D}_{\mathbf{k}_\mathfrak{p}}(\Lambda_\mathfrak{p})^0$,*

$$\int_{\mathcal{L}_\mathfrak{p}} f d\rho_{\mathbf{k}_\mathfrak{p}^y}\mu \in I_\mathfrak{p} \subset \Lambda_\mathfrak{p}.$$

Indeed, since f has values in $\mathcal{O}_{F_\mathfrak{p}}^\times$, we have that $\rho_1 f = 1$. Hence

$$\rho_1\left(\int_{\mathcal{L}_\mathfrak{p}} f d\rho_{\mathbf{k}_\mathfrak{p}^y}\mu\right) = \int_{\mathcal{L}_\mathfrak{p}} 1 d\rho_1\mu = 0,$$

by the definition of $\mathbb{D}_{\mathbf{k}_\mathfrak{p}}(\Lambda_\mathfrak{p})^0$.

Lemma 5.4.6. *Given* $f_1 \in C_{\mathbf{k}_\mathfrak{p}^{y_1}}(\mathcal{L}_\mathfrak{p}, \mathcal{O}_{F_\mathfrak{p}}^\times)$ *and* $f_2 \in C_{\mathbf{k}_\mathfrak{p}^{y_2}}(\mathcal{L}_\mathfrak{p}, \mathcal{O}_{F_\mathfrak{p}}^\times)$*, we have*

$$\int_{\mathcal{L}_\mathfrak{p}} (f_1 \cdot f_2)\, d\rho_{\mathbf{k}_\mathfrak{p}^{y_1+y_2}}\mu = \int_{\mathcal{L}_\mathfrak{p}} f_1 d\rho_{\mathbf{k}_\mathfrak{p}^{y_1}}\mu + \int_{\mathcal{L}_\mathfrak{p}} f_2 d\rho_{\mathbf{k}_\mathfrak{p}^{y_2}}\mu \mod I_\mathfrak{p}^2$$

for any $\mu \in \mathbb{D}_{\mathbf{k}_\mathfrak{p}}(\Lambda_\mathfrak{p})^0$.

Proof. Let $f_0 \in C_{\mathbf{k}_\mathfrak{p}}(\mathcal{L}_\mathfrak{p}, \mathcal{O}_{F_\mathfrak{p}}^\times)$. For example, we can choose

$$f_0(c,d) = \begin{cases} \mathbf{k}_\mathfrak{p}(c), & \text{if } (c,d) \in \mathcal{O}_{F_\mathfrak{p}}^\times \times \mathcal{O}_{F_\mathfrak{p}} \\ \mathbf{k}_\mathfrak{p}(d), & \text{otherwise.} \end{cases}$$

If $\varphi : \mathcal{L}_\mathfrak{p} \longrightarrow \mathbb{P}^1(F_\mathfrak{p})$ is the natural projection,

$$\begin{aligned} \int_{\mathcal{L}_\mathfrak{p}} f_i d\rho_{\mathbf{k}_\mathfrak{p}^{y_i}}\mu &= \lim_{\mathcal{U}\in \mathrm{Cov}(\mathbb{P}^1(F_\mathfrak{p}))} \sum_{U\in\mathcal{U}} \left(\frac{f_i}{f_0^{y_i}}\right)(x_U) \int_{\varphi^{-1}(U)} f_0^{y_i} d\rho_{\mathbf{k}_\mathfrak{p}^{y_i}}\mu \\ &= \lim_{\mathcal{U}\in \mathrm{Cov}(\mathbb{P}^1(F_\mathfrak{p}))} \sum_{U\in\mathcal{U}} \left(\frac{f_i}{f_0^{y_i}}\right)(x_U) \cdot \rho_{\mathbf{k}_\mathfrak{p}^{y_i}} \int_{\varphi^{-1}(U)} f_0 d\mu, \end{aligned}$$

since $\rho_{\mathbf{k}_\mathfrak{p}^{y_i}} f_0 = f_0^{y_i}$. For any $\beta \in I_\mathfrak{p}$, write $\overline{\beta}$ for its image in $I_\mathfrak{p}/I_\mathfrak{p}^2$, and write $M = \mu(f_0) \in I_\mathfrak{p}$. Since $\overline{\alpha\beta} = \rho_1(\alpha)\cdot\overline{\beta}$ for any $\alpha \in \Lambda_\mathfrak{p}$, we obtain

$$\overline{\int_{\mathcal{L}_\mathfrak{p}} f_i d\rho_{\mathbf{k}_\mathfrak{p}^{y_i}}\mu} - \overline{\rho_{\mathbf{k}_\mathfrak{p}^{y_i}} M} = \lim_{\mathcal{U}\in \mathrm{Cov}(\mathbb{P}^1(F_\mathfrak{p}))} \sum_{U\in\mathcal{U}} \overline{\left(\left(\frac{f_i}{f_0^{y_i}}\right)(x_U) - 1\right)} \cdot \rho_1\mu(1_U).$$

Hence

$$\begin{aligned} \overline{\int_{\mathcal{L}_\mathfrak{p}} (f_1 \cdot f_2)\, d\rho_{\mathbf{k}_\mathfrak{p}^{y_1+y_2}}\mu} - \overline{\rho_{\mathbf{k}_\mathfrak{p}^{y_1+y_2}} M} &= \lim_{\mathcal{U}\in \mathrm{Cov}(\mathbb{P}^1(F_\mathfrak{p}))} \sum_{U\in\mathcal{U}} \overline{\left(\left(\frac{f_1\cdot f_2}{f_0^{y_1+y_2}}\right)(x_U) - 1\right)} \cdot \rho_1\mu(1_U) \\ &= \lim_{\mathcal{U}\in \mathrm{Cov}(\mathbb{P}^1(F_\mathfrak{p}))} \sum_{U\in\mathcal{U}} \left(\overline{\left(\left(\frac{f_1}{f_0^{y_1}}\right)(x_U) - 1\right)} + \overline{\left(\left(\frac{f_2}{f_0^{y_2}}\right)(x_U) - 1\right)}\right) \cdot \rho_1\mu(1_U) \\ &= \overline{\int_{\mathcal{L}_p} f_1 d\rho_{\mathbf{k}_\mathfrak{p}^{y_1}}\mu} - \overline{\rho_{\mathbf{k}_\mathfrak{p}^{y_1}} M} + \overline{\int_{\mathcal{L}_p} f_2 d\rho_{\mathbf{k}_\mathfrak{p}^{y_2}}\mu} - \overline{\rho_{\mathbf{k}_\mathfrak{p}^{y_2}} M}. \end{aligned}$$

Finally, result follows from the fact that $\overline{\rho_{\mathbf{k}_\mathfrak{p}^y} M} = y\overline{M}$. Indeed, under the group isomorphism $\hat{\mathcal{O}}_{F_\mathfrak{p}}^\times \simeq I_\mathfrak{p}/I_\mathfrak{p}^2$ the specialization $\rho_{\mathbf{k}_\mathfrak{p}^y}$ on $I_\mathfrak{p}/I_\mathfrak{p}^2$ corresponds to raising to the y-th power on $\hat{\mathcal{O}}_{F_\mathfrak{p}}^\times$. □

Remark 5.4.7. *Since* $\mathcal{O}_{F_\mathfrak{p}}^\times = \mu_{q_\mathfrak{p}-1}\cdot(1+\mathfrak{p})$*, where* $\mu_{q_\mathfrak{p}-1}$ *are the* $(q_\mathfrak{p}-1)$*-th roots of unity, we have*

$$\hat{\mathcal{O}}_{F_\mathfrak{p}}^\times = \mathcal{O}_{F_\mathfrak{p}}^\times \otimes_\mathbb{Z} \mathbb{Z}_p \simeq 1+\mathfrak{p} \subset \mathcal{O}_{F_\mathfrak{p}}^\times \overset{\mathbf{k}_\mathfrak{p}}{\hookrightarrow} \Lambda_\mathfrak{p}^\times.$$

Thus, we can think of $\ell_{\mathbf{a}_\mathfrak{p}}$ *and the functions in* $\mathcal{E}(\ell_{\mathbf{a}_\mathfrak{p}})$ *as having values in* $\hat{\mathcal{O}}_{F_\mathfrak{p}}^\times \subset \Lambda_\mathfrak{p}^\times$.

Lemma 5.4.8. *We have a well defined $G(F_\mathfrak{p})$-equivariant morphism*

$$r_\mathfrak{p} : \mathbb{D}_{\mathbf{k}_\mathfrak{p}^{-2}}(\Lambda_\mathfrak{p})^0_{\mathbf{a}_\mathfrak{p}} \longrightarrow \mathbf{Hom}_{\mathbb{Z}_p}(\mathcal{E}(\ell^2_{\mathbf{a}_\mathfrak{p}}), I_\mathfrak{p}/I_\mathfrak{p}^2)$$

given by

$$r_\mathfrak{p}(\mu)(\phi, y) = \int_{\mathcal{L}_\mathfrak{p}} \phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} d\rho_{\mathbf{k}^{-y}}\mu(c,d) \quad \mod I_\mathfrak{p}^2,$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \boldsymbol{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$.

Proof. The morphism is well defined since for any other $\left(\begin{smallmatrix} a' & b' \\ \lambda c & \lambda d \end{smallmatrix}\right) \in \mathbf{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$ there exist $\left(\begin{smallmatrix} t & x \\ & \lambda \end{smallmatrix}\right) \in \mathbf{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$ such that $\left(\begin{smallmatrix} a' & b' \\ \lambda c & \lambda d \end{smallmatrix}\right) = \left(\begin{smallmatrix} t & x \\ & \lambda \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Moreover $r_\mathfrak{p}(\mu)$ is a group homomorphism by Lemma 5.4.6.

For any $(\phi, y) \in \mathcal{E}(\ell^2_{\mathbf{a}_\mathfrak{p}})$, we write $\hat\phi(c,d) = \phi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in C_{\mathbf{k}_\mathfrak{p}^{2y}}(\mathcal{L}_\mathfrak{p}, \mathcal{O}_{F_\mathfrak{p}}^\times)$. If we fix $(c,d) \in \mathcal{L}_\mathfrak{p}$, and $g \in G(F_\mathfrak{p})$, we choose $k \in \mathbf{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) g = \left(\begin{smallmatrix} x_1 & y \\ & x_2 \end{smallmatrix}\right) k$ for some $x_1, y, x_2 \in F_\mathfrak{p}$. Thus, we obtain

$$\begin{aligned}
(g\hat\phi)(c,d) &= ({\varphi_\mathfrak{p}}^{-1} g \varphi_\mathfrak{p}\hat\phi)(c,d) = g\varphi_\mathfrak{p}\hat\phi\begin{pmatrix} a & b \\ c & d \end{pmatrix}\mathbf{k}_\mathfrak{p}(ad-bc)^y \\
&= \varphi_\mathfrak{p}\hat\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} g\right)\mathbf{k}_\mathfrak{p}(ad-bc)^y = \varphi_\mathfrak{p}\hat\phi\left(\begin{pmatrix} x_1 & y \\ & x_2 \end{pmatrix} k\right)\mathbf{k}_\mathfrak{p}(ad-bc)^y \\
&= \hat{\mathbf{k}}_\mathfrak{p}\left(\frac{x_1}{x_2}\right)^{-y}\phi(k)\cdot\mathbf{k}_\mathfrak{p}(\det(k))^{-y}\mathbf{k}_\mathfrak{p}(ad-bc)^y = \hat{\mathbf{k}}_\mathfrak{p}(x_2)^{2y}\,\phi(k)\cdot\hat{\mathbf{k}}_\mathfrak{p}(\det(g))^{-y} \\
&= \phi\left(\begin{pmatrix} x_1 & y \\ & x_2 \end{pmatrix} k\right)\cdot\hat{\mathbf{k}}_\mathfrak{p}(\det(g))^{-y} = (g\phi)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\hat{\mathbf{k}}_\mathfrak{p}(\det(g))^{-y}.
\end{aligned}$$

Since $\rho_1\hat{\mathbf{k}}_\mathfrak{p}(\det(g))^y = 1$, we obtain,

$$\begin{aligned}
r_\mathfrak{p}(g\mu)(\phi,y) &= \overline{\int_{\mathcal{L}_\mathfrak{p}}\phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} d\rho_{\mathbf{k}^{-y}}g\mu(c,d)} = \overline{\int_{\mathcal{L}_\mathfrak{p}}(g^{-1}\hat\phi)(c,d)d\rho_{\mathbf{k}^{-y}}\mu(c,d)} \\
&= \rho_1\hat{\mathbf{k}}_\mathfrak{p}(\det(g))^y\cdot\overline{\int_{\mathcal{L}_\mathfrak{p}}(g^{-1}\phi)\begin{pmatrix} a & b \\ c & d \end{pmatrix} d\rho_{\mathbf{k}^{-y}}\mu(c,d)} = \iota(\mu)(g^{-1}\phi, y) \\
&= (gr_\mathfrak{p}(\mu))(\phi,y).
\end{aligned}$$

Hence the $G(F_\mathfrak{p})$-equivariance follows. □

5.4.3 Relation between Iwasawa algebras

At the beginning of §5.4 we have introduced the general Iwasawa algebra Λ_F, and in §4.1.1 we have defined the $\mathfrak{p}$-Iwasawa algebra $\Lambda_\mathfrak{p}$. In this section we will explore relations between both.

Since we have a natural continuous character

$$\mathcal{O}_{F_\mathfrak{p}}^\times \stackrel{\alpha\mapsto(\alpha,1)}{\longrightarrow} \mathcal{O}_{F_p}^\times \stackrel{\mathbf{k}_p}{\longrightarrow} \Lambda_F^\times,$$

we obtain using the universal property of $\Lambda_\mathfrak{p}$ an algebra morphism $\iota_\mathfrak{p} : \Lambda_\mathfrak{p} \longrightarrow \Lambda_F$. Similarly, the character

$$\mathcal{O}_{F_p}^\times \stackrel{\alpha\mapsto\alpha_\mathfrak{p}}{\longrightarrow} \mathcal{O}_{F_\mathfrak{p}}^\times \stackrel{\mathbf{k}_\mathfrak{p}}{\longrightarrow} \Lambda_\mathfrak{p}^\times,$$

provides another algebra morphism $d_{\mathfrak{p}}:\Lambda_F\to\Lambda_{\mathfrak{p}}$. It is clear that $\iota_{\mathfrak{p}}$ is a section of $d_{\mathfrak{p}}$, namely, $d_{\mathfrak{p}}\circ\iota_{\mathfrak{p}}=\mathrm{id}$. The existence of such sections implies that $\Lambda_F^\times\simeq\prod_{\mathfrak{p}|p}\Lambda_{\mathfrak{p}}^\times$.

If $I\subset\Lambda_F$ is the augmentation ideal

$$I=\ker\left(\Lambda_F\xrightarrow{\rho_1}\mathbb{Z}_p\right),$$

then $\iota_{\mathfrak{p}}(I_{\mathfrak{p}})\subseteq I$. Indeed, the morphism $\Lambda_{\mathfrak{p}}\xrightarrow{\iota_{\mathfrak{p}}}\Lambda_F\xrightarrow{\rho_1}\mathbb{Z}_p$ corresponds to the character $1:\mathcal{O}_{F_{\mathfrak{p}}}^\times\to 1$, hence $\rho_1\circ\iota_{\mathfrak{p}}=\rho_1$. Thus, for any $\iota_{\mathfrak{p}}(s)\in\iota_{\mathfrak{p}}(I_{\mathfrak{p}})$, we have

$$\rho_1(\iota_{\mathfrak{p}}(s))=\rho_1\circ\iota_{\mathfrak{p}}(s)=\rho_1 s=0,$$

therefore $\iota_{\mathfrak{p}}(s)\in I$.

We have a natural morphism

$$\bigotimes_{\mathfrak{p}|p}C_{\mathbf{k}_{\mathfrak{p}}}(\mathcal{L}_{\mathfrak{p}},\Lambda_{\mathfrak{p}})\longrightarrow C_{\mathbf{k}_p}(\mathcal{L}_p,\Lambda_F),\qquad\bigotimes_{\mathfrak{p}|p}f_{\mathfrak{p}}\longmapsto\prod_{\mathfrak{p}|p}\iota_{\mathfrak{p}}(f_{\mathfrak{p}})$$

Let $\underline{\mathbf{a}}=(\mathbf{a}_{\mathfrak{p}})_{\mathfrak{p}}$, with $\mathbf{a}_{\mathfrak{p}}\in\Lambda_F^\times$ providing an extension $\hat{\mathbf{k}}_p$ of the universal character $\mathbf{k}_p$. We will assume throughout the next chapters that $\mathbf{a}_{\mathfrak{p}}\in\Lambda_{\mathfrak{p}}^\times\subseteq\Lambda_F^\times$. This will imply that the induced morphism

$$\bigotimes_{\mathfrak{p}|p}C_{\mathbf{k}_{\mathfrak{p}}}(\mathcal{L}_{\mathfrak{p}},\Lambda_{\mathfrak{p}})_{\mathbf{a}_{\mathfrak{p}}}\longrightarrow C_{\mathbf{k}_p}(\mathcal{L}_p,\Lambda_F)_{\underline{\mathbf{a}}}$$

is $G(F_p)$-equivariant. In fact, under the assumption

$$\hat{\mathbf{k}}_p(x)=\prod_{\mathfrak{p}|p}\iota_{\mathfrak{p}}\left(\hat{\mathbf{k}}_{\mathfrak{p}}(x_{\mathfrak{p}})\right),\qquad\text{for all } x=(x_{\mathfrak{p}})_{\mathfrak{p}}\in F_p^\times.$$

For any set S of primes dividing p and any $n_S=(n_{\mathfrak{p}})_{\mathfrak{p}\in S}\in\mathbb{Z}^S$, write

$$\mathbf{k}_p^{n_S}:\mathcal{O}_{F_p}^\times\longrightarrow\prod_{\mathfrak{p}\in S}\mathcal{O}_{F_{\mathfrak{p}}}^\times\xrightarrow{\alpha_{\mathfrak{p}}\mapsto\alpha_{\mathfrak{p}}^{n_{\mathfrak{p}}}}\prod_{\mathfrak{p}\in S}\mathcal{O}_{F_{\mathfrak{p}}}^\times\xrightarrow{\mathbf{k}_p}\Lambda_F^\times.$$

Thus, we can consider the subspaces

$$\mathbb{D}_{\mathbf{k}_p}(\Lambda_F)^S=\{\mu\in\mathbb{D}_{\mathbf{k}_p}(\Lambda_F):\ \rho_{\mathbf{k}_p^{\underline{1}^{\mathfrak{p}}}}\mu=0,\ \text{for all }\mathfrak{p}\in S\},$$

where $\underline{1}^{\mathfrak{p}}\in\mathbb{Z}^{\{\mathfrak{q}|p,\mathfrak{q}\neq\mathfrak{p}\}}$ is the element with all components 1. By Lemma 5.4.4, if we assume that $\rho_1(\mathbf{a}_{\mathfrak{p}})=1$ for all $\mathfrak{p}\in S$, this defines a $G(F_p)$-invariant subspace denoted by $\mathbb{D}_{\mathbf{k}_p}(\Lambda_F)^S_{\underline{\mathbf{a}}}$.

Proposition 5.4.9. *Write $\rho_1\hat{\mathbf{k}}_p=(\hat{\chi}_{\mathfrak{p}})_{\mathfrak{p}}$ and assume that $\hat{\chi}_{\mathfrak{p}}=1$ for all $\mathfrak{p}\in S$ (equivalently $\rho_1(\mathbf{a}_{\mathfrak{p}})=1$). We have a well defined $G(F_p)$-equivariant morphism*

$$r_S:\mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)^S_{\underline{\mathbf{a}}}\longrightarrow\mathbf{Hom}_{\mathbb{Z}_p}\left(\bigotimes_{\mathfrak{p}\notin S}\mathrm{Ind}_P^G(\hat{\chi}_{\mathfrak{p}})\otimes\bigotimes_{\mathfrak{p}\in S}\mathcal{E}(\ell^2_{\mathbf{a}_{\mathfrak{p}}}),I^r/I^{r+1}\right)$$

given by

$$r_S(\mu)\left(\bigotimes_{\mathfrak{p}\notin S}\varphi_{\mathfrak{p}}(f_{\mathfrak{p}})\otimes\bigotimes_{\mathfrak{p}\in S}(\phi_{\mathfrak{p}},y_{\mathfrak{p}})\right)=\int_{\mathcal{L}_p}\prod_{\mathfrak{p}\notin S}f_{\mathfrak{p}}(c_{\mathfrak{p}},d_{\mathfrak{p}})\prod_{\mathfrak{p}\in S}\phi_{\mathfrak{p}}\begin{pmatrix}a_{\mathfrak{p}}&b_{\mathfrak{p}}\\c_{\mathfrak{p}}&d_{\mathfrak{p}}\end{pmatrix}d\rho_{\mathbf{k}_p^{-\underline{y}}}\mu(c,d)\mod I^{r+1},$$

where $\left(\begin{smallmatrix}a_{\mathfrak{p}}&b_{\mathfrak{p}}\\c_{\mathfrak{p}}&d_{\mathfrak{p}}\end{smallmatrix}\right)\in\boldsymbol{GL}_2(\mathcal{O}_{F_{\mathfrak{p}}})$, $r=\#S$ and $\underline{y}:=(y_{\mathfrak{p}})_{\mathfrak{p}\in S}\in\mathbb{Z}^S$.

Proof. We write $\mathcal{L}_{p\setminus\mathfrak{p}} := \prod_{\mathfrak{q}\neq\mathfrak{p}} \mathcal{L}_{\mathfrak{p}}$. Note that

$$\int_{\mathcal{L}_p} \prod_{\mathfrak{p}\notin S} f_{\mathfrak{p}}(c_{\mathfrak{p}}, d_{\mathfrak{p}}) \prod_{\mathfrak{p}\in S} \phi_{\mathfrak{p}} \begin{pmatrix} a_{\mathfrak{p}} & b_{\mathfrak{p}} \\ c_{\mathfrak{p}} & d_{\mathfrak{p}} \end{pmatrix} d\rho_{\mathbf{k}^{-\underline{y}}}\mu(c,d)$$

$$= \int_{\mathcal{L}_{p\setminus\mathfrak{p}}} \prod_{\mathfrak{p}\notin S} f_{\mathfrak{p}}(c_{\mathfrak{p}}, d_{\mathfrak{p}}) \prod_{\mathfrak{q}\neq\mathfrak{p}} \phi_{\mathfrak{q}} \begin{pmatrix} a_{\mathfrak{q}} & b_{\mathfrak{q}} \\ c_{\mathfrak{q}} & d_{\mathfrak{q}} \end{pmatrix} \cdot \int_{\mathcal{L}_{\mathfrak{p}}} \phi_{\mathfrak{p}} \begin{pmatrix} a_{\mathfrak{p}} & b_{\mathfrak{p}} \\ c_{\mathfrak{p}} & d_{\mathfrak{p}} \end{pmatrix} d\rho_{\mathbf{k}_p^{-\underline{y}}}\mu(c,d),$$

and it is clear by definition that $d\rho_{\mathbf{k}_p^{-\underline{y}}}\mu(c,d)\mid_{\mathcal{L}_{\mathfrak{p}}} \in \mathbb{D}_{\mathbf{k}_{\mathfrak{p}}^{2y_{\mathfrak{p}}}}(\Lambda_{\mathfrak{p}})^0 \otimes_{\Lambda_{\mathfrak{p}}} \Lambda_F$. Thus

$$\int_{\mathcal{L}_{\mathfrak{p}}} \phi_{\mathfrak{p}} \begin{pmatrix} a_{\mathfrak{p}} & b_{\mathfrak{p}} \\ c_{\mathfrak{p}} & d_{\mathfrak{p}} \end{pmatrix} d\rho_{\mathbf{k}_p^{-\underline{y}}}\mu(c,d)\Big|_{\mathcal{L}_{\mathfrak{q}}} \in \iota_{\mathfrak{p}}(I_{\mathfrak{p}})\Lambda_F \otimes_{\Lambda_{\mathfrak{q}}} \mathbb{D}_{\mathbf{k}_{\mathfrak{q}}^{2y_{\mathfrak{q}}}}(\Lambda_{\mathfrak{q}})^0, \qquad \mathfrak{q} \in S\setminus\{\mathfrak{p}\}.$$

Applying a straightforward induction we obtain that

$$\int_{\mathcal{L}_p} \prod_{\mathfrak{p}\notin S} f_{\mathfrak{p}}(c_{\mathfrak{p}}, d_{\mathfrak{p}}) \prod_{\mathfrak{p}\in S} \phi_{\mathfrak{p}} \begin{pmatrix} a_{\mathfrak{p}} & b_{\mathfrak{p}} \\ c_{\mathfrak{p}} & d_{\mathfrak{p}} \end{pmatrix} d\rho_{\mathbf{k}_p^{-\underline{y}}}\mu(c,d) \in \prod_{\mathfrak{p}\in S} \iota_{\mathfrak{p}}(I_{\mathfrak{p}})\Lambda_F \subset I^r.$$

Moreover, by Lemma 5.4.8 the expression r_S defines a well defined $G(F_p)$-equivariant group homomorphism. □

Write $\Delta_{F_{\mathfrak{p}}} \subset \Delta_{\mathfrak{p}}$ for the subgroup of $\mathrm{Gal}(K_{\mathfrak{p}}/F_{\mathfrak{p}})$-invariant divisors, namely, the even degree divisors generated by those of the form $\tau + \bar{\tau}$. Write $\Delta_{F_{\mathfrak{p}}}^0$ for the degree zero subgroup. Let

$$\hat{\mathcal{O}}_S^\times := \bigotimes_{\mathfrak{p}\in S} \hat{\mathcal{O}}_{F_{\mathfrak{p}}}^\times, \quad \mathcal{E}(\ell_{\mathbf{a}_S}^2) := \bigotimes_{\mathfrak{p}\in S} \mathcal{E}(\ell_{\mathbf{a}_{\mathfrak{p}}}^2), \quad \hat{F}_S^\times = \bigotimes_{\mathfrak{p}\in S} (F_{\mathfrak{p}}^\times \otimes_{\mathbb{Z}} \mathbb{Z}_p), \quad \mathrm{St}_{\mathbb{Z}_p}(F_S) := \bigotimes_{\mathfrak{p}\in S} \mathrm{St}_{\mathbb{Z}_p}(F_{\mathfrak{p}}),$$

where tensor products are taken with respect to $\mathbb{Z}_p^\times$, and

$$\Delta_{F_S} := \bigotimes_{\mathfrak{p}\in S} \Delta_{F_{\mathfrak{p}}}, \quad \Delta_{F_S}^0 := \bigotimes_{\mathfrak{p}\in S} \Delta_{F_{\mathfrak{p}}}^0, \quad \mathrm{St}_{F_S^\times} := \bigotimes_{\mathfrak{p}\in S} \mathrm{St}_{F_{\mathfrak{p}}^\times}, \quad \mathcal{E}_{K_S^\times} := \bigotimes_{\mathfrak{p}\in S} \mathcal{E}_{K_{\mathfrak{p}}^\times},$$

where the tensor products are with respect to $\mathbb{Z}$. Since $\hat{\mathcal{O}}_{F_{\mathfrak{p}}}^\times \simeq I_{\mathfrak{p}}/I_{\mathfrak{p}}^2$, we have a well defined morphism

$$\hat{\mathcal{O}}_S^\times \longrightarrow I^r/I^{r+1}.$$

This provides via $\ell_{\mathbf{a}_{\mathfrak{p}}} : F_{\mathfrak{p}}^\times \to \hat{\mathcal{O}}_{F_{\mathfrak{p}}}^\times$ a morphism

$$\ell_{\mathbf{a}_S} : \hat{F}_S^\times \xrightarrow{\bigotimes \ell_{\mathbf{a}_{\mathfrak{p}}}} \hat{\mathcal{O}}_S^\times \longrightarrow I^r/I^{r+1}. \tag{5.20}$$

Note that φ_{unv} and ev of (4.4) and (4.6) extend to

$$\varphi_{\mathrm{unv}} : \mathbf{Hom}(\mathrm{St}_{\mathbb{Z}_p}(F_S), \mathbb{Z}_p) \longrightarrow \mathbf{Hom}\left(\mathrm{St}_{F_S^\times}, \hat{F}_S^\times\right), \qquad \mathrm{ev}_S : \Delta_S \longrightarrow \mathcal{E}_{K_S^\times}$$

Similarly as in (4.3), the composition $\mathrm{ev}_S^* \circ \varphi_{\mathrm{unv}}$ provides a morphism

$$\begin{array}{ccc} \mathbf{Hom}(\mathrm{St}_{\mathbb{Z}_p}(F_S), \mathbb{Z}_p) & \longrightarrow & \mathbf{Hom}(\Delta_S^0, \hat{K}_S^\times) \\ \psi & \longmapsto & \left(\bigotimes_{\mathfrak{p}\in S}(z_{2,\mathfrak{p}} - z_{1,\mathfrak{p}}) \mapsto \lim_{\{\mathcal{U}_{\mathfrak{p}}\}_{\mathfrak{p}} \in \mathrm{Cov}'(\mathbb{P}^1(F_S))} \bigotimes_{\mathfrak{p}\in S} \prod_{U_{\mathfrak{p}}\in\mathcal{U}_{\mathfrak{p}}} \left(\frac{x_{U_{\mathfrak{p}}} - z_{2,\mathfrak{p}}}{x_{U_{\mathfrak{p}}} - z_{1,\mathfrak{p}}}\right)^{\psi(\mathbb{1}_{U_{\mathfrak{p}}})}\right), \end{array} \tag{5.21}$$

where $\mathrm{Cov}'(\mathbb{P}^1(F_S)) := \prod_{\mathfrak{p}\in S} \mathrm{Cov}(\mathbb{P}^1(F_{\mathfrak{p}}))$ and each $x_{U_{\mathfrak{p}}} \in U_{\mathfrak{p}}$. For any $z_{\mathfrak{p}} \in \Delta_{\mathfrak{p}}$, we recall the functions $\phi_{z_{\mathfrak{p}}} : G(F_{\mathfrak{p}}) \longrightarrow K_{\mathfrak{p}}$ of (4.6).

Proposition 5.4.10. *Write* $\rho_1 \hat{\mathbf{k}}_p = (\hat{\chi}_\mathfrak{p})_\mathfrak{p}$, $V^S = \bigotimes_{\mathfrak{p}\notin S} \mathrm{Ind}_P^G(\hat{\chi}_\mathfrak{p})$, *and assume that* $\hat{\chi}_\mathfrak{p} = 1$ *for all* $\mathfrak{p} \in S$. *We have* $G(F_p)$*-equivariant morphisms*

$$\bar{\mathrm{ev}}_S : \mathbf{Hom}\left(V^S \otimes \mathcal{E}(\ell^2_{\mathbf{a}_S}), I^r/I^{r+1}\right) \longrightarrow \mathbf{Hom}\left(V^S \otimes \Delta_{F_S}, I^r/I^{r+1}\right),$$

$$\varphi \longmapsto \left(v^S \otimes \bigotimes_{\mathfrak{p}\in S}(z_\mathfrak{p} + \bar{z}_\mathfrak{p}) \mapsto \varphi\left(v^S \otimes \bigotimes_{\mathfrak{p}\in S}(\ell_{\mathbf{a}_\mathfrak{p}}(\phi_{z_\mathfrak{p}} \cdot \phi_{\bar{z}_\mathfrak{p}}), 1)\right)\right),$$

making the following diagram commutative

$$\begin{array}{ccccc}
 & \overset{\bar{\mathrm{ev}}_S \circ r_S}{\nearrow} & \mathbf{Hom}(V^S \otimes \Delta_{F_S}, I^r/I^{r+1}) & \longrightarrow & \mathbf{Hom}(V^S \otimes \Delta^0_{F_S}, I^r/I^{r+1}) \\
\mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)^S_{\underline{\mathbf{a}}} & & & & \uparrow \ell_{\mathbf{a}_S} \\
 & \underset{\rho_1}{\searrow} & \mathbf{Hom}_{\mathbb{Z}_p}(V^S \otimes \mathrm{St}_{\mathbb{Z}_p}(F_S), \mathbb{Z}_p) & \xrightarrow{\mathrm{ev}^*_S \circ \varphi_{\mathrm{unv}}} & \mathbf{Hom}(V^S \otimes \Delta^0_{F_S}, \hat{F}^\times_S)
\end{array}$$

Proof. The $G(F_p)$-equivariance of $\bar{\mathrm{ev}}_S$ is clear by the definitions. Given $\mu \in \mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)^S_{\underline{\mathbf{a}}}$, we compute

$$\ell_{\mathbf{a}_S}\left(\left(\mathrm{ev}^*_S \circ \varphi_{\mathrm{unv}} \circ \rho_1\right)(\mu)\left(v^S \otimes \bigotimes_{\mathfrak{p}\in S}(z_{\mathfrak{p},1} - z_{\mathfrak{p},2} + \bar{z}_{\mathfrak{p},1} - \bar{z}_{\mathfrak{p},2})\right)\right) =$$

$$= \lim_{\{\mathcal{U}_\mathfrak{p}\}_\mathfrak{p}\in \mathrm{Cov}'(\mathbb{P}^1(F_S))} \ell_{\mathbf{a}_S}\left(\bigotimes_{\mathfrak{p}\in S} \prod_{U_\mathfrak{p}\in\mathcal{U}_\mathfrak{p}} \left(\frac{\phi_{z_{\mathfrak{p},1}}\phi_{\bar{z}_{\mathfrak{p},1}}}{\phi_{z_{\mathfrak{p},2}}\phi_{\bar{z}_{\mathfrak{p},2}}}\right)(x_{U_\mathfrak{p}})^{\rho_1(\mu)(v^S\otimes \mathbb{1}_{U_\mathfrak{p}})}\right)$$

$$= \lim_{\{\mathcal{U}_\mathfrak{p}\}_\mathfrak{p}\in \mathrm{Cov}'(\mathbb{P}^1(F_S))} \prod_{\mathfrak{p}\in S} \iota_\mathfrak{p}\ell_{\mathbf{a}_\mathfrak{p}}\left(\prod_{U_\mathfrak{p}\in\mathcal{U}_\mathfrak{p}} \left(\frac{\phi_{z_{\mathfrak{p},1}}\phi_{\bar{z}_{\mathfrak{p},1}}}{\phi_{z_{\mathfrak{p},2}}\phi_{\bar{z}_{\mathfrak{p},2}}}\right)(x_{U_\mathfrak{p}})^{\rho_1(\mu)(v^S\otimes \mathbb{1}_{U_\mathfrak{p}})}\right) \mod I^{r+1}$$

$$= \int_{\mathbb{P}^1(F_S)} \prod_{\mathfrak{p}\in S} \ell_{\mathbf{a}_\mathfrak{p}}\left(\frac{\phi_{z_{\mathfrak{p},1}}\phi_{\bar{z}_{\mathfrak{p},1}}}{\phi_{z_{\mathfrak{p},2}}\phi_{\bar{z}_{\mathfrak{p},2}}}\right) d\rho_1(\mu)(v^S) \mod I^{r+1}$$

$$= r_S(\mu)\left(v^S \otimes \bigotimes_{\mathfrak{p}\in S}\left(\ell_{\mathbf{a}_\mathfrak{p}}\left(\frac{\phi_{z_{\mathfrak{p},1}}\phi_{\bar{z}_{\mathfrak{p},1}}}{\phi_{z_{\mathfrak{p},2}}\phi_{\bar{z}_{\mathfrak{p},2}}}\right), 0\right)\right) = \bar{\mathrm{ev}}_S \circ r_S(\mu)\left(v^S \otimes \bigotimes_{\mathfrak{p}\in S}(z_{\mathfrak{p},1} - z_{\mathfrak{p},2} + \bar{z}_{\mathfrak{p},1} - \bar{z}_{\mathfrak{p},2})\right).$$

Hence, we obtain the required commutativity of the diagram. □

5.4.4 p-adic periods and L-invariants

Fix a prime $\mathfrak{p} \mid p$ and let $\phi^\mathfrak{p}_\lambda \in H^u(G(F)_+, \mathcal{A}^{\mathfrak{p}\cup\infty}(\mathrm{St}_\mathbb{Z}(F_\mathfrak{p}), \mathbb{Z}))^\lambda$ be a modular symbol associated with an elliptic curve E/F with multiplicative reduction at $\mathfrak{p}$. The short exact sequence

$$0 \longrightarrow \Delta^0_\mathfrak{p} \longrightarrow \Delta_\mathfrak{p} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

provides a connection morphism

$$H^u(G(F)_+, \mathcal{A}^{\mathfrak{p}\cup\infty}(\Delta^0_\mathfrak{p}, K^\times_\mathfrak{p}))^\lambda \xrightarrow{c} H^{u+1}(G(F)_+, \mathcal{A}^{\mathfrak{p}\cup\infty}(K^\times_\mathfrak{p}))^\lambda.$$

Moreover, the commutative diagram (4.7) shows that

$$c(\mathrm{ev}^* \circ \varphi_{\mathrm{unv}})\phi^\mathfrak{p}_\lambda \in H^{u+1}(G(F)_+, \mathcal{A}^{\mathfrak{p}\cup\infty}(q^\mathbb{Z}_\mathfrak{p}))^\lambda \subset H^{u+1}(G(F)_+, \mathcal{A}^{\mathfrak{p}\cup\infty}(F^\times_\mathfrak{p}))^\lambda,$$

for some $q_{\mathfrak{p}} \in F_{\mathfrak{p}}^{\times}$. The generalized Oda's conjecture ([GMM17, Conjecture 3.8]) asserts that, in fact, the elliptic curve $E/F_{\mathfrak{p}}$ is isogenous to the Tate curve defined by the quotient $\bar{F}_{\mathfrak{p}}/q_{\mathfrak{p}}^{\mathbb{Z}}$.

Theorem 5.4.11. *Let $\Phi_{\lambda}^{\mathfrak{p}} \in H^{u}(G(F)_{+}, \mathcal{A}^{\mathfrak{p}\cup\infty}(\mathbb{D}_{\mathbf{k}_{\mathfrak{p}}^{-2}}(\Lambda_{\mathfrak{p}}))_{\mathbf{a}_{\mathfrak{p}}})^{\lambda}$ be a family lifting $\phi_{\lambda}^{\mathfrak{p}}$. Then*

$$\ell_{\mathbf{a}_{\mathfrak{p}}}(q_{\mathfrak{p}}) = 1 \in \hat{\mathcal{O}}_{F_{\mathfrak{p}}}^{\times}.$$

Proof. Since $\Phi_{\lambda}^{\mathfrak{p}}$ specializes to the Steinberg representation, we have in fact

$$\Phi_{\lambda}^{\mathfrak{p}} \in H^{u}(G(F)_{+}, \mathcal{A}^{\mathfrak{p}\cup\infty}(\mathbb{D}_{\mathbf{k}_{\mathfrak{p}}^{-2}}(\Lambda_{\mathfrak{p}}))_{\mathbf{a}_{\mathfrak{p}}}^{0})^{\lambda}.$$

If we consider the exact sequence

$$H^{u}(G(F)_{+}, \mathcal{A}^{\mathfrak{p}\cup\infty}(\Delta_{F_{\mathfrak{p}}}, F_{\mathfrak{p}}^{\times}))^{\lambda} \xrightarrow{\text{res}} H^{u}(G(F)_{+}, \mathcal{A}^{\mathfrak{p}\cup\infty}(\Delta_{F_{\mathfrak{p}}}^{0}, F_{\mathfrak{p}}^{\times}))^{\lambda} \xrightarrow{c} H^{u+1}(G(F)_{+}, \mathcal{A}^{\mathfrak{p}\cup\infty}(F_{\mathfrak{p}}^{\times}))^{\lambda},$$

by Proposition 5.4.10,

$$\text{res}(\bar{\text{ev}} \circ r_{\mathfrak{p}})\Phi_{\lambda}^{\mathfrak{p}} = \ell_{\mathbf{a}_{\mathfrak{p}}}(\text{ev}^{*} \circ \varphi_{\text{unv}})\phi_{\lambda}^{\mathfrak{p}} \mid_{\Delta_{F_{\mathfrak{p}}}^{0}} .$$

This implies that

$$\ell_{\mathbf{a}_{\mathfrak{p}}} c(\text{ev}^{*} \circ \varphi_{\text{unv}})\phi_{\lambda}^{\mathfrak{p}} \mid_{\Delta_{F_{\mathfrak{p}}}^{0}} = 0.$$

Since $c(\text{ev}^{*} \circ \varphi_{\text{unv}})\phi_{\lambda}^{\mathfrak{p}} \mid_{\Delta_{F_{\mathfrak{p}}}^{0}}$ lies in $H^{u+1}(G(F)_{+}, \mathcal{A}^{\mathfrak{p}\cup\infty}(q_{\mathfrak{p}}^{2\mathbb{Z}}))^{\lambda}$, we deduce $\ell_{\mathbf{a}_{\mathfrak{p}}}(q_{\mathfrak{p}}) = 1$ from the fact that $\hat{\mathcal{O}}_{F_{\mathfrak{p}}}^{\times}$ is torsion free. □

Remark 5.4.12. *As showed in §5.4.1, the elements $\mathbf{a}_{\mathfrak{p}} \in \Lambda_{F}$ are the eigenvalues of the $U_{\mathfrak{p}}$-operators acting on the Hida family. By definition*

$$\ell_{\mathbf{a}_{\mathfrak{p}}}(\varpi_{\mathfrak{p}}) = \mathbf{a}_{\mathfrak{p}}^{-1} - 1 \in I_{\mathfrak{p}}/I_{\mathfrak{p}}^{2}.$$

Thus, the relation $\ell_{\mathbf{a}_{\mathfrak{p}}}(q_{\mathfrak{p}}) = 1$ implies

$$\text{ord}_{\mathfrak{p}}(q_{\mathfrak{p}}) \left(\mathbf{a}_{\mathfrak{p}}^{-1} - 1\right) + \log_{\mathfrak{p}}(q_{\mathfrak{p}}) \equiv 0 \mod I_{\mathfrak{p}}^{2}.$$

We obtain that the image of $\mathbf{a}_{\mathfrak{p}} - 1$ in $I_{\mathfrak{p}}/I_{\mathfrak{p}}^{2}$ is given by the L-invariant $\mathcal{L}_{\mathfrak{p}} = \log_{\mathfrak{p}}(q_{\mathfrak{p}})/\text{ord}_{\mathfrak{p}}(q_{\mathfrak{p}})$. With the formalism previously described, we have showed that the derivative of $\mathbf{a}_{\mathfrak{p}} - 1$ with respect to the weight variable is given by the L-invariant $\mathcal{L}_{\mathfrak{p}}$. In the classical setting, this is a key result due to Greenberg-Stevens towards the exceptional zero conjecture (see [GS93]).

5.4.5 2 variable p-adic L-functions

Let $\phi_{\lambda}^{p} \in H^{u}\left(G(F)_{+}, \mathcal{A}^{p\cup\infty}(V_{p}, V(\underline{k})_{\mathbb{C}_{p}})\right)^{\lambda}$ be an ordinary cohomology class generating an automorphic representation π. By Proposition 5.3.6, ϕ_{λ}^{p} provides an overconvergent cohomology class

$$\hat{\phi}_{\lambda}^{p} \in H^{u}\left(G(F)_{+}, \mathcal{A}^{p\cup\infty}(\mathbb{D}_{\chi_{p}^{-2}}(\mathbb{C}_{p})_{\underline{\alpha}^{*}})\right)^{\lambda},$$

for some locally polynomial character ${\chi_p}^{-1}$. Assume that we have the extension $\hat{\mathbf{k}}_p$ of the universal character $\mathbf{k}_p$ and an element

$$\Phi_\lambda^p \in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathbb{D}_{\mathbf{k}_p^{-2}}(\Lambda_F)_{\underline{\mathbf{a}}})\right)^\lambda$$

mapping to $\hat{\phi}_\lambda^p$ through the morphism ρ_{χ_p} of (5.16). Similarly as in Theorem 5.3.7, we can define the distribution $\mu_{\Phi_\lambda^p}$ as

$$\int_{\mathcal{G}_T} f d\mu_{\Phi_\lambda^p} = \left((\rho^* f) \cap \eta\right) \cap \delta_p^*(\Phi_\lambda^p), \qquad f \in C(\mathcal{G}_T, \Lambda_F)$$

where δ_p is defined as in (5.8):

$$\delta_p : C_c(T(F_p), \Lambda_F) \longrightarrow \mathrm{Ind}_P^G(\hat{\mathbf{k}}_p) \overset{\varphi_p}{\simeq} C_{\mathbf{k}_p^{-2}}(\mathcal{L}_p, \Lambda_F).$$

Indeed, we can think $\delta_p^*(\Phi_\lambda^p)$ and $\rho^* f \cap \eta$ as elements

$$\begin{aligned} \delta_p^*(\Phi_\lambda^p) &\in H^u\left(G(F)_+, \mathcal{A}^{p\cup\infty}(\mathrm{Meas}(T(F_p), \Lambda_F))\right)^\lambda, \\ \rho^* f \cap \eta &\in H_u(T(F), C_c(T(\mathbb{A}_F), \Lambda_F)) = H_u\left(T(F), \mathrm{Ind}_{T(F)_+}^{T(F)} C_c^0\left(T(\mathbb{A}_F^{p\cup\infty}), C_c(T(F_p), \Lambda_F)\right)\right), \end{aligned}$$

hence the pairing (3.15) applies and the cap-product is well defined. The measure $\mu_{\Phi_\lambda^p}$ is considered as the 2-variable p-adic L-function since its specialization at different weights $\theta_p : \mathcal{O}_{F_p}^\times \to \mathbb{C}_p$ provides different measures $\rho_{\theta_p}(\mu_{\Phi_\lambda^p})$ of $\mathcal{G}_T$. In particular

$$\rho_{\chi_p}(\mu_{\Phi_\lambda^p}) = \mu_{\phi_\lambda^p},$$

by Theorem 5.3.7. Thus $\mu_{\Phi_\lambda^p}$ interpolates $\mu_{\phi_\lambda^p}$ as the weight varies.

Chapter 6

Main results

Throughout this chapter we will assume that the automorphic representation is attached to an elliptic curve E/F.

6.1 A p-adic Gross-Zagier formula

As in §4.1, let us assume that $S = \{\mathfrak{p}\}$ and $G(F_\mathfrak{p}) = \mathbf{PGL}_2(F_\mathfrak{p})$, $V_\mathfrak{p}^{\mathbb{Z}} = \mathrm{St}_{\mathbb{Z}}(F_\mathfrak{p})$ and T does not split at $\mathfrak{p}$. We have seen in §5.2 that we can construct a measure $\mu_{\phi_\lambda^\mathfrak{p}}$ attached to a modular symbol $\phi_\lambda^\mathfrak{p}$. In this setting the distribution is 0-admissible, hence bounded.

Let $\xi \in C(\mathcal{G}_T, \mathbb{C})$ be as above with $\rho^*\xi|_{T(F_\infty)} = \lambda$. If we also assume that $\xi_\mathfrak{p} = 1$, we have by Remark 5.1.10

$$\int_{\mathcal{G}_T} \xi d\mu_{\phi_\lambda^\mathfrak{p}} = 0.$$

Write $\mathrm{Meas}(\mathcal{G}_T, \mathbb{C}_p)$ for the space of measures of $C(\mathcal{G}_T, \mathbb{C}_p)$, i.e. bounded distributions of $\mathcal{G}$ endowed with the natural group law

$$\int_{\mathcal{G}_T} f d(\mu_1 \star \mu_2) = \int_{\mathcal{G}_T}\int_{\mathcal{G}_T} f(\alpha\cdot\beta) d\mu_1(\alpha) d\mu_2(\beta). \tag{6.1}$$

Thus, $\mu_{\phi_\lambda^\mathfrak{p}}$ lies in the kernel I_ξ of the algebra morphism ϵ_ξ given by integration, defined by the exact sequence

$$0 \longrightarrow I_\xi \longrightarrow \mathrm{Meas}(\mathcal{G}_T, \mathbb{C}_p) \xrightarrow[\mu\mapsto\int_{\mathcal{G}_T}\xi d\mu]{\epsilon_\xi} \mathbb{C}_p \longrightarrow 0 \tag{6.2}$$

Since $\xi_\mathfrak{p} = 1$ the character ξ descends to a character of $\mathcal{G}_T^\mathfrak{p}$, denoted the same way. In 4.1.3 we have constructed Darmon points $P_\xi^\lambda \in E(K_\mathfrak{p}) \otimes_{\mathbb{Z}} \bar{\mathbb{Z}}$ associated with ξ. We aim to relate the image of $\mu_{\phi_\lambda^\mathfrak{p}}$ in I_ξ/I_ξ^2 with P_ξ^λ. In order to do that, we introduce the following well-defined morphism

$$\varphi : \mathcal{G}_T \otimes_{\mathbb{Z}} \bar{\mathbb{Z}} \longrightarrow I_\xi/I_\xi^2; \qquad \textstyle\int_{\mathcal{G}_T} f d\varphi(\sigma) = \xi(\sigma)^{-1}\cdot f(\sigma) - f(1), \qquad \sigma \in \mathcal{G}_T. \tag{6.3}$$

Denote by $(x \mapsto \overline{x})$ the nontrivial element of $\mathrm{Gal}(K_\mathfrak{p}/F_\mathfrak{p})$. Since $q \in F_\mathfrak{p}$ we can regard the difference $\overline{P}_\xi^\mathfrak{p} - P_\xi^\mathfrak{p}$ as

$$\overline{P}_\xi^\mathfrak{p} - P_\xi^\mathfrak{p} \in T(F_\mathfrak{p}) \otimes_{\mathbb{Z}} \bar{\mathbb{Z}}, \qquad T(F_\mathfrak{p}) = K_\mathfrak{p}^\times/F_\mathfrak{p}^\times \simeq \{x \in K_\mathfrak{p}^\times;\ \bar{x}\cdot x = 1\} \subset K_\mathfrak{p}^\times, \tag{6.4}$$

and we can consider its image through the natural morphism provided by the Artin map

$$\mathrm{rec}_\mathfrak{p} : T(F_\mathfrak{p}) \otimes_{\mathbb{Z}} \bar{\mathbb{Z}} \longrightarrow \mathcal{G}_T \otimes_{\mathbb{Z}} \bar{\mathbb{Z}}. \tag{6.5}$$

Theorem 6.1.1. *The image of* $\mu_{\phi_\lambda^{\mathfrak{p}}}$ *in* I_ξ/I_ξ^2 *is given by*

$$\mu_{\phi_\lambda^{\mathfrak{p}}} \equiv [O_+^{\mathfrak{p}} : O_+]^{-1} \cdot \varphi \circ \mathrm{rec}_{\mathfrak{p}} \left(\overline{P}_\xi^{\mathfrak{p}} - P_\xi^{\mathfrak{p}}\right) \mod I_\xi^2.$$

Proof. Let $\Delta_T := \mathbb{Z}[G(F)/T(F)]$ and Δ_T^0 the kernel of the degree map $\deg : \Delta_T \to \mathbb{Z}$. The normalizer of $T(F)$ in $G(F)$ is $T(F) \rtimes J$ where $J^2 \in T(F)$, moreover, $\overline{\tau}_{\mathfrak{p}} = J\tau_{\mathfrak{p}}$. Since $\tau_{\mathfrak{p}}$ is $T(F)$-invariant one can construct a $G(F)$-module morphism

$$\begin{array}{ccc} \Delta_T^0 & \hookrightarrow & \Delta_{\mathfrak{p}}^0 \\ \sum_i g_i \cdot T(F) & \longmapsto & \sum_i g_i \cdot \tau_{\mathfrak{p}} \end{array}$$

The morphism in (4.3) restricts to a $G(F)$-equivariant morphism $\mathbf{Hom}(\mathrm{St}_{\mathbb{Z}}(F_{\mathfrak{p}}), \mathbb{Z}) \to \mathbf{Hom}(\Delta_T^0, K_{\mathfrak{p}}^\times)$. Thus, we obtain the following diagram for $M = K_{\mathfrak{p}}^\times$ or $K_{\mathfrak{p}}^\times/q^{\mathbb{Z}}$

$$\begin{array}{ccc} & & \phi_\lambda^{\mathfrak{p}} \in H^u(G(F), \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\mathrm{St}_{\mathbb{Z}}(F_{\mathfrak{p}}), \mathbb{Z})(\lambda)) \\ & & \downarrow i_M \\ H^u(T(F), \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(M)(\lambda)) & \xrightarrow{\mathrm{res}} & H^u(G(F), \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_T^0, M)(\lambda)) \end{array}$$

since $\mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_T, M)(\lambda) = \mathrm{coInd}_{T(F)}^{G(F)}(\mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(M)(\lambda))$.

By definition $\mathrm{res}(\psi_\lambda^{\mathfrak{p}}|_{\tau_p}) = i_{K_{\mathfrak{p}}^\times}(\phi_\lambda^{\mathfrak{p}}) \mod q^{\mathbb{Z}}$, and $P_\xi^{\mathfrak{p}} = \eta^{\mathfrak{p}} \cap \xi_\lambda \cap \psi_\lambda^{\mathfrak{p}}|_{\tau_p}$. By the same construction of 4.1.3, let $\overline{P}_\xi^{\mathfrak{p}} = \eta^{\mathfrak{p}} \cap \xi_\lambda \cap \psi_{\mathfrak{p}}^\lambda|_{\overline{\tau}_p}$ where $\cdot|_{\overline{\tau}_p}$ is defined as in (4.10) but the evaluation is at $\overline{\tau}_{\mathfrak{p}}$ instead. Thus,

$$\overline{P}_\xi^{\mathfrak{p}} - P_\xi^{\mathfrak{p}} = \eta^{\mathfrak{p}} \cap \xi_\lambda \cap \mathrm{EV}(i_{K_{\mathfrak{p}}^\times}\phi_\lambda^{\mathfrak{p}}),$$

where $\overline{P}_\xi^{\mathfrak{p}} - P_\xi^{\mathfrak{p}}$ is seen in $T(F_{\mathfrak{p}}) \otimes \bar{\mathbb{Z}}$,

$$\begin{array}{rcl} \mathrm{EV} : \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\Delta_T^0, K_{\mathfrak{p}}^\times)(\lambda) & \longrightarrow & \mathrm{coInd}_{T(F)}^{G(F)}\left(\mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(K_{\mathfrak{p}}^\times)(\lambda)\right) \\ f & \longmapsto & \mathrm{EV}(f) : (h, g^{\mathfrak{p}}) \mapsto \lambda(h) \cdot f(h^{-1}g^{\mathfrak{p}})(h^{-1}JT(F) - h^{-1}T(F)) \end{array}$$

and $h, g^{\mathfrak{p}} \in G(F) \times G(\mathbb{A}^{\{\mathfrak{p}\}\cup\infty})$. Then one can take the composition

$$\mathrm{EV} \circ i_{K_{\mathfrak{p}}^\times} : \mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(\mathrm{St}_{\mathbb{Z}}(F_{\mathfrak{p}}), \mathbb{Z})(\lambda) \to \mathrm{coInd}_{T(F)}^{G(F)}\left(\mathcal{A}^{\{\mathfrak{p}\}\cup\infty}(T(F_{\mathfrak{p}}))(\lambda)\right)$$

and

$$\left(\mathrm{EV} \circ i_{K_{\mathfrak{p}}^\times}(\phi)\right)(h)(g^{\mathfrak{p}}) = \lambda(h) \cdot \times\!\!\!\!\!\!\int_{\mathbb{P}^1(F_{\mathfrak{p}})} \frac{x - h^{-1}J\tau_{\mathfrak{p}}}{x - h^{-1}\tau_{\mathfrak{p}}} d\mu_{\phi(h^{-1}g^{\mathfrak{p}})}.$$

Since $\ell := \varphi \circ \mathrm{rec}_{\mathfrak{p}}$ is a group homomorphism, by Lemma 5.1.6,

$$\begin{aligned} \left(\ell \circ \mathrm{EV} \circ i_{K_{\mathfrak{p}}^\times}(\phi)\right)(g^{\mathfrak{p}}) &= \ell \times\!\!\!\!\!\!\int_{\mathbb{P}^1(F_{\mathfrak{p}})} \frac{x - \overline{\tau}_{\mathfrak{p}}}{x - \tau_{\mathfrak{p}}} d\mu_{\phi(g^{\mathfrak{p}})} = \lim_{\mathcal{U} \in \mathrm{Cov}(\mathbb{P}^1(F_{\mathfrak{p}}))} \ell \prod_{U \in \mathcal{U}} \left(\frac{x_U - \overline{\tau}_{\mathfrak{p}}}{x_U - \tau_{\mathfrak{p}}}\right)^{\phi(g^{\mathfrak{p}})(\mathbb{1}_U)} \\ &= \lim_{\mathcal{U} \in \mathrm{Cov}(\mathbb{P}^1(F_{\mathfrak{p}}))} \sum_{U \in \mathcal{U}} \phi(g^{\mathfrak{p}})(\mathbb{1}_U) \cdot \ell\left(\frac{x_U - \overline{\tau}_{\mathfrak{p}}}{x_U - \tau_{\mathfrak{p}}}\right) = \phi(g^{\mathfrak{p}})(\delta_{\mathfrak{p}}\ell) = \delta_{\mathfrak{p}}^*\phi(g^{\mathfrak{p}})(\ell), \end{aligned}$$

where $\delta_{\mathfrak{p}}^*\phi(g^{\mathfrak{p}})$ is extended to a p-adic measure of $T(F_{\mathfrak{p}})$ using Riemann sums, and ℓ is seen as an element of $C(T(F_{\mathfrak{p}}), I_\chi/I_\chi^2)$.

For any $h^{\mathfrak{p}} \in C_c^0(T(\mathbb{A}_F^{\{\mathfrak{p}\}\cup\infty}), \mathbb{C}_p)$ the pairing $\langle h^{\mathfrak{p}}, \ell \circ \mathrm{EV} \circ i_{K_{\mathfrak{p}}^\times}(\phi)\rangle_+ \in I_\chi/I_\chi^2$ of (3.14) is represented by the distribution that maps $f \in C^0(\mathcal{G}_T, \mathbb{C}_p)$ to

$$\int_{T(\mathbb{A}_F^{\mathfrak{p}})} h^{\mathfrak{p}}(t^{\mathfrak{p}}) \cdot \delta_{\mathfrak{p}}^*\phi(t^{\mathfrak{p}})(\rho^* f_{\mathfrak{p}} - \rho^* f_{\mathfrak{p}}(1)) d^\times t^{\mathfrak{p}} = \langle h^{\mathfrak{p}} \otimes \rho^* f_{\mathfrak{p}}, \delta_{\mathfrak{p}}^*\phi\rangle_+.$$

since $\delta_{\mathfrak{p}}^*\phi(t^{\mathfrak{p}})(f_{\mathfrak{p}}(1)) = f_{\mathfrak{p}}(1) \cdot \phi(t^{\mathfrak{p}})(1) = 0$. Thus,

$$\varphi \circ \mathrm{rec}_{\mathfrak{p}}\left(\overline{P}_\xi^{\mathfrak{p}} - P_\xi^{\mathfrak{p}}\right) = \ell(\eta^{\mathfrak{p}} \cap \xi_\lambda \cap \mathrm{EV}(i_{K_{\mathfrak{p}}^\times}\phi_\lambda^{\mathfrak{p}})) = \eta^{\mathfrak{p}} \cap \xi_\lambda \cap \ell \circ \mathrm{EV}(i_{T(F_{\mathfrak{p}})}\phi_\lambda^{\mathfrak{p}}).$$

This implies that given $f \in C^0(\mathcal{G}_T, \mathbb{C}_p)$ such that $\rho^* f \mid_{T(\mathbb{A}_F^{\mathfrak{p}})} = \rho^*\xi \mid_{T(\mathbb{A}_F^{\mathfrak{p}})}$, by lemma 3.1.4

$$\begin{aligned}\int_{\mathcal{G}_T} f d\left(\varphi \circ \mathrm{rec}_{\mathfrak{p}}\left(\overline{P}_\xi^{\mathfrak{p}} - P_\xi^{\mathfrak{p}}\right)\right) &= \left((\eta^{\mathfrak{p}} \cap \xi_\lambda^{\mathfrak{p}}) \cap \rho^* f_{\mathfrak{p}}\right) \cap \delta_{\mathfrak{p}}^*\phi_\lambda^{\mathfrak{p}} \\ &= \left((\eta^{\mathfrak{p}} \cap 1_{T(F_{\mathfrak{p}})}) \cap (\xi_\lambda^{\mathfrak{p}} \otimes \rho^* f_{\mathfrak{p}}\right)) \cap \delta_{\mathfrak{p}}^*\phi_\lambda^{\mathfrak{p}} \\ &= [\mathcal{O}_+^{\mathfrak{p}} : \mathcal{O}_+](\eta \cap (\rho^* f \mid_{T(A_F^\infty)}) \cap \delta_{\mathfrak{p}}^*\phi_\lambda^{\mathfrak{p}} \stackrel{\text{(a)}}{=} [\mathcal{O}_+^{\mathfrak{p}} : \mathcal{O}_+] \int_{\mathcal{G}_T} f d\mu_{\phi_\lambda^{\mathfrak{p}}},\end{aligned}$$

where the cap product $(\eta^{\mathfrak{p}} \cap \xi_\lambda^{\mathfrak{p}}) \cap \rho^* f_{\mathfrak{p}}$ is taken with respect to (3.13) and (a) follows from (3.17). This shows that a representative of $\varphi \circ \mathrm{rec}_{\mathfrak{p}}\left(\overline{P}_\xi^{\mathfrak{p}} - P_\xi^{\mathfrak{p}}\right)$ agrees with $\mu_{\phi_\lambda^{\mathfrak{p}}}$ at all functions $f \in C(\mathcal{G}_T, \mathbb{C}_p)$ satisfying $\rho^* f \mid_{T(\mathbb{A}_F^{\mathfrak{p}})} = \rho^*\xi \mid_{T(\mathbb{A}_F^{\mathfrak{p}})}$. Thus the difference $\mu' := \mu_{\phi_\lambda^{\mathfrak{p}}} - \varphi \circ \mathrm{rec}_{\mathfrak{p}}\left(\overline{P}_\xi^{\mathfrak{p}} - P_\xi^{\mathfrak{p}}\right)$ vanishes at the subspace

$$C_\xi(\mathcal{G}_T, \mathbb{C}_p) = \left\{f \in C(\mathcal{G}_T, \mathbb{C}_p), \quad \rho^* f \mid_{T(\mathbb{A}_F^{\mathfrak{p}})} \in \mathbb{C}_p \rho^*\xi \mid_{T(\mathbb{A}_F^{\mathfrak{p}})}\right\}.$$

Clearly, $\mu' \in I_\xi$, but it is easy to show that

$$\mu' = \mu' \star \delta_\xi, \qquad \int_{\mathcal{G}_T} f d\delta_\xi = \begin{cases} f(1), & f \notin C_\xi(\mathcal{G}_T, \mathbb{C}_p); \\ 0, & f \in C_\xi(\mathcal{G}_T, \mathbb{C}_p). \end{cases} \tag{6.6}$$

Since clearly $\delta_\xi \in I_\xi$, we conclude that $\mu' \in I_\xi^2$ and the result follows. □

Remark 6.1.2. *The morphism $\varphi \circ \mathrm{rec}_{\mathfrak{p}} : T(F_{\mathfrak{p}}) \otimes_{\mathbb{Z}} \bar{\mathbb{Z}} \longrightarrow I_\xi/I_\xi^2$ vanishes at $T(F)$. Indeed, we have seen in the above proof that the class of a measure in I_ξ/I_ξ^2 depends on the image of functions in $C_\xi(\mathcal{G}_T, \mathbb{C}_p)$ (indeed such a class is a local attribute), and for such a $f \in C_\xi(\mathcal{G}_T, \mathbb{C}_p)$*

$$\int_{\mathcal{G}_T} f d\left(\varphi \circ \mathrm{rec}_{\mathfrak{p}}(\gamma)\right) = \rho^*\xi_{\mathfrak{p}}(\gamma)^{-1} \cdot f(\mathrm{rec}_{\mathfrak{p}}(\gamma)) - f(1) = \rho^*\xi_{\mathfrak{p}}(\gamma)^{-1} \cdot f(\mathrm{rec}^{\mathfrak{p}}(\gamma^{-1})) - f(1) = 0,$$

for all $\gamma \in T(F)$, where $\mathrm{rec}^{\mathfrak{p}} : T(\mathbb{A}_F^{\mathfrak{p}}) \longrightarrow \mathcal{G}_T$ is the natural morphism.

6.2 Exceptional zero formulas

Let S be a set of primes above p and assume that $G(F_S) = \mathbf{PGL}_2(F_S)$ and $V_\mathfrak{p}$ is ordinary at any $\mathfrak{p} \in S$. Let $\mu_{\phi_\lambda^S}$ be the measure of $\mathcal{G}_T$ constructed in §5.2. We can write

$$V_S^{\mathbb{Z}} = \bigotimes_{\mathfrak{p}\in S_+} \mathrm{St}_{\mathbb{Z}}(F_\mathfrak{p})(\varepsilon_\mathfrak{p}) \otimes \bigotimes_{\mathfrak{p}\in S_-} V_\mathfrak{q}^{\mathbb{Z}}, \qquad \text{where} \quad V_\mathfrak{q}^{\mathbb{Z}} \not\simeq \mathrm{St}_{\mathbb{Z}}(F_\mathfrak{q})(\pm), \text{ for any } \mathfrak{q} \in S_-.$$

If we write $r := \#S_+$, Remark 5.1.10 makes us think that $\mu_{\phi_\lambda^S} \in I_\xi^r$, for all $\xi \in C^0(\mathcal{G}_T, \bar{\mathbb{Z}})^{\varepsilon_{S_+}}$ with $\xi \mid_{T(F_\infty)} = \lambda$. Moreover, Mazur-Tate interpretation realizes the image of $\mu_{\phi_\lambda^S}$ in I_ξ^r / I_ξ^{r+1} as the r-th derivative of the corresponding p-adic L-function. Hence following the philosophy of the p-adic BSD conjecture, such image must be related to the extended Mordell-Weil group of E. Precisely, this is the content of Theorem 6.1.1 when $S = \{\mathfrak{p}\}$, $V_\mathfrak{p}^{\mathbb{Z}} = \mathrm{St}_{\mathbb{Z}}(F_\mathfrak{p})$ and T does not split at $\mathfrak{p}$, since Darmon points are supposed to generate the extended Mordell-Weil group in these rank 1 situations. In [FG21, Theorem A], Fornea and Gehrmann prove a similar result with plectic points, when $S = S_+$ and T is inert at any $\mathfrak{p} \in S$. Our aim in this section is to adapt the proof of Theorem 6.1.1 to establish the general result for arbitrary S, V_S and T.

Write $S_+ = S_+^1 \cup S_+^2$, where

$$S_+^1 := \{\mathfrak{p} \in S_+,\ T \text{ splits in } \mathfrak{p}\}, \qquad S_+^2 = \{\mathfrak{q} \in S_+,\ T \text{ does not split in } \mathfrak{q}\}.$$

Recall that the construction of ϕ_λ^S (and thus $\mu_{\phi_\lambda^S}$) depends on the choice of $x^S \in \pi^{S\cup\infty}$. For any $\mathfrak{q} \in S_-$ we choose $x_\mathfrak{q} \in V_\mathfrak{q}^{\mathbb{Z}}$ such that $\alpha(x_\mathfrak{q}) = 1$, where as in proof of Theorem 5.1.9

$$\alpha(x_\mathfrak{q}) = \frac{L(1,\psi_\mathfrak{q})}{\xi_\mathfrak{q}(1)\cdot L(1/2,\pi_\mathfrak{q},\xi_\mathfrak{q})} \int_{T(F_\mathfrak{q})} \xi_\mathfrak{q}(t)\langle \pi_\mathfrak{q}(t)x_\mathfrak{q}, \pi_\mathfrak{q}(J_\mathfrak{q})x_\mathfrak{q}\rangle_\mathfrak{q} d^\times t,$$

is the local factor appearing in Waldspurger's formula. Recall that this can be done because the Euler factor at $\mathfrak{q}$ does not vanish. For any $\mathfrak{p} \in S_+^1$, we choose $x_\mathfrak{p} \in \mathrm{St}_{\mathbb{Z}}(F_\mathfrak{p})(\varepsilon_\mathfrak{p})$ to be the image of $\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}} \in C^0(\mathbb{P}^1(F_\mathfrak{p}), \mathbb{Z})$, once we identify $T(F_\mathfrak{p}) \simeq F_\mathfrak{p}^\times$ as a subset of $\mathbb{P}^1(F_\mathfrak{p})$ by means of $t \mapsto \iota(t) * \infty$. Thus, we can write

$$x^{S_+^2} := x^S \otimes \bigotimes_{\mathfrak{p}\in S_+^1} x_\mathfrak{p} \otimes \bigotimes_{\mathfrak{q}\in S_-} x_\mathfrak{q} \in \rho^{S_+^2\cup\infty} \subset \pi^{S_+^2\cup\infty}.$$

Write $P_\xi^{S_+^2} \in \hat{E}(K_S)_{\varepsilon_{S_+^2}} \otimes_{\mathbb{Z}} \overline{\mathbb{Z}}$ for the plectic point associated with $x^{S_+^2} \in \pi^{S_+^2\cup\infty}$. For $\mathfrak{p} \in S_+^2$, there exists an automorphism $\sigma_\mathfrak{p} \in \mathrm{Gal}(H_\mathfrak{p}/F_\mathfrak{p})$ that coincides on $E(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}} \simeq K_\mathfrak{p}^\times / q_\mathfrak{p}^{\mathbb{Z}}$ with the non-trivial automorphism of $K_\mathfrak{p}$. Hence, the following point can be seen as an element of $\hat{T}(F_{S_+^2}) := \bigotimes_{\mathfrak{p}\in S_+^2} \widehat{T(F_\mathfrak{p})}$

$$Q_\xi^{S_+^2} := \left(\prod_{\mathfrak{p}\in S_+^2} (\sigma_\mathfrak{p} - 1)\right) P_\xi^{S_+^2} \in \hat{T}(F_{S_+^2}),$$

where again $\widehat{(\cdot)}$ stands for the p-adic completion of the non-torsion part. In case $S_+^2 = \emptyset$, we write $Q_\xi^\emptyset := L(1/2,\pi,\rho^*\xi)^{\frac{1}{2}} \cdot \prod_{\mathfrak{p}\in S_+^1} \frac{q_\mathfrak{p}^{-1}(1+q_\mathfrak{p}^{-1})}{(1-q_\mathfrak{p}^{-1})^3}$.

The product of morphisms $\varphi \circ \mathrm{rec}_{\mathfrak{p}}$ given in (6.3) and (6.5) provides a morphism

$$\varphi \circ \mathrm{rec}_{S_+} : \hat{T}(F_{S_+}) \otimes_{\mathbb{Z}} \bar{\mathbb{Z}} \longrightarrow I_\xi^r / I_\xi^{r+1}. \tag{6.7}$$

If we write $\epsilon_{S_-}(\pi_{S_-}, \xi_{S_-})$ for the epsilon factor appearing in Theorem 5.1.9 at primes $\mathfrak{q} \in S_-$, we obtain:

Theorem 6.2.1. *For all $\xi \in C^0(\mathcal{G}_T, \bar{\mathbb{Z}})^{\varepsilon_{S_+}}$ with $\xi\mid_{T(F_\infty)} = \lambda$, we have that $\mu_{\phi_\lambda^S} \in I_\xi^r$ and the image of $\mu_{\phi_\lambda^S}$ in I_ξ^r / I_ξ^{r+1} is given by*

$$\mu_{\phi_\lambda^S} \equiv (-1)^s \cdot [\mathcal{O}_+^{S_+^2} : \mathcal{O}_+]^{-1} \cdot \epsilon_{S_-}(\pi_{S_-}, \xi_{S_-}) \cdot \varphi \circ \mathrm{rec}_{S_+}\left(q_{S_+^1} \otimes Q_\xi^{S_+^2}\right) \mod I_\xi^{r+1},$$

where $q_{S_+^1} = \bigotimes_{\mathfrak{p} \in S_+^1} q_{\mathfrak{p}} \in \hat{T}(F_{S_+^1})$ is the product of Tate periods and $s = \# S_+^1$.

Proof. Step 1: Let us consider the subspace of functions

$$C_\xi(\mathcal{G}_T, \mathbb{C}_p) = \left\{ f \in C(\mathcal{G}_T, \mathbb{C}_p), \quad \rho^* f\mid_{T(\mathbb{A}_F^S)} \in \mathbb{C}_p \rho^* \xi\mid_{T(\mathbb{A}_F^S)} \right\}.$$

Recall that ϕ_λ^S is by construction the image of $x \in \pi^{S \cup \infty}$ through Φ_λ of (2.6). Let $\phi_\lambda^{S_+}$ be the image of $x \otimes \bigotimes_{\mathfrak{q} \in S_-} x_{\mathfrak{q}} \in \pi^{S_+^2 \cup \infty}$. Using Theorem 5.1.9, since locally constant characters are dense and $\alpha(x_{\mathfrak{q}}) = 1$, we deduce

$$\int_{\mathcal{G}_T} f \, d\mu_{\phi_\lambda^S} = \epsilon_{S_-}(\pi_{S_-}, \xi_{S_-}) \cdot \int_{\mathcal{G}_T} f \, d\mu_{\phi_\lambda^{S_+}}, \tag{6.8}$$

for all $f \in C_\xi(\mathcal{G}_T, \mathbb{C}_p)$ such that $f\mid_{T(F_{S_-})} = \xi_{S_-}$. Over $C_\xi(\mathcal{G}_T, \mathbb{C}_p)$, this characterizes $\mu_{\phi_\lambda^S}$ in terms of $\mu_{\phi_\lambda^{S_+}}$.

Step 2: For any $\mathfrak{p} \in S_+$, any group M and any finite rank $\mathbb{Z}_p$-module N, we can consider the diagram

$$\begin{array}{ccc} H^m(G(F)_+, \mathcal{A}^{S_+ \cup \infty}(M \otimes \mathrm{St}_{\mathbb{Z}}(F_{\mathfrak{p}})(\varepsilon_{\mathfrak{p}}), N))^\lambda & & \\ \downarrow i_{\mathfrak{p}} & \searrow c_{\mathfrak{p}}^m & \\ H^m(G(F)_+, \mathcal{A}^{S_+ \cup \infty}(M \otimes_{\mathbb{Z}} \Delta_{\mathfrak{p}}^0, \hat{K}_{\mathfrak{p}}^\times \otimes_{\mathbb{Z}_p} N)(\varepsilon_{\mathfrak{p}}))^\lambda & \longrightarrow & H^{m+1}(G(F)_+, \mathcal{A}^{S_+ \cup \infty}(M, \hat{K}_{\mathfrak{p}}^\times \otimes_{\mathbb{Z}_p} N)(\varepsilon_{\mathfrak{p}}))^\lambda, \end{array}$$

where the vertical arrow arises from (5.21) and the horizontal arrow is the connection morphism of the degree long exact sequence. We write i_{S_+} for the composition of $i_{\mathfrak{p}}$, for all $\mathfrak{p} \in S_+^2$, and $c_{\mathfrak{p}_1}^u$, $c_{\mathfrak{p}_2}^{u+1}$, $\cdots$, $c_{\mathfrak{p}_s}^{u+s-1}$, where $S_+^1 = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_s\}$. Thus,

$$i_{S_+} \phi_\lambda^{S_+} \in H^{u+s}(G(F)_+, \mathcal{A}^{S_+ \cup \infty}(\Delta_{S_+^2}^0, \hat{K}_{S_+^2}^\times \otimes \hat{F}_{S_+^1}^\times)(\varepsilon_{S_+}))^\lambda.$$

Let us consider the $G(F)$-equivariant morphism

$$\begin{aligned} \mathrm{EV} : \mathcal{A}^{S_+ \cup \infty}(\Delta_{S_+^2}^0, \hat{K}_{S_+^2}^\times \otimes \hat{F}_{S_+^1}^\times)(\lambda \cdot \varepsilon_{S_+}) &\longrightarrow \mathrm{coInd}_{T(F)}^{G(F)} \left(\mathcal{A}^{S_+ \cup \infty}(\hat{K}_{S_+^2}^\times \otimes \hat{F}_{S_+^1}^\times)(\lambda \cdot \varepsilon_{S_+}) \right) \\ f &\longmapsto (h, g^{S_+}) \mapsto \varepsilon_{S_+}(h) \cdot \lambda(h) \cdot f(h^{-1} g^{S_+}) \left(\bigotimes_{\mathfrak{p} \in S_+^2} (h^{-1} \bar{\tau}_{\mathfrak{p}} - h^{-1} \tau_{\mathfrak{p}}) \right), \end{aligned}$$

where $h, g^{S_+} \in G(F)\times G(\mathbb{A}^{S_+\cup\infty})$. Since we are evaluating at divisors of the form $h^{-1}\bar{\tau}_{\mathfrak{p}} - h^{-1}\tau_{\mathfrak{p}}$ it is easy to show that the image of $\phi_\lambda^{S_+}$ lies in $\hat{T}(F_{S_+})$. Thus, by Shapiro,

$$\mathrm{EV}i_{S_+}\phi_\lambda^{S_+} \in H^{u+s}(T(F)_+, \mathcal{A}^{S_+\cup\infty}(\hat{T}(F_{S_+}))(\varepsilon_{S_+}))^\lambda.$$

We claim that

$$q_{S_+^1}\otimes Q_\xi^{S_+^2} = \#\left(\mathcal{O}_+^{S_+}/\mathcal{O}_+^{S_+^2}\right)_{\mathrm{tor}}^{-1}\cdot(\eta^{S_+}\cap\xi_\lambda)\cap \mathrm{EV}i_{S_+}\phi_\lambda^{S_+} \in \hat{T}(F_{S_+})\otimes_{\mathbb{Z}}\overline{\mathbb{Z}}. \tag{6.9}$$

Indeed, by functoriality and the description of the multiplicative integral given in §4.1.1, we have that

$$\mathrm{EV}i_{S_+}\phi_\lambda^{S_+} = \mathrm{EV}i_{S_+^2}\varphi_{\mathrm{unv}}\phi_\lambda^{S_+}\cup\bigcup_{\mathfrak{p}\in S_+^1}\mathrm{res}_{T(F)_+}^{G(F)_+}c_{\mathfrak{p}},$$

where φ_{unv} is the morphism of (4.4), $c_{\mathfrak{p}}\in H^1(G(F),\mathrm{St}_{F_{\mathfrak{p}}^\times})$ is the class associated with the extension $\mathcal{E}_{F_{\mathfrak{p}}^\times}$ of (4.5), and

$$\varphi_{\mathrm{unv}}\phi_\lambda^{S^+}\in H^u(G(F)_+,\mathcal{A}^{S_+\cup\infty}(V_{S_+^2}^{\mathbb{Z}}\otimes\mathrm{St}_{F_{S_+^1}^\times},\hat{F}_{S_+^1}^\times(\varepsilon_{S_+^1})))^\lambda,\qquad \mathrm{St}_{\hat{F}_{S_+^1}^\times} := \bigotimes_{\mathfrak{p}\in S_+^1}\mathrm{St}_{F_{\mathfrak{p}}^\times}$$

is the corresponding push-forward. In fact, [GMM17, Conjecture 3.8] can be interpreted as

$$\varphi_{\mathrm{unv}}\phi_\lambda^{S_+}\cup\bigcup_{\mathfrak{p}\in S_+^1}c_{\mathfrak{p}} = q_{S_+^1}\otimes\left(\phi_\lambda^{S_+}\cup\bigcup_{\mathfrak{p}\in S_+^1}c_{\mathfrak{p}}^{\mathrm{ord}}\right),$$

where $c_{\mathfrak{p}}^{\mathrm{ord}}\in H^1(G(F),\mathrm{St}_{\mathbb{Z}}(F_{\mathfrak{p}}))$ is the class associated with the extension (see [Bla19, §6.1])

$$\begin{aligned} \mathrm{St}_{\mathbb{Z}}(F_{\mathfrak{p}}) &\hookrightarrow \mathcal{E}_{\mathbb{Z}}(F_{\mathfrak{p}})\\ f &\longmapsto (f(g^{-1}*\infty),0)\end{aligned}$$

and

$$\mathcal{E}_{\mathbb{Z}}(F_{\mathfrak{p}}) := \left\{(\phi,y)\in C(\mathbf{GL}_2(F_{\mathfrak{p}}),\mathbb{Z})\times\mathbb{Z} : \phi\left(\begin{pmatrix} s & x\\ & t\end{pmatrix}g\right) = y\cdot v_{\mathfrak{p}}(t)+\phi(g)\right\}/(\mathbb{Z},0).$$

For $\mathfrak{p}\in S_+^1$, write $\tau_{\mathfrak{p}}$ and $\bar{\tau}_{\mathfrak{p}}$ for the points in $\mathbb{P}^1(F_{\mathfrak{p}})$ fixed by $T(F_{\mathfrak{p}})$ as in §4.1.1. Thus, if we write

$$\phi_1\begin{pmatrix} a & b\\ c & d\end{pmatrix} = v_{\mathfrak{p}}(d+\bar{\tau}_{\mathfrak{p}}\cdot c) - \left((v_{\mathfrak{p}}+1)\cdot\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}\right)\left(\frac{d+\bar{\tau}_{\mathfrak{p}}\cdot c}{d+\tau_{\mathfrak{p}}\cdot c}\right), \tag{6.10}$$

the element $(\phi_1,1)\in\mathcal{E}_{\mathbb{Z}}(F_{\mathfrak{p}})$ is a generator mapping to 1 under the natural $G(F)$-morphism $\mathcal{E}_{\mathbb{Z}}(F_{\mathfrak{p}})\longrightarrow\mathbb{Z}$. Hence the cocycle $\mathrm{res}_{T(F)}^{G(F)}c_{\mathfrak{p}}^{\mathrm{ord}}$ has representative

$$\begin{aligned} c_{\mathfrak{p}}^{\mathrm{ord}}(t)(x) &= (1-t)\phi_1(x) = (1-t)\left((v_{\mathfrak{p}}+1)\cdot\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}\right)\left(\frac{x-\bar{\tau}_{\mathfrak{p}}}{x-\tau_{\mathfrak{p}}}\right)\\ &= \left(\int_{T(F_{\mathfrak{p}})}\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}(\tau)\left(\tau\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}-t\tau\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}\right)d^\times\tau\right)\left(\frac{x-\bar{\tau}_{\mathfrak{p}}}{x-\tau_{\mathfrak{p}}}\right),\end{aligned}$$

for all $t\in T(F)$. We compute, recalling the definition of $z_{\mathfrak{p}}$ of Lemma 3.1.4,

$$\begin{aligned}(\eta^{S_+}\cap\xi_\lambda)\cap\mathrm{EV}i_{S_+}\phi_\lambda^{S_+} &= q_{S_+^1}\otimes(\eta^{S_+}\cap\xi_\lambda)\cap\left(\mathrm{EV}i_{S_+^2}\phi_\lambda^{S_+}\cup\bigcup_{\mathfrak{p}\in S_+^1}\mathrm{res}_{T(F)_+}^{G(F)_+}c_{\mathfrak{p}}^{\mathrm{ord}}\right)\\ &= q_{S_+^1}\otimes\left(\eta^{S_+}\cap\bigcup_{\mathfrak{p}\in S_+^1}\mathrm{res}_{T(F)_+}^{T(F_{\mathfrak{p}})}z_{\mathfrak{p}}\cap\xi_\lambda\right)\cap\mathrm{EV}i_{S_+^2}\phi_\lambda^{S_+^2}\\ &= \#\left(\mathcal{O}_+^{S_+}/\mathcal{O}_+^{S_+^2}\right)_{\mathrm{tor}}\cdot q_{S_+^1}\otimes\left(\eta^{S_+^2}\cap\xi_\lambda\right)\cap\mathrm{EV}i_{S_+^2}\phi_\lambda^{S_+^2},\end{aligned}$$

where the third equality follows from Lemma 3.1.4 and the second from the fact that

$$\begin{aligned}\Phi(x^{S\cup\mathfrak{p}})(c_{\mathfrak{p}}^{\mathrm{ord}}(t)) &= \int_{T(F_{\mathfrak{p}})}\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}(\tau)\cdot\Phi(x^{S\cup\mathfrak{p}})\left(\tau\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}-t\tau\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}\right)d^\times\tau\\ &= \int_{T(F_{\mathfrak{p}})}\left(\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}(\tau)-t\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}(\tau)\right)\cdot\Phi(x^{S\cup\mathfrak{p}})\left(\tau\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}\right)d^\times\tau\\ &= \int_{T(F_{\mathfrak{p}})}z_{\mathfrak{p}}(t)(\tau)\cdot\Phi(x^{S\cup\mathfrak{p}}\otimes\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}})(\tau)d^\times\tau,\end{aligned}$$

for any $\Phi\in\mathcal{A}^{\{\mathfrak{p}\}\cup S\cup\infty}(\mathrm{St}_{\mathbb{Z}}(F_{\mathfrak{p}}),M)_\rho=\mathcal{A}^{S\cup\infty}(M)_\rho$ and $x^{S\cup\mathfrak{p}}\in\rho^{S\cup\{\mathfrak{p}\}}$. If $S_+^2=\emptyset$, the same arguments used in the interpolation formula (Theorem 5.1.9) together with the fact that $\alpha(\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}})=\frac{q_{\mathfrak{p}}^{-1}(1+q_{\mathfrak{p}}^{-1})}{(1-q_{\mathfrak{p}}^{-1})^3}$ (as seen in the proof of [Bla19, Theorem 7.5]) proves the claim (6.9) in this case. If $S_+^2\neq\emptyset$, we apply the same arguments used in the proof of Theorem 6.1.1 to obtain $\left(\eta^{S_+^2}\cap\xi_\lambda\right)\cap\mathrm{EV}i_{S_+^2}\phi_\lambda^{S_+^2}=Q_\xi^{S_+^2}$, and to deduce the claim (6.9).

Step 3: Similarly as in (6.10), if $\mathfrak{p}\in S_+^1$ and we write

$$\bar{\phi}_1\begin{pmatrix}a&b\\c&d\end{pmatrix}=(d+\bar{\tau}_{\mathfrak{p}}\cdot c)\cdot\left(\frac{d+\tau_{\mathfrak{p}}\cdot c}{d+\bar{\tau}_{\mathfrak{p}}\cdot c}\right)^{\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}\left(\frac{d+\bar{\tau}_{\mathfrak{p}}\cdot c}{d+\tau_{\mathfrak{p}}\cdot c}\right)},\tag{6.11}$$

the element $(\bar{\phi}_1,1)\in\mathcal{E}_{F_{\mathfrak{p}}^\times}$ is a generator mapping to 1 under the natural $G(F)$-morphism $\mathcal{E}_{F_{\mathfrak{p}}^\times}\longrightarrow\mathbb{Z}$. Hence the cocycle $\mathrm{res}_{T(F)}^{G(F)}c_{\mathfrak{p}}$ has representative

$$\begin{aligned}c_{\mathfrak{p}}(t)(x)=(1-t)\bar{\phi}_1(x)&=(1-t)\left(\left(\frac{x-\tau_{\mathfrak{p}}}{x-\bar{\tau}_{\mathfrak{p}}}\right)^{\mathbb{1}_{\mathcal{O}_{F,\mathfrak{p}}}\left(\frac{x-\bar{\tau}_{\mathfrak{p}}}{x-\tau_{\mathfrak{p}}}\right)}\right)\\ =t^{\mathbb{1}_{t\mathcal{O}_{F,\mathfrak{p}}}\left(\frac{x-\bar{\tau}_{\mathfrak{p}}}{x-\tau_{\mathfrak{p}}}\right)}\cdot\left(\frac{x-\tau_{\mathfrak{p}}}{x-\bar{\tau}_{\mathfrak{p}}}\right)^{z_{\mathfrak{p}}(t)\left(\frac{x-\bar{\tau}_{\mathfrak{p}}}{x-\tau_{\mathfrak{p}}}\right)}&\in T(F)\cdot\left(\frac{x-\tau_{\mathfrak{p}}}{x-\bar{\tau}_{\mathfrak{p}}}\right)^{z_{\mathfrak{p}}(t)\left(\frac{x-\bar{\tau}_{\mathfrak{p}}}{x-\tau_{\mathfrak{p}}}\right)},\end{aligned}$$

for all $t\in T(F)$. Therefore, we can consider the composition

$$\mathrm{EV}\circ i_{K_{\mathfrak{p}}^\times}:\mathcal{A}^{S_+\cup\infty}(V_{S_+}^{\mathbb{Z}},\mathbb{Z})(\lambda)\longrightarrow\mathrm{coInd}_{T(F)}^{G(F)}\left(\mathcal{A}^{S^+\cup\infty}(\mathrm{St}_{F_{S_+^1}^\times},\hat{T}(F_{S_+^2})(\varepsilon_{S_+}))(\lambda)\right),$$

and the cocycle $c_{S_+^1}=\bigotimes_{\mathfrak{p}\in S_+^1}c_{\mathfrak{p}}:T(F)^s\longrightarrow\mathrm{St}_{F_{S_+^1}^\times}$, obtaining

$$\left(\mathrm{EV}\circ i_{K_{\mathfrak{p}}^\times}(\phi)\right)(1)(g^{S_+})\left(c_{S_+^1}(\underline{\gamma})\right)=\lim_{\mathcal{U}\in\mathrm{Cov}(\mathbb{P}^1(F_{S_+}))}\prod_{U\in\mathcal{U}}\bigotimes_{\mathfrak{p}\in S_+}\alpha_{\mathfrak{p}}\left(\frac{x_U-\bar{\tau}_{\mathfrak{p}}}{x_U-\tau_{\mathfrak{p}}}\right)^{s(\mathfrak{p})\cdot z_{\mathfrak{p}}(\gamma_{\mathfrak{p}})\left(\frac{x_U-\bar{\tau}_{\mathfrak{p}}}{x_U-\tau_{\mathfrak{p}}}\right)\cdot\phi(g^{S_+})(\mathbb{1}_U)},$$

where $\underline{\gamma} = (\gamma_\mathfrak{p})_{\mathfrak{p}\in S_+^1} \in T(F)^s$, $\alpha_\mathfrak{p} \in T(F)$; and $s(\mathfrak{p}) = -1$ if $\mathfrak{p} \in S_+^1$ and 1 otherwise (recall that $z_\mathfrak{p} = \mathbb{1}_{T(F_\mathfrak{p})}$ for $\mathfrak{p} \in S_+^2$). Since $\ell = \prod_{\mathfrak{p}\in S_+} \ell_\mathfrak{p} := \varphi \circ \mathrm{rec}_{S_+}$ is a product of group morphisms $\ell_\mathfrak{p}$ that vanish at $T(F)$ by Remark 6.1.2,

$$\ell \circ \mathrm{EV} \circ i_{K_\mathfrak{p}^\times}(\phi)(g^{S_+})\left(c_{S_+^1}(\underline{\gamma})\right) = \lim_{\mathcal{U}\in \mathrm{Cov}(\mathbb{P}^1(F_{S_+}))} (-1)^s \sum_{U\in\mathcal{U}} \phi(g^{S_+})(\mathbb{1}_U)\cdot \prod_{\mathfrak{p}\in S_+} (z_\mathfrak{p}(\gamma_\mathfrak{p})\cdot \ell_\mathfrak{p})\left(\frac{x_U - \overline{\tau}_\mathfrak{p}}{x_U - \tau_\mathfrak{p}}\right)$$

$$= (-1)^s \phi(g^{S_+})\left(\bigotimes_{\mathfrak{p}\in S_+} \delta_\mathfrak{p}(\ell_\mathfrak{p} \cap z_\mathfrak{p})(\gamma_\mathfrak{p})\right) = (-1)^s \delta_{S_+}^*\phi(g^{S_+})\left(\prod_{\mathfrak{p}\in S_+} (\ell_\mathfrak{p} \cap z_\mathfrak{p})(\gamma_\mathfrak{p})\right),$$

where $\delta_{S_+}^*\phi(g_\mathfrak{p})$ is extended to a p-adic measure of the compactly supported functions in $T(F_\mathfrak{p})$ using Riemann sums, and ℓ is seen as an element of $H^0(T(F), C(T(F_\mathfrak{p}), I_\xi/I_\xi^2))$ by Remark 6.1.2. Since φ is a group homomorphism, we have that

$$\int_{\mathcal{G}_T} f d\left(\prod_{\mathfrak{p}\in S_+} \ell_\mathfrak{p}(t_\mathfrak{p})\right) = \rho^*\xi(t_{S_+})^{-1}\cdot \rho^* f(t_{S_+}) - f(1) = \varepsilon_{S_+}(t_{S_+})^{-1}\cdot \rho^* f(t_{S_+}) - f(1),$$

where $t_{S_+} = (t_\mathfrak{p})_\mathfrak{p} \in T(F_{S_+})$. Thus, for any $h^{S_+} \in C_c^0(T(\mathbb{A}_F^{S_+\cup\infty}), \mathbb{C}_p)$ the element

$$(-1)^s \left\langle h^{S_+}, \ell \circ \mathrm{EV} \circ i_{K_\mathfrak{p}^\times}(\phi)\left(c_{S_+^1}(\underline{\gamma})\right)\right\rangle_+ \in I_\xi^r/I_\xi^{r+1}$$

obtained by means of (3.14) is represented by the distribution that maps $f \in C^0(\mathcal{G}_T, \mathbb{C}_p)$ to

$$\int_{T(\mathbb{A}_F^{S_+})} h^{S_+}(t^{S_+})\cdot \delta_{S_+}^*\phi(t^{S_+})\left(\bigotimes_{\mathfrak{p}\in S_+} z_\mathfrak{p}(\underline{\gamma})(\varepsilon_{S_+}{}^{-1}\rho^* f_{S_+} - f(1))\right) d^\times t^\mathfrak{p}$$

$$= \left\langle h^{S_+} \otimes (\varepsilon_{S_+}{}^{-1}\rho^* f_{S_+} - f(1)) \bigotimes_{\mathfrak{p}\in S_+} z_\mathfrak{p}(\underline{\gamma}), \delta_{S_+}^*\phi\right\rangle_+ .$$

This implies that, for any $f \in C_\xi^{S_+}(\mathcal{G}_T, \mathbb{C}_p) := \{f \in C_\xi(\mathcal{G}_T, \mathbb{C}_p),\ \rho^* f\mid_{T(F_{S_-})} = \rho^*\xi\mid_{T(F_{S_-})}\}$,

$$\int_{\mathcal{G}_T} f d\left(\varphi \circ \mathrm{rec}_{S_+}\left(q_{S_+^1} \otimes Q_\xi^{S_+^2}\right)\right)$$
$$= \#\left(\mathcal{O}_+^{S_+}/\mathcal{O}_+^{S_+^2}\right)_{\mathrm{tor}}^{-1}\cdot \int_{\mathcal{G}_T} f d\left((\eta^{S^+} \cap \xi_\lambda) \cap \ell \circ \varphi_{\mathrm{univ}} \circ \mathrm{EV}\left(i_{S_+^2}\phi_\lambda^{S_+} \cap c_{S_+^1}\right)\right)$$
$$= \frac{(-1)^s}{\#\left(\mathcal{O}_+^{S_+}/\mathcal{O}_+^{S_+^2}\right)_{\mathrm{tor}}}\cdot \left(\left((\eta^{S^+} \cap \xi_\lambda)\right) \cap \left(\bigotimes_{\mathfrak{p}\in S_+} z_\mathfrak{p} \cap (\varepsilon_{S_+}{}^{-1}\rho^* f_{S_+} - f(1))\right)\right) \cap \delta_{S_+}^*\phi_\lambda^{S_+}$$
$$= (-1)^s\cdot [\mathcal{O}_+^{S_+^2} : \mathcal{O}_+]\cdot\left((\eta \cap \rho^* f) \cap \delta_{S_+}^*\phi_\lambda^{S_+} - f(1)\cdot(\eta \cap \rho^*\xi) \cap \delta_{S_+}^*\phi_\lambda^{S_+}\right)$$
$$= (-1)^s\cdot [\mathcal{O}_+^{S_+^2} : \mathcal{O}_+]\cdot \int_{\mathcal{G}_T} f d\mu_{\phi_\lambda^{S_+}},$$

by Lemma 3.1.4 and (3.17), since $(\eta \cap \rho^*\xi) \cap \delta^*_{S_+}\phi_\lambda^{S_+} = \int_{\mathcal{G}_T} \xi d\mu_{\phi_\lambda^{S_+}} = 0$.

Step 4: Analogously as in Theorem 6.1.1, combining the previous calculation with (6.8), we deduce that $\mu' := \mu_{\phi_\lambda^S} - (-1)^s \cdot [\mathcal{O}_+^{S_+^2} : \mathcal{O}_+]^{-1} \cdot \epsilon_{S_-}(\pi_{S_-}, \xi_{S_-}) \cdot \varphi \circ \mathrm{rec}_{S_+}\left(q_{S_+^1} \otimes Q_\xi^{S_+^2}\right)$ vanishes at $C_\xi(\mathcal{G}_T, \mathbb{C}_p)$. Clearly, $\mu' \in I_\xi$ and $\mu' = \mu' \star \delta_\xi$, where

$$\int_{\mathcal{G}_T} f d\delta_\xi = \begin{cases} f(1), & f \notin C_\xi(\mathcal{G}_T, \mathbb{C}_p); \\ 0, & f \in C_\xi(\mathcal{G}_T, \mathbb{C}_p). \end{cases}.$$

Since clearly $\delta_\xi \in I_\xi$, we conclude that $\mu' \in I_\xi^{r+1}$ and the result follows. □

6.3 The Hida-Rankin p-adic L-function

Let $\xi : \mathcal{G}_T \to \bar{\mathbb{Z}}^\times$ be a locally constant character such that $\rho^*\xi \mid_{T(F_\infty)} = \lambda$. Using the natural ring homomorphism $\bar{\mathbb{Z}} \subset \Lambda_F \otimes \bar{\mathbb{Z}}$, it can be seen as a function in $C(\mathcal{G}_T, \Lambda_F \otimes \bar{\mathbb{Z}})$. We consider

$$L_p(\phi_\lambda^p, \xi, \underline{k}) := \int_{\mathcal{G}_T} \xi d\mu_{\Phi_\lambda^p} \in \Lambda_F \otimes_{\mathbb{Z}} \bar{\mathbb{Z}}. \tag{6.12}$$

For any weight $\theta_p \in \Lambda_F(\mathbb{C}_p) = \mathbf{Hom}(\Lambda_F, \mathbb{C}_p)$, we write

$$L_p(\phi_\lambda^p, \xi, \theta_p) := \rho_{\theta_p}(L_p(\phi_\lambda^p, \xi, \underline{k})) \in \mathbb{C}_p.$$

We can think of $L_p(\phi_\lambda^p, \xi, \underline{k})$ as a restriction of the two variable p-adic L-function $\mu_{\Phi_\lambda^p}$ to the weight variable.

Assume as in the above section that ϕ_λ^p is ordinary (where $S = p$), and let $\hat{\phi}_\lambda^p$ be its associated overconvergent modular symbol. We will also assume that there exists a family Φ_λ^p passing through $\hat{\phi}_\lambda^p$, hence we can construct the Hida-Rankin p-adic L-function $L_p(\phi_\lambda^p, \xi, \underline{k})$ associated with Φ_λ^p. Recall the augmentation ideals $I, I_\mathfrak{p}$ fitting in the exact sequences

$$0 \longrightarrow I \longrightarrow \Lambda_F \xrightarrow{\rho_1} \mathbb{Z}_p \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow I_\mathfrak{p} \longrightarrow \Lambda_\mathfrak{p} \xrightarrow{\rho_1} \mathbb{Z}_p \longrightarrow 0$$

Let S be a set of primes $\mathfrak{p}$ above p such that:

- T does not split at any $\mathfrak{p} \in S$.
- The representation $V_\mathfrak{p}^{\mathbb{Z}}$ is $\mathrm{St}_{\mathbb{Z}}(F_\mathfrak{p})(\varepsilon_\mathfrak{p})$ for all $\mathfrak{p} \in S$.
- We have that $\rho^*\xi \mid_{T(F_S)} = \varepsilon_S \mid_{T(F_S)}$.

As discussed in §5.4.4, for any $\mathfrak{p} \in S$ one can identify $E(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}}$ with $K_\mathfrak{p}^\times / q_\mathfrak{p}^{\mathbb{Z}}$. For any point $P \in E(K_\mathfrak{p})_{\varepsilon_\mathfrak{p}}$, we can think its trace $P + \bar{P} \in E(F_\mathfrak{p})_{\varepsilon_\mathfrak{p}}$ as an element

$$P + \bar{P} \in F_\mathfrak{p}^\times / q_\mathfrak{p}^{2\mathbb{Z}} \subset E(F_\mathfrak{p})_{\varepsilon_\mathfrak{p}} := \{P \in E(H_{\varepsilon_\mathfrak{p}});\ P^\tau = \varepsilon_\mathfrak{p}(\tau) \cdot P,\ \bar{P} = P\},$$

where $H_{\varepsilon_\mathfrak{p}}/K_\mathfrak{p}$ is the extension cut out by $\varepsilon_\mathfrak{p}$. Hence, by Theorem 5.4.11 it makes sense to consider

$$\ell_{\mathbf{a}_\mathfrak{p}} \circ \mathrm{Tr}(P) := \ell_{\mathbf{a}_\mathfrak{p}}(P + \bar{P}) \in I_\mathfrak{p}/I_\mathfrak{p}^2.$$

This implies that, given a plectic point $P_\xi^S \in \hat{E}(K_S)_{\varepsilon_S} \otimes_{\mathbb{Z}} \bar{\mathbb{Z}}$, we can consider

$$\ell_{\underline{\mathbf{a}}} \circ \mathrm{Tr}(P_\xi^S) := \left(\prod_{\mathfrak{p} \in S} \ell_{\mathbf{a}_\mathfrak{p}} \circ \mathrm{Tr}\right)(P_\xi^S) \in I^r / I^{r+1} \otimes_{\mathbb{Z}} \bar{\mathbb{Z}}.$$

where $r := \#S$.

Remark 6.3.1. *Recall that* $\phi_\lambda^p \in H^u(G(F), \mathcal{A}^{p\cup\infty}(V^S \otimes_{\mathbb{Z}_p} V_S^{\mathbb{Z}_p}, \mathbb{Z}_p)(\lambda))$*, where* $V_S^{\mathbb{Z}_p}$ *is a product of twisted Steinberg representations and* $V^S = \bigotimes_{\mathfrak{q}\notin S} V_\mathfrak{q}^{\mathbb{Z}_p}$. *Since for any* $v^S \in V^S$ *we have a* $G(F)$*-equivariant morphism*

$$\mathcal{A}^{p\cup\infty}(V^S \otimes_{\mathbb{Z}_p} V_S^{\mathbb{Z}_p}, \mathbb{Z}_p) \longrightarrow \mathcal{A}^{S\cup\infty}(V_S^{\mathbb{Z}_p}, \mathbb{Z}_p); \qquad \phi \longmapsto \phi(v^S),$$

with

$$\phi(v^S)(g^S)(v_S) := \phi(g^p)(g_{p\setminus S} v^S \otimes v_S), \qquad v_S \in V_S, \quad g^S = (g_{p\setminus S}, g^p) \in G(\mathbb{A}^{S\cup\infty}),$$

we can consider $\phi_\lambda^p(v^S) \in H^u(G(F), \mathcal{A}^{S\cup\infty}(V_S, \mathbb{Z}_p)(\lambda))$. *By means of* $\phi_\lambda^p(v^S)$ *and a character* ξ *we can construct the corresponding plectic point (depending on* v^S*)*

$$P_\xi^S(v^S) \in \hat{E}(K_S)_{\varepsilon_S} \otimes_{\mathbb{Z}} \bar{\mathbb{Z}}.$$

Let $P_\xi^S = P_\xi^S(v_0^S)$ be the plectic point corresponding to $v_0^S \in V^S$ such that $\alpha(v_{0,\mathfrak{q}}^S) = 1$ for all $\mathfrak{q} \in p\setminus S$, where the pairing α is that of (5.12). Recall the Euler factors $\epsilon_\mathfrak{p}(\pi_\mathfrak{p}, \xi_\mathfrak{p})$ introduced in §5.1.5. The following theorem is a p-adic Gross-Zagier formula for the Hida-Rankin p-adic L-function:

Theorem 6.3.2. *Assume that* $r = \#S \neq 0$. *Then we have that*

$$L_p(\phi_\lambda^p, \xi, \underline{k}) \in I^r \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}.$$

Moreover,

$$L_p(\phi_\lambda^p, \xi, \underline{k}) \equiv \frac{(-1)^r}{[\mathcal{O}_+^S : \mathcal{O}_+]} \cdot \prod_{\mathfrak{p}\notin S} \epsilon_\mathfrak{p}(\pi_\mathfrak{p}, \xi_\mathfrak{p}) \cdot \ell_{\mathbf{a}_S} \circ \mathrm{Tr}(P_\xi^S) \pmod{I^{r+1} \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}}. \tag{6.13}$$

Proof. Since $\Phi_\lambda^p \in H^u(G(F), \mathcal{A}^{p\cup\infty}(\mathbb{D}_{\mathbf{k}_p-2}(\Lambda_F)_{\underline{\mathbf{a}}})(\lambda))$ is such that

$$\rho_1(\Phi_\lambda^p) \in H^u(G(F), \mathcal{A}^{p\cup\infty}(V^S \otimes_{\mathbb{Z}_p} \mathrm{St}_{\mathbb{Z}_p}(F_S)(\varepsilon_S), \mathbb{Z}_p)(\lambda)); \qquad V^S = \bigotimes_{\mathfrak{q}\notin S} V_\mathfrak{q},$$

the family lies in fact in

$$\Phi_\lambda^p \in H^u(G(F), \mathcal{A}^{p\cup\infty}(\mathbb{D}_{\mathbf{k}_p-2}(\Lambda_F)_{\underline{\mathbf{a}}^*}^S(\varepsilon_S))(\lambda)), \qquad \underline{\mathbf{a}}^* = \left(\varepsilon_\mathfrak{p}\left(\begin{smallmatrix} \varpi_\mathfrak{p} & \\ & 1\end{smallmatrix}\right) \cdot \mathbf{a}_\mathfrak{p}\right)_\mathfrak{p}.$$

Hence we can apply r_S and $\bar{\mathrm{ev}}_S$ of Propositions 5.4.9 and 5.4.10 and we obtain

$$\bar{\mathrm{ev}}_S \circ r_S\left(\Phi_\lambda^p\right) \in H^u(G(F), \mathcal{A}^{p\cup\infty}(V^S \otimes_{\mathbb{Z}_p} \Delta_{F_S}, I^r/I^{r+1})(\varepsilon_S)(\lambda))$$

Let $\Delta_T := \mathbb{Z}[G(F)/T(F)]$. Since $\tau_\mathfrak{p}$ and $\overline{\tau}_\mathfrak{p}$ are $T(F)$-invariant, one can construct a $G(F)$-module morphism

$$\begin{array}{ccc} \Delta_T & \hookrightarrow & \Delta_{F_S} = \bigotimes_{\mathfrak{p}\in S} \Delta_{F_\mathfrak{p}} \subset \Delta_S = \bigotimes_{\mathfrak{p}\in S} \Delta_\mathfrak{p} \\ \sum_i g_i \cdot T(F) & \longmapsto & \sum_i \bigotimes_{\mathfrak{p}\in S} g_i \cdot (\tau_\mathfrak{p} + \overline{\tau}_\mathfrak{p}) \end{array}$$

The morphism $\mathrm{ev}_S^*\circ\varphi_{\mathrm{unv}}$ restricts to a $G(F)$-equivariant morphism $\mathbf{Hom}(\mathrm{St}_{\mathbb{Z}_p}(F_S)(\varepsilon_S),\mathbb{Z}_p)\to\mathbf{Hom}(\Delta_T^0,\hat{F}_S^\times)(\varepsilon_S)$, where Δ_T^0 is the set of degree zero divisors in Δ_T. Thus, for all $v^S\in V^S$ we obtain the following diagram for $M=\hat{F}_S^\times$ or $M=\bigotimes_{\mathfrak{p}\in S}\hat{F}_\mathfrak{p}^\times/q_\mathfrak{p}^{2\mathbb{Z}}\subseteq\bigotimes_{\mathfrak{p}\in S}\hat{E}(F_\mathfrak{p})_{\varepsilon_\mathfrak{p}}=:\hat{E}(F_S)_{\varepsilon_S}$

$$\begin{array}{ccc} & & \phi_\lambda^p(v^S)\in H^u(G(F),\mathcal{A}^{S\cup\infty}(\mathrm{St}_{\mathbb{Z}_p}(F_S)(\varepsilon_S),\mathbb{Z}_p)(\lambda)) \\ & & \downarrow{\scriptstyle\varphi^{\mathrm{univ}}} \\ H^u(G(F),\mathcal{A}^{S\cup\infty}(\mathcal{E}_{F_S^\times},M)(\varepsilon_S)(\lambda)) & \longrightarrow & H^u(G(F),\mathcal{A}^{S\cup\infty}(\mathrm{St}_{F_S^\times},M)(\varepsilon_S)(\lambda)) \\ \downarrow{\scriptstyle\mathrm{ev}|_{\Delta_T}} & & \downarrow \\ H^u(T(F),\mathcal{A}^{S\cup\infty}(M)(\varepsilon_S)(\lambda)) & \xrightarrow{\mathrm{res}} & H^u(G(F),\mathcal{A}^{S\cup\infty}(\Delta_T^0,M)(\varepsilon_S)(\lambda)) \end{array}$$

since $\mathcal{A}^{S\cup\infty}(\Delta_T,M)(\lambda)=\mathrm{coInd}_{T(F)}^{G(F)}(\mathcal{A}^{S\cup\infty}(M)(\lambda))$. If $M=\bigotimes_{\mathfrak{p}\in S}\hat{F}_\mathfrak{p}^\times/q_\mathfrak{p}^{2\mathbb{Z}}$, we have that $\varphi^{\mathrm{univ}}\phi_\lambda^p(v^S)$ extends to an element $\bar{\phi}_\lambda^p(v^S)\in H^u(G(F),\mathcal{A}^{S\cup\infty}(\mathcal{E}_{F_S^\times},M)(\varepsilon_S)(\lambda))$. It is clear that

$$\mathrm{Tr}\left(P_\xi^S(v^S)\right)=\eta^S\cap\xi_\lambda\cap\mathrm{ev}\,|_{\Delta_T}\,\bar{\phi}_\lambda^p(v^S)\in\hat{E}(F_S)_{\varepsilon_S}\otimes\bar{\mathbb{Z}}.$$

If we apply the logarithm $\ell_{\mathbf{a}_S}:M\longrightarrow I^r/I^{r+1}$, by Proposition 5.4.10 we have that

$$\ell_{\mathbf{a}_S}\circ\mathrm{Tr}\left(P_\xi^S(v^S)\right)=\eta^S\cap\xi_\lambda\cap\ell_{\mathbf{a}_S}\mathrm{ev}\,|_{\Delta_T}\,\bar{\phi}_\lambda^p(v^S)=\eta^S\cap\xi_\lambda\cap\mathrm{ev}\,|_{\Delta_T}\,r_S\Phi_\lambda^p(v^S).$$

where $\mathrm{ev}\,|_{\Delta_T}\,r_S\Phi_\lambda^p(v^S)\in H^u(T(F),\mathcal{A}^{S\cup\infty}(I^r/I^{r+1})(\varepsilon_S)(\lambda))$ is defined as in Remark 6.3.1.

Write $\overline{(\cdot)}$ for the reduction modulo I^{r+1}. Let $\Phi\in\mathcal{A}^{p\cup\infty}(\mathbb{D}_{\mathbf{k}_p-2}(\Lambda_F)_{\mathrm{a}_p}^S)$ and let $f^S=f_{p\setminus S}\otimes f^p\in C_c^0(T(\mathbb{A}_F^{S\cup\infty}),\bar{\mathbb{Z}})$, where $f_{p\setminus S}\in C_c^0(T(F_{p\setminus S}),\bar{\mathbb{Z}})$ and $f^p\in C_c^0(T(\mathbb{A}_F^{p\cup\infty}),\bar{\mathbb{Z}})$. Moreover, assume that $H\subseteq T(F_{p\setminus S})$ is a compact subgroup small enough so that $f_{p\setminus S}=\sum_x f_{p\setminus S}(x)\cdot\mathbb{1}_{xH}$. If we write $\delta_p^S:=\bigotimes_{\mathfrak{p}\in p\setminus S}\delta_\mathfrak{p}$, we compute the pairing (3.14):

$$\langle f^S,\mathrm{ev}\,|_{\Delta_T}\,r_S\Phi(\delta_p^S(\mathbb{1}_H))\rangle_+=\int_{T(\mathbb{A}_F^{S\cup\infty})}\left(f_{p\setminus S}\otimes f^p\right)(t)\cdot\mathrm{ev}\,|_{\Delta_T}\,r_S\Phi(\delta_p^S(\mathbb{1}_H))(t)d^\times t$$

$$\begin{aligned}
&= \int_{T(\mathbb{A}_F^{p\cup\infty})}f^p(z)\int_{T(F_{p\setminus S})}f_{p\setminus S}(x)\cdot(\bar{\mathrm{ev}}_S\circ r_S)(\Phi)(z)\left(x\delta_p^S(\mathbb{1}_H)\otimes\bigotimes_{\mathfrak{p}\in S}(\tau_\mathfrak{p}+\bar{\tau}_\mathfrak{p})\right)d^\times x d^\times z\\
&= \mathrm{vol}(H)\int_{T(\mathbb{A}_F^{p\cup\infty})}f^p(z)\cdot r_S\Phi(z)\left(\delta_p^S(f_{p\setminus S})\otimes\bigotimes_{\mathfrak{p}\in S}(\ell_{\mathrm{a}_\mathfrak{p}}(\phi_{\tau_\mathfrak{p}}\phi_{\bar{\tau}_\mathfrak{p}}),1)\right)d^\times z\\
&= \mathrm{vol}(H)\int_{T(\mathbb{A}_F^{p\cup\infty})}f^p(z)\cdot\overline{\int_{\mathcal{L}_p}\delta_p^S(f_{p\setminus S})\cdot\prod_{\mathfrak{p}\in S}\ell_{\mathrm{a}_\mathfrak{p}}\left((c\tau_\mathfrak{p}+d)(c\bar{\tau}_\mathfrak{p}+d)\right)d\rho_{\mathbf{k}_p-1_S}\Phi(z)}d^\times z\\
&\overset{(a)}{=} \frac{\mathrm{vol}(H)}{(-1)^{\#S}}\int_{T(\mathbb{A}_F^{p\cup\infty})}f^p(z)\cdot\overline{\int_{\mathcal{L}_p}\delta_p^S(f_{p\setminus S})\cdot\prod_{\mathfrak{p}\in S}\ell_{\mathrm{a}_\mathfrak{p}}{}^{-1}\left((c\tau_\mathfrak{p}+d)(c\bar{\tau}_\mathfrak{p}+d)\right)d\rho_{\mathbf{k}_p^{1_S}}\Phi(z)}d^\times z
\end{aligned}$$

$$\overset{(b)}{=} \frac{\mathrm{vol}(H)}{(-1)^r}\int_{T(\mathbb{A}_F^{p\cup\infty})} f^p(z)\cdot \overline{\int_{\mathcal{L}_p}\delta_p^S(f_{p\setminus S})\cdot\prod_{\mathfrak{p}\in S}\mathbf{k}_{\mathfrak{p}}{}^{-1}\left((c\tau_{\mathfrak{p}}+d)(c\bar{\tau}_{\mathfrak{p}}+d)\right)d\rho_{\mathbf{k}_p^{1_S}}\Phi(z)d^\times z}$$

$$\overset{(c)}{=} \frac{\mathrm{vol}(H)}{(-1)^r}\int_{T(\mathbb{A}_F^{p\cup\infty})} f^p(z)\cdot\overline{\Phi(z)\left(\delta_p\left(f_{p\setminus S}\otimes\mathbb{1}_{T(F_S)}\right)\right)}d^\times z = \frac{\mathrm{vol}(H)}{(-1)^r}\overline{\langle f^S\otimes\mathbb{1}_{T(F_S)},\delta_p^*\Phi\rangle_+},$$

where (a) follows from Corollary 5.4.6, (b) follows from the fact that $\mathbf{k}_{\mathfrak{a}_\mathfrak{p}}(x)-\ell_{\mathfrak{a}_\mathfrak{p}}(x)\in I_\mathfrak{p}^2$ by definition of $\ell_{\mathfrak{a}_\mathfrak{p}}$, and (c) follows from Lemma 5.1.6. Hence by Lemma 3.1.4 and relation (3.17), we obtain

$$\overline{L_p(\phi_\lambda^p,\xi,\underline{k})} = \overline{((\rho^*\xi)\cap\eta)\cap\delta_p^*(\Phi_\lambda^p)} = [\mathcal{O}_+^S:\mathcal{O}_+]^{-1}\overline{((\rho^*\xi^S\otimes\varepsilon_S)\cap(\eta^S\cap\mathbb{1}_{T(F_S)}))\cap\delta_p^*(\Phi_\lambda^p)}$$

$$= \frac{(-1)^r}{[\mathcal{O}_+^S:\mathcal{O}_+]\cdot\mathrm{vol}(H)}\cdot\eta^S\cap\xi_\lambda\cap\mathrm{ev}\,|_{\Delta_T}\, r_S\Phi_\lambda^p(\delta_p^S(\mathbb{1}_H)) = \frac{(-1)^r}{[\mathcal{O}_+^S:\mathcal{O}_+]\cdot\mathrm{vol}(H)}\cdot\ell_{\mathfrak{a}_S}\circ\mathrm{Tr}\left(P_\xi^S(\delta_p^S(\mathbb{1}_H))\right).$$

Since $\phi_\lambda^p(tv^S) = t\phi_\lambda^p(v_S)$, for all $t\in T(F_{p\setminus S})$, the morphism

$$\psi^S: V^S\longrightarrow I^r/I^{r+1}\otimes_{\mathbb{Z}}\bar{\mathbb{Q}},\qquad v^S\longmapsto \ell_{\mathfrak{a}_S}\circ\mathrm{Tr}\left(P_\xi^S(v^S)\right),$$

satisfies $\psi^S(tv^S) = \rho^*\xi(t)^{-1}\cdot\psi^S(v^S)$ for all $t\in T(F_{p\setminus S})$. By the results of Saito and Tunnel (Theorem 2.7.1, see also [Sai93],[Tun83]), the space

$$\mathbf{Hom}_{T(F_{p\setminus S})}(V^S\otimes\rho^*\xi, I^r/I^{r+1}\otimes_{\mathbb{Z}}\bar{\mathbb{Q}})$$

is at most one dimensional. Moreover, in §5.1.5 we have introduced the pairing

$$\alpha(v_1,v_2) = \int_{T(F_{p\setminus S})}\rho^*\xi(t)\langle tv_1, Jv_2\rangle d^\times t\in\mathbf{Hom}_{T(F_{p\setminus S})}(V^S\otimes\rho^*\xi,\bar{\mathbb{Q}})^{\otimes 2}\simeq\bar{\mathbb{Q}}.$$

By Saito-Tunnel, we have that

$$\psi^S(\delta_p^S(\mathbb{1}_H)) = \psi^S(v_0^S)\cdot\alpha(\delta_p^S(\mathbb{1}_H),v_0^S).$$

Moreover, also by Saito-Tunnel,

$$\alpha(v^S,\delta_p^S(\mathbb{1}_H)) = \alpha(v_0^S,\delta_p^S(\mathbb{1}_H))\cdot\alpha(v^S,v_0^S).$$

From the symmetry of α, we deduce $\alpha(\delta_p^S(\mathbb{1}_H),v_0^S) = \alpha(\delta_p^S(\mathbb{1}_H),\delta_p^S(\mathbb{1}_H))^{\frac{1}{2}}$, and the result follows from the computations of §5.1.4. □

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