Óscar Rivero · Victor Rotger

Derived Beilinson–Flach elements and the arithmetic of the adjoint of a modular form

Abstract. Kings, Lei, Loeffler and Zerbes constructed in [LLZ], [KLZ1] a three-variable Euler system $\kappa(\mathbf{g}, \mathbf{h})$ of Beilinson–Flach elements associated to a pair of Hida families $(\mathbf{g}, \mathbf{h})$ and exploited it to obtain applications to the arithmetic of elliptic curves, extending the earlier work [BDR]. The aim of this article is to show that this Euler system also encodes arithmetic information concerning the group of units of the associated number fields. The setting becomes specially novel and intriguing when $\mathbf{g}$ and $\mathbf{h}$ specialize in weight 1 to p-stabilizations of eigenforms such that one is dual to the other. We encounter an exceptional zero phenomenon which forces the specialization of $\kappa(\mathbf{g}, \mathbf{h})$ to vanish and we are led to study the *derivative* of this class. The main result we obtain is the proof of the main conjecture of [DLR4] on iterated integrals and the main conjecture of [DR1] for Beilinson–Flach elements in the adjoint setting. The main point of this paper is that the methods of [DLR1], [DLR4] and [CH], where the above conjectures are proved when the weight 1 eigenforms have CM, do not apply to our setting and new ideas are required. In the previous works, a crucial ingredient is a factorization of p-adic L-functions, which in our scenario is not available due to the lack of critical points. Instead we resort to the principle of *improved* Euler systems and p-adic L-functions to reduce our problems to questions which can be resolved using Galois deformation theory. We expect this approach may be adapted to prove other cases of the Elliptic Stark Conjecture and of its generalizations that appear in the literature.

Keywords. p-adic L-functions, Hida–Rankin convolution, special values, Beilinson–Flach elements, exceptional zeros

Contents

O. Rivero: Departament de Matemàtiques, Universitat Politècnica de Catalunya,
C. Jordi Girona 1-3, 08034 Barcelona, Spain; e-mail: oscar.rivero@upc.edu

V. Rotger (corresponding author): Departament de Matemàtiques, Universitat Politècnica de
Catalunya, C. Jordi Girona 1-3, 08034 Barcelona, Spain; e-mail: victor.rotger@upc.edu

1. Introduction

The purpose of this article is to prove two results conjectured by H. Darmon, A. Lauder and one of the present authors in [DLR4] and [DR1] respectively, concerning the interplay between the arithmetic of units in number fields, the theory of Coleman iterated integrals, and the Rankin p-adic L-function and Euler system of Beilinson–Flach elements associated to a pair $(\mathbf{g}, \mathbf{h})$ of Hida families of modular forms.

   The first theorem of this article is Theorem A (together with its equivalent form given in Theorem A′), which yields a proof of the main conjecture of [DLR4] in the adjoint setting. Although Beilinson–Flach elements are certainly behind the scenes, we find it interesting to remark that this Euler system is involved neither in the statement nor in the proof we provide of Theorems A, A′, and can be phrased in purely analytic terms by means of p-adic iterated integrals and p-adic L-functions. It is only later in the note that we explore the consequences that Theorem A has on the weight one specializations of the Euler system of Beilinson–Flach elements, as described in Theorems B and C.

   Let χ be a Dirichlet character of level $N \geq 1$ and let $M_k(N, \chi)$ (resp. $S_k(N, \chi)$) denote the space of (resp. cuspidal) modular forms of weight k, level N and nebentype χ. Let $g = \sum_{n \geq 1} a_n q^n \in S_1(N, \chi)$ be a normalized newform and let $g^* = g \otimes \chi^{-1}$ denote its twist by the inverse of its nebentype. Let

$$\varrho_g : \mathrm{Gal}(H_g/\mathbb{Q}) \hookrightarrow \mathrm{GL}(V_g) \simeq \mathrm{GL}_2(L),$$

$$\varrho_{\mathrm{ad}^0(g)} : \mathrm{Gal}(H/\mathbb{Q}) \hookrightarrow \mathrm{GL}(\mathrm{ad}^0(g)) \simeq \mathrm{GL}_3(L)$$

denote the Artin representations associated to g and its adjoint, respectively. Here $H_g \supseteq H$ denote the finite Galois extensions of $\mathbb{Q}$ cut out by these representations, and L is a sufficiently large finite extension of $\mathbb{Q}$ containing their traces. The three-dimensional representation $\mathrm{ad}^0(g)$ may be identified with the subspace of $\mathrm{End}(V_g)$ of null-trace endomorphisms, on which $G_{\mathbb{Q}}$ acts by conjugation. There is a natural decomposition of $L[G_{\mathbb{Q}}]$-modules $\mathrm{ad}(g) := \mathrm{End}(V_g) = L \oplus \mathrm{ad}^0(g)$, where L stands for the trivial representation.

   Fix a prime $p \nmid N$ and let $S_k^{\mathrm{oc}}(N, \chi)$ denote the space of overconvergent p-adic modular forms of weight k, tame level N and character χ. Fix an embedding $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$ and let L_p denote the completion of L in $\bar{\mathbb{Q}}_p$.

   Label and order the roots of the p-th Hecke polynomial of g as $X^2 - a_p(g)X + \chi(p) = (X - \alpha)(X - \beta)$. We assume throughout that

(H1) the reduction of ϱ_g mod p is irreducible;

(H2) g is p-distinguished, i.e. $\alpha \neq \beta \pmod{p}$, and

(H3) ϱ_g is not induced from a character of a real quadratic field in which p splits.

Let

$$g_\alpha(q) = g(q) - \beta g(q^p)$$

denote the p-stabilization of g on which the Hecke operator U_p acts with eigenvalue α. Enlarge L if necessary so that it contains all Fourier coefficients of g_α.

As shown in [BeDi] and [DLR1], the above hypotheses ensure that any generalized overconvergent modular form with the same generalized eigenvalues as g_α is classical, and hence simply a multiple of g_α. This allows one to define a canonical projector

$$e_{g_\alpha} : S_1^{\text{oc}}(N, \chi) \to L_p$$

which extracts from an overconvergent modular form its coefficient at g_α with respect to an orthonormal basis: cf. [DLR1] for more details.

One can attach a p-adic invariant $I_p(g) \in L_p$ to g_α as follows. Let

$$Y := Y_1(N) \quad \subset \quad X := X_1(N)$$

denote the models over $\mathbb{Q}$ of the (affine and projective, respectively) modular curves classifying pairs (A, P) where A is a (generalized) elliptic curve and P is a point of order N on A.

For $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, let $\mathfrak{g}_{a;N}$ be Kato's *Siegel unit* whose q-expansion is given by

$$\mathfrak{g}_{a;N} := (1 - \zeta_N^a)q^{1/12} \prod_{n=1}^\infty (1 - \zeta_N^a q^n)(1 - \zeta_N^{-a} q^n), \tag{1}$$

where ζ_N is a fixed primitive N-th root of unity. The unit $\mathfrak{g}_{a;N}$ can naturally be viewed as belonging to $\mathbb{Q} \otimes \mathcal{O}_Y^\times$, and its q-expansion is defined over $\mathbb{Q}(\mu_N)$.

Set

$$\mathfrak{g} = \frac{1}{2} \otimes \prod_{a=1}^N \mathfrak{g}_{a;N} \in \mathbb{Q} \otimes \mathcal{O}_Y^\times, \tag{2}$$

and define

$$E_0 = \log_p \mathfrak{g}.$$

This is a locally analytic modular form of weight 0 on Y. It follows from (1) and (2) that the logarithmic derivative of $\mathfrak{g}$ is

$$dE_0 = E_2 \frac{dq}{q} \in \Omega_Y^1, \tag{3}$$

where

$$E_2 = \frac{1}{2} \cdot \zeta(-1) + \sum_{n=1}^\infty \left(\sum_{d|n} d \right) q^n$$

is the classical Eisenstein series of weight 2.

Let $E_0^{[p]} := \sum_{p \nmid n} a_n(E_0)q^n$ denote the p-depletion of E_0; this is an overconvergent modular form and we may define

$$I_p(g) := e_{g_\alpha}(E_0^{[p]} \cdot g_\alpha). \tag{4}$$

The invariant $I_p(g)$ can be recast as a p-adic iterated integral, denoted $I_p'(E_2, g, g^*)$ in [DLR4]. The first main result of this note, conjectured in [DLR4], asserts that $I_p(g)$ is equal to the following motivic expression. As shown in [DLR4, Lemma 1.1], we have

$$\dim_L(\mathcal{O}_H^\times \otimes \text{ad}^0(g))^{G_\mathbb{Q}} = 1, \quad \dim_L(\mathcal{O}_H[1/p]^\times/p^\mathbb{Z} \otimes \text{ad}^0(g))^{G_\mathbb{Q}} = 2.$$

Fix a generator u of $(\mathcal{O}_H^{\times} \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}}}$ and also an element v of $(\mathcal{O}_H[1/p]^{\times} \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}}}$ in such a way that $\{u, v\}$ is a basis of $(\mathcal{O}_H[1/p]^{\times}/p^{\mathbb{Z}} \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}}}$. The element v may be chosen to have p-adic valuation $\mathrm{ord}_p(v) = 1$, and we do so.

Viewed as a $G_{\mathbb{Q}_p}$-module, $\mathrm{ad}^0(g)$ decomposes as $\mathrm{ad}^0(g) = L \oplus L^{\alpha \otimes \bar{\beta}} \oplus L^{\beta \otimes \bar{\alpha}}$, where each line is characterized by the property that the arithmetic Frobenius Fr_p acts on it with eigenvalue 1, α/β and β/α, respectively[1]. Let H_p denote the completion of H in $\bar{\mathbb{Q}}_p$ and let

$$u_1, \ u_{\alpha \otimes \bar{\beta}}, \ u_{\beta \otimes \bar{\alpha}}, \ v_1, \ v_{\alpha \otimes \bar{\beta}}, \ v_{\beta \otimes \bar{\alpha}} \in H_p^{\times} \otimes_{\mathbb{Q}} L \ (\mathrm{mod} \ L^{\times})$$

denote the projection of the elements u and v in $(H_p^{\times} \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}_p}}$ to the above lines: cf. (50) for more details. By construction we have $u_1, v_1 \in \mathbb{Q}_p^{\times}$ and

$$\mathrm{Fr}_p(u_{\alpha \otimes \bar{\beta}}) = \frac{\beta}{\alpha} u_{\alpha \otimes \bar{\beta}}, \quad \mathrm{Fr}_p(v_{\alpha \otimes \bar{\beta}}) = \frac{\beta}{\alpha} v_{\alpha \otimes \bar{\beta}},$$

$$\mathrm{Fr}_p(u_{\beta \otimes \bar{\alpha}}) = \frac{\alpha}{\beta} u_{\beta \otimes \bar{\alpha}}, \quad \mathrm{Fr}_p(v_{\beta \otimes \bar{\alpha}}) = \frac{\alpha}{\beta} v_{\beta \otimes \bar{\alpha}}.$$

Let

$$\log_p : H_p^{\times} \otimes L \to H_p \otimes L$$

denote the usual p-adic logarithm.

Theorem A. *The following equality holds in $L_p \subset H_p \otimes L$ (mod $L^{\times}$):*

$$I_p(g) = \frac{1}{\log_p(u_{\alpha \otimes \bar{\beta}})} \times \det \begin{pmatrix} \log_p u_1 & \log_p(u_{\alpha \otimes \bar{\beta}}) \\ \log_p v_1 & \log_p(v_{\alpha \otimes \bar{\beta}}) \end{pmatrix}. \tag{5}$$

This is [DLR4, Conjecture 1.2] specialized to the pair (g, g^*): note that both statements are equivalent because $(\mathcal{O}_H^{\times} \otimes V_g^{\alpha})^{G_{\mathbb{Q}_p}} \simeq \mathrm{Hom}_{G_{\mathbb{Q}_p}}(V_g^{\beta}, \mathcal{O}_H^{\times})$.

Let $\mathcal{L}(\mathrm{ad}^0(g_{\alpha}))$ be Hida's analytic $\mathcal{L}$-invariant attached to the adjoint Galois representation of g_{α}; cf. (23). As we recall in Proposition 2.5, there are several equivalent definitions of this invariant.

These $\mathcal{L}$-invariants are in general difficult to compute, and explicit expressions for them are rather rare. The proof of the above result, which we describe further below in this introduction, shows that Theorem A may be recast as a formula for $\mathcal{L}(\mathrm{ad}^0(g_{\alpha}))$. Since this might be of independent interest, specially for the reader less acquainted with the Stark elliptic conjectures of [DLR1], [DLR4] and more familiar with the theory of $\mathcal{L}$-invariants, we quote it below as a separate statement:

Theorem A'. *The analytic $\mathcal{L}$-invariant of $\mathrm{ad}^0(g_{\alpha})$ is*

$$\mathcal{L}(\mathrm{ad}^0(g_{\alpha})) = \frac{\log_p(u_1) \log_p(v_{\alpha \otimes \bar{\beta}}) - \log_p(u_{\alpha \otimes \bar{\beta}}) \log_p(v_1)}{\log_p(u_{\alpha \otimes \bar{\beta}})} \ (\mathrm{mod} \ L^{\times}). \tag{6}$$

---

[1] The decomposition of $\mathrm{ad}^0(g)$ as the direct sum of three canonical lines is also available when $\alpha/\beta = \beta/\alpha = -1$: see (45) and (50) for details.

Remark 1.1. The prototypical case where an explicit formula for the $\mathcal{L}$-invariant is known arises of course when $E/\mathbb{Q}$ is an elliptic curve of split multiplicative reduction at p. In that case $\mathcal{L}_p(E) = \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)}$ where $q_E \in p\mathbb{Z}_p$ is a Tate period for E. Observe that this expression and the one in Theorem A$'$ are in fact very similar, as both may be recast as

$$\mathcal{L} = \frac{\log_p(\kappa)}{\mathrm{ord}_p(\kappa)}, \tag{7}$$

where κ is an element in $H^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ arising from some global class in motivic co-homology. In the classical case one has $\kappa = q_E$, which might be regarded as an element in the *extended* Mordell-Weil group $\tilde{E}(\mathbb{Q})$, or Nekovář's extended Selmer group of E. In Theorem A$'$ one has, up to scalar, $\kappa = \log_p(v_{\alpha \otimes \bar{\beta}})u_1 - \log_p(u_{\alpha \otimes \bar{\beta}})v_1$ in the unit group of H; note that recipe (7) indeed gives rise to (6) because $\mathrm{ord}_p(u_1) = 0$ and $\mathrm{ord}_p(v_1) = 1$.

The second main result of this note may be regarded as a conceptual explanation of Theorem A, as it establishes a connection between the iterated integral $I_p(g)$, a special value of Hida's p-adic Rankin L-function associated to the pair (g, g^*), and the generalized Kato classes arising from the three-variable Euler system of Beilinson–Flach elements constructed in [KLZ1].

In order to describe it more precisely, let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ denote the Iwasawa algebra and denote $\mathcal{W} = \mathrm{Spf}(\Lambda)$ the associated weight space. As shown by Wiles [Wi] (cf. also [KLZ1, §7.2] for the normalizations we adopt), associated to g_α there are:

(1) a finite flat extension $\Lambda_{\mathbf{g}}$ of Λ, giving rise to a covering $\mathrm{w} : \mathcal{W}_{\mathbf{g}} = \mathrm{Spf}(\Lambda_{\mathbf{g}}) \to \mathcal{W}$;
(2) a family of overconvergent p-adic ordinary modular forms $\mathbf{g}$ with coefficients in $\Lambda_{\mathbf{g}}$ specializing to g_α at some point $y_0 \in \mathcal{W}_{\mathbf{g}}$ of weight $\mathrm{w}(y_0) = 1$; our running assumptions imply that the above Hida family passing through g_α is unique by [BeDi];
(3) a Galois representation $\varrho_{\mathbf{g}} : G_\mathbb{Q} \to \mathrm{GL}_2(\Lambda_{\mathbf{g}})$ characterized by the property that all its classical specializations coincide with Deligne's Galois representation associated to the corresponding specialization of the Hida family.

Let $\mathbb{V}_{\mathbf{g}}$ denote the rank two $\Lambda_{\mathbf{g}}$-module realizing the Galois representation $\varrho_{\mathbf{g}}$ as specified e.g. in [KLZ1, §7.2], where this is denoted $M(\mathbf{g})^*$. Let $\mathbf{g}^* := \mathbf{g} \otimes \chi^{-1}$ denote the twist of $\mathbf{g}$ by the inverse of its tame nebentype. Note that $\mathbf{g}^*$ specializes at y_0 to the eigenform $g_\alpha \otimes \chi^{-1} = g^*_{1/\beta}$, namely the p-stabilization of g^* on which U_p acts with eigenvalue $1/\beta$. Let

$$\varepsilon_{\mathrm{cyc}} : G_\mathbb{Q} \to \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \to \mathbb{Z}_p^\times$$

denote the p-adic cyclotomic character. Let $\underline{\varepsilon}_{\mathrm{cyc}}$ be the composition of $\varepsilon_{\mathrm{cyc}}$ with the natural inclusion $\mathbb{Z}_p^\times \subset \Lambda^\times$ taking z to the group-like element $[z]$ in $\Lambda^\times$.

Define the three-variable Iwasawa algebra $\Lambda_{\mathbf{g}\mathbf{g}^*} := \Lambda_{\mathbf{g}} \,\hat{\otimes}_{\mathbb{Z}_p}\, \Lambda_{\mathbf{g}^*} \,\hat{\otimes}_{\mathbb{Z}_p}\, \Lambda$ and the $\Lambda_{\mathbf{g}\mathbf{g}^*}[G_\mathbb{Q}]$-module

$$\mathbb{V}_{\mathbf{g}\mathbf{g}^*} := \mathbb{V}_{\mathbf{g}} \,\hat{\otimes}_{\mathbb{Z}_p}\, \mathbb{V}_{\mathbf{g}^*} \,\hat{\otimes}_{\mathbb{Z}_p}\, \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}).$$

The article [KLZ1] attaches to $(\mathbf{g}, \mathbf{g}^*)$ a Λ-adic global cohomology class

$$\kappa(\mathbf{g}, \mathbf{g}^*) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{g}\mathbf{g}^*})$$

parametrized by the triple product of weight spaces $\mathcal{W}_{\mathbf{gg}^*} := \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}$, where the first two variables are afforded by the weight of the Hida families $\mathbf{g}$ and $\mathbf{g}^*$, and the third one is the cyclotomic variable.

The common key strategy in several recent works on the arithmetic of elliptic curves ([BDR], [DR2], [LLZ], [KLZ1], [CH]) consists in proving a *reciprocity law* relating a suitable specialization of a Λ-adic class as the one above to the critical value of the underlying classical L-function. The non-vanishing of the appropriate (classical or p-adic) L-value can then be invoked to show that the associated global cohomology class is not trivial, and one can derive striking arithmetic consequences from this (cf. e.g. [DR2, Theorems A and B], [KLZ1, Corollary C]).

Our setting differs in several key aspects from the previous ones. Namely, while it is also natural to consider the specialization

$$\kappa(g_\alpha, g^*_{1/\beta}) := \kappa(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 0) \in H^1(\mathbb{Q}, V_g \otimes V_{g^*}(1)) \tag{8}$$

of the Euler system of Beilinson–Flach elements at the point $(y_0, y_0, 0)$, one can show that this global cohomology class is trivial and thus no arithmetic information can be extracted directly from it: cf. Theorem 3.5.

The fact that $\kappa(g_\alpha, g^*_{1/\beta}) = 0$ might be regarded as an *exceptional zero phenomenon*, albeit quite different from the ones one typically encounters in e.g. the pioneering work [MTT] of Mazur, Tate and Teitelbaum, because our case corresponds to a point lying *outside* the classical region of interpolation of the associated Hida–Rankin p-adic L-function. But it still keeps the same flavor, because the vanishing of the global cohomology class (8) is caused by the cancellation of the Euler-like factor at p arising in the interpolation process of the construction of the Euler system. To be more precise, the situation is more delicate than that: for classical points $y \in \mathcal{W}_{\mathbf{g}}$ of weights $\ell > 1$, the triples $(y, y, \ell - 1)$ do lie in the region of *geometric* interpolation of the Euler system of Beilinson–Flach elements, and the main theorem of [KLZ1] applies and implies that $\kappa(\mathbf{g}, \mathbf{g}^*)(y, y, \ell - 1) = 0$. The vanishing of $\kappa(g_\alpha, g^*_{1/\beta})$ then follows by a density argument exploiting various Λ-adic Perrin–Riou regulators; we refer to §3.2 for more details.

We are hence placed to work with the *derived class* $\kappa'(g_\alpha, g^*_{1/\beta}) \in H^1(\mathbb{Q}, \mathrm{ad}^0(g)(1))$, which is introduced in §3.3 as the derivative along the weight of $\mathbf{g}^*$ of the Λ-adic class $\kappa(\mathbf{g}, \mathbf{g}^*)$ at $(y_0, y_0, 0)$.

In [DR1] a conjecture was laid proposing an explicit description of the generalized Kato classes arising from the Euler systems of diagonal cycles of [DR2]. This conjecture, together with the expected (but so far also unproved) behavior of certain periods arising from Hida theory, is shown to imply the main conjecture of [DLR1] for twists of elliptic curves by Artin representations. This is seen to provide a conceptual interpretation of the numerical examples computed in [DLR1].

At the time of writing [DR1], the authors had in mind that a parallel formulation should also hold for the Euler system of Beilinson–Flach elements attached to pairs (g, h) of eigenforms of weights $(1, 1)$ in connection with [DLR4]. This is indeed the case for arbitrary pairs (g, h) such that $h \neq g^*$, as there are no exceptional zero phenomena and the constructions and conjectures [DR1] can be adapted to the setting of units of number fields. We refer to [RR] for the details.

However, pairs (g, g^*) present a completely different scenario, as we already hinted at above. Not only $\kappa(g_\alpha, g_{1/\beta}^*) = 0$, but it also turns out that the derived class $\kappa'(g_\alpha, g_{1/\beta}^*)$ is *not* crystalline at p, as opposed to the set-up of [DR1] and [RR]. Yet it is still possible to formulate an analogous conjecture, proposing an explicit description of $\kappa'(g_\alpha, g_{1/\beta}^*)$ in terms of the $\mathrm{ad}^0(g)$-isotypical component of the group of p-units of H. We can in fact prove it, giving rise to the second main result of this paper.

Let $H_{\mathrm{f},p}^1(\mathbb{Q}, \mathrm{ad}^0(g)(1))$ denote the subspace of $H^1(\mathbb{Q}, \mathrm{ad}^0(g)(1))$ consisting of classes that are de Rham at p and unramified at all remaining places. Kummer theory (cf. Proposition 3.11 below) gives rise to a canonical isomorphism

$$\mathcal{O}_H[1/p]^\times[\mathrm{ad}^0(g)] \otimes L_p \xrightarrow{\simeq} H_{\mathrm{f},p}^1(\mathbb{Q}, \mathrm{ad}^0(g)(1)) \tag{9}$$

and we shall use (9) throughout to identify these two spaces.

Theorem B. *Assume that* $\mathcal{L}(\mathrm{ad}^0(g_\alpha)) \neq 0$. *The equality*

$$\kappa'(g_\alpha, g_{1/\beta}^*) = \mathcal{L}(\mathrm{ad}^0(g_\alpha)) \cdot \frac{\log_p(v_{\alpha\otimes\bar\beta})u - \log_p(u_{\alpha\otimes\bar\beta})v}{\log_p(u_{\alpha\otimes\bar\beta})} \tag{10}$$

holds in $\frac{\mathcal{O}_H[1/p]^\times}{p^{\mathbb{Z}}}[\mathrm{ad}^0(g)] \otimes L_p \pmod{L^\times}$.

Here it again becomes apparent that there are notable differences between the phenomena occurring in this article and say [BDR], [DR2], [KLZ1]. In the cited works, the non-vanishing of the central critical L-value is shown to imply the triviality of the Mordell–Weil group; conversely, the vanishing of the central L-value should imply the non-triviality of the Mordell–Weil group.

In contrast, in our setting we have $L(g, g^*, 1) \neq 0$ and nonetheless the associated motivic groups $\mathcal{O}_H^\times[\mathrm{ad}^0(g)]$ and $\mathcal{O}_H[1/p]^\times[\mathrm{ad}^0(g)]$ have positive rank. This is ultimately explained by the fact that $L(g, g^*, s) = L(\mathrm{ad}^0(g), s) \cdot \zeta(s)$ factors as the product of two zeta functions having a simple zero and a simple pole at $s = 1$, respectively.

Remark 1.2. As we show in §5, it is actually possible to prove that $\mathcal{L}(\mathrm{ad}^0(g_\alpha)) \neq 0$ in many dihedral cases, where ϱ_g is induced from a finite order character of the Galois group of a (real or imaginary) quadratic field. We refer to the cited papers for more details. On the other hand, when g is exotic, we still expect the $\mathcal{L}$-invariant not to vanish systematically, but proving this properly appears to be a less accessible question in the theory of transcendental p-adic numbers.

The main ingredients in the proof of Theorems A, A$'$ and B are the following:

(I) Let $L_p(\mathbf{g}, \mathbf{g}^*)$ be the three-variable Hida–Rankin p-adic L-function as introduced in [Hi1] or [Das2, §3.6], and let $L_p(g_\alpha, g_{1/\beta}^*, s) := L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, s)$ denote the restriction to $\{y_0, y_0\} \times \mathcal{W}$. This function satisfies a functional equation relating the values at s and $1 - s$.

As explained in [DLR4], the p-adic iterated integral introduced in (4) may be recast as a special value of a p-adic L-function, namely [DLR4, Lemma 4.2] asserts that under our running assumptions H1 to H3 that are in force throughout the article, we have

$$I_p(g) = L_p(g_\alpha, g_{1/\beta}^*, 0) = L_p(g_\alpha, g_{1/\beta}^*, 1) \pmod{L^\times}.$$

Define $\mathcal{S}_{\mathrm{Hida}} \subset \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}$ as the surface for which the collection of crystalline points (y, z, s) of weights (ℓ, m, m) is Zariski dense. The Euler-like factor at p showing up in the interpolation formula of Hida–Rankin's p-adic L-function turns out to be a rigid-analytic Iwasawa function when restricted to $\mathcal{S}_{\mathrm{Hida}}$. In §2.3 we recall how this observation allows one to conclude thanks to [Das2] that

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) := \alpha_{\mathbf{g}}'(y_0) = I_p(g) \pmod{L^\times}. \tag{11}$$

Computing the latter derivative is a question which can be reduced to a problem in Galois deformation theory as in [DLR2], although the methods there do not apply directly and need to be adapted in order to cover our setting. This is carried out in §4, and allows us to prove Theorems A and A', as we spell out in detail in §5.

(II) The main result of §3 is Theorem 3.7, which we state here in slightly rough terms (as the Bloch–Kato logarithm map appearing below has not been specified; cf. loc. cit. for more details):

Theorem C (Theorem 3.7). *The derived Beilinson–Flach element satisfies*

$$\log_p(\kappa'(g_\alpha, g_{1/\beta}^*)) = \mathcal{L}(\mathrm{ad}^0(g_\alpha)) \cdot L_p(g_\alpha, g_{1/\beta}^*, 0) \pmod{L^\times}. \tag{12}$$

We may see this as an *exceptional zero formula*, reminiscent of the main theorem of [GS], which asserts that for an elliptic curve $E/\mathbb{Q}$ with split multiplicative reduction at p, we have

$$L_p'(E, 1) = \mathcal{L}_p(E) \cdot L_{\mathrm{alg}}(E, 1).$$

We may thus invoke Theorem A, which combined with (12) implies that

$$\log_p(\kappa'(g_\alpha, g_{1/\beta}^*)) = \left(\frac{\log_p(u_1) \log_p(v_{\alpha\otimes\bar\beta}) - \log_p(u_{\alpha\otimes\bar\beta}) \log_p(v_1)}{\log_p(u_{\alpha\otimes\bar\beta})} \right)^2 \pmod{L^\times}.$$

In light of the general properties satisfied by Beilinson–Flach elements established in [KLZ1], this allows us to prove in §5 that the following equality holds in $\mathcal{O}_H[1/p]^\times[\mathrm{ad}^0(g)] \otimes L_p$:

$$\kappa'(g_\alpha, g_{1/\beta}^*) = \frac{\log_p(u_1) \log_p(v_{\alpha\otimes\bar\beta}) - \log_p(u_{\alpha\otimes\bar\beta}) \log_p(v_1)}{\log_p(u_{\alpha\otimes\bar\beta})^2}$$
$$\times \left(\log_p(v_{\alpha\otimes\bar\beta})u - \log_p(u_{\alpha\otimes\bar\beta})v \right)$$

up to a factor in $L^\times$, as claimed.

We close the paper in §6 by pointing out the connection between our derived Beilinson –Flach classes, Darmon–Dasgupta units and the Artin p-adic L-functions associated to the motives in play. We prove a factorization theorem in the CM case, and explain how our main results shed some light on the much more intriguing RM setting.

As a concluding remark, we hope the approach introduced in this paper may be adapted to prove other instances of variants of the Elliptic Stark Conjecture in the presence of exceptional zeroes. We also hope that the derived global cohomology classes appearing in this work may also be useful for the on-going research project [DHRV] of Darmon, Harris, Rotger and Venkatesh on derived Hecke algebras in weight one.

2. Preliminary concepts

2.1. Hida families

Fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. For a number field K, let $G_K := \mathrm{Gal}(\bar{\mathbb{Q}}/K)$ denote its absolute Galois group. Fix a prime p and an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, and let ord_p denote the resulting p-adic valuation on $\bar{\mathbb{Q}}^\times$, normalized in such a way that $\mathrm{ord}_p(p) = 1$.

Let g be a newform of weight $k \geq 1$, level N and character χ, with Fourier coefficients in a finite extension L of $\mathbb{Q}$. Label the roots of the p-th Hecke polynomial of g as α_g, β_g with $\mathrm{ord}_p(\alpha_g) \leq \mathrm{ord}_p(\beta_g)$. By enlarging L if necessary, we shall assume throughout that L contains α_g, β_g, the N-th roots of unity and the pseudo-eigenvalue $\lambda_N(g)$ with respect to the Atkin–Lehner operator W_N (cf. [AL]).

Let L_p denote the completion of L in $\bar{\mathbb{Q}}_p$ and let V_g denote the two-dimensional representation of $G_{\mathbb{Q}}$ with coefficients in L_p associated to g as defined e.g. in [KLZ1, §2.8], where it is denoted $M_{L_p}(g)^*$. If g is ordinary at p, there is an exact sequence of $G_{\mathbb{Q}_p}$-modules

$$0 \to V_g^+ \to V_g \to V_g^- \to 0, \qquad V_g^+ \simeq L_p(\varepsilon_{\mathrm{cyc}}^{k-1} \chi \psi_g^{-1}), \quad V_g^- \simeq L_p(\psi_g), \quad (13)$$

where ψ_g is the unramified Galois character of $G_{\mathbb{Q}_p}$ sending Fr_p to α_g.

The formal spectrum $\mathcal{W} = \mathrm{Spf}(\Lambda)$ of the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ is called the *weight space attached to* Λ, and their A-valued points over a p-adic ring A are given by

$$\mathcal{W}(A) = \mathrm{Hom}_{\mathrm{alg}}(\Lambda, A) = \mathrm{Hom}_{\mathrm{grp}}(\mathbb{Z}_p^\times, A^\times).$$

The weight space is equipped with a distinguished class of *arithmetic points* $\nu_{s,\varepsilon}$ indexed by integers s and Dirichlet characters $\varepsilon : (\mathbb{Z}/p^r\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ of p-power conductor. The point $\nu_{s,\varepsilon} \in \mathcal{W}$ is defined by the rule

$$\nu_{s,\varepsilon}(n) = \varepsilon(n)n^s.$$

We let $\mathcal{W}^{\mathrm{cl}}$ denote the subset of $\mathcal{W}$ formed by such arithmetic points. When $\varepsilon = 1$ is the trivial character, we denote the point $\nu_{s,1}$ simply as ν_s or even s by a slight abuse of notation.

If $\tilde{\Lambda}$ is a finite flat algebra over Λ, there is a natural finite map $\tilde{\mathcal{W}} := \mathrm{Spf}(\tilde{\mathcal{W}}) \overset{\mathrm{w}}{\to} \mathcal{W}$, and we say that a point $x \in \tilde{\mathcal{W}}$ is *arithmetic* of weight s and character ε if $\mathrm{w}(x) = \nu_{s,\varepsilon}$. As

in the introduction, let $\varepsilon_{\mathrm{cyc}}$ denote the p-adic cyclotomic character and $\underline{\varepsilon}_{\mathrm{cyc}}$ be the composition of $\varepsilon_{\mathrm{cyc}}$ with the natural inclusion $\mathbb{Z}_p^{\times} \subset \Lambda^{\times}$ taking z to the group-like element $[z]$ in $\Lambda^{\times}$.

Let $N \geq 1$ be an integer not divisible by p, and let $\chi : (\mathbb{Z}/N\mathbb{Z})^{\times} \to \mathbb{C}_p^{\times}$ be a Dirichlet character. Let $\mathbf{g}$ be a Hida family of tame level N and tame character χ as defined and normalized in [KLZ1, §7.2]. Let $\Lambda_{\mathbf{g}}$ be the associated Iwasawa algebra (cf. Def. 7.2.5 of loc. cit.), which is finite and flat over Λ. As in [KLZ1, §7.3], we may specialize $\mathbf{g}$ at any arithmetic point $x \in \mathcal{W}_{\mathbf{g}} = \mathrm{Spf}(\Lambda_{\mathbf{g}})$ of weight $k \geq 2$ and character $\varepsilon : (\mathbb{Z}/p^r\mathbb{Z})^{\times} \to L_p^{\times}$ and obtain a classical p-ordinary eigenform g_x in the space $S_k(Np^r, \chi\varepsilon)$ of cusp forms of weight k, level Np^r and nebentype $\chi\varepsilon$.

Definition 2.1. Let $x \in \mathcal{W}_{\mathbf{g}}$ be an arithmetic point of weight $k \geq 1$ and character ε. We say x is *crystalline* if $\varepsilon = 1$ and there exists an eigenform g_x° of level N such that g_x is the ordinary p-stabilization of g_x° (which given the previous condition, is automatic if $k > 2$, but not necessarily if $k \leq 2$). We denote by $\mathcal{W}_{\mathbf{g}}^{\circ}$ the set of crystalline arithmetic points of $\mathcal{W}_{\mathbf{g}}$.

As shown by Wiles [Wi], [KLZ1, §7.2] and already recalled in the introduction, a Hida family $\mathbf{g}$ as above comes equipped with a free $\Lambda_{\mathbf{g}}[G_{\mathbb{Q}}]$-module $\mathbb{V}_{\mathbf{g}}$ of rank two, yielding a Galois representation $\varrho_{\mathbf{g}} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\Lambda_{\mathbf{g}})$, all whose classical specializations recover the p-adic Galois representation associated to the classical specialization of $\mathbf{g}$. We assume throughout that the mod p residual representation $\bar{\varrho}_{\mathbf{g}}$ associated to any of its classical specializations is irreducible, and that the semi-simplification of $\bar{\varrho}_{|G_{\mathbb{Q}_p}}$ is non-scalar, as in hypotheses (H1)–(H2) in the introduction.

Similarly to (13), the restriction to $G_{\mathbb{Q}_p}$ of $\mathbb{V}_{\mathbf{g}}$ admits a filtration

$$0 \to \mathbb{V}_{\mathbf{g}}^+ \to \mathbb{V}_{\mathbf{g}} \to \mathbb{V}_{\mathbf{g}}^- \to 0, \tag{14}$$

where $\mathbb{V}_{\mathbf{g}}^+$ and $\mathbb{V}_{\mathbf{g}}^-$ are flat $\Lambda_{\mathbf{g}}[G_{\mathbb{Q}_p}]$-modules, free of rank one over $\Lambda_{\mathbf{g}}$. If we let $\psi_{\mathbf{g}}$ denote the unramified character of $G_{\mathbb{Q}_p}$ taking the arithmetic Frobenius element Fr_p to $a_p(\mathbf{g})$, then

$$\mathbb{V}_{\mathbf{g}}^+ \simeq \Lambda_{\mathbf{g}}(\psi_{\mathbf{g}}^{-1}\chi\varepsilon_{\mathrm{cyc}}^{-1}\underline{\varepsilon}_{\mathrm{cyc}}), \quad \mathbb{V}_{\mathbf{g}}^- \simeq \Lambda_{\mathbf{g}}(\psi_{\mathbf{g}}). \tag{15}$$

Let $\widehat{\mathbb{Z}_p^{\mathrm{ur}}}$ be the ring of integers of the completion of the maximal unramified extension of $\mathbb{Q}_p$. Given a finite-dimensional $G_{\mathbb{Q}_p}$-module V with coefficients over a finite extension L_p of $\mathbb{Q}_p$, the de Rham Dieudonné module associated to V is defined as

$$D(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_{\mathbb{Q}_p}},$$

where B_{dR} is Fontaine's field of de Rham periods. If V is unramified there is a further canonical isomorphism

$$D(V) \simeq (V_{\mathrm{int}} \hat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}_p^{\mathrm{ur}}})^{\mathrm{Fr}_p=1}[1/p],$$

where V_{int} is an integral lattice in V. Similarly, if $\mathbb{V}$ is a free module over an Iwasawa algebra equipped with an unramified action of $G_{\mathbb{Q}_p}$, we may define

$$\mathbb{D}(\mathbb{V}) := (\mathbb{V} \hat{\otimes}_{\mathbb{Z}_p} \widehat{\mathbb{Z}_p^{\mathrm{ur}}})^{\mathrm{Fr}_p=1}. \tag{16}$$

Given a newform $g \in S_k(N, \chi)$ as at the beginning of this section, $D(V_g)$ is an L_p-filtered vector space of rank 2. Set $g^* = g \otimes \bar{\chi} \in S_k(N, \bar{\chi})$. Poincaré duality induces a perfect pairing

$$\langle \, , \, \rangle : D(V_g(-1)) \times D(V_{g^*}) \to L_p. \tag{17}$$

If g is ordinary at p, then (13) gives rise to an exact sequence of Dieudonné modules

$$0 \to D(V_g^+) \xrightarrow{i} D(V_g) \xrightarrow{\pi} D(V_g^-) \to 0$$

where $D(V_g^+)$ and $D(V_g^-)$ have rank 1 over L_p.

If $k \geq 2$, Faltings' comparison theorem allows one to associate to g a regular differential form $\omega_g \in \mathrm{Fil}(D(V_g))$, which induces a linear form

$$\omega_g : D(V_{g^*}^+) \to L_p, \quad \eta \mapsto \langle \omega_g, \eta \rangle.$$

There is also the differential form η_g, characterized by the properties that it spans the line $D(V_g^+)$ and $\langle \eta_g, \omega_{g^*} \rangle = 1$. It again gives rise to a linear functional

$$\eta_g : D(V_{g^*}^-) \to L_p, \quad \omega \mapsto \langle \pi^{-1}(\omega), \eta_g \rangle.$$

As shown in [Oh] and [KLZ1], the differential forms (or linear functionals) ω_g and η_g vary in families. In order to recall this more precisely, let $\mathbf{g}$ be a Hida family of tame level N and tame character χ as above. Set $\mathbf{g}^* = \mathbf{g} \otimes \bar{\chi}$. Let $\mathcal{Q}_{\mathbf{g}}$ denote the fraction field of $\Lambda_{\mathbf{g}}$, and set $\mathbb{U}_{\mathbf{g}}^+ := \mathbb{V}_{\mathbf{g}}^+(\chi^{-1}\varepsilon_{\mathrm{cyc}}\varepsilon_{\mathrm{cyc}}^{-1})$; denote by $U_{g_y}^+$ its specialization at a point $y \in \mathcal{W}_{\mathbf{g}}^\circ$. By [KLZ1, Proposition 10.1.1], there exist:

- A homomorphism of $\Lambda_{\mathbf{g}}$-modules

$$\langle \, , \omega_{\mathbf{g}} \rangle : \mathbb{D}(\mathbb{U}_{\mathbf{g}^*}^+) \to \Lambda_{\mathbf{g}} \tag{18}$$

such that for every $y \in \mathcal{W}_{\mathbf{g}}^\circ$, the specialization of $\omega_{\mathbf{g}}$ at y is the linear form

$$y \circ \langle \, , \omega_{\mathbf{g}} \rangle = \langle \, , \mathrm{Pr}^{\alpha*}(\omega_{g_y^\circ}) \rangle : D(U_{g_y^*}^+) \to L_p$$

where $\mathrm{Pr}^{\alpha*}$ is the p-stabilization pull-back isomorphism defined in loc. cit. Note that this makes sense: since L_p is assumed to contain the N-th roots of unity, $D(U_{g_y^*}^+)$ and $D(V_{g_y^*}^+)$ are the same module up to a shift in their filtration.

- A homomorphism of $\Lambda_{\mathbf{g}}$-modules

$$\langle \, , \eta_{\mathbf{g}} \rangle : \mathbb{D}(\mathbb{V}_{\mathbf{g}^*}^-) \to \mathcal{Q}_{\mathbf{g}} \tag{19}$$

such that for every $y \in \mathcal{W}_{\mathbf{g}}^\circ$ we have

$$y \circ \langle \, , \eta_{\mathbf{g}} \rangle = \frac{\mathrm{Pr}^{\alpha*}(\eta_{g_y^\circ})}{\lambda_N(g_y^\circ)\mathcal{E}_0(g_y^\circ)\mathcal{E}_1(g_y^\circ)} : D(V_{g_y^*}^-) \to L_p,$$

where $\lambda_N(g_y^\circ)$ stands for the pseudo-eigenvalue of g_y°, and

$$\mathcal{E}_0(g_y^\circ) = 1 - \chi^{-1}(p)\beta_{g_y^\circ}^2 p^{1-k}, \quad \mathcal{E}_1(g_y^\circ) = 1 - \chi(p)\alpha_{g_y^\circ}^{-2} p^{k-2}.$$

2.2. Hida–Rankin's three-variable p-adic L-function

Let $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$, $\mathbf{h} \in \Lambda_{\mathbf{h}}[[q]]$ be a pair of Hida families of tame level N and tame characters χ_g, χ_h respectively. As in the introduction, set $\Lambda_{\mathbf{gh}} := \Lambda_{\mathbf{g}} \hat{\otimes}_{\mathbb{Z}_p} \Lambda_{\mathbf{h}} \hat{\otimes}_{\mathbb{Z}_p} \Lambda$ and $\mathcal{W}_{\mathbf{gh}} :=$ Spf$(\Lambda_{\mathbf{gh}})$. Let $\mathcal{Q}_{\mathbf{gh}}$ denote the fraction field of $\Lambda_{\mathbf{gh}}$, and set $\mathcal{W}_{\mathbf{gh}}^{\circ} := \mathcal{W}_{\mathbf{g}}^{\circ} \times \mathcal{W}_{\mathbf{h}}^{\circ} \times \mathcal{W}^{cl}$.

Definition 2.2. The *critical range* is the set of points $(y, z, \sigma) \in \mathcal{W}_{\mathbf{gh}}^{\circ}$ of weights (ℓ, m, s) such that $\ell, m \geq 2$ and $m \leq s < \ell$.

The *geometric range* is defined to be the set of points $(y, z, \sigma) \in \mathcal{W}_{\mathbf{gh}}^{\circ}$ of weights (ℓ, m, s) such that $\ell, m \geq 2$ and $1 \leq s < \min(\ell, m)$.

Hida [Hi1], [Hi2] constructed a three-variable p-adic Rankin L-function $L_p(\mathbf{g}, \mathbf{h})$ on $\mathcal{W}_{\mathbf{gh}}$, interpolating the algebraic parts of the critical values $L(g_y^{\circ}, h_z^{\circ}, s)$ for every triple of classical points (y, z, s) in $\mathcal{W}_{\mathbf{gh}}^{\circ}$ lying in the critical range. More precisely, [Hi2, Theorem 5.1d] asserts the following.

Theorem 2.3 (Hida). *There exists a unique element $L_p(\mathbf{g}, \mathbf{h}) \in \mathcal{Q}_{\mathbf{gh}}$ whose value at any $(y, z, s) \in \mathcal{W}_{\mathbf{gh}}^{\circ}$ of weights (ℓ, m, s) in the critical range is well-defined and equal to*

$$L_p(\mathbf{g}, \mathbf{h})(y, z, s) = \frac{C \cdot \mathcal{E}(y, z, s)}{(2\pi i)^{2s-m+1} \langle g_y^{\circ}, g_y^{\circ} \rangle} \times L(g_y^{\circ}, h_z^{\circ}, s)$$

where C is a non-zero algebraic number in the finite extension $\mathbb{Q}(g_y^{\circ}, h_z^{\circ})$ generated by the Fourier coefficients of g_y° and h_z°, $\langle g_y^{\circ}, g_y^{\circ} \rangle$ is the Petersson norm as normalized in loc. cit., and

$$\mathcal{E}(y, z, s) = \left(1 - \frac{p^{s-1}}{\alpha_{g_y^{\circ}} \alpha_{h_z^{\circ}}}\right)\left(1 - \frac{p^{s-1}}{\alpha_{g_y^{\circ}} \beta_{h_z^{\circ}}}\right)\left(1 - \frac{\beta_{g_y^{\circ}} \alpha_{h_z^{\circ}}}{p^s}\right)\left(1 - \frac{\beta_{g_y^{\circ}} \beta_{h_z^{\circ}}}{p^s}\right). \qquad (20)$$

Let g and h be classical specializations of the families $\mathbf{g}$ and $\mathbf{h}$ at some point $(y_0, z_0) \in \mathcal{W}_{\mathbf{g}}^{cl} \times \mathcal{W}_{\mathbf{h}}^{cl}$ of weight (ℓ, m). We denote by $L_p(g, h, s)$ the restriction of $L_p(\mathbf{g}, \mathbf{h})(y, z, s)$ to the line (y_0, z_0, s). As stated e.g. in [Das2, §9.2], this p-adic L-function satisfies the functional equation

$$L_p(g, h, \ell + m - 1 - s) = \epsilon(g, h, s) L_p(g^*, h^*, s) \qquad (21)$$

where $\epsilon(g, h, \psi, s) = A \cdot B^s$ with $A \in \mathbb{Q}(g_y, h_z)^{\times}$ and $B \in \mathbb{Q}^{\times}$.

2.3. Improved p-adic L-functions

As before, let $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$ be a Hida family of tame level N and tame character $\chi = \chi_g$, and set $\mathbf{h} = \mathbf{g}^* := \mathbf{g} \otimes \chi^{-1}$. As in the introduction, define the surface

$$\mathcal{S}_{\text{Hida}} := \mathcal{S}_{\ell, m, m} := \{(y, z, \sigma) \in \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W} : w(z) = \sigma\}. \qquad (22)$$

Note that this is a subvariety of $\mathcal{W}_{\mathbf{gg}^*}$ all of whose crystalline arithmetic points have weights (ℓ, m, m) for some $\ell, m \geq 1$.

The restriction to $\mathcal{S}_{\text{Hida}}$ of the *second multiplier* in the Euler-like factor (20) appearing in the interpolation formula for Hida–Rankin's p-adic L-function is

$$1 - \frac{p^{s-1}}{\alpha_{g_y^\circ}\beta_{h_z^\circ}} = 1 - \frac{\alpha_{g_z^\circ}}{\alpha_{g_y^\circ}}.$$

This expression interpolates to an Iwasawa function in $\Lambda_{\mathbf{g}} \times \Lambda_{\mathbf{g}}$, which by abuse of notation we continue to denote with the same symbol. One naturally expects $1 - \alpha_{g_z^\circ}/\alpha_{g_y^\circ}$ should divide the two-variable p-adic L-function $L_p(\mathbf{g}, \mathbf{g}^*)(y, z, s)$, where (y, z) vary in $\mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{g}}$ and $s = \mathrm{w}(z)$ is determined by the weight of z. Hida [Hi2] proved the following stronger statement:

Theorem 2.4 (Hida). *Let $\hat{L}_p(\mathbf{g}, \mathbf{g}^*)$ be the unique element in the fraction field of $\Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{g}}$ such that*

$$L_p(\mathbf{g}, \mathbf{g}^*)(y, z, \mathrm{w}(z)) = \left(1 - \frac{\alpha_{g_z^\circ}}{\alpha_{g_y^\circ}}\right) \cdot \hat{L}_p(\mathbf{g}, \mathbf{g}^*)(y, z).$$

Then, for fixed $y_0 \in \mathcal{W}_{\mathbf{g}}^\circ$, the one-variable meromorphic function

$$\hat{L}_p(\mathbf{g}, \mathbf{g}^*)(y_0, z)$$

on $\mathcal{W}_{\mathbf{g}}$ has a simple pole at $z = y_0$ whose residue is a non-zero explicit rational number.

Starting from the adjoint representation attached to any classical specialization $g = g_{y_0}$ of the Hida family $\mathbf{g}$ at some arithmetic point $y_0 \in \mathcal{W}_{\mathbf{g}}^{\text{cl}}$, Hida defined an analytic $\mathcal{L}$-invariant, which can be recast in several equivalent ways (cf. the works of Hida, Harron, Citro and Dasgupta (cf. [Hi3], [Ci], [Das2]). We may define it for instance as

$$\mathcal{L}(\mathrm{ad}^0(g_{y_0})) := \frac{-\alpha'_{\mathbf{g}}(y_0)}{\alpha_{\mathbf{g}}(y_0)}, \tag{23}$$

where recall $\alpha_{\mathbf{g}} = a_p(\mathbf{g}) \in \Lambda_{\mathbf{g}}$ is the Iwasawa function given by the eigenvalue of the Hecke operator U_p acting on $\mathbf{g}$, and $\alpha'_{\mathbf{g}}$ is its derivative.

Let $L'_p(\mathrm{ad}^0(g_{y_0}), s)$ denote Hida–Schmidt's p-adic L-function associated to the adjoint of the ordinary eigenform g_{y_0} (cf. [Sc], [Hi3]). The argument below is mainly due to Citro and Dasgupta, but since in their works they often assume that $\ell \geq 2$, we include it in order to ensure that it holds as well at weight 1, which is the case we mostly focus on. The main point is that the objects in play all vary in Hida families.

Proposition 2.5. *For a crystalline classical point $y_0 \in \mathcal{W}_{\mathbf{g}}^\circ$ of weight $\ell \geq 1$, we have*

$$\mathcal{L}(\mathrm{ad}^0(g_{y_0})) = L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, \ell) = L'_p(\mathrm{ad}^0(g_{y_0}), \ell),$$

up to a non-zero rational constant.

Proof. The first equality follows from Theorem 2.4, which amounts to saying that

$$L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, \ell) = \lim_{z \to y_0} \left(1 - \frac{\alpha_{g_z^\circ}}{\alpha_{g_{y_0}^\circ}}\right) \cdot \hat{L}_p(\mathbf{g}, \mathbf{g}^*)(y_0, z).$$

Since $1 - \alpha_{g_z}^{\circ}/\alpha_{g_{y_0}^{\circ}}$ vanishes at $z = y_0$ and $\hat{L}_p(\mathbf{g}, \mathbf{g}^*)(y_0, z)$ has a pole at $z = y_0$ given by a non-zero rational number, the value of the previous limit agrees, modulo $L^{\times}$, with the derivative of the first factor, i.e.,

$$L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, \ell) = \frac{-\alpha_{\mathbf{g}}'(y_0)}{\alpha_{\mathbf{g}}(y_0)}.$$

In addition, Dasgupta's factorization proved in [Das2] asserts that $L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, s)$ $= \zeta_p(s - \ell + 1)L_p(\mathrm{ad}^0(g_{y_0}), s)$. Here $\zeta_p(s)$ is the p-adic zeta function, which has a pole at $s = 1$ with non-zero rational residue. The second factor vanishes at $s = \ell$ and it follows that $L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, \ell) = L_p'(\mathrm{ad}^0(g_{y_0}), \ell) \pmod{\mathbb{Q}^{\times}}$.   □

3. Derived Beilinson–Flach elements

3.1. The three-variable Euler system of Kings, Lei, Loeffler and Zerbes

Let $\mathbf{g}$ and $\mathbf{h}$ be a pair of p-adic cuspidal Hida families of tame conductor N and tame nebentype χ_g and χ_h as in §2.1. As in the Introduction, and keeping the notations of the previous section, define the $\Lambda_{\mathbf{gh}}$-module

$$\mathbb{V}_{\mathbf{gh}} := \mathbb{V}_{\mathbf{g}} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{V}_{\mathbf{h}} \hat{\otimes}_{\mathbb{Z}_p} \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}). \tag{24}$$

This Λ-adic Galois representation is characterized by the property that for any $(y, z, \sigma) \in \mathcal{W}_{\mathbf{gh}}^{\circ}$ with $\mathrm{w}(\sigma) = \nu_s$ with $s \in \mathbb{Z}$, (24) specializes to

$$\mathbb{V}_{\mathbf{gh}}(y, z, \sigma) = V_{g_y} \otimes V_{h_z}(1 - s),$$

the $(1 - s)$-th Tate twist of the tensor product of the Galois representations attached to g_y and h_z.

Fix $c \in \mathbb{Z}_{>1}$ such that $(c, 6pN_gN_h) = 1$. [KLZ1, Theorem A] yields a three-variable Λ-adic global Galois cohomology class

$$\kappa^c(\mathbf{g}, \mathbf{h}) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{gh}})$$

that is referred to as the *Euler system* of Beilinson–Flach elements associated to $\mathbf{g}$ and $\mathbf{h}$. We denote by $\kappa_p^c(\mathbf{g}, \mathbf{h}) \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gh}})$ the image of $\kappa^c(\mathbf{g}, \mathbf{h})$ under the restriction map.

Since c is fixed throughout, we may sometimes drop it from the notation. This constant does make an appearance in fudge factors accounting for the interpolation properties satisfied by the Euler system, but in all cases we are interested in these fudge factors do not vanish and hence do not pose any problem for our purposes.

Given a crystalline arithmetic point $(y, z, s) \in \mathcal{W}_{\mathbf{gh}}^{\circ}$ of weights (ℓ, m, s), set for notational simplicity throughout this section $g = g_y^{\circ}$, $h = h_z^{\circ}$. With these notations, g_y (resp. h_z) is the p-stabilization of g (resp. h) with U_p-eigenvalue α_g (resp. α_h).

Define

$$\kappa(g_y, h_z, s) := \kappa(\mathbf{g}, \mathbf{h})(y, z, s) \in H^1(\mathbb{Q}, V_{g_y} \otimes V_{h_z}(1 - s)) \tag{25}$$

as the specialization of $\kappa(\mathbf{g}, \mathbf{h})$ at (y, z, s).

If one further assumes that (y, z, s) lies in the geometric range, Kings, Loeffler and Zerbes [KLZ2] showed that the cohomology group appearing in (25) also hosts a canonical *Rankin–Eisenstein class*, denoted

$$\mathrm{Eis}_{\mathrm{et}}^{[g,h,s]} \in H^1(\mathbb{Q}, V_g \otimes V_h(1-s)). \tag{26}$$

This class is attached to the classical pair (g, h) and can be constructed purely by geometric methods, without appealing to the variation of (g_y, h_z) in p-adic families. It is for this reason that in fact the classes $\mathrm{Eis}_{\mathrm{et}}^{[g,h,s]}$ lie in the Bloch–Kato Selmer subgroup

$$H_{\mathrm{f}}^1(\mathbb{Q}, V_g \otimes V_h(1-s)) \subset H^1(\mathbb{Q}, V_g \otimes V_h(1-s)).$$

We refer to [KLZ2, §5] and [KLZ1, Definition 3.3.2] for the precise statements.

Since both $\kappa(g_y, h_z, s)$ and $\mathrm{Eis}_{\mathrm{et}}^{[g,h,s]}$ live in the same space, it makes sense to ask whether they are related. This is the content of [KLZ1, Theorem A (8.1.3)]:

Theorem 3.1. *Assume* $(y, z, s) \in \mathcal{W}_{\mathbf{gh}}^{\circ}$ *lies in the geometric range. Then*

$$\kappa(g_y, h_z, s) = \mathcal{E}(g, h, s) \cdot \mathrm{Eis}_{\mathrm{et}}^{[g,h,s]} \tag{27}$$

where

$$\mathcal{E}(g, h, s) = \frac{\left(1 - \frac{p^{s-1}}{\alpha_g \alpha_h}\right)\left(1 - \frac{\alpha_g \beta_h}{p^s}\right)\left(1 - \frac{\beta_g \alpha_h}{p^s}\right)\left(1 - \frac{\beta_g \beta_h}{p^s}\right)(c^2 - c^{2s-\ell-m+2})}{(-1)^{s-1}(s-1)!\binom{\ell-2}{s-1}\binom{m-2}{s-1}}. \tag{28}$$

In particular $\kappa(g_y, h_z, s)$ *lies in* $H_{\mathrm{f}}^1(\mathbb{Q}, V_g \otimes V_h(1-s))$.

The following proposition recalls the existence of the so-called Perrin–Riou big logarithm, interpolating the Bloch–Kato logarithm $\log_{\mathrm{BK}}$ and dual exponential map $\exp_{\mathrm{BK}}^*$ associated to the classical specializations of a Λ-adic representation of $G_{\mathbb{Q}_p}$. We refer to [BK] and [Bel] for an introduction to p-adic Hodge theory and the definitions of these maps.

Recall the unramified character $\psi_{\mathbf{g}}$ of $G_{\mathbb{Q}_p}$ taking a Frobenius element Fr_p to $a_p(\mathbf{g})$, and as before, let $\varepsilon_{\mathbf{g}}$ be the composition of the cyclotomic character $\underline{\varepsilon}_{\mathrm{cyc}}$ with the natural inclusion $\Lambda^{\times} \subset \Lambda_{\mathbf{g}}^{\times}$. Define the $G_{\mathbb{Q}_p}$-subquotient

$$\mathbb{V}_{\mathbf{gh}}^{-+} := \mathbb{V}_{\mathbf{g}}^{-} \hat{\otimes} \mathbb{V}_{\mathbf{h}}^{+}$$

of $\mathbb{V}_{\mathbf{g}} \hat{\otimes} \mathbb{V}_{\mathbf{h}}$ of rank one over the two-variable Iwasawa algebra $\Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}}$. In light of (15), the Galois action on $\mathbb{V}_{\mathbf{gh}}^{-+}$ is given by the character

$$\eta_{\mathbf{h}}^{\mathbf{g}} := \varepsilon_{\mathrm{cyc}}^{-1} \chi_h \cdot \psi_{\mathbf{g}} \otimes \psi_{\mathbf{h}}^{-1} \varepsilon_{\mathbf{h}}. \tag{29}$$

It follows that $\mathbb{U}_{\mathbf{gh}}^{-+} := \mathbb{V}_{\mathbf{gh}}^{-+}(\varepsilon_{\mathrm{cyc}} \varepsilon_{\mathbf{h}}^{-1})$ is an unramified $G_{\mathbb{Q}_p}$-module and we can thus invoke its Λ-adic Dieudonné module as defined in (16).

Proposition 3.2 ([KLZ1, Theorem 8.2.8]). *There is an injective morphism of $\Lambda_{\mathbf{gh}}$-modules*

$$\mathcal{L}_{\mathbf{gh}}^{-+} : H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gh}}^{-+} \, \hat{\otimes} \, \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1})) \to \mathbb{D}(\mathbb{U}_{\mathbf{gh}}^{-+}) \, \hat{\otimes} \, \Lambda$$

such that for all $\kappa_p \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gh}}^{-+} \, \hat{\otimes} \, \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1}))$ and all $(y, z, s) \in \mathcal{W}_{\mathbf{gh}}^{\circ}$ of weights (ℓ, m, s):

- *if $s < m$, $(\mathcal{L}_{\mathbf{gh}}^{-+}(\kappa_p))_{y,z,s} = \left(1 - \frac{p^{s-1}}{\alpha_g \beta_h}\right)\left(1 - \frac{\alpha_g \beta_h}{p^s}\right)^{-1} \cdot \frac{(-1)^{m-s+1}}{(m-s+1)!} \cdot \log_{\mathrm{BK}}(\kappa_p(y, z, s))$;*
- *if $s \geq m$, $(\mathcal{L}_{\mathbf{gh}}^{-+}(\kappa_p))_{y,z,s} = \left(1 - \frac{p^{s-1}}{\alpha_g \beta_h}\right)\left(1 - \frac{\alpha_g \beta_h}{p^s}\right)^{-1}(s - m)! \cdot \exp_{\mathrm{BK}}^*(\kappa_p(y, z, s))$.*

Here, $\log_{\mathrm{BK}}$ and $\exp_{\mathrm{BK}}^*$ stand for the Bloch–Kato logarithm (resp. dual exponential) associated to the Dieudonné module of the p-adic representation $V_{g_y}^- \otimes U_{h_z}^+(1 - s)$.

Remark 3.3. In the case where $\alpha_g \beta_h = p^s$, we implicitly understand that the Euler factor in the denominator appears on the left hand side of the equality.

As shown in [KLZ1, Theorem 8.1.7], there is an injection

$$H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^- \, \hat{\otimes} \, \mathbb{V}_{\mathbf{h}}^+ \, \hat{\otimes} \, \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1})) \hookrightarrow H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^- \, \hat{\otimes} \, \mathbb{V}_{\mathbf{h}} \, \hat{\otimes} \, \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1})).$$

If we denote by $\kappa_p^{--}(\mathbf{g}, \mathbf{h})$ the projection of $\kappa_p(\mathbf{g}, \mathbf{h})$ to $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^- \hat{\otimes} \mathbb{V}_{\mathbf{h}}^- \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1}))$, it is further shown in loc. cit. that

$$\kappa_p^{--}(\mathbf{g}, \mathbf{h}) = 0. \tag{30}$$

As a consequence, the projection of $\kappa_p(\mathbf{g}, \mathbf{h})$ to $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^- \, \hat{\otimes} \, \mathbb{V}_{\mathbf{h}} \, \hat{\otimes} \, \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1}))$ actually lies in $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^- \, \hat{\otimes} \, \mathbb{V}_{\mathbf{h}}^+ \, \hat{\otimes} \, \Lambda(\varepsilon_{\mathrm{cyc}} \underline{\varepsilon}_{\mathrm{cyc}}^{-1}))$ and we may hence denote it $\kappa_p^{-+}(\mathbf{g}, \mathbf{h})$.

Let $\lambda_N(\mathbf{g})$ denote the Λ-adic pseudo-eigenvalue of $\mathbf{g}$ as defined in [KLZ1, §10], interpolating the Atkin–Lehner pseudo-eigenvalues of the classical specializations of $\mathbf{g}$. Recall from (18) and (19) Ohta's families of differential forms $\eta_{\mathbf{g}} \in \mathbb{D}(\mathbb{U}_{\mathbf{g}}^+)$ and $\omega_{\mathbf{h}} \in \mathbb{D}(\mathbb{V}_{\mathbf{h}}^-)$. It follows from the properties recalled in loc. cit. that there exists a homomorphism of $\Lambda_{\mathbf{gh}}$-modules

$$\langle \, , \eta_{\mathbf{g}} \otimes \omega_{\mathbf{h}} \rangle : \mathbb{D}(\mathbb{U}_{\mathbf{gh}}^{-+}) \, \hat{\otimes} \, \Lambda \to \mathcal{Q}_{\mathbf{gh}} \otimes \mathbb{Q}_p(\mu_N)$$

such that for all $\delta \in \mathbb{D}(\mathbb{U}_{\mathbf{gh}}^{-+}) \, \hat{\otimes} \, \Lambda$ and all $(y, z, s) \in \mathcal{W}_{\mathbf{gh}}^{\circ}$,

$$\nu_{y,z,s}(\langle \delta, \eta_{\mathbf{g}} \otimes \omega_{\mathbf{h}} \rangle) = \frac{1}{\lambda_N(g)\mathcal{E}_0(g)\mathcal{E}_1(g)} \cdot \langle \nu_{y,z,s}(\delta), \mathrm{Pr}^{\alpha*}(\eta_{g_y^{\circ}}) \otimes \mathrm{Pr}^{\alpha*}(\omega_{h_z^{\circ}}) \rangle$$

$$= \frac{1}{\lambda_N(g)\mathcal{E}_0(g)\mathcal{E}_1(g)} \cdot \langle \mathrm{Pr}_*^{\alpha}(\nu_{y,z,s}(\delta)), \eta_{g_y^{\circ}} \otimes \omega_{h_z^{\circ}} \rangle, \tag{31}$$

where, recall again,

$$\mathcal{E}_0(g) = 1 - \chi_g^{-1}(p)\beta_g^2 p^{1-\ell}, \quad \mathcal{E}_1(g) = 1 - \chi_g(p)\alpha_g^{-2}p^{\ell-2}.$$

The following explicit reciprocity law is [KLZ1, Theorem 10.2.2].

Theorem 3.4. *Define the Iwasawa function*

$$\mathcal{A}(\mathbf{g}, \mathbf{h}) := \lambda_N(\mathbf{g})^{-1}(-1)^s(c^2 - c^{-(\ell+m-2-2s)}\varepsilon_{\mathbf{g}}(c)^{-1}\varepsilon_{\mathbf{h}}(c)^{-1}) \tag{32}$$

in $\Lambda_{\mathbf{gh}}$. Then

$$\langle \mathcal{L}_{\mathbf{gh}}^{-+}(\kappa_p^{-+}(\mathbf{g}, \mathbf{h})), \eta_{\mathbf{g}} \otimes \omega_{\mathbf{h}} \rangle = \mathcal{A}(\mathbf{g}, \mathbf{h}) \cdot L_p(\mathbf{g}, \mathbf{h}). \tag{33}$$

3.2. *The self-dual case*

Let $\mathbf{g}$ be a Hida family of tame conductor N and tame nebentype χ, and set again $\mathbf{h} = \mathbf{g}^* = \mathbf{g} \otimes \bar{\chi}$. Define the curve

$$\mathcal{C} := \mathcal{C}_{\ell,\ell,\ell-1} = \{(y, z, \sigma) \in \mathcal{W}_{\mathbf{g}\mathbf{g}^*} : y = z, \ \alpha_{g_y} \neq \beta_{g_y}, \ \mathrm{w}(z) = \sigma \cdot \varepsilon_{\mathrm{cyc}}\}. \qquad (34)$$

Note that $\mathcal{C}$ is a finite cover of the line in $\mathcal{W}^3$ given as the set of regular points in the Zariski closure of the set of points of weights $(\ell, \ell, \ell - 1)$ for some $\ell \geq 1$.

Theorem 3.5. *The restriction of $\kappa_p(\mathbf{g}, \mathbf{g}^*)$ to $\mathcal{C}$ is zero.*

Proof. Recall firstly from (30) that $\kappa_p^{--}(\mathbf{g}, \mathbf{g}^*) = 0$. We also claim that $\kappa_p^{-+}(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}} = \kappa_p^{+-}(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}} = 0$. To see this, observe that the $\Lambda_{\mathbf{g}}[G_{\mathbb{Q}_p}]$-module $\mathbb{V}_{\mathbf{g}\mathbf{g}^*}^{-+} \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1})_{|\mathcal{C}}$ is isomorphic to $\Lambda_{\mathbf{g}}(1)$. This follows directly from (29) and (34), because $\frac{\alpha_{g_y^{\circ}}}{\alpha_{g_y^{\circ}}} \cdot \varepsilon_{\mathrm{cyc}}^{\ell-1} \cdot \varepsilon_{\mathrm{cyc}}^{2-\ell} = \varepsilon_{\mathrm{cyc}}$. Hence

$$H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}\mathbf{g}^*}^{-+} \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1})_{|\mathcal{C}}) \simeq H^1(\mathbb{Q}_p, \Lambda_{\mathbf{g}}(1))$$

$$\simeq H^1(\mathbb{Q}_p, \mathbb{Z}_p(1)) \hat{\otimes} \Lambda_{\mathbf{g}} \overset{(\mathrm{ord}_p, \log_p)}{\simeq} \Lambda_{\mathbf{g}} \oplus \Lambda_{\mathbf{g}} \qquad (35)$$

and $\kappa_p^{-+}(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}}$ vanishes if and only if infinitely many of its specializations are zero. But this is true for any crystalline classical point $(y, y, \ell - 1)$ on $\mathcal{C}$ with $\ell > 1$. Indeed, the factor $\mathcal{E}(g_y, h_z, s)$ of Theorem 3.1 vanishes, as $\alpha_g \beta_g = \chi(p) p^{\ell-1}$, and hence

$$\alpha_h = \alpha_g \cdot \chi^{-1}(p) = \frac{\alpha_g p^{\ell-1}}{\alpha_g \beta_g} = \frac{p^{\ell-1}}{\beta_g}. \qquad (36)$$

By (27) this shows that the specialization of the global cohomology class $\kappa(\mathbf{g}, \mathbf{g}^*)(y, y, \ell - 1)$ is zero for all $y \in \mathcal{W}_{\mathbf{g}}^{\circ}$ of weight $\ell > 1$, and a fortiori $\kappa_p^{-+}(\mathbf{g}, \mathbf{g}^*)(y, y, \ell - 1) = 0$. We conclude that $\kappa_p^{-+}(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}} = 0$ and likewise $\kappa_p^{+-}(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}} = 0$ by a symmetric reasoning.

Finally, note that there is an exact sequence

$$0 \to H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}\mathbf{g}^*}^{++} \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1})_{|\mathcal{C}}) \to H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}\mathbf{g}^*|\mathcal{C}})$$

$$\to H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}\mathbf{g}^*}/(\mathbb{V}_{\mathbf{g}\mathbf{g}^*}^{++} \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}))_{|\mathcal{C}}). \qquad (37)$$

The first map above is injective since $H^0(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}\mathbf{g}^*}/(\mathbb{V}_{\mathbf{g}\mathbf{g}^*}^{++} \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1}))_{|\mathcal{C}}) = 0$, as follows again from the description of $\mathbb{V}_{\mathbf{g}\mathbf{g}^*}^{-+}$ given in (29) (and similarly for $\mathbb{V}_{\mathbf{g}\mathbf{g}^*}^{+-}$ and $\mathbb{V}_{\mathbf{g}\mathbf{g}^*}^{--}$).

Since we have already shown that $\kappa_p^{--}(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}} = \kappa_p^{-+}(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}} = \kappa_p^{+-}(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}} = 0$, this implies that the image of $\kappa(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}}$ in the rightmost term of (37) vanishes. Hence, $\kappa(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}}$ lies in $H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{g}}^+ \hat{\otimes} \mathbb{V}_{\mathbf{g}^*}^+ \hat{\otimes} \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}^{-1})_{|\mathcal{C}})$. It follows from [KLZ1, Theorem 8.2.3, Remark 8.2.4] that the latter space is isomorphic to $\Lambda_{\mathbf{g}}$, and thus $\kappa(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}}$ is zero if and only if infinitely many of its specializations are, which is the case as already argued above. $\square$

3.3. A derived system of Beilinson–Flach elements

Keep the notations and assumptions as in previous sections. Theorem 3.5 above establishes the vanishing of the local cohomology class $\kappa_p(\mathbf{g}, \mathbf{g}^*)$ along $\mathcal{C}$ and it is thus natural to ask about the existence of a *derived* cohomology class $\kappa'_p(\mathbf{g}, \mathbf{g}^*)$ on a proper subspace of three-dimensional weight space $\mathcal{W}_{\mathbf{gg}^*}$ containing $\mathcal{C}$, bearing a reciprocity law with Hida–Rankin's improved p-adic L-function.

The purpose of this section is to make this construction explicit. Consider the surface

$$\mathcal{S} := \mathcal{S}_{\ell,m,m-1} = \{(y, z, \sigma) \in \mathcal{W}_{\mathbf{gg}^*} : \mathrm{w}(z) = \sigma \cdot \varepsilon_{\mathrm{cyc}}\},$$

which is a finite cover of the plane in $\mathcal{W}^3$ arising as the Zariski closure of points of weights $(\ell, m, m - 1)$ for some $\ell, m \geq 1$. Note that $\mathcal{S}$ obviously contains the curve $\mathcal{C}$.

Let $\mathbb{V}_{\mathbf{gg}^*|\mathcal{S}}$ denote the restriction of $\mathbb{V}_{\mathbf{gg}^*}$ to the surface $\mathcal{S}$, and let $\mathbb{V}_{\mathbf{gg}^*|\mathcal{C}}$ denote its restriction to $\mathcal{C}$. The following proposition establishes the existence of a class $\kappa'_p(\mathbf{g}, \mathbf{g}^*) \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{S}})$ that may be regarded as the derivative of $\kappa_p(\mathbf{g}, \mathbf{g}^*)$ along the z-direction.

We shrink the weight space $\mathcal{W}$ to a rigid-analytic open disk $\mathcal{U} \subset \mathcal{W}$ centered at 1 at which the finite cover $\mathrm{w} : \mathcal{W}_{\mathbf{g}} \to \mathcal{W}$ restricts to an isomorphism $\mathrm{w} : \mathcal{U}_{\mathbf{g}} \xrightarrow{\sim} \mathcal{U}$ with $y_0 \in \mathcal{U}_{\mathbf{g}}$. Let $\Lambda_{\mathcal{U}_{\mathbf{g}}} = \mathcal{O}(\mathcal{U}_{\mathbf{g}})$ denote the Iwasawa algebra of analytic functions on $\mathcal{U}_{\mathbf{g}}$ whose supremum norm is bounded by 1. Shrink likewise $\mathcal{C}$ and $\mathcal{S}$ so that projection to the weight space restricts to an isomorphism with $\mathcal{U}$ and $\mathcal{U} \times \mathcal{U}$ respectively. Then the associated Iwasawa algebras are respectively $\mathcal{O}(\mathcal{C}) = \Lambda_{\mathcal{U}_{\mathbf{g}}} \simeq \mathbb{Z}_p[[Z]]$ and $\mathcal{O}(\mathcal{S}) = \Lambda_{\mathcal{U}_{\mathbf{g}}} \hat{\otimes} \Lambda_{\mathcal{U}_{\mathbf{g}}} \simeq \mathbb{Z}_p[[Y, Z]]$. The isomorphism $\Lambda_{\mathcal{U}_{\mathbf{g}}} \simeq \mathbb{Z}_p[[Z]]$ is not canonical and depends on the choice of an element $\gamma \in \Lambda^\times_{\mathcal{U}_{\mathbf{g}}}$ which is sent to $1 + Z$.

Consider the short exact sequence of $\mathbb{Z}_p$-modules

$$0 \to \mathbb{Z}_p[[Y, Z]] \xrightarrow{\cdot(Z-Y)} \mathbb{Z}_p[[Y, Z]] \to \mathbb{Z}_p[[Z]] \to 0.$$

Under the above identifications the sequence may be recast as

$$0 \to \mathcal{O}_{\mathcal{S}} \xrightarrow{\delta} \mathcal{O}_{\mathcal{S}} \to \mathcal{O}_{\mathcal{C}} \to 0$$

with $\delta = 1 \otimes (\gamma - 1) - (\gamma - 1) \otimes 1$ in $\mathcal{O}_{\mathcal{S}} \simeq \Lambda_{\mathcal{U}_{\mathbf{g}}} \hat{\otimes} \Lambda_{\mathcal{U}_{\mathbf{g}}}$.

Proposition 3.6. *There exists a unique local class $\kappa'_{p,\gamma}(\mathbf{g}, \mathbf{g}^*) \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{S}})$ such that*

$$\kappa_p(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{S}} = \delta \cdot \kappa'_{p,\gamma}(\mathbf{g}, \mathbf{g}^*).$$

Proof. The short exact sequence of $G_{\mathbb{Q}_p}$-modules

$$0 \to \mathbb{V}_{\mathbf{gg}^*|\mathcal{S}} \xrightarrow{\delta} \mathbb{V}_{\mathbf{gg}^*|\mathcal{S}} \to \mathbb{V}_{\mathbf{gg}^*|\mathcal{C}} \to 0$$

gives rise to the long exact sequence

$$H^0(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{C}}) \to H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{S}}) \xrightarrow{\delta} H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{S}}) \to H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{C}}).$$

Since $H^0(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{C}}) = 0$ as already argued in the proof of Theorem 3.5, the vanishing of $\kappa_p(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{C}}$ implies the existence of a unique element $\kappa'_{p,\gamma}(\mathbf{g}, \mathbf{g}^*) \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{S}})$ satisfying the claim. □

We are interested in the restriction of $\kappa'_{p,\gamma}(\mathbf{g}, \mathbf{g}^*)$ to $\mathcal{C}$; although it depends on the choice of the topological generator γ, the class $\kappa'_p(\mathbf{g}, \mathbf{g}^*) \in H^1(\mathbb{Q}_p, \mathbb{V}_{\mathbf{gg}^*|\mathcal{C}})$ defined by

$$\kappa'_p(\mathbf{g}, \mathbf{g}^*) = \frac{\kappa'_{p,\gamma}(\mathbf{g}, \mathbf{g}^*)}{\log_p(\gamma)}$$

is independent of γ.

Since in our setting the Euler factor $1 - \alpha_{g^\circ_y}\beta_{h^\circ_z}/p^s$ appearing in Proposition 3.2 is equal to $1 - \alpha_{g^\circ_y}/\alpha_{g^\circ_z}$, it is natural to introduce a modified p-adic L-function on the surface $\mathcal{S}$, defined as

$$\tilde{L}_p(\mathbf{g}, \mathbf{g}^*)(y, z, s) = \left(1 - \frac{\alpha_{g^\circ_y}}{\alpha_{g^\circ_z}}\right) \times L_p(\mathbf{g}, \mathbf{g}^*)(y, z, s). \tag{38}$$

Fix a point $(y, z, s) \in \mathcal{W}^\circ_{\mathbf{gh}} \cap \mathcal{S}$. Set $L = \mathbb{Q}(g^\circ_y, h^\circ_z, \lambda_N(g^\circ_y))$ and let L_p denote the p-adic completion of L. Define

$$\log^{-+} : H^1(\mathbb{Q}_p, \mathbb{V}_{g_y} \otimes V_{h_z}(1-s)) \xrightarrow{\mathrm{pr}^{-+}} H^1(\mathbb{Q}_p, \mathbb{V}^-_{g_y} \otimes V^+_{h_z}(1-s)) \to L_p, \tag{39}$$

where the first map is the projection onto $H^1(\mathbb{Q}_p, \mathbb{V}^-_{g_y} \otimes V^+_{h_z}(1-s))$ and the last one is the composition of the Bloch–Kato logarithm with the pairing with the differential $\eta_{g^\circ_y} \otimes \omega_{h^\circ_z}$.

It follows from Theorems 3.2 and 3.4 that for all $(y, z, s) \in \mathcal{W}^\circ_{\mathbf{gh}} \cap \mathcal{S}$,

$$\tilde{L}_p(\mathbf{g}, \mathbf{g}^*)(y, z, s) = \left(1 - \frac{\alpha_{g^\circ_z}}{p\alpha_{g^\circ_y}}\right) \cdot \log^{-+}(\kappa_p(\mathbf{g}, \mathbf{g}^*)(y, z, s)), \tag{40}$$

up to multiplication by the c-factor we have described in Theorem 3.4 and which does not affect our discussion since we always work modulo $L^\times$.

Observe that the function $\tilde{L}_p(\mathbf{g}, \mathbf{g}^*)$ vanishes along the curve $\mathcal{C}$, so the restriction of its derivative to that line is exactly zero. Recall that $\tilde{L}_p(\mathbf{g}, \mathbf{g}^*)$ is a two-variable function, determined by the values of y and z, since the third variable σ is automatically determined by z. Hence,

$$\frac{\partial}{\partial y}\tilde{L}_p(\mathbf{g}, \mathbf{h}) + \frac{\partial}{\partial z}\tilde{L}_p(\mathbf{g}, \mathbf{h}) = 0.$$

Recall that $\alpha_{\mathbf{g}}$ is an analytic function defined over $\Lambda_{\mathbf{g}}$; we denote by $\alpha'_{\mathbf{g}}$ its derivative. We may consider as before the $\mathcal{L}$-invariant attached to the adjoint representation of $\mathbf{g}$,

$$\mathcal{L}(\mathrm{ad}^0(\mathbf{g})) = -\alpha'_{\mathbf{g}}/\alpha_{\mathbf{g}}.$$

Observe that its specializations at classical points agree with the definitions given before in (23).

Now, we can compute the partial derivatives of $\tilde{L}_p(\mathbf{g}, \mathbf{h})$ at a crystalline point $(y, y, \ell - 1)$ of the curve $\mathcal{C}$. Using (38), one gets

$$\frac{\partial}{\partial y} \tilde{L}_p(\mathbf{g}, \mathbf{h})(y, y, \ell - 1) = \mathcal{L}(\mathrm{ad}^0(g_y)) \cdot L_p(\mathbf{g}, \mathbf{g}^*)(y, y, \ell - 1). \tag{41}$$

Then, using (40), we first observe that $\kappa_p(\mathbf{g}, \mathbf{g}^*)(y, z, s)$ vanishes at $(y, y, \ell - 1)$ and its derivative in the z-direction is precisely the logarithm of the derived cohomology class we have previously computed in Proposition 3.6; hence,

$$\frac{\partial}{\partial z} \tilde{L}_p(\mathbf{g}, \mathbf{h})(y, y, \ell - 1) = (1 - p^{-1}) \cdot (\log^{-+}(\kappa_p'(\mathbf{g}, \mathbf{g}^*)(y, y, \ell - 1))). \tag{42}$$

We have thus proved the following result:

Theorem 3.7. *For any crystalline point* $(y, y, \ell - 1)$ *on* $\mathcal{C}$,

$$\mathcal{L}(\mathrm{ad}^0(g_y)) \cdot L_p(\mathbf{g}, \mathbf{g}^*)(y, y, \ell - 1) = \log^{-+}(\kappa_p'(\mathbf{g}, \mathbf{g}^*)(y, y, \ell - 1)) \ (\mathrm{mod}\ L^{\times}).$$

3.4. Weight one modular forms

Let $g \in S_1(N, \chi_g)$ and $h \in S_1(N, \chi_h)$ be two cuspidal eigenforms of weight one. Let V_g and V_h denote the Artin representations over a finite extension L of $\mathbb{Q}$ attached to g and h. Let α_g, β_g (resp. α_h, β_h) denote the roots of the p-th Hecke polynomial of g (resp. of h). We assume throughout that $\alpha_g \neq \beta_g$ and $\alpha_h \neq \beta_h$. We also assume L is large enough as specified in §2.1.

Definition 3.8. Let $\mathbf{g}$ and $\mathbf{h}$ be Hida families passing through p-stabilizations g_α, h_α of g, h at some point $(y_0, z_0) \in \mathcal{W}_{\mathbf{g}}^\circ \times \mathcal{W}_{\mathbf{h}}^\circ$ of weights $(1, 1)$. Define

$$\kappa(g_\alpha, h_\alpha) := \kappa(\mathbf{g}, \mathbf{h})(y_0, z_0, 0) \in H^1(\mathbb{Q}, V_{gh} \otimes L_p(1))$$

as the specialization of $\kappa(\mathbf{g}, \mathbf{h})$ at the point $(y_0, z_0, 0)$.

This procedure yields four a priori different global cohomology classes:

$$\kappa(g_\alpha, h_\alpha), \quad \kappa(g_\alpha, h_\beta), \quad \kappa(g_\beta, h_\alpha), \quad \kappa(g_\beta, h_\beta), \tag{43}$$

one for each choice of pair of roots of the p-th Hecke polynomials of g and h.

Given a p-adic representation V of $G_{\mathbb{Q}_p}$ with coefficients in $\mathbb{Q}_p$, Bloch and Kato introduced in [BK] a collection of subspaces of the local Galois cohomology group $H^1(\mathbb{Q}_p, V)$, denoted respectively

$$0 \subset H_e^1(\mathbb{Q}_p, V) \subset H_{\mathrm{f}}^1(\mathbb{Q}_p, V) \subset H_g^1(\mathbb{Q}_p, V) \subset H^1(\mathbb{Q}_p, V).$$

Definition 3.9. Let V be a representation of $G_{\mathbb{Q}}$ with coefficients in $\mathbb{Q}_p$. The group of classes that are de Rham at p (i.e. the restriction to $\mathbb{Q}_p$ lies in $H_g^1(\mathbb{Q}_p, V)$) and unramified at all primes $\ell \neq p$ is denoted as $H_{\mathrm{f}, p}^1(\mathbb{Q}, V)$.

The group of classes that are crystalline at p and unramified at any other prime $q \neq p$ is denoted as $H_{\mathrm{f}}^1(\mathbb{Q}, V)$.

Proposition 3.10. *The four classes in* (43) *lie in* $H^1_{\mathrm{f},p}(\mathbb{Q}, V_{gh} \otimes L_p(1))$.

Proof. The dimensions of $H^1_g(\mathbb{Q}_p, V_{gh}(1))$ and $H^1(\mathbb{Q}_p, V_{gh}(1))$ are equal, according to the discussion of [Da, Section 1.4]; hence, the classes are de Rham at p. Furthermore, the restriction of these classes to $\mathbb{Q}_q$ is 0 for $q \neq p$. In fact a stronger fact holds true: the local Λ-adic classes $\kappa_q(\mathbf{g}, \mathbf{h}) \in H^1(\mathbb{Q}_q, \mathbb{V}_{\mathbf{gh}})$ are 0 at all $q \neq p$. This can be argued for instance by fixing weights (ℓ, m, s) large enough so that for every triple $(\epsilon_1, \epsilon_2, \epsilon_3)$ of characters of arbitrary p-power conductor and for any point (x, y, z) above $(\nu_{\ell,\epsilon_1}, \nu_{m,\epsilon_2}, \nu_{s,\epsilon_3})$, $\mathbb{V}(\mathbf{g}, \mathbf{h})(x, y, z)$ contains no subquotient isomorphic to either $\mathbb{Q}_p$ or $\mathbb{Q}_p(1)$. It then follows from Tate's local Euler characteristic formula (cf. [Nek, 2.5]) that $H^1(\mathbb{Q}_q, \mathbb{V}_{\mathbf{gh}}(x, y, z)) = 0$. From this it follows that $H^1(\mathbb{Q}_q, \mathbb{V}_{\mathbf{gh}}) = 0$ arguing as in e.g. [KLZ1, Prop. 8.1.7 or Lemma 8.2.6]. $\square$

Recall that we let H denote the Galois extension of $\mathbb{Q}$ cut out by $V_g \otimes V_h$.

Proposition 3.11. *There are natural identifications*

$$
\begin{aligned}
H^1_{\mathrm{f}}(\mathbb{Q}, V_{gh} \otimes_L L_p(1)) &= (\mathcal{O}_H^\times \otimes_{\mathbb{Z}} V_{gh} \otimes_L L_p)^{G_{\mathbb{Q}}}, \\
H^1_{\mathrm{f},p}(\mathbb{Q}, V_{gh} \otimes_L L_p(1)) &= (\mathcal{O}_H[1/p]^\times \otimes_{\mathbb{Z}} V_{gh} \otimes_L L_p)^{G_{\mathbb{Q}}}.
\end{aligned}
\tag{44}
$$

Proof. This follows from the same arguments as in e.g. [Bel, Prop. 2.9]. $\square$

Since $\alpha_g \neq \beta_g$, V_g decomposes as a $G_{\mathbb{Q}_p}$-module as $V_g = V_g^{\alpha_g} \oplus V_g^{\beta_g}$, where $V_g^{\alpha_g}$ and $V_g^{\beta_g}$ are the $G_{\mathbb{Q}_p}$-invariant lines on which Fr_p acts with eigenvalue α_g and β_g respectively. We similarly have $V_h = V_h^{\alpha_h} \oplus V_h^{\beta_h}$, and we may define

$$
V_{gh}^{\alpha\alpha} := V_g^{\alpha_g} \otimes V_h^{\alpha_h}, \quad \ldots, \quad V_{gh}^{\beta\beta} := V_g^{\beta_g} \otimes V_h^{\beta_h}.
$$

Note that these four $G_{\mathbb{Q}_p}$-invariant lines in V_{gh} are linearly independent even though some of the eigenvalues $\alpha_g\alpha_h$, $\alpha_g\beta_h$, $\beta_g\alpha_h$, $\beta_g\beta_h$ might be equal. Hence there is a decomposition of $G_{\mathbb{Q}_p}$-modules

$$
V_{gh} := V_g \otimes V_h = V_{gh}^{\alpha\alpha} \oplus \cdots \oplus V_{gh}^{\beta\beta}.
\tag{45}
$$

It follows that the local class $\kappa_p(g_\alpha, h_\alpha)$ may be decomposed as

$$
\kappa_p(g_\alpha, h_\alpha) = \kappa_p^{++}(g_\alpha, h_\alpha) + \kappa_p^{+-}(g_\alpha, h_\alpha) + \kappa_p^{-+}(g_\alpha, h_\alpha) + \kappa_p^{++}(g_\alpha, h_\alpha),
$$

where

$$
\begin{aligned}
\kappa_p^{++}(g_\alpha, h_\alpha) &\in H^1(\mathbb{Q}_p, V_{gh}^{\beta\beta} \otimes L_p(1)), \quad \kappa_p^{+-}(g_\alpha, h_\alpha) \in H^1(\mathbb{Q}_p, V_{gh}^{\beta\alpha} \otimes L_p(1)), \\
\kappa_p^{-+}(g_\alpha, h_\alpha) &\in H^1(\mathbb{Q}_p, V_{gh}^{\alpha\beta} \otimes L_p(1)), \quad \kappa_p^{--}(g_\alpha, h_\alpha) \in H^1(\mathbb{Q}_p, V_{gh}^{\alpha\alpha} \otimes L_p(1)).
\end{aligned}
\tag{46}
$$

As before, we are specially interested in the case where $h = g^* = g \otimes \chi^{-1}$, and we impose on g the assumptions (H1)–(H3) listed in the introduction; if we denote by $\{\alpha, \beta\}$ the p-th Hecke eigenvalues of g, the p-th Hecke eigenvalues of h are $\{1/\beta, 1/\alpha\}$.

Let $\mathbf{g}$ and $\mathbf{h} = \mathbf{g}^* := \mathbf{g} \otimes \chi^{-1}$ denote the Hida families over $\mathcal{W}_{\mathbf{g}} = \mathcal{W}_{\mathbf{h}}$ passing through g_α and $(g_\alpha)^* = (g^*)_{1/\beta}$ respectively at some point $y_0 \in \mathcal{W}_{\mathbf{g}}$ in the weight space. Thanks to our running assumptions, the main theorem of [BeDi] ensures that the weight map $\mathcal{W}_{\mathbf{g}} \to \mathcal{W}$ is étale at the point associated to g_α. It is therefore possible to fix an open subset in $\mathcal{W}_{\mathbf{g}}$ around g_α on which the weight map is an isomorphism. This way we are entitled to work under the simplifying assumptions posed in §3.2 and the results in loc. cit. and §3.3 may be applied.

Proposition 3.12. *The global cohomology classes* $\kappa(g_\alpha, g^*_{1/\beta})$ *and* $\kappa(g_\beta, g^*_{1/\alpha})$ *are zero.*

Proof. The restriction map

$$\mathrm{res}_p : H^1(\mathbb{Q}, V_{gh} \otimes L_p(1)) \to H^1(\mathbb{Q}_p, V_{gh} \otimes L_p(1))$$

is injective. This follows from [Bel, Proposition 2.12], which asserts that there are natural isomorphisms

$$H^1(\mathbb{Q}, V_{gh} \otimes L_p(1)) \simeq (H^\times \otimes V_{gh})^{G_\mathbb{Q}}, \quad H^1(\mathbb{Q}_p, V_{gh} \otimes L_p(1)) \simeq (H_p^\times \otimes V_{gh})^{G_{\mathbb{Q}_p}},$$

and hence the restriction map corresponds to the natural inclusion $H \hookrightarrow H_p$. From Theorem 3.5, it follows that $\kappa_p(g_\alpha, g^*_{1/\beta})$ and $\kappa_p(g_\beta, g^*_{1/\alpha})$ are both zero, and the result follows. $\square$

Define the curve

$$\mathcal{D} := \{(y, z, \sigma) \in \mathcal{W}_{\mathbf{gg}^*} : y = y_0, \ \mathrm{w}(z) = \sigma \cdot \varepsilon_{\mathrm{cyc}}\} \subset \mathcal{S}.$$

Proposition 3.13. *There exists a unique global class* $\kappa'_\gamma(\mathbf{g}, \mathbf{g}^*) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{gg}^*|\mathcal{D}})$ *such that*

$$\kappa'_\gamma(\mathbf{g}, \mathbf{g}^*)_{|\mathcal{D}} = (\gamma - 1) \cdot \kappa'_\gamma(\mathbf{g}, \mathbf{g}^*).$$

Proof. The short exact sequence of $G_\mathbb{Q}$-modules

$$0 \to \mathbb{V}_{\mathbf{gg}^*|\mathcal{D}} \xrightarrow{\delta} \mathbb{V}_{\mathbf{gg}^*|\mathcal{D}} \to V_{gh} \otimes L_p(1) \to 0$$

gives rise to the long exact sequence

$$H^0(\mathbb{Q}, V_{gh} \otimes L_p(1)) \to H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{gg}^*|\mathcal{D}}) \xrightarrow{\delta} H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{gg}^*|\mathcal{D}}) \to H^1(\mathbb{Q}, V_{gh} \otimes L_p(1)).$$

Since $H^0(\mathbb{Q}, V_{gh} \otimes L_p(1)) = 0$, the vanishing of $\kappa(g_\alpha, g^*_{1/\beta})$ proved in Proposition 3.12 implies the existence of a unique element $\kappa'_\gamma(\mathbf{g}, \mathbf{g}^*) \in H^1(\mathbb{Q}, \mathbb{V}_{\mathbf{gg}^*|\mathcal{D}})$ satisfying the claim. $\square$

Remark 3.14. The *global* cohomology class in Proposition 3.13 only makes sense along the curve $\mathcal{D}$. However, the *local* cohomology class constructed in Proposition 3.6 exists along the whole surface $\mathcal{S}$. Hence one cannot define the latter on the surface $\mathcal{S}$ as the restriction at p of the former, although this is indeed true after restricting to $\mathcal{D}$, a fact that we shall apply right below.

Set

$$\kappa'(g_\alpha, g_{1/\beta}^*) = \frac{1}{\log_p(\gamma)} \kappa'_\gamma(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 0). \tag{47}$$

Since the construction of this class coincides with the one performed in the previous section once we localize at p and restrict to the curve $\mathcal{D}$, Theorem 3.7 applies and we deduce that

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) \cdot L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 0) = \log^{-+}(\kappa'_p(g_\alpha, g_{1/\beta}^*)) \pmod{L^\times}. \tag{48}$$

Recall from (21) that $L_p(\mathbf{g}, \mathbf{g}^*)$ satisfies a functional equation relating the values at $s = 0$ and $s = 1$ up to a simple non-zero rational constant. Together with Proposition 2.5 this implies that

$$L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 1)^2 = \mathcal{L}(\mathrm{ad}^0(g_{y_0}))^2 = \log^{-+}(\kappa'_p(g_\alpha, g_{1/\beta}^*)) \pmod{L^\times}. \tag{49}$$

This formula is the key input for deriving Theorem B, the second main result of this note.

4. Derivatives of Fourier coefficients via Galois deformation theory

As in previous sections, let $g \in S_1(N, \chi)$ satisfy the hypothesis of the introduction. Let

$$\varrho_g : \mathrm{Gal}(H_g/\mathbb{Q}) \hookrightarrow \mathrm{GL}(V_g) \simeq \mathrm{GL}_2(L),$$
$$\varrho_{\mathrm{ad}^0(g)} : \mathrm{Gal}(H/\mathbb{Q}) \hookrightarrow \mathrm{GL}(\mathrm{ad}^0(g)) \simeq \mathrm{GL}_3(L)$$

denote the Artin representations associated to g and its adjoint, respectively. Here L is a finite extension of $\mathbb{Q}$, and $H_g \supseteq H$ denote the finite Galois extensions of $\mathbb{Q}$ cut out by these representations. Let $\mathcal{P}$ denote the set of primes of H lying above p, and fix once for all a prime $\wp \in \mathcal{P}$, thus determining an embedding $H \subset H_p \subset \bar{\mathbb{Q}}_p$ of H into its completion H_p at $\wp$, and an arithmetic Frobenius $\mathrm{Fr}_p \in \mathrm{Gal}(H_p/\mathbb{Q}_p)$.

As in [DLR4], the regularity assumptions we have imposed on g imply by e.g. [Das1, Prop. 3.2.2] that

$$\dim_L (\mathcal{O}_H^\times \otimes V_{gg^*})^{G_\mathbb{Q}} = 1, \quad \dim_L (\mathcal{O}_H[1/p]^\times \otimes V_{gg^*})^{G_\mathbb{Q}} = 3,$$

and thus

$$\dim_L (\mathcal{O}_H[1/p]^\times/p^\mathbb{Z} \otimes V_{gg^*})^{G_\mathbb{Q}} = \dim_L (\mathcal{O}_H[1/p]^\times/p^\mathbb{Z} \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}} = 2.$$

Fix two linearly independent global cohomology classes

$$u \in H^1_{\mathrm{f}}(\mathbb{Q}, V_{gg^*} \otimes L_p(1)), \quad v \in H^1_{\mathrm{f},p}(\mathbb{Q}, V_{gg^*} \otimes L_p(1))$$

that under the identifications provided by Proposition 3.11 project to a basis of the two-dimensional space $(\mathcal{O}_H[1/p]^\times/p^\mathbb{Z} \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}$. By a slight abuse of notation, we continue to denote the resulting elements

$$u \in (\mathcal{O}_H^\times \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}, \quad v \in (\mathcal{O}_H[1/p]^\times/p^\mathbb{Z} \otimes \mathrm{ad}^0(g))^{G_\mathbb{Q}}.$$

Recall from (45) that V_{gh} admits a natural decomposition as $G_{\mathbb{Q}_p}$-module as the direct sum of the four different lines $V_{gg^*}^{\alpha\alpha}, \ldots, V_{gg^*}^{\beta\beta}$. Since $\mathrm{ad}^0(V_g)$ is the quotient of V_{gg^*} by the trivial representation, (45) descends to a decomposition of $\mathrm{ad}^0(V_g)$ as $G_{\mathbb{Q}_p}$-module as

$$\mathrm{ad}^0(g) = \mathrm{ad}^0(g)^1 \oplus \mathrm{ad}^0(g)^{\alpha\otimes\bar\beta} \oplus \mathrm{ad}^0(g)^{\beta\otimes\bar\alpha} = L \cdot e_1 \oplus L \cdot e_{\alpha\otimes\bar\beta} \oplus L \cdot e_{\beta\otimes\bar\alpha},$$

where $\mathrm{Fr}_p(e_1) = e_1$, $\mathrm{Fr}_p(e_{\alpha\otimes\bar\beta}) = \frac{\alpha}{\beta} \cdot e_{\alpha\otimes\bar\beta}$, $\mathrm{Fr}_p(e_{\beta\otimes\bar\alpha}) = \frac{\beta}{\alpha} \cdot e_{\beta\otimes\bar\alpha}$. Note that $\alpha/\beta \neq 1$ thanks to the regularity assumption. It could be that $\alpha = -\beta$ and hence $\alpha/\beta = \beta/\alpha = -1$, but the above decomposition is still available as explained in (45).

Restriction to the decomposition group at p allows us to regard u and v as elements in $H^1(\mathbb{Q}_p, \mathrm{ad}^0(g) \otimes L_p(1)) = (H_p^\times \otimes \mathrm{ad}^0(g))^{G_{\mathbb{Q}_p}}$, and as such we may write them as

$$\begin{aligned}
u &= u_1 \otimes e_1 + u_{\alpha\otimes\bar\beta} \otimes e_{\alpha\otimes\bar\beta} + u_{\beta\otimes\bar\alpha} \otimes e_{\beta\otimes\bar\alpha}, \\
v &= v_1 \otimes e_1 + v_{\alpha\otimes\bar\beta} \otimes e_{\alpha\otimes\bar\beta} + v_{\beta\otimes\bar\alpha} \otimes e_{\beta\otimes\bar\alpha},
\end{aligned} \tag{50}$$

where $u_1, v_1, u_{\alpha\otimes\bar\beta}, v_{\alpha\otimes\bar\beta}, u_{\beta\otimes\bar\alpha}, v_{\beta\otimes\bar\alpha} \in H_p^\times$ satisfy

$$\mathrm{Fr}_p(u_1) = u_1, \quad \mathrm{Fr}_p(u_{\alpha\otimes\bar\beta}) = \frac{\beta}{\alpha} \cdot u_{\alpha\otimes\bar\beta}, \quad \mathrm{Fr}_p(u_{\beta\otimes\bar\alpha}) = \frac{\alpha}{\beta} \cdot u_{\beta\otimes\bar\alpha},$$

and similarly for v.

We can now provide the last step in our proof of Theorems A, A′ and B in the introduction. As shown in Corollary 2.5, Theorem A may be reduced to the computation of the derivative of the Fourier coefficient $a_p(\mathbf{g})$ at y_0, where $\mathbf{g}$ stands for the unique Hida family passing through g_α.

Let $\tilde\rho_g : G_{\mathbb{Q}} \to \mathrm{GL}_2(L_p[\varepsilon])$ be the unique first order α-ordinary deformation of ρ_g such that

$$\det \tilde\rho_g = \chi_g(1 + \log_p \chi_{\mathrm{cyc}} \cdot \varepsilon).$$

This representation, whose existence follows from [BeDi], satisfies

$$\tilde\rho_g = (1 + \varepsilon \cdot \kappa_g) \cdot \rho_g$$

for some cohomology class $\kappa_g : G_{\mathbb{Q}} \to \mathrm{ad}(\rho)$. Considering a diagonal basis for the Frobenius action (where we take the first vector to have eigenvalue α), the matrix form of $\kappa(\sigma)$ can be expressed as

$$\kappa(\sigma) = \begin{pmatrix} \kappa_1(\sigma) & \kappa_2(\sigma) \\ \kappa_3(\sigma) & \kappa_4(\sigma) \end{pmatrix}.$$

We denote by $\tilde\rho_{g,p}$ the restriction of $\tilde\rho_g$ to the decomposition group at p; in the same way, the restriction of κ to the decomposition group at p is denoted by κ_p, and similarly we denote by $\kappa_{i,p}$ the restriction of κ_i to $G_{\mathbb{Q}_p}$.

In [BeDi, Lemmas 2.3 and 2.5] the authors determine the tangent space to a deformation problem which can be seen to be equivalent to ours. They conclude that there is a

natural bijection between this tangent space and a certain subspace of $H^1(\mathbb{Q}, \mathrm{ad}(\rho_g))$; in this case, it consists of those classes κ whose matrix representation satisfies

$$\kappa_{3,p}(\sigma) = 0, \qquad \kappa_{1,p}(\sigma)|_{I_p} = 0, \tag{51}$$

where I_p is the inertia group at p. In [DLR3], the restriction of κ to the inertia group at p was determined, and the identifications of class field theory allow us to extend this to the whole decomposition group. A similar setting, where also a related deformation problem arises, is exploited in [BDP] to treat the case of weight one Eisenstein points.

Let V_g^α be the étale subspace on which the action of the Frobenius is unramified for the Λ-adic representation. Restricting to the decomposition group at p, we find that

$$\tilde{\rho}_{g,p}|_{V_g^\alpha} = (1 + \varepsilon \cdot \kappa_{1,p}) \cdot \alpha_g,$$

where $\kappa_{1,p} \in H^1(\mathbb{Q}_p, \mathrm{Hom}(V_g^\alpha, V_g^\alpha))$. If g_α' stands for the derivative of **g** evaluated at y_0, then

$$a_p(g_\alpha) + \varepsilon \cdot a_p(g_\alpha') = \alpha_g + \varepsilon \cdot \kappa_{1,p}(\mathrm{Fr}_p) \cdot \alpha_g,$$

and consequently we have the following.

Proposition 4.1. *Let* **g** *be the Hida family through* g_α. *Then*

$$a_p(g_\alpha') = \kappa_{1,p}(\mathrm{Fr}_p) \ (\mathrm{mod} \ L^\times).$$

Taking into account the identifications provided by class field theory, one can make $\kappa_1(\mathrm{Fr}_p)$ explicit.

From [BeDi, Section 3.2], there is an exact sequence

$$0 \to \mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}((\mathcal{O}_H \otimes \mathbb{Q}_p)^\times, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathcal{O}_H^\times \otimes \mathbb{Q}_p, \bar{\mathbb{Q}}_p).$$

Similarly, one has another exact sequence

$$0 \to \mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}((H \otimes \mathbb{Q}_p)^\times, \bar{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}_p, \bar{\mathbb{Q}}_p).$$

Consequently, we have identifications

$$H^1(\mathbb{Q}, \bar{\mathbb{Q}}_p) \otimes \mathrm{ad}(\rho) \simeq (H^1(H, \bar{\mathbb{Q}}_p) \otimes \mathrm{ad}(\rho))^{G_\mathbb{Q}} \simeq (\mathrm{Hom}(G_H, \bar{\mathbb{Q}}_p) \otimes \mathrm{ad}(\rho))^{G_\mathbb{Q}},$$

and this corresponds to the subspace of homomorphisms of

$$(\mathrm{Hom}((H \otimes \mathbb{Q}_p)^\times, \bar{\mathbb{Q}}_p) \otimes \mathrm{ad}(\rho))^{G_\mathbb{Q}}$$

vanishing at $\mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}_p$.

We now recall some results which allow us to determine each of the $\kappa_{i,p}$. Since there are isomorphisms

$$(\mathcal{O}_H \otimes \mathbb{Q}_p)^\times \simeq \prod_{\mathfrak{q} \in \mathcal{P}} \mathcal{O}_{H_\mathfrak{q}}^\times, \qquad (H \otimes \mathbb{Q}_p)^\times \simeq \prod_{\mathfrak{q} \in \mathcal{P}} H_\mathfrak{q}^\times,$$

we may write elements in $(\mathcal{O}_H \otimes \mathbb{Q}_p)^\times$ (or $(H \otimes \mathbb{Q}_p)^\times$) as tuples $(x_i)_{i \in \mathcal{P}}$. The action of the Galois group $G = \mathrm{Gal}(H/\mathbb{Q})$ is transitive on $\mathcal{P}$, so any Galois equivariant homomorphism from $(H \otimes \mathbb{Q}_p)^\times$ (resp. $(\mathcal{O}_H \otimes \mathbb{Q}_p)^\times$) to $\bar{\mathbb{Q}}_p$ is completely determined by its values on $H_p^\times$ (resp. $\mathcal{O}_{H_p}^\times$).

From class field theory, one has two distinguished elements in $H^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p)$:

(1) The class κ_{nr}, which is the unique homomorphism

$$\kappa_{\mathrm{nr}} \in \mathrm{Hom}(\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p), \bar{\mathbb{Q}}_p)$$

taking Fr_p to 1.

(2) The restriction to $G_{\mathbb{Q}_p}$ of the logarithm of the cyclotomic character

$$\kappa_{\mathrm{cyc}} := \log_p(\varepsilon_{\mathrm{cyc}}),$$

which gives a ramified element of $H^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p)$.

Furthermore, $H^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p)$ is identified with $\mathrm{Hom}(G_{H_p}, \bar{\mathbb{Q}}_p)^{\mathrm{Gal}(H_p/\mathbb{Q}_p)}$ as explained above. In the latter space, we denote by κ_{nr} the morphism that takes Fr_p to 1, and by κ_{cyc} the ramified element defined by $\log_p(\varepsilon_{\mathrm{cyc}})$.

Now we can explicitly determine the element $\kappa \in H^1(\mathbb{Q}, \mathrm{ad}(\rho))$, by constructing a homomorphism

$$\Phi_g(x) : (H \otimes \mathbb{Q}_p)^\times \to H_p \otimes \mathrm{ad}(\rho)$$

corresponding to κ via the previous identifications. In particular, it vanishes when evaluated at the basis $\{u, v\}$ of units for the adjoint. Moreover, $\Phi_g(\pi_p^{-1})$ leaves invariant the one-dimensional space V_g^α, where π_p^{-1} is the idèle which is equal to the inverse of a local uniformizer of H_p for the fixed prime $\wp$ above p, and to 1 everywhere else. The eigenvalue for the action of $\Phi_g(\pi_p^{-1})$ on that subspace is precisely $\kappa_1(\mathrm{Fr}_p)$.

Let $u_g^\times$ be any generator of $(\mathcal{O}_H^\times \otimes \mathrm{ad}(\rho))^{G_\mathbb{Q}}$, and let $v_g^\times$ be any element of the space $(\mathcal{O}_H[1/p]^\times \otimes \mathrm{ad}(\rho))^{G_\mathbb{Q}}$ such that $\{u_g^\times, v_g^\times\}$ is a basis of $(\mathcal{O}_H[1/p]^\times/p^\mathbb{Z} \otimes \mathrm{ad}(\rho))^{G_\mathbb{Q}}$. Let

$$u_g := (\log_p \otimes \mathrm{Id})(u_g^\times), \quad v_g := (\log_p \otimes \mathrm{Id})(v_g^\times), \quad \tilde{v}_g := (\mathrm{ord}_p \otimes \mathrm{Id})(v_g^\times) \quad \in H_p \otimes \mathrm{ad}(\rho).$$

Consider the element

$$A_g \in (H_p \otimes \mathrm{ad}(\rho))^{G_{\mathbb{Q}_p}},$$

which on an eigenbasis for the Frobenius takes the form

$$A_g : \begin{pmatrix} 0 & \frac{\log_p(u_1)}{\log_p(u_{\alpha \otimes \bar\beta})} \\ 0 & 1 \end{pmatrix}.$$

Consider also $J \in (H_p \otimes \mathrm{ad}(\rho))^{G_{\mathbb{Q}_p}}$, which in the same basis is expressed as

$$J = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where $a, b, c \in H_p$. These choices give rise to the $G_{\mathbb{Q}_p}$-equivariant homomorphism $\Phi_g : H^\times \to H_p \otimes \mathrm{ad}(\rho)$ given by the rule

$$\Phi_g(x) := \sum_{\sigma \in G} \log_p(^\sigma x) \cdot (\sigma^{-1} \cdot A_g) + \sum_{\sigma \in G} \mathrm{ord}_p(^\sigma x) \cdot (\sigma^{-1} \cdot J). \tag{52}$$

Here, $\sigma^{-1} \cdot A_g$ denotes the action of σ^{-1} by conjugation on the second factor of the tensor product, $\mathrm{ad}(\rho)$; then, this homomorphism can be extended to the whole $(H \otimes \mathbb{Q}_p)^{\times}$. The aim is to determine a suitable J such that $\Phi_g(x)$ corresponds to κ via class field theory. The following result explains the behavior of the terms coming from the log-part.

Proposition 4.2. *Let* $\mathrm{ad}(\rho)^{\mathrm{ord}} := \mathrm{Hom}(V_g / V_g^{\alpha}, V_g)$. *The homomorphism* Φ_g *vanishes on* $\mathcal{O}_H^{\times} \otimes \mathbb{Q}_p$, *and* $\Phi_g(x) \subset H_p \otimes \mathrm{ad}(\rho)^{\mathrm{ord}}$ *for any* x *of the form* $(x_p, 1, \ldots, 1)$, *where* $x_p \in \mathcal{O}_{H_p}^{\times}$. *Moreover,* $\Phi_g(x)$ *fixes* V_g^{α} *for all* x *of the form* $(x_p, 1, \ldots, 1)$ *with* $x_p \in H_p^{\times}$.

Proof. This follows from [DLR3, Lemma 1.6], where Φ_g is defined in the same way but without the ord-terms. It is clear that the behavior at the $\mathcal{O}_H$-units is not affected by the presence of these extra terms. The last part of the statement follows from the definition of A_g and J. $\qquad\square$

To determine the remaining parameters of the matrix J (a, b and c), we first observe that the homomorphism Φ_g must vanish identically on $\mathcal{O}_H[1/p]^{\times} \otimes \mathbb{Q}_p$. The endomorphism v_g is represented by a matrix of the form

$$v_g : \begin{pmatrix} \log_p(v_1) & \log_p(v_{\beta \otimes \bar{\alpha}}) \\ \log_p(v_{\alpha \otimes \bar{\beta}}) & -\log_p(v_1) \end{pmatrix},$$

and hence

$$\mathrm{Tr}(A_g v_g) = \frac{\log_p(u_1)\log_p(v_{\alpha \otimes \bar{\beta}}) - \log_p(u_{\alpha \otimes \bar{\beta}})\log_p(v_1)}{\log_p(u_{\alpha \otimes \bar{\beta}})}. \tag{53}$$

Furthermore, the restrictions of both $\kappa_{1,p}$ and $\kappa_{4,p}$ to the decomposition group at p belong to

$$H^1(\mathbb{Q}_p, \bar{\mathbb{Q}}_p) \simeq H^1(H_p, \bar{\mathbb{Q}}_p)^{G_{\mathbb{Q}_p}};$$

this space is two-dimensional and it is generated by κ_{cyc} and κ_{nr}; in the same way, $\kappa_{2,p}$ is a cohomology class in the one-dimensional space

$$H^1(H_p, \bar{\mathbb{Q}}_p(\beta/\alpha))^{G_{\mathbb{Q}_p}}.$$

It is clear that

$$\kappa_1|_{I_p} = 0, \quad \kappa_2|_{I_p} = \frac{\log_p(u_1)}{\log_p(u_{\alpha \otimes \bar{\beta}})}\kappa_{\mathrm{cyc}}, \quad \kappa_4|_{I_p} = \kappa_{\mathrm{cyc}}.$$

Then, to determine the restriction of the cohomology classes to the whole decomposition group, we impose these three conditions:

(1) The fact that the trace of the representation is prescribed (and is equal to κ_{cyc}) forces that

$$\kappa_1 = \lambda \cdot \kappa_{\mathrm{nr}}, \quad \kappa_4 = \kappa_{\mathrm{cyc}} - \lambda \cdot \kappa_{\mathrm{nr}}.$$

(2) The fact that $H^1(H_p, \bar{\mathbb{Q}}_p(\beta/\alpha))^{G_{\mathbb{Q}_p}}$ is one-dimensional (since $\alpha \neq \beta$) yields

$$\kappa_2 = \frac{\log_p(u_1)}{\log_p(u_{\alpha\otimes\bar{\beta}})}\kappa_{\text{cyc}}$$

(i.e., there is no contribution coming from the Frobenius).

(3) The remaining parameter λ can be determined by considering the associated matrix J giving rise to the endomorphism $\Phi_g(x)$. This is the content of the following proposition.

Proposition 4.3. *There exists a unique λ such that the homomorphism $\Phi_g(x)$ vanishes identically on $\mathcal{O}_H[1/p]^\times \otimes \mathbb{Q}_p$.*

Proof. Picking $w \in \mathcal{O}_H[1/p]^\times$ and an arbitrary $B \in \text{ad}(\rho)$, it is enough to see that

$$\text{Tr}(\Phi_g(w) \cdot B) = 0,$$

due to the non-degeneracy of the H_p-valued trace pairing on $H_\wp \otimes \text{ad}(\rho)$. Set

$$w_g^\times := \sum_{\sigma \in G} {}^\sigma w \otimes (\sigma \cdot B) \in (\mathcal{O}_H^\times \otimes \text{ad}(\rho))^{G_{\mathbb{Q}}}.$$

Observe that $w_g^\times$ can be expressed in terms of the basis $u_g^\times$ and $v_g^\times$. In particular, let

$$w_g := (\log_p \otimes \text{Id})(w_g^\times) = \lambda \cdot u_g + \mu \cdot v_g, \qquad \tilde{w}_g := (\text{ord}_p \otimes \text{Id})(w_g^\times) = \mu \cdot \tilde{v}_g.$$

Then

$$\text{Tr}(\Phi_g(w) \cdot B) = \lambda \cdot \text{Tr}(A_g \cdot u_g) + \mu \cdot \text{Tr}(A_g v_g - J \cdot \tilde{v}_g).$$

Observe that

$$J = \begin{pmatrix} -\lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

and from (53) one sees that

$$\lambda = \frac{\log_p(u_1)\log_p(v_{\alpha\otimes\bar{\beta}}) - \log_p(v_1)\log_p(u_{\alpha\otimes\bar{\beta}})}{2 \cdot \log_p(u_{\alpha\otimes\bar{\beta}}) \cdot \text{ord}_p(v_1)}. \qquad \square$$

Hence, we conclude that

$$\kappa_1 = -\frac{\log_p(u_1)\log_p(v_{\alpha\otimes\bar{\beta}}) - \log_p(v_1)\log_p(u_{\alpha\otimes\bar{\beta}})}{2 \cdot \log_p(u_{\alpha\otimes\bar{\beta}}) \cdot \text{ord}_p(v_1)} \cdot \kappa_{\text{nr}},$$

and evaluating at Fr_p we finally obtain the formula we anticipated below:

$$a_p(g_\alpha') = \frac{\log_p(u_1)\log_p(v_{\alpha\otimes\bar{\beta}}) - \log_p(v_1)\log_p(v_{\alpha\otimes\bar{\beta}})}{\log_p(u_{\alpha\otimes\bar{\beta}})} \pmod{L^\times}, \qquad (54)$$

as claimed.

5. Proof of main results

5.1. Proof of Theorems A and A′

Recall that by [DLR4, Proposition 4.2], the p-adic iterated integral of the statement agrees, up to multiplication by a scalar in $L^\times$, with the special value $L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 1)$. Further, applying the relation between $L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 1)$ and $a_p(g'_\alpha)$ as described in Corollary 2.5, we find that

$$L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 1) = L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 0) = \mathcal{L}(\mathrm{ad}^0(g_\alpha)) = a_p(g'_\alpha) \ (\mathrm{mod} \ L^\times).$$

The derivative of the Fourier coefficient was computed in the previous section, and it follows from (54) that

$$L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 1) = \frac{\log_p(u_1) \log_p(v_{\alpha\otimes\bar\beta}) - \log_p(v_1) \log_p(u_{\alpha\otimes\bar\beta})}{\log_p(u_{\alpha\otimes\bar\beta})} \ (\mathrm{mod} \ L^\times). \quad (55)$$

This proves Theorems A and A′.

5.2. Proof of Theorem B

We may combine (49) with the above result to deduce that

$$\log^{-+}(\kappa'_p(g_\alpha, g^*_{1/\beta})) = \left(\frac{\log_p(u_1) \log_p(v_{\alpha\otimes\bar\beta}) - \log_p(u_{\alpha\otimes\bar\beta}) \log_p(v_1)}{\log_p(u_{\alpha\otimes\bar\beta})}\right)^2 \ (\mathrm{mod} \ L^\times).$$

Recall that in (39) we defined the map $\log^{-+}$ as the composition of the Perrin–Riou big logarithm of Proposition 3.2 specialized at weight $(y_0, y_0, 0)$ and the pairing with the class $\eta_{g_\alpha} \otimes \omega_{g^*_{1/\beta}}$ introduced in §2.1. These differential classes satisfy

$$\langle \eta_{g_\alpha}, \omega_{g^*_{1/\beta}} \rangle = \frac{1}{\lambda_N(g_\alpha)\mathcal{E}_0(g_\alpha)\mathcal{E}_1(g_\alpha)} \in L^\times$$

under the perfect pairing $D(V_g) \times D(V_{g^*}) \to L_p$. We may take decompositions of V_g and V_{g^*} as $G_{\mathbb{Q}_p}$-modules

$$V_g = L \cdot e^g_\alpha \oplus L \cdot e^g_\beta, \qquad V_{g^*} = L \cdot e^{g^*}_{1/\alpha} \oplus L \cdot e^{g^*}_{1/\beta},$$

respectively, where $\{e^g_\alpha, e^g_\beta\}$ and $\{e^{g^*}_{1/\alpha}, e^{g^*}_{1/\beta}\}$ are bases of V_g and V_{g^*}, dual to each other, and compatible with the choice of the basis for the tensor product V_{gg^*} considered in the previous section. As explained in [DR1, §2], one may define p-adic periods

$$\Xi_{g_\alpha} \in H_p^{\mathrm{Fr}_p=\beta^{-1}}, \qquad \Omega_{g^*_{1/\beta}} \in H_p^{\mathrm{Fr}_p=\beta} \quad (56)$$

satisfying

$$\Xi_{g_\alpha} \otimes e^g_\beta = \eta_{g_\alpha}, \qquad \Omega_{g^*_{1/\beta}} \otimes e^{g^*}_{1/\beta} = \omega_{g^*_{1/\beta}}.$$

The natural pairing between $D(V_g) = (H_p \otimes V_g)^{G_{\mathbb{Q}_p}}$ and $D(V_{g^*}) = (H_p \otimes V_{g^*})^{G_{\mathbb{Q}_p}}$ induces a duality between V_g and V_{g^*}, and hence the quantity

$$\langle \eta_{g_\alpha}, \omega_{g^*_{1/\beta}} \rangle = \Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\beta}} \cdot \langle e^g_\beta, e^{g^*}_{1/\beta} \rangle = \Xi_{g_\alpha} \cdot \Omega_{g^*_{1/\beta}}$$

belongs to $L^\times$. Consequently, the class

$$\kappa_\circ = \frac{\log_p(u_1)\log_p(v_{\alpha\otimes\bar\beta}) - \log_p(u_{\alpha\otimes\bar\beta})\log_p(v_1)}{\log_p(u_{\alpha\otimes\bar\beta})^2} \times \big(\log_p(v_{\alpha\otimes\bar\beta})u - \log_p(u_{\alpha\otimes\bar\beta})v\big)$$

satisfies $\log^{-+}(\kappa'_p(g_\alpha, g^*_{1/\beta})) = \log^{-+}(\mathrm{res}_p(\kappa_\circ))$.

We may write the cohomology class $\kappa'(g_\alpha, g^*_{1/\beta})$ as a linear combination

$$\kappa'(g_\alpha, g^*_{1/\beta}) = a \cdot u + b \cdot v + c \cdot p,$$

where $a, b, c \in L_p$. The condition for an element to lie in the kernel of the map $\log^{-+}$ is

$$a \cdot \log_p(u_{\alpha\otimes\bar\beta}) + b \cdot \log_p(v_{\alpha\otimes\bar\beta}) = 0.$$

Hence,

$$\kappa'(g_\alpha, g^*_{1/\beta}) = \kappa_\circ + \lambda\big(\log_p(v_{\alpha\otimes\bar\beta}) \cdot u - \log_p(u_{\alpha\otimes\bar\beta}) \cdot v\big) + \mu \cdot p.$$

Consider now the map

$$\log^{--} : H^1(\mathbb{Q}_p, V_{gh}(1)) \xrightarrow{\mathrm{pr}^{--}} H^1(\mathbb{Q}_p, V^{--}_{gh}(1)) \simeq H^1(\mathbb{Q}_p, \mathbb{Q}_p(\alpha/\beta)(1)) \xrightarrow{\log_{\mathrm{BK}}} L_p.$$

According to (30), the class $\kappa'_p(g_\alpha, g^*_{1/\beta})$ also lies in the kernel of $\log^{--}$. Hence, taking equalities up to periods,

$$\begin{aligned}
\log^{--}&(\kappa'_p(g_\alpha, g^*_{1/\beta})) \\
&= \log^{--}(\mathrm{res}_p(\kappa_\circ)) + \lambda\big(\log_p(v_{\alpha\otimes\bar\beta})\log_p(u_1) - \log_p(u_{\alpha\otimes\bar\beta})\log_p(v_1)\big) \\
&= \lambda\big(\log_p(v_{\alpha\otimes\bar\beta})\log_p(u_1) - \log_p(u_{\alpha\otimes\bar\beta})\log_p(v_1)\big) \\
&= \lambda \cdot \log_p(u_{\alpha\otimes\bar\beta}) \cdot L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 1) = 0.
\end{aligned}$$

The assumption $\mathcal{L}(\mathrm{ad}^0(g_\alpha)) \neq 0$ in Theorem B implies that $L_p(\mathbf{g}, \mathbf{g}^*)(y_0, y_0, 1) \neq 0$ by the first display in this section. Hence $\lambda = 0$ and Theorem B follows.

5.3. Non-vanishing of the $\mathcal{L}$-invariant

We conclude this article by noting that the non-vanishing of the $\mathcal{L}$-invariant can be proved in most dihedral cases, because the expression (55) simplifies considerably. Indeed, let K be a real or imaginary quadratic field of discriminant D and let $\psi : G_K \to L^\times$ be a finite

order character of conductor $\mathfrak{c} \subset \mathcal{O}_K$ (and of mixed signature at the two archimedean places if K is real). Then the theta series $g = \theta(\psi)$ attached to ψ is an eigenform of weight 1, level $N_g = |D| \cdot \mathbf{N}_{K/\mathbb{Q}}(\mathfrak{c})$ and nebentype $\chi_g = \chi_K \chi_\psi$, where χ_K is the quadratic character associated to $K/\mathbb{Q}$ and χ_ψ is the central character of ψ.

Let ψ' denote the $\mathrm{Gal}(K/\mathbb{Q})$-conjugate of ψ defined by the rule $\psi'(\sigma) = \psi(\sigma_0 \sigma \sigma_0^{-1})$ for any choice of $\sigma_0 \in \Gamma_\mathbb{Q} \setminus G_K$. If $\psi \neq \psi'$ then g is cuspidal.

Let $p \nmid N_g$ be a prime number and fix an embedding $\bar{\mathbb{Q}} \subset \bar{\mathbb{Q}}_p$. In line with the introduction, suppose that hypotheses (H1)–(H3) are fulfilled. This amounts to requiring that:

(i) $\psi \neq \psi' \pmod{p}$, so that g is cuspidal even residually at p.
(ii) If K is imaginary and $p = \wp\bar{\wp}$ splits in K, then $\psi(\wp) \neq \psi'(\wp) \pmod{p}$.
(iii) If K is real, p does not split. If K is imaginary and the field $H = \bar{K}^{\ker(\psi)}$ cut out by ψ has Galois group $\mathrm{Gal}(H/\mathbb{Q}) = D_4$, the dihedral group of order 8, then p does not split in the single real quadratic field contained in H.

Proposition 5.1. *If K is imaginary with p split, or K is real with p inert, then*

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) = \log_p(v_1) \pmod{L^\times}. \tag{57}$$

In particular $\mathcal{L}(\mathrm{ad}^0(g_\alpha)) \neq 0$.

Proof. If K is real (and thus p remains inert in it) then u_1 is the norm of the fundamental unit of K and hence its p-adic logarithm vanishes. If K is imaginary and p splits, then v is a p-unit in $K^\times$ and hence $\log_p(v_{\alpha \otimes \bar{\beta}}) = 0$. $\square$

Note that Theorem B simplifies considerably in the setting of the above proposition. The simplest scenario where it is not a priori obvious that the $\mathcal{L}$-invariant is non-zero arises when K is imaginary but p remains inert. Fix a prime $\wp$ in H above p and set $\psi_{\mathrm{ad}} = \psi/\psi'$. Note that ψ_{ad} is a ring class character, regardless of whether ψ is so or not. Let $u_{\psi_{\mathrm{ad}}}$ (resp. $v_{\psi_{\mathrm{ad}}}$) denote any element spanning the 1-dimensional L-vector space

$$\mathcal{O}_H^\times[\psi_{\mathrm{ad}}] := \{x \in \mathcal{O}_H^\times \otimes L : \sigma(x) = \psi_{\mathrm{ad}}(\sigma)x, \ \sigma \in \mathrm{Gal}(H/K)\},$$

respectively $\frac{\mathcal{O}_H[1/\wp]^\times[\psi_{\mathrm{ad}}]}{\mathcal{O}_H^\times[\psi_{\mathrm{ad}}]}$. Set also $u'_{\psi_{\mathrm{ad}}} = \mathrm{Fr}_\wp u_{\psi_{\mathrm{ad}}}$ and $v'_{\psi_{\mathrm{ad}}} = \mathrm{Fr}_\wp v_{\psi_{\mathrm{ad}}}$. Then it is a straightforward computation to check that the $\mathcal{L}$-invariant appearing in Theorems A and A$'$ is

$$\mathcal{L}(\mathrm{ad}^0(g_\alpha)) = \frac{\log_\wp(u_{\psi_{\mathrm{ad}}})\log_\wp(v'_{\psi_{\mathrm{ad}}}) - \log_\wp(u'_{\psi_{\mathrm{ad}}})\log_\wp(v_{\psi_{\mathrm{ad}}})}{\log_\wp(u_{\psi_{\mathrm{ad}}}) - \log_\wp(u'_{\psi_{\mathrm{ad}}})} \pmod{L^\times}. \tag{58}$$

6. Darmon–Dasgupta units and factorization of p-adic L-functions

6.1. Darmon–Dasgupta units and Gross' conjecture

Let us place ourselves again in the setting of §5.3, where K is real and p remains inert in it. In this scenario Darmon and Dasgupta [DD] associated to the ring class character ψ_{ad} a local unit $v_{\mathrm{DD}}[\psi_{\mathrm{ad}}] \in K_p^\times$ and conjectured that $v_{\mathrm{DD}}[\psi_{\mathrm{ad}}]$ actually belongs

to $\mathcal{O}[1/\wp]^{\times}[\psi_{\mathrm{ad}}]$. The combination of [Park, Theorem 4.4] and Darmon–Dasgupta–Pollack's [DDP, Theorem 2] provides strong evidence for this conjecture, as putting these results together it follows that, in our notations,

$$\log_p(\mathbf{N}_{K_p/\mathbb{Q}_p}(v_{\mathrm{DD}}[\psi_{\mathrm{ad}}])) = \log_p(v_1) \ (\mathrm{mod}\ L^{\times}). \tag{59}$$

The above equality together with Theorems B and C yields a formula relating the derived Beilinson–Flach elements of this note to Darmon–Dasgupta units, much in the spirit of [BSV, Theorem A] and [Da, Theorem C] for diagonal cycles versus Stark–Heegner points. On account of the decomposition introduced in (50), the element $\kappa'(g_{\alpha}, g^*_{1/\beta})_1$ belongs to $\mathbb{Q}_p^{\times} \otimes L_p$.

Corollary 6.1. *Let $g = \theta(\psi)$ be the theta series associated to a finite order character ψ of mixed signature of a real quadratic field K. Let p be a prime that remains inert in K. Then*

$$\kappa'(g_{\alpha}, g^*_{1/\beta})_1 = \log_p(\mathbf{N}_{K_p/\mathbb{Q}_p}(v_{\mathrm{DD}}[\psi_{\mathrm{ad}}])) \cdot \mathbf{N}_{K_p/\mathbb{Q}_p}(v_{\mathrm{DD}}[\psi_{\mathrm{ad}}]) \ (\mathrm{mod}\ L^{\times}).$$

In spite of the ostensible parallelism between the above formula and [BSV, Theorem A] and [Da, Theorem C], the proof of Corollary 6.1 follows quite a different route from [BSV] and [Da]. The main reason is that in the last two references a factorization of p-adic L-functions was crucially exploited, which follows from a comparison of critical values.

In our setting one still expects to have an analogous factorization, but proving it appears to be far less trivial. Since this issue poses intriguing questions, and our results shed some light on them, we discuss it in more detail below.

6.2. Artin p-adic L-functions

Let $g = \theta(\psi) \in S_1(N, \chi)$ be a theta series of a quadratic field K and p be a prime which is split (resp. inert) if K is imaginary (resp. real). Keep the assumptions of Proposition 5.1, and set $h = g^* = \theta(\psi^{-1})$ and $\psi_{\mathrm{ad}} = \psi/\psi'$ as usual.

Let $L_p(g_{\alpha}, g^*_{1/\beta}, s)$ denote the cyclotomic p-adic Rankin–Hida L-function associated to the pair $(g_{\alpha}, g^*_{1/\beta})$ of p-stabilizations of g and h, as in §2.2. Note that this p-adic L-function has no critical points. Nevertheless, since

$$V_g \otimes V_{g^*} \simeq 1 \oplus \chi_K \oplus \mathrm{Ind}_{\mathbb{Q}}^K(\psi_{\mathrm{ad}}), \tag{60}$$

one may still wonder whether $L_p(g_{\alpha}, g^*_{1/\beta}, s)$ admits a factorization mirroring the one satisfied by its classical counterpart:

$$L(g, h, s) = \zeta(s) \cdot L(\chi_K, s) \cdot L(K, \psi_{\mathrm{ad}}, s). \tag{61}$$

While (61) follows directly from (60) by Artin formalism, a putative analogous factorization of $L_p(g_{\alpha}, g^*_{1/\beta}, s)$ is far less trivial. In the CM case one can prove it by a method which is nowadays standard, but rather deep as one needs firstly to extend $L_p(g_{\alpha}, g^*_{1/\beta}, s)$

to a two-variable p-adic L-function, prove a factorization in this scenario, and then invoke Gross's theorem [Gr]. Since we did not find it in the literature and the output is not precisely what one would naïvely expect,[2] we provide it below.

We say that an element $\mathfrak{f}$ in a finite algebra over $\Lambda^{\otimes n}$ for some $n \geq 0$ is an L-*rational fudge factor* if it is a rational function with coefficients in L which extends to an Iwasawa function with neither poles nor zeroes at crystalline classical points. Let also L_p^{Katz} denote Katz's p-adic L-function on the space of Hecke characters of an imaginary quadratic field. If ξ is one such character of K, we write $L_p^{\text{Katz}}(\xi, s) = L_p^{\text{Katz}}(\xi \cdot \mathbf{N}^s)$ where $\mathbf{N}$ stands for the Hecke character of infinity type $(1, 1)$ induced by the norm from K to $\mathbb{Q}$.

Theorem 6.2. *Assume K is imaginary and p splits in it. Then there exists an L-rational fudge factor $\mathfrak{f} \in \Lambda$ such that*

$$L_p(g, g^*, s) = \frac{\mathfrak{f}(s)}{\log_p(u_{\psi_{\text{ad}}})} \cdot \zeta_p(s) \cdot L_p(\chi_K \omega, s) \cdot L_p^{\text{Katz}}(\psi_{\text{ad}}, s).$$

Proof. We follow the notations and normalizations adopted in [DLR1, §3] and [DLR4, §4]. Fix a prime $\wp$ of K above p. Take a Hecke character λ with image in $\mathbb{Z}_p^\times$ of infinity type $(0, 1)$ and conductor $\bar{\wp}$. For every integer $\ell \geq 1$ define $\psi_{g, \ell-1}^{(p)} = \psi_g \langle \lambda \rangle^{\ell-1}$ and let $\psi_{g, \ell-1}$ be the Hecke character given by

$$\psi_{g, \ell-1}(\mathfrak{q}) = \begin{cases} \psi_{g, \ell-1}^{(p)}(\mathfrak{q}) & \text{if } \mathfrak{q} \neq \bar{\wp}, \\ \chi(p) p^{\ell-1} / \psi_{g, \ell-1}^{(p)}(\wp) & \text{if } \mathfrak{q} = \bar{\wp}. \end{cases}$$

As explained in [DLR1, Section 3.2], there is a p-adic family $\psi_{\mathbf{g}}$ of Hecke characters whose weight ℓ specialization is $\psi_{g, \ell-1}^{(p)}$ and such that the Hida family $\mathbf{g}$ passing through g_α satisfies $g_\ell^\circ = \theta(\psi_{g, \ell-1})$.

Given a pair (ℓ, s) of classical weights, define the Hecke characters

$$\Psi_{gh}(\ell, s) = \psi_{g, \ell-1}^{-1} \cdot \psi_g' \cdot \mathbf{N}^s, \qquad \Psi_{gh'}(\ell, s) = \psi_{g, \ell-1}^{-1} \cdot \psi_g \cdot \mathbf{N}^s, \qquad \Psi_g(\ell) = \psi_{g, \ell-1}^{-2} \chi \mathbf{N}^\ell.$$

All pairs (ℓ, s) such that $\ell > s \geq 1$ belong to the region of interpolation of both Rankin–Hida's p-adic L-function $L_p(\mathbf{g}, g_{1/\beta}^*, s)$ and Katz's p-adic L-functions $L_p^{\text{Katz}}(\psi_{\mathbf{g}h}(\ell, s))$ and $L_p^{\text{Katz}}(\psi_{\mathbf{g}h'}(\ell, s))$. At such critical pairs, it is readily verified that the following factorization of classical L-values holds up to an L-rational fudge factor:

$$L(g_\ell, h, s) = L(\psi_{gh}(\ell, s)^{-1}, 0) L(\psi_{gh'}(\ell, s)^{-1}, 0).$$

Using this identity, the same computations as in [DLR4, Theorem 4.2] show that there is an L-rational fudge factor $\mathfrak{f}(\ell, s) \in \Lambda^{\otimes 2}$ such that

$$L_p(\mathbf{g}, h)(\ell, s) \cdot L_p^{\text{Katz}}(\psi_g(\ell)) = \mathfrak{f}(\ell, s) \cdot L_p^{\text{Katz}}(\psi_{gh}(\ell, s)) \cdot L_p^{\text{Katz}}(\psi_{gh'}(\ell, s)). \quad (62)$$

---

[2] Indeed, the formula $L_p(g, g^*, s) = \mathfrak{f}(s) \cdot \zeta_p(s) \cdot L_p(\chi_K \omega, s) \cdot L_p^{\text{Katz}}(\psi_{\text{ad}} \cdot \mathbf{N}^s)$ is not correct, in spite of being the direct analogue of (61). For one thing, this formula would contradict Theorem A.

If we now restrict to $\ell = 1$ and invoke Katz's p-adic analogue of Kronecker's limit formula which asserts that $L_p^{\text{Katz}}(\psi) = \log_p(u_{\psi_{\text{ad}}}) \pmod{L^\times}$, it follows that

$$L_p(g, g^*, s) = \frac{\mathfrak{f}(s)}{\log_p(u_{\psi_{\text{ad}}})} \cdot L_p^{\text{Katz}}(\mathbf{N}^s) \cdot L_p^{\text{Katz}}(\psi_{\text{ad}}, s). \tag{63}$$

Finally, Gross's main theorem in [Gr] together with the functional equation for Kubota–Leopoldt's p-adic L-function asserts that

$$L_p^{\text{Katz}}(s) = \zeta_p(s) \cdot L_p(\chi_K \omega, s) \tag{64}$$

up to a rational fudge factor. This yields the theorem. $\square$

Assume now that $g = \theta(\psi)$ is the theta series of a character of a real quadratic field in which p is inert. In light of Theorem 6.2 it is natural to pose the following question:

Question 6.3. Assume K is real and p remains inert in K. Let u_K be a fundamental unit of K and let $L_p(\psi_{\text{ad}}\omega, s)$ denote the Deligne–Ribet p-adic L-function attached to $\psi_{\text{ad}}\omega$. Is it true that

$$L_p(g, g^*, s) \overset{?}{=} \frac{1}{\log_p(u_K)} \cdot \zeta_p(s) \cdot L_p(\chi_K, s) \cdot L_p(\psi_{\text{ad}}\omega, s) \tag{65}$$

up to an L-rational fudge factor?

Note that the results of this paper, combined with Darmon–Dasgupta–Pollack's [DDP, Theorem 2] prove that the above factorization holds when evaluated at $s = 0$ and $s = 1$. Indeed, Theorem A in this setting takes the simple form

$$L_p(g, g^*, 1) = \log_p(v_1) \pmod{L^\times},$$

while [DDP, Theorem 2] asserts that $L_p(\psi_{\text{ad}}\omega, s)$ vanishes at $s = 0$ and

$$L_p'(\psi_{\text{ad}}\omega, 0) = \log_p(v_1) \pmod{L^\times}.$$

Moreover, $\zeta_p(s)$ has a simple pole at $s = 0$ whose residue is a non-zero rational number, and Leopoldt's formula asserts that $L_p(\chi_K, 0) = \log_p(u_K) \pmod{L^\times}$. Putting all together shows that (65) is true at $s = 0$. The functional equations satisfied by each of the p-adic L-functions in play ensure that the same is true at $s = 1$. This of course falls short from establishing (65).

References

[AL]   Atkin, A. O. L., Li, W. C. W.: Twists of newforms and pseudo-eigenvalues of W-operators. Invent. Math. **48**, 221–243 (1978)   Zbl 0369.10016   MR 508986

[Bel]   Bellaiche, J.: An introduction to the conjecture of Bloch and Kato. http://people.brandeis.edu/~jbellaic

[BeDi]   Bellaïche, J., Dimitrov, M.: On the eigencurve at classical weight 1 points. Duke Math. J. **165**, 245–266 (2016)   Zbl 1404.11047   MR 3457673

[BDR]   Bertolini, M., Darmon, H., Rotger, V.: Beilinson–Flach elements and Euler systems II: the Birch–Swinnerton-Dyer conjecture for Hasse–Weil–Artin L-series. J. Algebraic Geom. **24**, 569–604 (2015)   Zbl 1328.11073   MR 3344765

[BDP]   Betina, A., Dimitrov, M., Pozzi, A.: On the Gross–Stark conjecture. Preprint

[BK]   Bloch, S., Kato, K.: L-functions and Tamagawa numbers of motives. In: The Grothendieck Festschrift, Vol. I, Progr. Math. 86, Birkhäuser Boston, Boston, MA, 333–400 (1990)   Zbl 0768.14001   MR 1086888

[CH]   Castellà, F., Hsieh, M.-L.: On the On the non-vanishing of generalized Kato classes for elliptic curves of rank two. Preprint

[Ci]   Citro, C.: $\mathscr{L}$-invariants of adjoint square Galois representations coming from modular forms. Int. Math. Res. Notices **2008**, art. rnn048, 12 pp.   Zbl 1218.11054   MR 2440330

[BSV]   Bertolini, M., Seveso, M., Venerucci, R.: Balanced diagonal classes and rational points on elliptic curves. In: Heegner Points, Stark–Heegner Points and Diagonal Classes (to appear)

[Da]   Darmon, H.: Heegner points, Stark–Heegner points, and values of L-series. In: Int. Congress of Mathematicians (Madrid, 2006), Vol. II, Eur. Math. Soc., Zürich, 313–345 (2006)   Zbl 1157.11023   MR 2275600

[DD]   Darmon, H., Dasgupta, S.: Elliptic units for real quadratic fields. Ann. of Math. (2) **163**, 301–346 (2006)   Zbl 1130.11030   MR 2195136

[DHRV]   Darmon, H., Harris, M., Rotger, V., Venkatesh, A.: Derived Hecke algebra for dihedral weight one forms. In progress

[DLR1]   Darmon, H., Lauder, A., Rotger, V.: Stark points and p-adic iterated integrals attached to modular forms of weight one. Forum Math. Pi **3**, art. e8, 95 pp. (2015)   Zbl 1392.11034   MR 3456180

[DLR2]   Darmon, H., Lauder, A., Rotger, V.: Overconvergent generalised eigenforms of weight one and class fields of real quadratic fields. Adv. Math. **283**, 130–142 (2015) Zbl 1393.11038   MR 3383798

[DLR3]   Darmon, H., Lauder, A., Rotger, V.: First order p-adic deformations of weight one newforms. In: L-functions and Automorphic Forms, Contrib. Math. Comput. Sci. 10, Springer, Cham, 39–80 (2017)   Zbl 1425.11082   MR 3931448

[DLR4]   Darmon, H., Lauder, A., Rotger, V.: Gross–Stark units and p-adic iterated integrals attached to modular forms of weight one. Ann. Math. Québec **40**, 325–354 (2016) Zbl 1417.11115   MR 3529185

[DR1]   Darmon, H., Rotger, V.: Elliptic curves of rank two and generalised Kato classes. Res. Math. Sci. **3**, art. 27, 32 pp. (2016)   Zbl 1416.11096   MR 3540085

[DR2]   Darmon, H., Rotger, V.: Diagonal cycles and Euler systems II: The Birch and Swinnerton-Dyer conjecture for Hasse–Weil–Artin L-functions. J. Amer. Math. Soc. **30**, 601–672 (2017)   Zbl 1397.11090   MR 3630084

[Das1]   Dasgupta, S.: Stark's Conjectures. Senior honors thesis, Harvard Univ. (1999)

[Das2]   Dasgupta, S.: Factorization of p-adic Rankin L-series. Invent. Math. **205**, 221–268 (2016) Zbl 1361.11076   MR 3514962

[DDP]   Dasgupta, S., Darmon, H., Pollack, R.: Hilbert modular forms and the Gross–Stark conjecture. Ann. of Math. (2) **174**, 439–484 (2011)   Zbl 1250.11099   MR 2811604

[GS]   Greenberg, R., Stevens, G.: On the conjecture of Mazur, Tate, and Teitelbaum. In: p-Adic Monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), Contemp. Math. 165, Amer. Math. Soc., Providence, RI, 183–211 (1994)   Zbl 0846.11030 MR 1279610

[Gr]   Gross, B. H.: On the factorization of p-adic L-series. Invent. Math. **57**, 83–95 (1980) Zbl 0472.12011   MR 564185

[Hi1]   Hida, H.: A p-adic measure attached to the zeta functions associated with two elliptic modular forms. I. Invent. Math. **79**, 159–195 (1985)   Zbl 0573.10020   MR 774534

[Hi2]   Hida, H.: A p-adic measure attached to the zeta functions associated with two elliptic modular forms. II. Ann. Inst. Fourier (Grenoble) **38**, no. 3, 1–83 (1988)   Zbl 0645.10028 MR 976685

[Hi3]   Hida, H.: Greenberg's $\mathscr{L}$-invariants of adjoint square Galois representations. Int. Math. Res. Notices **2004**, 3177–3189   Zbl 1081.11039   MR 2097038

[KLZ1]   Kings, G., Loeffler, D., Zerbes, S. L.: Rankin–Eisenstein classes and explicit reciprocity laws. Cambridge J. Math. **5**, 1–122 (2017)   Zbl 1428.11103   MR 3637653

[KLZ2]   Kings, G., Loeffler, D., Zerbes, S. L.: Rankin–Eisenstein classes for modular forms. Amer. J. Math. **142**, 79–138 (2020)   Zbl 07176031   MR 4060872

[LLZ]   Lei, A., Loeffler, D., Zerbes, S. L.: Euler systems for Rankin–Selberg convolutions of modular forms. Ann. of Math. (2) **180**, 653–771 (2014)   Zbl 1315.11044   MR 3224721

[LZ]   Loeffler, D., Zerbes, S. L.: Iwasawa theory for the symmetric square of a modular form. J. Reine Angew. Math. **752**, 179–210 (2019)   Zbl 1430.11149   MR 3975641

[MTT]   Mazur, B., Tate, J., Teitelbaum, J.: On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer. Invent. Math. **84**, 1–48 (1986)   Zbl 0699.14028   MR 830037

[Nek]   Nekovář, J.: p-adic Abel–Jacobi maps and p-adic heights. In: The Arithmetic and Geometry of Algebraic Cycles (Banff, AB, 1998), CRM Proc. Lecture Notes 24, Amer. Math. Soc., Providence, RI, 367–379 (2000)   Zbl 0983.14009   MR 1738867

[Och]   Ochiai, T.: A generalization of the Coleman map for Hida deformations. Amer. J. Math. **125**, 849–892 (2003)   Zbl 1057.11048   MR 1993743

[Oh]   Ohta, M.: Ordinary p-adic étale cohomology groups attached to towers of elliptic modular curves. II. Math. Ann. **318**, 557–583 (2000)   Zbl 0967.11016   MR 1800769

[Park]   Park, J.: The Darmon–Dasgupta units over genus fields and the Shimura correspondence. J. Number Theory **130**, 2610–2627 (2010)   Zbl 1206.11078   MR 2678865

[RR]   Rivero, O., Rotger, V.: Beilinson–Flach elements, Stark units and p-adic iterated integrals. Forum Math. **31**, 1517–1531 (2019)   Zbl 07143579   MR 4026467

[Sc]   Schmidt, C.-G.: p-adic measures attached to automorphic representations of GL(3). Invent. Math. **92**, 597–631 (1988)   Zbl 0656.10023   MR 939477

[Wi]   Wiles, A.: On ordinary λ-adic representations associated to modular forms. Invent. Math. **94**, 529–573 (1988)   Zbl 0664.10013   MR 969243