Vague and Fuzzy t-norms and t-conorms

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Abstract

t-norms and t-conorms are the natural connectives "and" and "or" in fuzzy logic. The unit interval with a t-norm or a t-conorm is a special monoid and some submonoids like discrete t-norms and t-conorms have been proved useful in many cases.

In the first part of this article these submonoids will be fuzzified to fuzzy t-subnorms and fuzzy t-subconorms in order to deal with imprecision. As particular examples we will provide fuzzifications of the classical and the Łukasiewicz three-valued conjunctions.

The second part of the article will define and study vague t-norms and t-conorms as fuzzy operations $\tilde{\circ} : [0,1]^3 \to [0,1]$ where $\tilde{\circ}(x,y,z)$ is the degree in which z is $T(x,y)$ ($S(x,y)$ respectively) where T (S) is a t-norm (t-conorm).

Keywords: t-norms, t-conorm, indistinguishability operator, fuzzy monoid, fuzzy t-subnorm, fuzzy t-subconorm, vague monoid, vague t-norm, vague t-conorm.

1 Introduction

t-norms and t-conorms are essential tools in fuzzy logic because they model the conjunction and disjunction as well as they allow us to evaluate the intersection and union of fuzzy subsets. They have been generalized to different algebraic structures such as Gödel algebras, MV-algebras, GL-monoids and complete residuated lattices. They have also been generalized to interval-valued, type-2, multiset and hesitant t-norms and t-conorms among others. Also discrete and finite-valued t-norms and t-conorms [9] can be seen as discrete submonoids of them.

There are situations where imprecision, lack of accuracy or noise have to be taken into account or must be added to the problems to be solved. In these cases, the relaxation or fuzzification of logic connectives can be useful to tackle these situations.

In this paper we will develop the fuzzification of t-norms and t-conorms from two perspectives:

1. Fuzzifying the concept of submonoid, so generalizing the idea of discrete and finite-valued t-norm and t-conorm.

2. Introducing vague t-norms and t-conorms, i.e.: considering a fuzzy operation compatible with them and with a given fuzzy equivalence relation.

Regarding the first point, there is a big amount of papers fuzzifying algebraic structures such as groups, rings, actions, [2, 11]. Following this direction, in the first part of the paper we will fuzzify the concept of submonoid and then apply it to the monoids $([0,1],T)$ and $([0,1],S)$, T and S a t-norm and a t-conorm respectively to obtain what we call T-fuzzy t-subnorms and T-fuzzy t-subconorms. They generalize discrete and finite-valued t-norms and t-conorms among others. For example the crisp conjunction $\wedge$ (considered a discrete t-norm as in Examples 3.6 and 3.13) can be fuzzified to $\overline{\wedge}$ by $x\overline{\wedge}y = \max(2x+2y-3,0)$ for all $x,y \in [0,1]$. Also in Example 3.14 Łukasiewicz three-valued conjunction is fuzzified.

The second part of the paper studies vague t-norms and t-conorms. A vague operation is a fuzzy operation $\tilde{\circ} : M \times M \times M \to [0,1]$ on a set M, where $\tilde{\circ}(x,y,z)$ is interpreted as the degree in which z is $x \circ y$, and imposing compatibility of $\tilde{\circ}$ with a given fuzzy equivalence relation on M [6] [1]. For this we first study general vague monoids and then apply our results to vague monoids on t-norms and t-conorms, called T-vague t-norms and t-conorms.

[1]The term vague refers to a vague algebraic structure as in [6] and has nothing to do with vague sets as in [7].

In particular, for a t-norm T we can generate a T-vague t-norm $\widetilde{T}$ where $\widetilde{T}(x, y, z)$ can be interpreted as the degree in which $T(x, y)$ is z and similarly for t-conorms.

An interesting result is that for a given t-norm T we have built a family $\{\widetilde{T_r}\}_{r>0}$ that allows us to relax or stress the degree in which we want to fuzzify T in the sense that if $r \to 0$ then $\widetilde{T_r}$ tends to T and if $r \to \infty$ then $\widetilde{T_r}$ tends to $\widetilde{T_\infty}(x, y, z) = 1$ for all $x, y, z \in [0, 1]$ meaning that $T(x, y)$ can be anything obtaining the greatest possible imprecision. Similar results are obtained for t-conorms.

There is a nice link between fuzzy t-subnorms (t-subconorms) and vague t-norms (vague t-conorms): The kernel of homomorphisms between vague t-norms (t-conorms) are fuzzy t-subnorms (t-subconorms).

2 Preliminaries

This section contains some definitions and properties related to t-norms and t-conorms that will be needed in the article. A good book on the topic is [8].

Definition 2.1. [8] *A t-norm is a binary operation* $T : [0, 1] \times [0, 1] \to [0, 1]$ *which satisfies the following properties for all* $x, y, z, t \in [0, 1]$

- $T(x, y) = T(y, x)$
- $T(x, y) \leq T(z, t)$ *if* $x \leq z$ *and* $y \leq t$
- $T(x, T(y, z)) = T(T(x, y), z)$
- $T(x, 1) = x$

Example 2.2.

- *The minimum t-norm* T_M *defined by* $T_M(x, y) = \min(x, y)$ *for all* $x, y \in [0, 1]$.
- *The t-norm* T_L *of Łukasiewicz defined by* $T_L(x, y) = \max(x + y - 1, 0)$.
- *The product t-norm* $T_P(x, y) = x \cdot y$.

Definition 2.3. [8] *A t-conorm is a binary operation* $S : [0, 1] \times [0, 1] \to [0, 1]$ *which satisfies the following properties for all* $x, y, z, t \in [0, 1]$

- $S(x, y) = S(y, x)$
- $S(x, y) \leq S(z, t)$ *if* $x \leq z$ *and* $y \leq t$
- $S(x, S(y, z)) = S(S(x, y), z)$
- $S(x, 0) = x$

Example 2.4.

- *The maximum t-conorm* S_M *defined by* $S_M(x, y) = \max(x, y)$ *for all* $x, y \in [0, 1]$.
- *The t-conorm* S_L *of Łukasiewicz defined by* $S_L(x, y) = \min(x + y, 1)$.
- *The probabilistic sum t-conorm* $S_P(x, y) = x + y - x \cdot y$.

Definition 2.5. *For a t-norm* T *(t-conorm* S*)* $x \in [0, 1]$ *is an idempotent element if and only if* $T(x, x) = x$ *(*$S(x, x) = x$*).* $E(T)$ *(*$E(S)$*) will be the set of idempotent elements of* T *(*S*).*

Definition 2.6. *A continuous t-norm* T *(t-conorm* S*) is Archimedean if and only if* $E(T) = \{0, 1\}$ *(*$E(S) = \{0, 1\}$*).*

Example 2.7. *The Łukasiewicz t-norm and t-conorm, the Product t-norm and the Probabilistic sum t-conorm are Archimedean t-norms, while the minimum t-norm and the maximum t-conorm are not.*

Definition 2.8. *For a t-norm* T *(t-conorm* S*),* $x \in [0, 1]$ *is nilpotent if and only if there exists* $n \in N$ *such that* $T^n(x) = 0$ *(*$S^n(x) = 1$*).* $Nil(T)$ *(*$Nil(S)$*) will be the set of nilpotent elements of* T *(*S*). (*T^n *(*S^n*) is defined recursively:* $T^n(x) = T(T^{n-1}(x), x)$ *(*$S^n(x) = S(S^{n-1}(x), x)$*)).*

Proposition 2.9. *If a t-norm* T *(t-conorm* S*) is continuous Archimedean, then* $Nil(T)$ *is* $[0, 1)$ *or* $\{0\}$ *(*$Nil(S)$ *is* $(0, 1]$ *or* $\{1\}$*). In the first case,* T *(*S*) is called nilpotent. In the second case it is called strict.*

Proposition 2.10 (Ling's Theorem)**.** *A continuous t-norm* T *is Archimedean if and only if there exists a continuous and strictly decreasing function* $t : [0, 1] \to [0, \infty]$ *with* $t(1) = 0$ *such that*

$$T(x, y) = t^{[-1]}(t(x) + t(y))$$

where $t^{[-1]}$ is the pseudo inverse of t, defined by

$$t^{[-1]}(x) = \begin{cases} t^{-1}(x) & \text{if } x \in [0, t(0)] \\ 0 & \text{otherwise.} \end{cases}$$

T is strict if $t(0) = \infty$ and nilpotent otherwise. t is called an additive generator of T and two generators of the same t-norm differ only by a positive multiplicative constant.

Example 2.11.

1. *$t(x) = 1 - x$ is an additive generator of the t-norm of Łukasiewicz.*

2. *$t(x) = -\log(x)$ is an additive generator of the Product t-norm.*

Proposition 2.12 (Ling's Theorem)**.** *A continuous t-conorm S is Archimedean if and only if there exists a continuous and strictly increasing function $s : [0,1] \to [0,\infty]$ with $s(0) = 0$ such that*

$$S(x,y) = s^{[-1]}(s(x) + s(y))$$

where $s^{[-1]}$ is the pseudo inverse of s, defined by

$$s^{[-1]}(x) = \begin{cases} s^{-1}(x) & \text{if } x \in [0, s(1)] \\ 1 & \text{otherwise.} \end{cases}$$

S is strict if $s(1) = \infty$ and nilpotent otherwise. s is called an additive generator of S and two generators of the same t-conorm differ only by a positive multiplicative constant.

Example 2.13.

1. *$s(x) = x$ is an additive generator of the t-conorm of Łukasiewicz.*

2. *$s(x) = -\log(1 - x)$ is an additive generator of the Probabilistic sum t-conorm.*

Definition 2.14. *A strong negation φ is defined as a strictly decreasing, continuous function $\varphi : [0,1] \to [0,1]$ with boundary conditions $\varphi(0) = 1$ and $\varphi(1) = 0$ such that φ is involutive (i.e., $\varphi(\varphi(x)) = x$ holds for any $x \in [0,1]$).*

Definition 2.15. *Let T be a t-norm and φ a strong negation. Then $S = \varphi \circ T \circ \varphi$ is the dual t-conorm of T with respect to φ. In this case (T, S, φ) is called a De Morgan triplet.*

3 Fuzzy t-norms and Fuzzy t-conorms

In this section we will introduce and study fuzzy t-subnorms and fuzzy t-subconorms. t-norms and t-conorms are special monoids and fuzzy t-subnorms and fuzzy t-subconorms will fuzzify their submonoids. We will start with the general definition of monoid, define T-fuzzy submonoid and then specify our results to t-norms and t-conorms. As examples of fuzzy t-subnorms and fuzzy t-subconorms we will fuzzify discrete t-norms (Definition 3.5) and in particular the crisp conjunction and the Łukasiewicz three-valued one.

Definition 3.1. *A monoid $(M, \circ)$ consists of a set M with a binary operation $\circ : M \times M \to M$ that*

- *is associative*
- *has an identity element.*

Definition 3.2. *A submonoid is a subset of a monoid closed by the operation and containing the identity element.*

Definition 3.3. *Let T be a t-norm, $(M, \circ)$ a monoid, e its identity element and μ a fuzzy subset of M. μ is a T-fuzzy submonoid of M if and only if*

- $T(\mu(x), \mu(y)) \leq \mu(x \circ y)\ \forall x, y \in M$.
- $\mu(e) = 1$.

Proposition 3.4. *Let $(M, \circ)$ be a monoid and μ a T-fuzzy submonoid of M. Then the core H of μ (i.e., the set of elements x of M such that $\mu(x) = 1$) is a submonoid of M.*

Proof. Let $x, y \in H$.

$$1 = T(\mu(x), \mu(y)) \leq \mu(x \circ y)$$

and therefore $x \circ y \in H$. □

The most important monoids in fuzzy logic are the unit interval with a t-norm or a t-conorm.

In diverse situations it can be useful to consider special submonoids of a t-norm or of a t-conorm. Among them, the so-called discrete t-norms and t-conorms are of special interest.

Definition 3.5. *A discrete t-norm is a submonoid of a t-norm of finite cardinality containing* 0*.*

Example 3.6.

1. *Consider the Łukasiewicz t-norm* T_L*. For every* $n \in \mathbb{N}$ *the set* $L_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n} = 1\}$ *is a discrete submonoid of* $([0,1], T_L)$*.*

2. *Consider the minimum t-norm* T_M*. Every finite subset of* $[0,1]$ *containing* 1 *and* 0 *is a discrete submonoid of* $([0,1], \min)$*.*

3. *Consider the Product t-norm* T_P*. For a fixed* $a \in (0,1]$ *the set of all numbers of the form* a^n*,* $n \in \mathbb{N}$*, together with* 0 *and* 1 *is a submonoid of* $([0,1], T_P)$ *of rational cardinality.*

Definition 3.7. *A discrete t-conorm is a submonoid of a t-norm of finite cardinality containing* 1*.*

Example 3.8.

- *Consider the Łukasiewicz t-conorm* S_L*. For every* $n \in \mathbb{N}$ *the set* $L_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n} = 1\}$ *is a discrete submonoid of* $([0,1], S_L)$*.*

- *Consider the maximum t-conorm* S_M*. Every finite or countable subset of* $[0,1]$ *containing* 1 *and* 0 *is a discrete submonoid of* $([0,1], \max)$*.*

Until the rest of the section we will consider fuzzy submonoids of a given t-norm or t-conorm.

Definition 3.9. *Let* T_1 *and* T_2 *be t-norms. A* T_1*-fuzzy submonoid of* $([0,1], T_2)$ *will be called a* T_1*-fuzzy t-subnorm of* T_2*.*

More explicitly, this means that a fuzzy subset μ of $[0,1]$ is a T_1-fuzzy t-subnorm of T_2 when $\mu(0) = 1$ and $T_1(\mu(x), \mu(y)) \leq \mu(T_2(x,y))$ for all $x, y \in [0,1]$.

Definition 3.10. *Let* T *and* S *be a t-norm and a t-conorm respectively. A* T*-fuzzy submonoid of* $([0,1], S)$ *will be called a* T*-fuzzy t-subconorm of* S*.*

In other words, a fuzzy subset μ of $[0,1]$ is a T-fuzzy t-subconorm of S when $\mu(1) = 1$ and $T(\mu(x), \mu(y)) \leq \mu(S(x,y))$ for all $x, y \in [0,1]$.

The submonoid 3.6.1 can be fuzzified in the following way.

Example 3.11. *Given $n \in \mathbb{N}$, we define the fuzzy subset μ of $[0,1]$ in the following way: For $i = 1, 2, ..., n$,*

$$\mu(x) = -nx + i \text{ if } x \in \left[\frac{i-1}{n}, \frac{i-\frac{1}{2}}{n}\right]$$

$$\mu(x) = nx - i + 1 \text{ if } x \in \left[\frac{i-\frac{1}{2}}{n}, \frac{i}{n}\right].$$

Proposition 3.12. *μ is a T_L-fuzzy t-subnorm of T_L, where T_L is the Łukasiewicz t-norm.*

Proof. We must prove that for all $x, y \in [0,1]$,

$$\max(\mu(x) + \mu(y) - 1, 0) \leq \mu(\max(x + y - 1, 0)). \tag{1}$$

Case a) There exist $i, j = 1, 2, ..., n$ such that $x \in \left[\frac{i-1}{n}, \frac{i-\frac{1}{2}}{n}\right]$ and $y \in \left[\frac{j-1}{n}, \frac{j-\frac{1}{2}}{n}\right]$. If $\mu(x)+\mu(y)-1 \leq 0$ or $x+y-1 \leq 0$, then the inequality (1) is satisfied trivially. Otherwise,

$$x + y - 1 \in \left[\frac{i+j-2-n}{n}, \frac{i+j-1-n}{n}\right] \tag{2}$$

and

$$\mu(x+y-1) = \begin{cases} -n(x+y-1) + (i+j-1-n) & \text{if } x+y-1 \in \left[\frac{i+j-2-n}{n}, \frac{i+j-\frac{3}{2}-n}{n}\right] \\ n(x+y-1) - (i+j-1-n) + 1 & \text{if } x+y-1 \in \left[\frac{i+j-\frac{3}{2}-n}{n}, \frac{i+j-1-n}{n}\right] \end{cases}$$

– In the first case, we must prove

$$-nx + i - ny + j - 1 \leq -nx - ny + n + i + j - 1 - n$$

which is trivially true.

– In the second case, we must prove

$$-nx + i - ny + j - 1 \leq nx + ny - n - i - j + 1 + n + 1$$

or equivalently

$$2nx + 2ny \geq 2i + 2j - 3$$

or equivalently

$$x + y - 1 \geq \frac{i + j - \frac{3}{2} - n}{n}$$

which is true by (2).

Case b) There exist $i, j = 1, 2, ..., n$ such that $x \in \left[\frac{i-\frac{1}{2}}{n}, \frac{i}{n}\right]$ and $y \in \left[\frac{j-\frac{1}{2}}{n}, \frac{j}{n}\right]$. If $\mu(x)+\mu(y)-1 \leq 0$ or $x+y-1 \leq 0$, then the inequality (1) is satisfied trivially. Otherwise,

$$x+y-1 \in \left[\frac{i+j-1-n}{n}, \frac{i+j-n}{n}\right]$$

and

$$\mu(x+y-1) = \begin{cases} -n(x+y-1)+(i+j-n) & \text{if} x+y-1 \in \left[\frac{i+j-1-n}{n}, \frac{i+j-\frac{1}{2}-n}{n}\right] \\ n(x+y-1)-(i+j-n)+1 & \text{if } x+y-1 \in \left[\frac{i+j-\frac{1}{2}-n}{n}, \frac{i+j-n}{n}\right] \end{cases} \quad (3)$$

- In the first case, we must prove

$$nx - i + 1 + ny - j + 1 - 1 \leq -nx - ny + n + i + j - n$$

or equivalently

$$2nx + 2ny \leq 2i + 2j - 1$$

or equivalently

$$x + y - 1 \leq \frac{i+j-\frac{1}{2}-n}{n}$$

which is true by (3).

- In the second case, we must prove

$$nx - i + 1 + ny - j + 1 - 1 \leq nx + ny - n - i - j + n + 1$$

which is trivially true.

Case c) There exist $i, j = 1, 2, ..., n$ such that $x \in \left[\frac{i-1}{n}, \frac{i-\frac{1}{2}}{n}\right]$ (4) and $y \in \left[\frac{j-\frac{1}{2}}{n}, \frac{j}{n}\right]$ (5). If $\mu(x) + \mu(y) - 1 \leq 0$ or $x + y - 1 \leq 0$, then the inequality (1) is satisfied trivially. Otherwise,

$$x+y-1 \in \left[\frac{i+j-\frac{3}{2}-n}{n}, \frac{i+j-\frac{1}{2}-n}{n}\right]$$

and

$$\mu(x+y-1) = \begin{cases} -2n(x+y-1)+2(i+j-\frac{1}{2}-n)-1 & \text{if} x+y-1 \in \left[\frac{i+j-\frac{3}{2}-n}{n}, \frac{i+j-1-n}{n}\right] \\ 2n(x+y-1)-2(i+j-\frac{1}{2}-n)+1 & \text{if } x+y-1 \in \left[\frac{i+j-1-n}{n}, \frac{i+j-\frac{1}{2}-n}{n}\right] \end{cases}$$

– In the first case, the right hand side extreme of the interval containing $x + y - 1$ is $\frac{i+j-1-n}{n}$ (and NOT $\frac{i+j-\frac{1}{2}-n}{n}$) and we must prove

$$-nx + i + ny - j + 1 - 1 \leq -nx - ny + n + i + j - n$$

or equivalently

$$2ny \leq 2j$$

or equivalently

$$y \leq \frac{j}{n}.$$

which is true by (5)

– In the second case, the right hand side extreme of the interval containing $x + y - 1$ is $\frac{i+j-1-n}{n}$ (and NOT $\frac{i+j-\frac{1}{2}-n}{n}$) and we must prove

$$-nx + i + ny - j + 1 - 1 \leq nx + ny - n - i - j + 1 + n + 1$$

or equivalently

$$2nx \geq 2i - 2$$

or equivalently

$$x \geq \frac{i-1}{n}.$$

which is true from (4).

□

Example 3.13. *In particular, the classic conjunction* $\wedge$

$$x \wedge y = \begin{cases} 1 \ if \ x = y = 1 \\ 0 \ otherwise \end{cases}$$

corresponds to $n = 1$ *in 3.6.1. In this case,*

$$\mu(x) = \begin{cases} 1 - x & \text{if } x \in \left[0, \frac{1}{2}\right] \\ x & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Then the fuzzification of $\wedge$ *is*

$$Ł(x, y, \mu(x), \mu(y)) = \max(2x + 2y - 3, 0).$$

Example 3.14. *Example 3.6.1 with* $n = 2$ *is the definition of the conjunction* $\wedge_3$ *of Łukasiewicz three-valued logic:*

$$x \wedge_3 y = \begin{cases} 1 \; if \; x = y = 1 \\ \frac{1}{2} \; if \; x = 1 \; and \; y = \frac{1}{2} \\ \frac{1}{2} \; if \; x = \frac{1}{2} \; and \; y = 1 \\ 0 \; otherwise. \end{cases}$$

According to 3.11 the corresponding fuzzy subset μ *of* $[0, 1]$ *is represented in Figure 1 and the fuzzification of* $\wedge_3$ *is* $Ł(x, y, \mu(x), \mu(y))$ *represented in Figure 2. For example, the truth degree of the conjunction of* 0.47 *and* 0.97 *is* 0.32 *and the truth degree of* 0.9 *and* 0.95 *is* 0.55.

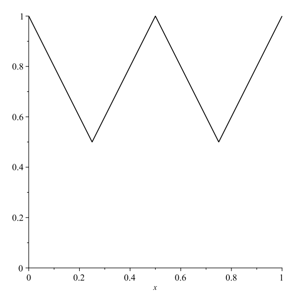

Figure 1:

Let us find the T_M-fuzzy t-subnorms of $([0, 1], T_M)$ and T_M-fuzzy t-subconorms of $([0, 1], S_M)$.

Proposition 3.15. *A fuzzy subset* μ *of* $[0, 1]$ *is a* T_M*-fuzzy t-subnorm of* $([0, 1], T_M)$ *if and only if* $\mu(1) = 1$.

Proof. Every fuzzy subset μ of $[0, 1]$ satisfies

$$\min(\mu(x), \mu(y)) \leq \mu(\min(x, y))$$

for every $x, y \in [0, 1]$. □

Proposition 3.16. *A fuzzy subset* μ *of* $[0, 1]$ *is a* T_M*-fuzzy t-subconorm of* $([0, 1], S_M)$ *if and only if* $\mu(0) = 1$.

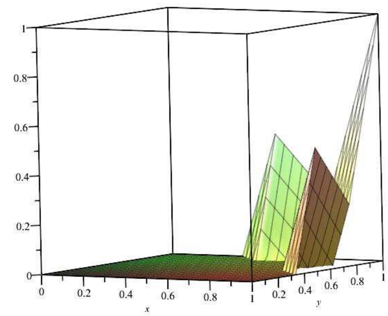

Figure 2: The fuzzification of $\wedge_3$.

Proof. Every fuzzy subset of $[0,1]$ satisfies

$$\min(\mu(x), \mu(y)) \leq \mu(\max(x, y))$$

for every $x, y \in [0,1]$. □

Let us now turn out our attention to continuous Archimedean t-norms and t-conorms.

Proposition 3.17. *Let T_1 and T_2 be two continuous Archimedean t-norms and t_1 and t_2 additive generators of T_1 and T_2 respectively. A fuzzy subset μ of $[0,1]$ is a T_1-fuzzy t-subnorm of $([0,1], T_2)$ if and only if the mapping $f : [0,\infty] \to [0,\infty]$ $f = t_1 \circ \mu \circ t_2^{[-1]}$ is subadditive.*

Proof. μ is a T_1-fuzzy t-subnorm of $([0,1], T_2)$ if and only if for all $x, y \in [0,1]$

$$T_1(\mu(x), \mu(y)) \leq \mu(T_2(x, y))$$

or

$$t_1^{[-1]}(t_1(\mu(x)) + t_1(\mu(y))) \leq \mu(t_2^{[-1]}(t_2(x) + t_2(y)))$$

which is equivalent to

$$t_1(\mu(x)) + t_1(\mu(y)) \geq t_1(\mu(t_2^{[-1]}(t_2(x) + t_2(y)))).$$

Putting $t_2(x) = a$ and $t_2(y) = b$,

$$t_1(\mu(t_2^{-1}(a))) + t_1(\mu(t_2^{-1}(b))) \geq t_1(\mu(t_2^{[-1]}(a + b))).$$

□

Corollary 3.18. *Let $f : [0, \infty] \to [0, \infty]$ be a subadditive mapping and T_1 and T_2 two continuous Archimedean t-norms with t_1 and t_2 additive generators of T_1 and T_2 respectively. Then $t_1^{[-1]} \circ f \circ t_2$ is a T_1-fuzzy t-subnorm of $([0, 1], T_2)$.*

Example 3.19. *If in the previous corollary $T_1 = T_2 = T$, t is an additive generator of T and f is the identity map, we get that the identity map on the unit interval is a T-fuzzy t-subnorm of $([0, 1], T)$.*

Remark: Regarding of t-norms, the most important case is when the two t-norms coincide. If $T_1 = T_2 = T$, we can consider different additive generators of T and from the same subadditive mapping f we can obtain different T-fuzzy t-subnorms.

Example 3.20. *Let $T_Ł$ be the Łukasiewicz t-norm, $t_1(x) = 1 - x$ and $t_2(x) = 2 - 2x$ two additive generators of Ł, and $f(x) = \sqrt{x}$. Then $\mu_1(x) = (t_1^{[-1]} \circ f \circ t_1)(x) = 1 - \sqrt{1 - x}$ and $\mu_2(x) = (t_2^{[-1]} \circ f \circ t_2)(x) = 1 - \sqrt{\frac{1-x}{2}}$ are two different T-fuzzy t-subnorms of T.*

In a similar way we can prove the next proposition and corollary.

Proposition 3.21. *Let T and S be a continuous Archimedean t-norm and t-conorm respectively and t and s additive generators of T and S respectively. A fuzzy subset μ of $[0, 1]$ is a T-fuzzy t-subconorm of $([0, 1], S)$ if and only if the mapping $f : [0, \infty] \to [0, \infty]$ $f = t \circ \mu \circ s^{[-1]}$ is subadditive.*

Corollary 3.22. *Let $f : [0, \infty] \to [0, \infty]$ be a subadditive mapping and T and S a continuous Archimedean t-norm and t-conorm respectively with t and s additive generators of T and S respectively. Then $t^{[-1]} \circ f \circ s$ is a T-fuzzy t-subconorm of $([0, 1], S)$.*

Again, different selections of the additive generators of T and S will lead to different T-fuzzy t-subconorms of S.

The duality between a t-conorm and the corresponding t-norm is desirable. If the t-norm is nilpotent, we can consider its corresponding negation and dual t-conorm.

Proposition 3.23. *[12] Let T be a nilpotent t-norm and t an additive generator of T. Then $\varphi_T : [0,1] \to [0,1]$ defined for all $x \in [0,1]$ by $\varphi_T(x) = t^{[-1]}(t(0) - t(x))$ is a strong negation.*

Proposition 3.24. *Let T be a nilpotent t-norm, t an additive generator of T, φ_T its associated negation and (T, S, φ_T) a De Morgan triplet. Then S is a nilpotent t-conorm and $s(x) = t(0) - t(x)$ and $s^{[-1]}(x) = t^{[-1]}(t(0) - x)$ are an additive generator of S and its corresponding pseudoinverse.*

Proof. We will calculate $S(x, y)$ as $S(x, y) = \varphi_T(T(\varphi_T(x), \varphi_T(y)))$ and as $S(x, y) = s^{[-1]}(s(x) + s(y))$ and see that they coincide.

$$\begin{aligned} S(x, y) &= \varphi_T(T(\varphi_T(x), \varphi_T(y))) \\ &= t^{[-1]}(t(0) - t(t^{[-1]}[t(t^{[-1]}(t(0) - t(x)) + t(t^{[-1]}(t(0) - t(y)))])) \\ &= t^{[-1]}(t(0) - t(t^{[-1]}((t(0) - t(x)) + (t(0) - t(y))))) \\ &= t^{[-1]}(t(0) - ((t(0) - t(x)) + (t(0) - t(y)))) \\ &= t^{[-1]}(t(x) + t(y) - t(0)). \end{aligned}$$

$$\begin{aligned} S(x, y) &= s^{[-1]}(s(x) + s(y)) \\ &= t^{[-1]}(t(0) - s(x) - s(y)) \\ &= t^{[-1]}(t(0) - t(0) + t(x) - t(0) + t(y)) \\ &= t^{[-1]}(t(x) + t(y) - t(0)). \end{aligned}$$

Moreover, S is nilpotent because $s(1) = t(0) - t(1) = t(0) < \infty$. □

As particular cases of Propositions 3.21 and Corollary 3.22 we obtain the following two results.

Proposition 3.25. *Let T be a nilpotent t-norm, t an additive generator of T, φ_T its associated negation and (T, S, φ_T) a De Morgan triplet. Then a fuzzy subset μ of $[0,1]$ is a T-fuzzy t-subconorm of $([0,1], S)$ if and only if $f(x) = t(\mu(t^{[-1]}(t(0) - x)))$ is subadditive.*

Proposition 3.26. *Let T be a nilpotent t-norm, t an additive generator of T, φ_T its associated negation, (T, S, φ_T) a De Morgan triplet and $f : [0,\infty] \to [0,\infty]$ a subadditive mapping. Then fuzzy subset μ of $[0,1]$ defined for all $x \in [0,1]$ by*

$$\mu(x) = t^{[-1]}(f(t(0) - t(x)))$$

is a T-fuzzy t-subconorm of S.

Example 3.27. *If in the last proposition we consider $f = id$, the identity map, then we get that φ_T is a T-fuzzy t-subconorm of S, the dual t-subconorm of T with respect to φ_T.*

In the strict case, we can consider the natural negation $n(x) = 1 - x$.

Proposition 3.28. *Let T be a strict t-norm, t an additive generator of T, n the natural negation and (T, S, n) a De Morgan triplet. Then S is a strict t-conorm and $s(x) = t(1-x)$ and $s^{-1}(x) = 1-t^{-1}(x)$ are an additive generator of S and its corresponding pseudoinverse.*

Proof. We will calculate $S(x, y)$ as $S(x, y) = n(T(n(x), n(y)))$ and as $S(x, y) = s^{-1}(s(x) + s(y))$ and see that they coincide.

$$\begin{aligned} S(x, y) &= n(T(n(x), n(y))) \\ &= 1 - T(1 - x, 1 - y) \\ &= 1 - t^{-1}(t(1 - x) + t(1 - y)). \end{aligned}$$

$$\begin{aligned} S(x, y) &= s^{-1}(s(x) + s(y)) \\ &= 1 - t^{-1}(t(1 - x) + t(1 - y)). \end{aligned}$$

Moreover, S is strict because $s(1) = t(0) = \infty$. □

As particular cases of Propositions 3.21 and Corollary 3.22 we obtain the following two results.

Proposition 3.29. *Let T be a strict t-norm, t an additive generator of T, n the natural negation and (T, S, n) a De Morgan triplet. Then a fuzzy subset μ of $[0, 1]$ is a T-fuzzy t-subconorm of $([0, 1], S)$ if and only if $f(x) = t(\mu(1 - t^{[-1]}(x)))$ is subadditive.*

Proposition 3.30. *Let T be a strict t-norm, t an additive generator of T, n the natural negation, (T, S, n) a De Morgan triplet and $f : [0, \infty] \to [0, \infty]$ a subadditive mapping. Then fuzzy subset μ of $[0, 1]$ defined for all $x \in [0, 1]$ by*

$$\mu(x) = t^{[-1]}(f(t(1 - x)))$$

is a T-fuzzy t-subconorm of S.

Example 3.31. *If in the last proposition we consider $f = id$, the identity map, then we get that n, the natural negation, is a T-fuzzy t-subconorm of S, the dual t-conorm of T with respect to n.*

4 Vague Monoids

In the crisp case, if $(M, \circ)$ is a set with an operation $\circ : M \times M \to M$ and $\sim$ is an equivalence relation on M, then $\circ$ is compatible with $\sim$ if and only if

$$x \sim x' \text{ and } y \sim y' \text{ implies } x \circ y \sim x' \circ y'.$$

In this case, an operation $\tilde{\circ}$ can be defined on $\overline{M} = M/\sim$ by

$$\overline{x} \tilde{\circ} \overline{y} = \overline{x \circ y}$$

where $\overline{x}$ and $\overline{y}$ are the equivalence classes of x and y with respect to $\sim$.

Demirci generalized this idea to the fuzzy framework by introducing the concept of vague algebra, which basically consists of fuzzy operations compatible with a given fuzzy equivalence relation [3].

In this section we will develop general results of vague monoids that will be applied to vague t-norms and t-conorms in Section 5.

The fuzzification of equivalence relations are fuzzy relations called T-fuzzy equivalences or T-indistinguishability operators. They are essential in fuzzy logic and have been widely studied [10]. We need to recall their definition.

Definition 4.1. *Let T be a t-norm and X a set. A fuzzy relation $E : X \times X \to [0, 1]$ is a T-indistinguishability operator if for all $x, y, z \in X$*

- $E(x, x) = 1$
- $E(x, y) = E(y, x)$
- $T(E(x, y), E(y, z)) \leq E(x, z)$.

If $E(x, y) = 1$ implies $x = y$, then it is said that E separates points.

Definition 4.2. *A fuzzy binary operation on a set M is a mapping $\tilde{\circ} : M \times M \times M \to [0, 1]$.*

$\tilde{\circ}(x, y, z)$ is interpreted as the degree in which z is $x \circ y$.

Definition 4.3. *Let E be T-indistinguishability operator on M. A T-vague binary operation on M is a fuzzy binary operation on M satisfying for all $x, y, z, x', y', z' \in M$*

a) $T(\tilde{\circ}(x, y, z), E(x, x'), E(y, y'), E(z, z')) \leq \tilde{\circ}(x', y', z')$.

b) $T(\tilde{\circ}(x,y,z), \tilde{\circ}(x,y,z')) \le E(z,z')$.

c) For all $x, y \in M$ *there exists* $z \in M$ *such that* $\tilde{\circ}(x,y,z) = 1$.

These conditions fuzzify the idea of compatibility between a binary operation and an equivalence relation on a set.

- Condition a) states that if z is the (vague) result of operating x and y and x', y', z' are indistinguishable from x, y, z, then z' is the vague result of operating x' and y'.
- b) asserts that if z and z' are vague results of the operation $x \circ y$, then they are indistinguishable.
- c) says that for every x and y there exists z that is exactly the result of operating x with y.

Proposition 4.4. *Let* E *be* T*-indistinguishability operator on* M *separating points and* $\tilde{\circ}$ *a vague binary operation on* M*. Then the* z *of Definition 4.3c) is unique.*

Proof. Let $z, z' \in M$ satisfy $\tilde{\circ}(x,y,z) = 1$ and $\tilde{\circ}(x,y,z') = 1$. From Definition 4.3b) $1 = T(\tilde{\circ}(x,y,z), \tilde{\circ}(x,y,z')) \le E(z,z')$. Hence $E(z,z') = 1$ and $z = z'$. □

Definition 4.5. *Let* $\tilde{\circ}$ *be a* T*-vague binary operation on* M *with respect to a* T*-indistinguishability operator* E *on* M*. Then* $(M, \tilde{\circ})$ *is a* T*-vague monoid if and only if it satisfies the following properties.*

1. *Associativity.* $\forall x, y, z, d, m, q, w, \in M$

$$T(\tilde{\circ}(y,z,d), \tilde{\circ}(x,d,m), \tilde{\circ}(x,y,q), \tilde{\circ}(q,z,w)) \le E(m,w).$$

2. *Identity. There exists a (two sided) identity element* $e \in M$ *such that*

$$T(\tilde{\circ}(e,x,x), \tilde{\circ}(x,e,x)) = 1$$

for each $x \in M$.

A T*-vague monoid is commutative if and only if*

$$\forall x, y, m, w \in M, T(\tilde{\circ}(x,y,m), \tilde{\circ}(y,x,w)) \le E(m,w).$$

For a given T-vague monoid $(M, \tilde{\circ})$, the identity is unique if the T-indistinguishability operator separates points.

Proposition 4.6. *Let T be a t-norm, E a T-indistinguishability operator separating points on a set M and $(M, \tilde{\circ})$ a T-vague monoid. Then the identity element is unique.*

Proof. Let e and e' be two identity elements of M. Then, since e is an identity element,

$$\tilde{\circ}(e, e', e') = 1$$

and since e' is an identity element,

$$\tilde{\circ}(e, e', e) = 1.$$

From Definition 4.3c),

$$1 = T(\tilde{\circ}(e, e', e'), \tilde{\circ}(e, e', e)) \leq E(e, e').$$

So $E(e, e') = 1$ and $e = e'$ because E separates points. □

Definition 4.7. *Let $\circ$ be a binary operation on M, and E a T-indistinguishability operator on M. E is regular with respect to $\circ$ if and only if for all $x, y, z \in M$*

$$E(x, y) \leq E(x \circ z, y \circ z)$$

and

$$E(x, y) \leq E(z \circ x, z \circ y).$$

Proposition 4.8. *Let E be a regular T-indistinguishability operator on M with respect to a binary operation $\circ$ on M.*

a) The fuzzy mapping $\tilde{\circ} : M \times M \times M \to [0, 1]$ defined for all $x, y, z \in M$ by

$$\tilde{\circ}(x, y, z) = E(x \circ y, z) \text{ for all } x, y, z \in M,$$

is a T-vague binary operation on M.

b) If $(M, \circ)$ is a monoid, then $(M, \tilde{\circ})$ is a T-vague monoid.

Proof.

a) We must prove that $\tilde{\circ}$ satisfies the properties of Definition 4.3. Let $x, y, z, x', y', z' \in M$.

4.3 a)

$$\begin{aligned}
&T(\tilde{\circ}(x,y,z), E(x,x'), E(y,y'), E(z,z'))\\
&= T(E(x\circ y, z), E(x,x'), E(y,y'), E(z,z'))\\
&\le T(E(x\circ y, z'), E(x,x'), E(y,y'))\\
&\le T(E(x\circ y, z'), E(x\circ y, x'\circ y), E(y,y'))\\
&\le T(E(x'\circ y, z'), E(y,y'))\\
&\le T(E(x'\circ y, z'), E(x'\circ y, x'\circ y'))\\
&\le E(x'\circ y', z') = \tilde{\circ}(x',y',z').
\end{aligned}$$

4.3 b)

$$\begin{aligned}
&T(\tilde{\circ}(x,y,z), \tilde{\circ}(x,y,z'))\\
&= T(E(x\circ y, z), E(x\circ y, z'))\\
&\le E(z,z')
\end{aligned}$$

4.3 c) For $x, y \in M$ we can consider $z = x \circ y$. Then

$$\tilde{\circ}(x,y,z) = E(x\circ y, x\circ y) = 1.$$

b)

Associativity. Using the associativity of $\circ$ and the regularity of E, $\forall x, y, z, d, m, q, w \in M$,

$$\begin{aligned}
&T(\tilde{\circ}(y,z,d), \tilde{\circ}(x,d,m), \tilde{\circ}(x,y,q), \tilde{\circ}(q,z,w))\\
&= T(E(y\circ z, d), E(x\circ d, m), E(x\circ y, q), E(q\circ z, w))\\
&\le T(E(x\circ(y\circ z), x\circ d), E(x\circ d, m), E((x\circ y)\circ z, q\circ z), E(q\circ z, w))\\
&\le T(E(x\circ(y\circ z), m), E((x\circ y)\circ z, w))\\
&\le E(m,w).
\end{aligned}$$

Identity element. Let e be the identity element of M. Then

$$\tilde{\circ}(x,e,x) = E(x\circ e, x) = E(x,x) = 1$$

$$\tilde{\circ}(e,x,x) = E(e\circ x, x) = E(x,x) = 1$$

and e is the identity element of $\tilde{\circ}$

$\square$

From the equality $\tilde{\circ}(x, y, z) = E(x \circ y, z)$, fixed a regular T-indistinguishability operator separating points on a monoid $(M, \circ)$ we obtain a vague monoid on M in a natural way. Reciprocally, from a vague monoid on M we obtain a monoid $(M, \circ)$.

Proposition 4.9. *Let $(M, \tilde{\circ})$ be a vague monoid with respect to a T-indistinguishability operator E separating points. Then $(M, \circ)$ is a monoid where $x \circ y$ is the unique $z \in M$ such that $\tilde{\circ}(x, y, z) = 1$.*

Proof.

Associativity

$$\begin{aligned} 1 =& T(\tilde{\circ}(y, z, y \circ z), \tilde{\circ}(x, y \circ z, x \circ (y \circ z)), \tilde{\circ}(x, y, x \circ y), \tilde{\circ}(x \circ y, z, (x \circ y) \circ z)) \\ =& T(E(y \circ z, y \circ z), E(x \circ (y \circ z), x \circ (y \circ z)), E(x \circ y, x \circ y), E((x \circ y) \circ z, (x \circ y) \circ z)) \\ \leq& E(x \circ (y \circ z), (x \circ y) \circ z)) \end{aligned}$$

and form this, $x \circ (y \circ z) = (x \circ y) \circ z$.

Identity

$$1 = \tilde{\circ}(x, e, x) = E(x \circ e, x)$$

$$1 = \tilde{\circ}(e, x, x) = E(e \circ x, x)$$

So, $x \circ e = x$ and $e \circ x = x$ and e is the identity element of $(M, \circ)$.

$\square$

Definition 4.10. *In the conditions of the previous proposition, we will say that $(M, \circ)$ is the monoid associated to the vague monoid $(M, \tilde{\circ})$.*

At this point, given a monoid $(M, \circ)$ and a t-norm T we have bijective maps between their T-vague monoids and their regular T-indistinguishability operators and a simple way to generate one of them knowing the other one:

- $\tilde{\circ}(x, y, z) = E(x \circ y, z)$.
- $E(x, y) = \tilde{\circ}(x, e, y)$.

$(M, \tilde{\circ})$ is commutative if and only if its associated monoid $(M, \circ)$ is commutative.

Proposition 4.11. *Let E be a T-indistinguishability operator separating points, $(M, \tilde{\circ})$ a T-vague operation and $(M, \circ)$ its associated operation ($x \circ y = z$ if and only if $\tilde{\circ}(x, y, z) = 1$). Then $(M, \tilde{\circ})$ is commutative if and only if $(M, \circ)$ is commutative.*

Proof.

$\Rightarrow$)

$$1 = T(\tilde{\circ}(x, y, x \circ y), \tilde{\circ}(y, x, y \circ x)) \leq E(x \circ y, y \circ x).$$

So, $E(x \circ y, y \circ x) = 1$ and $x \circ y = y \circ x$.

$\Leftarrow$)

$$\begin{aligned} T(\tilde{\circ}(x, y, z), \tilde{\circ}(y, x, z')) &= T(E(x \circ y, z), E(y \circ x, z')) \\ &= T(E(x \circ y, z), E(x \circ y, z')) \\ &\leq E(z, z'). \end{aligned}$$

□

In [4], [5] homomorphisms between T-vague algebras have been studied. In this section we will use them to relate vague monoids and fuzzy monoids.

Definition 4.12. *Let $(M, \tilde{\circ})$ and $(N, \tilde{*})$ be two T-vague monoids with respect to the T-indistinguishability operators E and F respectively. A map $f : M \to N$ is a homomorphism from M onto N if and only if*

$$\tilde{\circ}(x, y, z) \leq \tilde{*}(f(x), f(y), f(z)) \; \forall x, y, z \in M.$$

Lemma 4.13. *Let $(M, \tilde{\circ})$ and $(N, \tilde{*})$ be two T-vague monoids with respect to the T-indistinguishability operators E and F respectively and $f : M \to N$ a homomorphism from M onto N. If e is the identity element of M, then $f(e)$ is the identity element of N.*

Proof.

$$1 = \tilde{\circ}(x, e, x) \leq \tilde{*}(f(x), f(e), f(x))$$
$$1 = \tilde{\circ}(e, x, x) \leq \tilde{*}(f(e), f(x), f(x))$$

So, $\tilde{*}(f(x), f(e), f(x)) = \tilde{*}(f(e), f(x), f(x)) = 1$ and $f(e)$ is the identity element of N. □

Proposition 4.14. *Let $(M, \tilde{\circ})$ and $(M, \tilde{*})$ be two T-vague monoids with respect to the T-indistinguishability operators E and F respectively such that $E \leq F$. Then the identity map* id : $M \to M$ *is a homomorphism from $(M, \tilde{\circ})$ onto $(M, \tilde{*})$.*

Proof.

$$\tilde{\circ}(x, y, z) = E(x \circ y, z) \leq F(x \circ y, z) \leq \tilde{*}(x, y, z).$$

□

It is clear that a crisp monoid $(M, \circ)$ is a T-vague monoid defining $\circ(x, y, z) = 1$ if $x \circ y = z$ and 0 otherwise, and considering the crisp equality as the T-indistinguishability operator.

Corollary 4.15. *Let $(M, \tilde{\circ})$ be a T-vague monoid with respect to a T-indistinguishability operator E separating points. Then the identity map* id : $M \to M$ *is a homomorphism from $(M, \circ)$ onto $(M, \tilde{\circ})$.*

Definition 4.16. *Let $f : M \to N$ be a homomorphism from $(M, \tilde{\circ})$ onto $(N, \tilde{*})$. The kernel of f is the fuzzy subset μ of M defined by $\mu(x) = E(f(x), e')$ $\forall x \in M$ where e' is the identity element of $(N, \tilde{*})$.*

Proposition 4.17. *Let $(M, \tilde{\circ})$ be a T-vague monoid with respect to a regular T-indistinguishability operator E separating points. The kernel of the identity map* id : $M \to M$ *is a T-fuzzy submonoid of $(M, \circ)$.*

Proof.

a)

$$ker\text{id}(e) = E(e, e) = 1.$$

b)

$$\begin{aligned} T(ker\text{id}(x), ker\text{id}(y)) &= T(E(x, e), E(y, e)) \\ &\leq T(E(x \circ y, y), E(y, e)) \\ &\leq E(x \circ y, e). \end{aligned}$$

□

5 Vague t-norms and Vague t-conorms

The results of the previous section will be applied to the case of t-norms and t-conorms.

Given a t-norm T, we will consider always the T-indistinguishability operator E_T and their powers that separate points.

Definition 5.1. *Let T be a left continuous t-norm. Its residuation $\overrightarrow{T}$ is defined for all $x, y \in [0,1]$ by*

$$\overrightarrow{T}(x,y) = \sup\{\alpha \in [0,1] \mid T(x,\alpha) \leq y\}.$$

Definition 5.2. *Given a t-norm T, the natural indistinguishability operator E_T on $[0,1]$ associated to T, is defined for all $x, y \in [0,1]$ by*

$$E_T(x,y) = \min(\overrightarrow{T}(x,y), \overrightarrow{T}(y,x)).$$

Note. E_T is usually called biresiduation. Here we prefer natural indistinguishability operator to stress the fact that it is indeed an indistinguishability operator.

Example 5.3.

- *If T_M is the minimum t-norm, then*

$$E_{T_M}(x,y) = \begin{cases} \min(x,y) & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$$

- *If T is continuous Archimedean and t is an additive generator of T, then $E_T(x,y) = t^{-1}(|t(x) - t(y)|)$.*

  In particular

  - *If $T = T_L$ is the Łukasiewicz t-norm, then $E_T(x,y) = 1 - |x - y|$.*
  - *If T is the product t-norm, then $E_T(x,y) = \frac{\min(x,y)}{\max(x,y)}$ with the convention $\frac{0}{0} = 1$.*

We need to recall the concept of power with respect to a t-norm [8, 10] to define powers of E_T.

Definition 5.4. [10] *Given a continuous t-norm T, $T(\overbrace{x, x, ..., x}^{n\ times})$ -the n-th power of x- will be denoted by by $x^{(n)}$.*

The n-th root $x^{(\frac{1}{n})}$ of x with respect to T is defined by

$$x^{(\frac{1}{n})} = \sup\{z \in [0,1] \mid z_T^{(n)} \leq x\}$$

and for $m, n \in \mathbb{N}$, $x^{(\frac{m}{n})} = (x^{(\frac{1}{n})})^{(m)}$.

If $r \in \mathbb{R}^+$ is a positive real number, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of rational numbers with $\lim_{n\to\infty} a_n = r$. For any $x \in [0,1]$, the power $x^{(r)}$ is

$$x^{(r)} = \lim_{n\to\infty} x^{(a_n)}.$$

Proposition 5.5. [10] *Let T be a continuous Archimedean t-norm with additive generator t, $x \in [0,1]$ and $r \in \mathbb{R}^+$. Then*

$$x^{(r)} = t^{[-1]}(rt(x)).$$

Proposition 5.6. [10] *Let T be a continuous t-norm, E_T the natural indistinguishability operator associated to T and $r > 0$. Then $E_T^{(r)}$ defined for all $x, y, \in [0,1]$ by $E(x,y) = (E_T(x,y))^{(r)}$ is a T-indistinguishability operator.*

Example 5.7.

- *If T_M is the minimum t-norm, then $E_{T_M}^{(r)} = E_{T_M}$.*
- *If T is continuous Archimedean and t is an additive generator of T, then $E_T^{(r)}(x,y) = t^{[-1]}(r|t(x) - t(y)|)$.*

  In particular

  - *If $T = T_L$ is the Łukasiewicz t-norm, then $E_T^{(r)}(x,y) = \max(1 - r|x-y|, 0)$.*
  - *If $T = T_P$ is the product t-norm, then $E_T^{(r)}(x,y) = (\frac{\min(x,y)}{\max(x,y)})^r$.*

Proposition 5.8. *E_T is regular on $([0,1], T)$.*

Proof. Let T be a left continuous t-norm. We must prove that for all $x, y, z \in [0,1]$, $E_T(x,y) \leq E_T(T(x,z), T(y,z))$. Assuming without loss of generality that $x \geq y$, this is equivalent to prove $\overrightarrow{T}(x,y) \leq \overrightarrow{T}(T(x,z), T(y,z))$ and this is done in Proposition 2.2 of [1]. □

Corollary 5.9. *Let T be a continuous t-norm and $r > 0$. Then $E_T^{(r)}$ is regular.*

Proof. It is a consequence of the fact that if $x, y \in [0,1]$ with $x \le y$, then $x^{(r)} \le y^{(r)}$. $\square$

Definition 5.10. *Let T be a t-norm and E a T-indistinguishability operator on $[0,1]$ separating points. A T-vague monoid $([0,1], \widetilde{T})$ whose associated operation is T ($T(x,y) = z$ if and only if $\widetilde{T}(x,y,z) = 1$) is called a T-vague t-norm.*

Definition 5.11. *Let T be a continuous t-norm and $r > 0$. Then $\widetilde{T}_r$ is defined for all $x, y, z \in [0,1]$ by*

$$\widetilde{T}_r(x,y,z) = E_T^{(r)}(T(x,y), z).$$

Proposition 5.12. *$\widetilde{T}_r$ is a T-vague t-norm.*

Proof. Consequence of Proposition 4.8 because $E_T^{(r)}$ is regular. $\square$

Example 5.13.

- *If $T = T_M$ is the minimum t-norm, then*

$$\widetilde{T}_r(x,y,z) = \begin{cases} 1 & \text{if } z = \min(x,y) \\ \min(x,y,z) & \text{otherwise.} \end{cases}$$

- *If T is continuous Archimedean and t is an additive generator of T, then*

$$\widetilde{T}_r(x,y,z) = \begin{cases} t^{[-1]}(r|t(x) - t(y)| - t(z)) & \text{if } t(x) + t(y) \le t(0) \\ t^{[-1]}(r|t(0) - t(z)|) & \text{otherwise.} \end{cases}$$

  In particular

  – *If $T = T_Ł$ is the Łukasiewicz t-norm, then*

$$\widetilde{T}_r(x,y,z) = \begin{cases} z - r|x-y| & \text{if } x+y \ge 1 \text{ and } z + r|x-y| \le 2 \\ 0 & \text{if } x+y \ge 1 \text{ and } z + r|x-y| > 2 \\ 1 - rz & \text{if } x+y < 1 \text{ and } rz \le 1 \\ 0 & \text{if } x+y < 1 \text{ and } rz > 1. \end{cases}$$

– *If* $T = T_P$ *is the product t-norm, then*

$$\widetilde{T}_r(x, y, z) = (\frac{\min(xy, z)}{\max(xy, z)})^r.$$

Proposition 5.14. *Let* T *be a continuous t-norm and* $r \geq s > 0$. *Then the identity map* $id : [0,1] \to [0,1]$ *is a homomorphism from* $([0,1], \widetilde{T}_r)$ *onto* $([0,1], \widetilde{T}_s)$

Proof. If $r \geq s$, then $E_T^{(r)} \leq E_T^{(s)}$ and we can apply Proposition 4.14. □

Proposition 5.15. *Let* T *be a continuous t-norm and* $r > 0$. *Then the identity map* $id : [0,1] \to [0,1]$ *is a homomorphism from* $([0,1], T)$ *onto* $([0,1], \widetilde{T}_s)$

Proof. For every $r > 0$, $E_T^{(r)} \geq \mathrm{Id}$ where Id is the crisp equality on $[0,1]$ and we can apply Proposition 4.14. □

Proposition 5.16. *Let* T *be a continuous t-norm and* $r \geq s > 0$. *Then the kernel of the identity map* $id : [0,1] \to [0,1]$ *from* $([0,1], \widetilde{T}_r)$ *onto* $([0,1], \widetilde{T}_s)$ *is a* T*-fuzzy t-subnorm of* $([0,1], T)$.

Proof. It is a special case of Proposition 4.17. □

Proposition 5.17. *Let* T *be a continuous t-norm and* $r > 0$. *Then the kernel of the identity map* $id : [0,1] \to [0,1]$ *from* $([0,1], T)$ *onto* $([0,1], \widetilde{T}_s)$ *is a* T*-fuzzy t-subnorm of* $([0,1], T)$.

Proof. It is a special case of Proposition 4.17. □

These kernels are of the form $\mu(x) = E_T^{(r)}(x, 1)$. For example, if T is the Łukasiewicz t-norm, the graphics of these T-fuzzy T-subnorms of T are the segments in $[0,1]$ ending at the point $(1,1)$.

Until now, in this section we have studied vague t-norms. Analogously we can consider vague t-conorms.

First we will see that the powers of E_T are regular with respect to t-conorms.

Proposition 5.18. *Let* T_M *be the minimum t-norm and* S_M *the maximum t-conorm. Then* E_{T_M} *is regular with respect to* S_M.

Proof. We must prove that for all $x, y, z \in [0,1]$, $E_{T_M}(x,y) \leq E_{T_M}(\max(x,z), \max(y,z))$.

Without loss of generality we can assume $x \geq y$. There are three possibilities

a) $z \geq x \geq y$: In this case $E_{T_M}(\max(x,z), \max(y,z)) = E_{T_M}(z,z) = 1 \geq E_{T_M}(x,y)$.

b) $x \geq z \geq y$: In this case $E_{T_M}(\max(x,z), \max(y,z)) = E_{T_M}(x,z) \geq E_{T_M}(x,y)$.

c) $x \geq y \geq z$: In this case $E_{T_M}(\max(x,z), \max(y,z)) = E_{T_M}(x,y)$.

□

Proposition 5.19. *Let T be a nilpotent t-norm, φ_T its associated negation and (T, S, φ_T) a De Morgan triplet. Then E_T is regular with respect to S.*

Proof. We must prove that for all $x, y, z \in [0,1]$, $E_T(x,y) \leq E_T(S(x,z), S(y,z))$.

From Proposition 3.24,

If both $t(x) + t(z) - t(0) \geq 0$ and $t(y) + t(z) - t(0) \geq 0$, then

$$E_T(t^{[-1]}(t(x) + t(z) - t(0)), t^{[-1]}(t(y) + t(z) - t(0)))$$
$$= t^{[-1]}(|t(x) + t(z) - t(0) - (t(y) + t(z) - t(0))|)$$
$$= t^{[-1]}(|t(x) - t(y)|) = E_T(x,y).$$

If both $t(x) + t(z) - t(0) \leq 0$ and $t(y) + t(z) - t(0) \leq 0$, then

$$E_T(t^{[-1]}(t(x) + t(z) - t(0)), t^{[-1]}(t(y) + t(z) - t(0)))$$
$$= E_T(1,1) = 1 \geq E_T(x,y).$$

If $t(x) + t(z) - t(0) \leq 0$ and $t(y) + t(z) - t(0) \geq 0$, $(x > y)$, then

$$E_T(t^{[-1]}(t(x) + t(z) - t(0)), t^{[-1]}(t(y) + t(z) - t(0)))$$
$$= E_T(1, t^{[-1]}(t(y) + t(z) - t(0)))$$
$$= t^{[-1]}(t(y) + t(z) - t(0))$$

and we must prove

$$E_T(x,y) = t^{[-1]}(t(y) - t(x)) \leq t^{[-1]}(t(y) + t(z) - t(0))$$

or

$$t(y) - t(x) \geq t(y) + t(z) - t(0).$$

which is true if $t(x) + t(z) - t(0) \leq 0$.

The case $t(x) + t(z) - t(0) \geq 0$ and $t(y) + t(z) - t(0) \leq 0$, $(x < y)$ is analogous. □

Corollary 5.20. *Let T be a nilpotent t-norm, φ_T its associated negation, (T, S, φ_T) a De Morgan triplet and $r > 0$. Then $E_T^{(r)}$ is regular with respect to S.*

Proposition 5.21. *Let T_P be the product t-norm and S_P its dual t-conorm with respect to the natural negation n. Then E_{T_P} is regular with respect to S_P.*

Proof. $S_P(x, y) = x + y - xy$ and, assuming without loss of generality that $x \leq y$, we must prove $\frac{x}{y} \leq \frac{x+z-xz}{y+z-yz}$. This is equivalent to prove

$$x(y + z - yz) \leq y(x + z - xz)$$

which is true if $x \leq y$. □

Proposition 5.22. *Let T_P be the product t-norm, S_P its dual t-conorm with respect to the natural negation n and $r > 0$. Then $E_{T_P}^{(r)}$ is regular with respect to S_P.*

Definition 5.23. *Let T be a t-norm, E a T-indistinguishability operator on $[0, 1]$ separating points and S a t-conorm. An S-vague monoid $([0, 1], \widetilde{S})$ whose associated operation is S ($S(x, y) = z$ if and only if $\widetilde{S}(x, y, z) = 1$) is called an T-vague t-conorm.*

Definition 5.24. *Let T be a continuous t-norm, S a t-conorm and $r > 0$. Then $\widetilde{S}_r$ is defined for all $x, y, z \in [0, 1]$ by*

$$\widetilde{S}_r(x, y, z) = E_T^{(r)}(S(x, y), z).$$

As a consequence of Proposition 4.8 we have the following results concerning vague t-conorms.

Proposition 5.25. *Let T_M be the minimum t-norm and S_P the maximum t-conorm. Then*

$$\widetilde{S}_P(x, y, z) = \begin{cases} \min(\max(x, y), z) & \text{if } \max(x, y) \neq z \\ 1 & \text{otherwise.} \end{cases}$$

is a T_M-vague t-conorm of S_P.

Proposition 5.26. *Let T be a nilpotent t-norm, φ_T its associated negation, (T, S, φ_T) a De Morgan triplet and $r > 0$. Then*

$$\widetilde{S}_r(x,y,z) = \begin{cases} t^{[-1]}(r|t(x)+t(y)-t(z)-t(0)|) & \text{if } t(x)+t(y)-t(z) \geq 0 \\ z^{(r)} & \text{otherwise.} \end{cases}$$

is a T-vague t-conorm with respect to S.

Corollary 5.27. *If T_L and S_L are the t-norm and t-conorm of Łukasiewicz, then*

$$\widetilde{S}_r(x,y,z) = \begin{cases} \max(1-r|x+y-z|,0) & \text{if } x+y \leq 1 \\ \max(1-r+rz = x^{(r)}) & \text{otherwise.} \end{cases}$$

is a T_L-vague t-conorm with respect to S_L.

Proposition 5.28. *If T_P is the product t-norm and S_P the probabilistic sum, then*

$$\widetilde{S}_r(x,y,z) = \left(\frac{\min(x+y-xy,z)}{\max(x+y-xy,z)}\right)^r$$

is a T_P-vague t-conorm with respect to S_P.

Proposition 5.29. *Let T be a continuous t-norm and $r \geq s > 0$. Then the identity map $id : [0,1] \to [0,1]$ is a homomorphism from $([0,1], \widetilde{S}_r)$ onto $([0,1], \widetilde{S}_s)$*

Proposition 5.30. *Let T be a continuous t-norm and $r > 0$. Then the identity map $id : [0,1] \to [0,1]$ is a homomorphism from $([0,1], S)$ onto $([0,1], \widetilde{S}_s)$*

Proposition 5.31. *Let T be a continuous t-norm and $r \geq s > 0$. Then the kernel of the identity map $id : [0,1] \to [0,1]$ from $([0,1], \widetilde{S}_r)$ onto $([0,1], \widetilde{S}_s)$ is a T-fuzzy t-subconorm of $([0,1], S)$.*

Proposition 5.32. *Let T be a continuous t-norm and $r > 0$. Then the kernel of the identity map $id : [0,1] \to [0,1]$ from $([0,1], S)$ onto $([0,1], \widetilde{S}_s)$ is a T-fuzzy t-subconorm of $([0,1], S)$.*

6 Concluding Remarks

In this paper we have fuzzified t-norms and t-conorms in two different ways:

- by defining T-fuzzy t-subnorms and t-subconorms.
- by introducing T-vague t-norms and t-conorms.

These concepts are interrelated as kernels of T-vague t-norms and t-conorms are T-fuzzy t-subnorms and t-subconorms, respectively.

A fuzzy operation $\tilde{\circ}$ on a set M can be seen as a type-2 fuzzy subset of $M \times M$: If $[0,1]^M$ is the set of fuzzy subsets of M, then we can define $\mathbb{O} : M \times M \to [0,1]^M$ by $\mathbb{O}(x,y)(z) = \tilde{\circ}(x,y,z)$. In this way, a T-vague t-norm $\widetilde{T}$ can be seen as a type-2 t-norm $\mathbb{T} : [0,1] \times [0,1] \to [0,1]^{[0,1]}$: $\mathbb{T}(x,y)(z) = \widetilde{T}(x,y,z)$. The same applies to vague t-conorms.

It seems interesting the fuzzification of other connectives used in fuzzy logic such as implications. The authors will develop it in further works.

Acknowledgments

The second author acknowledges financial support from FEDER/Ministerio de Ciencia, Innovación y Universidades – Agencia Estatal de Investigación – Proyecto PGC2018-095709-B-C21.

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