Fifty Years of Similarity Relations: A Survey of Foundations and Applications

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ABSTRACT
On the occasion of the fiftieth anniversary of the publication of Zadeh's significant paper *Similarity Relations and Fuzzy Orderings*, an account of the development of similarity relations during this time will be given. Moreover the main topics related to these fuzzy relations will be reviewed.

KEYWORDS
Similarity relations; fuzzy equivalence relations; indistinguishability operators; t-norm; generalized metric space; extensionality

1. Introduction

Fifty years ago L.A. Zadeh published one of his most influential articles: *Similarity Relations and Fuzzy Orderings* (Zadeh 1971). These are the two most important types of fuzzy relations because they fuzzify the concepts of (crisp) equivalence relation and (crisp) preorder. Zadeh's article constitutes the start of its systematic study. Here we will focus on similarity relations by describing the relevance of the article and presenting the main advances and contributions during these fifty years.

In this introductory section we will show the importance of similarity relations and the most relevant results on Zadeh's article.

1.1. The Need of Fuzzy Equivalence Relations

The concept of equality is a fundamental notion in any theory since it is essential to the ability of discerning the objects to whom it concerns. Its need and importance come from the fact that it allows us to classify. Classifying is one of the most important processes of knowledge because it permits to relate, organize, generalize, find general laws, etc.

In many real situations, the objects of a universe of discourse do not verify a property completely but in general they satisfy it only at a certain degree and the properties become fuzzy. In the same way, we cannot talk about identical objects but equality between objects becomes fuzzy as well. Crisp equivalence relations are in these cases

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too rigid to model this kind of equalities and have to be replaced by some fuzzification where "equal" must be replaced by "indistinguishable".

As early as 1901, H. Poincaré (Poincaré 1902) was already concerned about this kind of disquisitions. He considered that in the physical (real) world, equal means indistinguishable because when we declare that two objects are equal what we can only be sure of is that they cannot be distinguished. This leads to the paradox that two objects A and B can be equal (indistinguishable), B equal to a third object C but A and C can be different (distinguishable) meaning the non-transitivity of equality relations in the real world. Fuzzy equivalence relations try to overcome this problem at least to a certain extent. These relations are now valued on a complete residuated lattice in many papers but in this survey we will restrict to the unit interval with a t-norm (mainly left-continuous). This is their definition.

Definition 1.1. Let T be a t-norm. A fuzzy relation E on a universe X is a fuzzy equivalence relation or T-fuzzy equivalence relation if for all $x, y, z \in X$,

1) $E(x, x) = 1$ (Reflexivity)
2) $E(x, y) = E(y, x)$ (Symmetry)
3) $T(E(x, y), E(y, z)) \leq E(x, z)$ (T-transitivity).

If moreover

4) $E(x, y) = 1$ implies $x = y$,

then E is called a fuzzy equality or a fuzzy equality relation. We also say that E separates points.

Definition 1.1 is the natural fuzzification of the concept of equivalence relation. Reflexivity and symmetry are translated in a straightforward way. Transitivity is more delicate and deserves a comment. Property 1.1.3) provides a lower bound to the degree of equivalence or indistinguishability $E(x, z)$ between x and z knowing the values $E(x, y)$ and $E(y, z)$. From a logical point of view, if the t-norm is left-continuous, 1.1.3) is equivalent to

$$T(E(x, y), E(y, z)) \rightarrow E(x, z) = 1$$

where $\rightarrow$ is the residuated implication associated to T and T is interpreted as a conjunctive connector.

The selection of a particular t-norm to model the transitivity depends on the type of logic we are using and provides flexibility to the definition of fuzzy equivalence relation.

Selecting the minimum t-norm we recover the original Zadeh's similarity relations. They are especially useful in taxonomy because their α-cuts (see Definition 4.4) generate hierarchical trees as we will see in Section 4. The next example provides a justification for using the minimum t-norm.

Example 1.2. Use of the minimum t-norm (Kruse, Gebhardt and Klawonn 1994).

Assume that the objects can only be observed through cloudy glasses of thickness ranging from 0 to 1. If after observing the objects x and y through a glass of thickness α we cannot distinguish them, we will say that they are $\alpha-$indistinguishable and define $E(x, y) = \sup\{1 - \alpha \mid x \text{ and } y \text{ are } \alpha-\text{indistinguishable}\}$. It is easy to check that E is min-transitive and hence a similarity relation.

The use of the product t-norm T_P was proposed in (Menger 1951) where he considers $E(x, y)$ as the probability of finding x and y equal as we will see in the next subsection.

E.H. Ruspini considers the use of the Łukasiewicz t-norm $Ł$ in (Ruspini 1977) mainly because if E is an $Ł$-fuzzy equivalence relation, then $1 - E$ is a pseudometric. The following example gives an intuitive justification of the use of this t–norm.

Example 1.3. Use of the Łukasiewicz t-norm (Sols 2012).
Consider a universe of objects defined by a set Ω of binary features. If we pick up one of this features ω we consider that two objects x and y are equal with respect to ω if $x(\omega) = y(\omega)$ and then define $E(x, y)$ as the proportion of features for which x and y are equal. Then

$$\begin{aligned} E(x,z) &= P(\{\omega \in \Omega \mid x(\omega) = z(\omega)\}) \\ &\geq P(\{\omega \in \Omega \mid x(\omega) = y(\omega),\ y(\omega) = z(\omega)\}) \\ &= P(\{\omega \in \Omega \mid x(\omega) = y(\omega)\}) + P(\{\omega \in \Omega \mid y(\omega) = z(\omega)\}) \\ &- P(\{\omega \in \Omega \mid x(\omega) = y(\omega) \text{ or } y(\omega) = z(\omega)\}) \\ &\geq E(x,y) + E(y,z) - 1 \end{aligned}$$

and E is transitive with respect to the t-norm of Łukasiewicz.

Another justification can be found in (Bezdek and Harris 1978).

Example 1.4. Use of t-norms of the Schweizer Sklar family (Sols 2012).
If in the last example we consider that the degree of indistinguishability $E(x, y)$ between two objects x and y is the probability of finding at least n features coinciding in x and y, then E is transitive with respect to the t-norm $T(x, y) = \max\{0, (x^{\frac{1}{n}} + y^{\frac{1}{n}} - 1)^n\}$ belonging to the Schweizer Sklar family.

1.2. Previous Works

Some previous works to Zadeh's article are to be commented going back to (Poincaré 1902). As previously said, Poincaré considers that in the observable physical continuum "equal" means "indistinguishable" because when we say that two objects are equal we only know that they cannot be distinguished. This leads to the paradox that for three objects A, B and C we can have

$$A = B,\ B = C \text{ but } A \neq C$$

failing the transitivity property.

To overcome this paradox, Karl Menger (Menger 1951) associates to any couple (a, b) a number $E(a, b)$ meaning the probability that a and b are equal and calls E a probabilistic relation if it satisfies the following axioms

1) $E(a, a) = 1$ for every a
2) $E(a, b) = E(b, a)$ for every a and b
3) $E(a, b) \cdot E(b, c) \leq E(a, c)$ for every a, b, c.

1) and 2) correspond to the reflexivity and the symmetry of the equality relation, 3) expresses a minimum of transitivity.

Menger realized that $-\log E = d$ is a pseudometric and that from a pseudometric d a probabilistic relation $E = e^{-d}$ can be obtained. These results were generalized later on to continuous Archimedean t-norms in (Valverde 1985).

Menger also contemplates that limitations in perceiving and measuring some properties would lead to an approximate equality. Let us consider the case of a particular appliance providing measurements on the real line with an error margin ε. We have then the following approximate equality:

$$a \approx b \text{ when } |a - b| \leq \varepsilon.$$

But then the relation $\approx$ is not transitive and hence not an equivalence relation. To solve this problem Menger considers the probabilistic relation $E(a, b) = e^{-|a-b|}$.

The problem with the imprecision with measurements was also treated in (Menger 1942) in the study of Probabilistic Metric Spaces. He again considers unrealistic in many situations to think that a precise number can be assigned to the distance between a pair of points p and q and rather associates a distribution function $F_{p,q}$ with p and q. The value $F_{p,q}(x)$ for $x \geq 0$ may then be interpreted as the probability that the distance between p and q is strictly less than x. It took a while before a good definition of the triangular inequality was given as it appears for example in (Schweizer and Sklar 1983) based on triangular functions. These functions can be defined as a convolution, obtaining what is called a Wald space, or by the use of a t-norm, and having what they are known as Menger spaces. [1]

1.3. Zadeh's Similarity Relations

In this subsection we will comment the most relevant aspects on similarity relation of Zadeh's original article.

Zadeh starts by acknowledging the relevance of the concepts of equivalence and similarity in many fields of pure and applied science. He also admits the existence of a number of techniques for dealing with problems involving equivalence, similarity, clustering, preference patterns, etc. as well as that many of these techniques are quite effective in numerical taxonomy, factor analysis, pattern classification and analysis of proximities. The purpose of the article is hence not to provide a new technique but rather to introduce a unifying point of view based on fuzzy sets and fuzzy relations. In 1971 he uses the already consolidated theory of fuzzy sets to develop the concept of similarity (fuzzy) relation and explore its properties and future applications, though the article is purely theoretic.

In Section 2, some basic definitions and properties of fuzzy relations are exposed starting with the definition of fuzzy (binary) relation R from universe X to universe Y as a fuzzy subset of $X \times Y$ or equivalently, a mapping $R : X \times Y \to [0, 1]$ and, for $x \in X$ a $y \in Y$, $R(x, y)$ is interpreted as the *strength* of the relation between x and y. He continues giving some definitions such as domain, range, height, support of a fuzzy relation, the inclusion or containment of a fuzzy relation R in a fuzzy relation S and the union and intersection of fuzzy relations defined with the use of max and min respectively.

If R is a fuzzy relation from X to Y and S a fuzzy relation from Y to Z, the $\sup - \min$ product $R \circ S$ of R and S is the fuzzy relation from X to Z defined for all

[1] It is worth noticing that t-norms appear for the first time in this context.

$x \in X$ and $z \in Z$ by

$$(R \circ S)(x, z) = \sup_{y \in Y}\{\min(R(x, y), S(y, z))\}.$$

Due to the associativity of $\circ$, if R is a fuzzy relation from X to X (simply a fuzzy relation on X), then for $n \in \mathbb{N}$ the power $R^n = \overbrace{R \circ R \circ ... \circ R}^{n \text{ times}}$ is well defined.

At this point Zadeh comments the possibility of replacing the minimum by another operation $*$ being at least associative and monotone non-decreasing in each of its arguments and gives the product as the easiest example. (The reader will notice that t-norms and t-subnorms satisfy these conditions).

Next he points out that for finite universes, a fuzzy relation R can be considered as a matrix where the corresponding (x, y) element is $R(x, y)$ and the $\sup - \min$ product of fuzzy relations coincides with the $\max - \min$ product of their matrices.

For a fuzzy relation from X to Y and $\alpha \in [0, 1]$, Zadeh defines the α-level-set R_α or α-cut R_α of R and proves in Proposition 1 that R can be recovered from its α-cuts by $R = \sup_{\alpha \in [0,1]}\{\alpha \cdot R_\alpha\}$

Section 3 is devoted to the specific study of similarity relations.

According to Zadeh, the definition of similarity relation is as follows.

Definition 1.5. A fuzzy relation R on a universe X is a similarity relation if for all $x, y, z \in X$,

1) $E(x, x) = 1$ (Reflexivity)
2) $E(x, y) = E(y, x)$ (Symmetry)
3) $\sup_{y \in X}\{\min(E(x, y), E(y, z))\} \leq E(x, z)$ (min-transitivity).

It is clear that Condition 1.5.3) is equivalent to

$$\min(E(x, y), E(y, z)) \leq E(x, z) \text{ for all } y \in X.$$

This is the way min-transitivity is usually written. Nevertheless, in Zadeh's article Condition 1.5.3) stresses that a fuzzy relation E is min-transitive if and only if $E \circ E \leq E$, where $\circ$ means the $\sup - \min$ product. The relationship between min-transitivity, the $\sup - \min$ product and the min-transitive closure of a fuzzy relation is established. (In Section 2.1 of the present article the results will be given for general left-continuous t-norms.)

The situation already analyzed by Menger that $a \approx b$ when $|a - b| \leq \varepsilon$ (see Subsection 1.2) is studied by Zadeh and he proves that $e^{-\beta|x-y|}$ with β a positive number is a fuzzy equivalence relation with respect to the product t-norm.

The important fact that E is a similarity relation if and only if $1 - E$ is a pseudoultrametric is proved. The most relevant consequence of this fact is that the α-cuts of a similarity relation E are (crisp) equivalence relations and if $\alpha < \beta$, then $E_\alpha \geq E_\beta$. Therefore there is a bijection between indexed trees and similarity relations if the universe of discourse has finite cardinality (see Section 4).

The study of similarity relations in (Zadeh 1971) ends with the definition of fuzzy equivalence class: If E is a similarity relation on a universe X and $x \in X$, then the fuzzy equivalence class E_x associated to x is the fuzzy subset E_x defined by $E_x(y) = E(x, y)$ for all $y \in Y$. If X is a finite universe, then E_x can be identified with the column x of E.

It is amazing that in the study of similarity relations Zadeh opened the door to almost all their relevant aspects. Namely,

- The max–min product.
- The min–transitivity property of a fuzzy relation.
- The min–transitive closure of a fuzzy relation.
- The definition of similarity relation as it is still valid now.
- The possibility of replacing the minimum with other operations of the unit interval to model transitivity including t-subnorms and t-norms.
- If E is a similarity relation transitive with respect to the product, then e^{-E} is a pseudo-metric.
- The fact that the α-cuts of a similarity relations form a nested family of (crisp) equivalence relations and hence a hierarchical tree.
- The relationship between similarity relations and pseudo-ultrametrics.
- The definition of fuzzy equivalence class.

Extensionality (see Section 3) is perhaps the only important property related to similarity relations that Zadeh does not consider in his article. Curiously enough this is the central property in one of his most relevant ideas: granularity.

As Zadeh points out in his article, in the same year a paper was published containing some of these results (Tamura, Higuchi and Tanaka 1971), which proves that great ideas evolve and get ripe at the same time in different places.

The rest of the article is structured as follows: In the next section we will study the way fuzzy equivalence relations can be generated. Section 3 is devoted to extensionality, the most important property that a fuzzy subset can fulfill with respect to a fuzzy equivalence relation. In Section 4 we will study similarity relations (transitive with respect to the minimum t-norm). The article ends with a section providing different aspects of the usefulness of fuzzy equivalence relations in different areas.

Different names have been given to fuzzy equivalence relations: indistinguishability operators, T-equivalences, fuzzy equalities, similarity relations, etc. Here we will use fuzzy equivalence relation or T-fuzzy equivalence relation.

Fuzzy equivalence relations have been generalized in different ways.

- They have been defined in more general structures like interval valued, type 2, intuitionistic, hesitant fuzzy sets and fuzzy equivalence relations valued on finite subsets of the unit interval.
- The definition has been relaxed in some ways. For example U. Hoehle (Hoehle 1988) replaces reflexivity with $E(x,x) \geq \lambda \in [0,1]$ and defines $E(x,x)$ as the degree of existence of x. J. Martín and G. Mayor consider multiindistinguishabilities, that is: fuzzy equivalence relations with more than two objects (Martín and Mayor 2009) a concept dual to Menger's n-dimensional metrics (Menger 1928).
- The t-norm has been replaced by other operations such as quasi-arithmetic means, conjunctors or semi-copulas.
- Fuzzy equivalence relations have also been valued in more general structures like MV and GL algebras and complete residuated lattices.
- They have also been studied in the setting of category theory (Hoehle 1988, 1992).

For the sake of simplicity in this article we will restrict to left-continuous t-norms to model the transitive property because this assures that the $\sup -T$ product is associative (Drewniak and Pękala 2007) and that the t-norm and its residuation satisfy the adjointness property (Boixader 1998).

2. Generating Fuzzy Equivalence Relations

One of the most interesting issues related to fuzzy equivalence relations is the way they can be generated, which depends on the way in which the data are given and on the use we want to make of them. It also helps to understand their structure. The four most common ways are:

- By calculating the T-transitive closure of a reflexive and symmetric fuzzy relation (a proximity or tolerance relation).
- By using the Representation Theorem.
- By generating a decomposable operator from a fuzzy subset.
- By obtaining a transitive opening of a proximity relation.

2.1. Transitive Closure

The results of this subsection can be found scattered in many places. A book that contains all the results with proofs is (Klir and Yuan 1995).

Given a t-norm T, the transitive closure of a reflexive and symmetric fuzzy relation R on a set X is the smallest T-fuzzy equivalence relation $\overline{R}$ on X greater than or equal to R.

It is instructive to see first what happens in the crisp case: If R is a reflexive and symmetric crisp relation on a finite set X, it can be represented by a graph G_R with vertices the elements of X and with two vertices x and y connected by an edge when xRy. Its transitive closure is the smallest graph that contains G_R and such that all its connected components are complete subgraphs. This makes $\overline{R}$ an equivalence relation. Figure 1 shows a graph corresponding to a reflexive and symmetric relation and the graph of its corresponding transitive closure.

Figure 1. A graph G_R corresponding to a crisp reflexive and symmetric relation R on $\{a, b, c, d, e, f, g\}$ on the left and the graph of its transitive closure $\overline{R}$ on the right.

This produces the well-known chain effect or chaining: two vertices of the graph connected by a chain but very distant belong to the same equivalence class of $\overline{R}$. In Figure 1, for instance, though that the vertices a and d are very far away in G_R, they are in the same connected subgraph of $G_{\overline{R}}$ and therefore related by $\overline{R}$. It is in

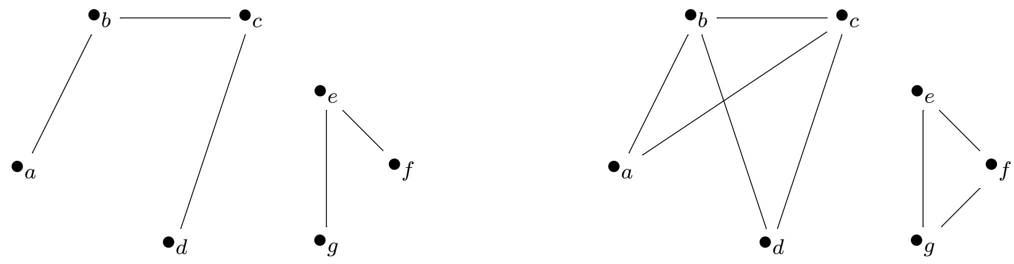


Figure 2. A graph G_R corresponding to a crisp reflexive and symmetric relation R on $\{a, b, c, d, e, f, g\}$ on the left and the graph of R^2 on the right.

general an undesirable property though in some cases can produce good results (see for instance the Example 13.3 of (Klir and Yuan 1995) borrowed from (Tamura, Higuchi and Tanaka 1971)).

Coming back to fuzzy relations, an interesting way to obtain their transitive closure is by using the $\sup -T$ product.

Definition 2.1. Let R and S be two fuzzy relations on X and T a t-norm. The $\sup -T$ product of R and S is the fuzzy relation $R \circ S$ on X defined for all $x, y \in X$ by

$$(R \circ S)(x, y) = \sup_{z \in X} \{T(R(x, z), S(z, y))\}.$$

If the t-norm is left-continuous, then the $\sup -T$ product is associative and the n^{th} power R^n of a fuzzy relation R is

$$R^n = \overbrace{R \circ ... \circ R}^{n \ times}.$$

If the relation R is crisp, then

$$xR^2y \text{ if and only if } \exists z \text{ such that } xRz \text{ and } zRy.$$

Figure 2 illustrates this with an example.

Lemma 2.2. *Let T be a left-continuous t-norm and R a fuzzy relation on a universe X. Then R is T-transitive if and only if $R^2 \leq R$.*[2]

Proof. For $x, z \in X$

$$R^2(x, z) = \sup_{y \in X} \{T(R(x, y), R(y, z))\} \leq R(x, z)$$

is equivalent to

$$T(R(x, y), R(y, z)) \leq R(x, z)$$

[2]Recall that given two fuzzy relations R, S on a universe X, $R \leq S$ when $R(x, y) \leq S(x, y)$ for all $x, y \in X$.

for all $y \in X$. □

Proposition 2.3. *Let T be a left-continuous t-norm and R a fuzzy relation on a universe X. Then the fuzzy relation $\sup_{n\in\mathbb{N}}\{R^n\}$ is the transitive closure $\overline{R}$ of R.*

Proposition 2.4. *Let T be a left-continuous t-norm and $(E_i)_{i\in I}$ a family of T-fuzzy equivalence relations on a universe X. Then $\inf_{i\in I}\{E_i\}$ is a T-equivalence relation on X.*

Proof. Reflexivity and symmetry is trivial. Let us prove T-transitivity.

$$T(\inf_{i\in I}\{E(x,z)\}, \inf_{i\in I}\{E(z,y)\}) \leq \inf_{i\in I}\{T(E(x,z), E(z,y))\} \leq \inf_{i\in I} E(x,y).$$

□

Corollary 2.5. *The T-transitive closure $\overline{R}$ of a reflexive and symmetric fuzzy relation R on a universe X is the intersection of all T-fuzzy equivalence relations on X greater than or equal to R.*

The next lemma proves the importance of the reflexivity property.

Lemma 2.6. *Let T be a left-continuous t-norm and R a reflexive fuzzy relation on X Then $R^n \leq R^m$ if $n \leq m$.*

If X is a finite set we get the following practical result.

Proposition 2.7. *Let T be a left-continuous t-norm and R a reflexive and symmetric fuzzy relation on a universe of finite cardinality $n \geq 2$. Then $\overline{R} = R^{n-1}$.*

Example 2.8. Let R be the fuzzy relation given by the matrix

$$\begin{pmatrix} 1 & 0.9 & 0.3 & 0.4 \\ 0.9 & 1 & 0.5 & 0.4 \\ 0.3 & 0.5 & 1 & 0.9 \\ 0.4 & 0.4 & 0.9 & 1 \end{pmatrix}$$

The transitive closure with respect to the Łukasiewicz, product and minimum t-norms respectively are

$$\begin{pmatrix} 1 & 0.9 & 0.4 & 0.4 \\ 0.9 & 1 & 0.5 & 0.4 \\ 0.4 & 0.5 & 1 & 0.9 \\ 0.4 & 0.4 & 0.9 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0.9 & 0.45 & 0.4 \\ 0.9 & 1 & 0.5 & 0.45 \\ 0.45 & 0.5 & 1 & 0.9 \\ 0.4 & 0.45 & 0.9 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0.9 & 0.5 & 0.5 \\ 0.9 & 1 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.9 \\ 0.5 & 0.5 & 0.9 & 1 \end{pmatrix}.$$

2.2. The Representation Theorem

As we will see in this subsection, from a fuzzy subset μ of a universe X a fuzzy equivalence relation E_μ on X can be generated in a very natural way. This is a very important result because it says that from a feature or characteristic represented by a fuzzy subset we can get a fuzzy equivalence relation. The Representation Theorem states that a fuzzy relation E on a universe X is a fuzzy equivalence relation if and only if it can be obtained as the infimum of fuzzy equivalence relations generated by a family of fuzzy subsets.

Apart from the important fact that every fuzzy equivalence relation can be obtained from a family of features, the Representation Theorem is very much related to the extensionality of fuzzy subsets as we will see in Section 3.

The Representation Theorem is much inspired in a similar (dual) result on metric spaces (Fréchet 1910). Given a mapping $f : X \to [0, \infty]$ on a universe X, $m_f : X \times X \to [0, \infty]$ defined for all $x, y \in X$ by $m_f(x, y) = |f(x) - f(y)|$ is a pseudometric and a representation theorem for metric spaces states that a mapping $m : X \times X \to [0, \infty]$ is a pseudometric on X if and only if there exists a family $(f_i)_{i \in I}$ of mappings $f_i : X \to [0, \infty]$ such that $m = \sup_{i \in I}\{m_{f_i}\}$.

Considering this result and that m is a pseudometric if and only if e^{-m} is a fuzzy equivalence relation with respect to the product t-norm, Ovchinnikov proved a Representation Theorem for these fuzzy relations (Ovchinnikov 1984) and for fuzzy preorders with respect to this t-norm.

The previous results were generalized in (Valverde 1985) to general continuous t-norms, though all results and proofs remain valid for complete residuated lattices and in particular for left-continuous t-norms.

(Fodor and Roubens 1995) proved a representation theorem for transitive fuzzy relations and in (Calvo and Recasens 2021) a representation theorem for local fuzzy equivalence relations (i.e., non-necessarily reflexive) can also be found.

We need to recall the definition of residuation and biresiduation of a left-continuous t-norm.

Definition 2.9. The residuation $\overrightarrow{T}$ of a left-continuous t-norm T is the mapping $\overrightarrow{T} : [0, 1] \times [0, 1] \to [0, 1]$ defined for all $x, y \in [0, 1]$ by

$$\overrightarrow{T}(x|y) = \sup\{\alpha \in [0, 1] \mid T(x, \alpha) \leq y\}.$$

Definition 2.10. The biresiduation $\overleftrightarrow{T}$ of a left-continuous t-norm T is the mapping $\overleftrightarrow{T} : [0, 1] \times [0, 1] \to [0, 1]$ defined for all $x, y \in [0, 1]$ by

$$\overleftrightarrow{T}(x, y) = \min(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)).$$

Example 2.11.

- If T is a continuous Archimedean t-norm with additive generator t, then $\overleftrightarrow{T}(x, y) = t^{-1}(|t(x) - t(y)|)$ for all $x, y \in [0, 1]$.
  As special cases,
  - If L is the Łukasiewicz t-norm, then $\overleftrightarrow{L}(x, y) = 1 - |x - y|$ for all $x, y \in [0, 1]$.
  - If T is the product t-norm, then $\overleftrightarrow{T}(x, y) = \min(\frac{x}{y}, \frac{y}{x})$ for all $x, y \in [0, 1]$ with the convention $\frac{z}{0} = 1$.
- If T is the minimum t-norm, then

$$\overleftrightarrow{T}(x, y) = \begin{cases} \min(x, y) & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 2.12. *Let T be a left-continuous t-norm. Then $\overleftrightarrow{T}$ is a T-fuzzy equivalence relation on $[0, 1]$ that separates points.*

Proposition 2.13. *Let μ be a fuzzy subset of a universe X and T a left-continuous t-norm. The fuzzy relation E_μ on X defined for all $x, y \in X$ by*

$$E_\mu(x,y) = \overleftrightarrow{T}(\mu(x), \mu(y))$$

is a T-fuzzy equivalence relation.

When $\mu = A$ is a crisp subset of X, E_A generates a partition of X into A and its complementary set $X - A$, since in this case $E_A(x,y) = 1$ if and only if x and y both belong to A or to $X - A$.

Representation Theorem 2.14. *(Valverde 1985) Let E be a fuzzy relation on a universe X and T a left-continuous t-norm. E is a T-fuzzy equivalence relation if and only if there exists a family $(\mu_i)_{i\in I}$ of fuzzy subsets of X such that for all $x, y \in X$*

$$E(x,y) = \inf_{i\in I} E_{\mu_i}(x,y).$$

$(\mu_i)_{i\in I}$ is called a generating family of E. A fuzzy subset belonging to a generating family of E is called a generator of E. A generating family of E with minimal cardinality is called a basis of E and the cardinality of the corresponding set of indexes its dimension.

The following result is trivial but will have important consequences in Section 3 as it shows the equivalence of being a generator and being extensional.

Lemma 2.15. *μ is a generator of E if and only if $E_\mu \geq E$.*

Example 2.16. Let $\mu = (0.3, 0.5, 0.8)$ be a fuzzy subset of $X = \{x_1, x_2, x_3\}$. The fuzzy equivalence relations generated by μ with respect to the minimum, product and Łukasiewicz t-norms are respectively

$$\begin{pmatrix} 1 & 0.3 & 0.3 \\ 0.3 & 1 & 0.5 \\ 0.3 & 0.5 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0.6 & 0.375 \\ 0.6 & 1 & 0.625 \\ 0.375 & 0.625 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0.8 & 0.5 \\ 0.8 & 1 & 0.7 \\ 0.5 & 0.7 & 1 \end{pmatrix}.$$

2.3. Decomposable Fuzzy Equivalence Relations

The study of decomposable fuzzy equivalence relations (Recasens 2008) is interesting basically for three reasons.

- They satisfy an interesting (though trivial) tetrahedral relation.
- They generate radial betweenness relations for continuous Archimedean t-norms as we will see in Section 5.1.
- t-norms are decomposable local (i.e.: non-reflexive) fuzzy equivalence relations (Calvo and Recasens 2021).

Definition 2.17. Let T be a t-norm and X a universe. The decomposable fuzzy relation E^μ generated by a fuzzy subset μ of X is defined for all $x, y \in X$ by

$$E^\mu(x,y) = T(\mu(x), \mu(y)).$$

The normalized decomposable fuzzy relation E_1^μ is

$$E_1^\mu(x,y) = \begin{cases} T(\mu(x),\mu(y)) & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 2.18. *Let T be a t-norm and μ a fuzzy subset of a universe X. Then E_1^μ is a T-fuzzy equivalence relation and E^μ a local T-fuzzy equivalence.*

Proposition 2.19. *Let T be a continuous Archimedean t-norm with additive generator t and μ a fuzzy subset of a universe X. If the decomposable local T-fuzzy equivalence relation E^μ on X generated by μ satisfies $E^\mu(x,y) \neq 0$ for all $x,y \in X$, then it generates the following tetrahedral relation on X: Given four different elements $x,y,z,u \in X$,*

$$T(E^\mu(x,y), E^\mu(z,u)) = T(E^\mu(x,z), E^\mu(y,u)).$$

Figure 3 shows this kind of relation.

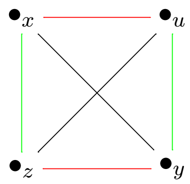


Figure 3. The tetrahedral relation of a decomposable fuzzy equivalence relation.

Definition 2.20. A fuzzy subset μ of a universe X is normal if there exists an $x \in X$ with $\mu(x) = 1$.

Proposition 2.21. *Let T be a continuous Archimedean t-norm with additive generator t and μ a normal fuzzy subset of a universe X with $a \in X$ and $\mu(a) = 1$. If the decomposable T-equivalence relation E^μ on X generated by μ satisfies $E^\mu(x,y) \neq 0$ for all $x,y \in X$, then for all $x,y \in X$,*

$$T(E^\mu(x,a), E^\mu(a,y)) = E^\mu(x,y) \text{ and}$$

$$T(E_1^\mu(x,a), E_1^\mu(a,y)) = E_1^\mu(x,y).$$

This means that the betweenness relation (see Subection 5.3) generated by E_1^μ is radial in the sense that a is between any pair of points of X and makes decomposable fuzzy equivalence relations suitable for modelling radial systems where there is a center, node or hub.

2.4. Transitive Openings

The T-transitive closure of a reflexive and symmetric fuzzy relation R is a T-fuzzy equivalence relation greater than or equal to R. It is the best upper approximation since the infimum of T-fuzzy equivalence relations is also a T-fuzzy equivalence relation. If we want a lower approximation, then the situation is more complicated since the supremum of fuzzy equivalence relations is not such a relation in general. What we can find is T-fuzzy equivalence relations maximal among the ones that are smaller than or equal to a given reflexive and symmetric fuzzy relation. These relations are called transitive openings and in general they are not unique, but there can be an infinite quantity of them, even in sets of finite cardinality.

In (Fodor and Roubens 1995), an algorithm to find maximal transitive openings of a given fuzzy relation is given, but the obtained openings are not symmetric in general and in the process of symmetrizing them, maximality can be lost. Heuristic methods to obtain T-fuzzy equivalence relations smaller than or equal to a given fuzzy relation close to maximal ones have been proposed in (De Baets and De Meyer 2003) and in (Boixader and Recasens 2011) a procedure is given for calculating all the transitive openings of a reflexive and symmetric fuzzy relation with rational values on a finite set X with respect to the Łukasiewicz t-norm.

The minimum t-norm is an exception because of the special behavior of min-fuzzy equivalence relations. In this case there are a number of algorithms to find at least some of the min-transitive openings of a given reflexive and symmetric fuzzy relation. A classic method is the complete linkage. Other algorithms can be found in (Dawyndt, De Meyer and De Baets 2005; Garmendia et al 2009).

2.4.1. The Łukasiewicz t-norm

In this subsection a method for calculating all the transitive openings of a reflexive and symmetric fuzzy relation with rational values on a finite set X with respect to the Łukasiewicz t-norm will be explained. The method is very effective when combined with heuristic algorithms such as the ones proposed in (De Baets and De Meyer 2003) (see Example 2.26).

Definition 2.22. A reflexive and symmetric fuzzy relation on a universe X will be called a proximity or tolerance relation on X.

Definition 2.23. Let R be a proximity relation on a universe X and T a t-norm. A T-fuzzy equivalence relation $\underline{R}$ on X is a T-transitive opening of R if

- $\underline{R} \leq R$
- If E is another T-fuzzy equivalence relation on X satisfying $E \leq R$, then $E \leq \underline{R}$.

Proposition 2.24. *Let R be a proximity relation on a finite set $X = \{r_1, r_2, ..., r_s\}$ of cardinality s and T a t-norm. S is a T-fuzzy equivalence relation smaller than or equal to R if and only if its entries satisfy the following system of inequalities for all $i, j, k = 1, 2, ..., s$.*

$$\begin{aligned} 0 \leq S(r_i, r_j) &\leq R(r_i, r_j) \\ T(S(r_i, r_j), S(r_j, r_k)) &\leq S(r_i, r_k) \\ S(r_i, r_j) &= S(r_j, r_i) \end{aligned}$$

Example 2.25. Let us consider the reflexive and symmetric relation R on $X =$

$\{s_1, s_2, s_3\}$ with matrix

$$R = \begin{pmatrix} 1 & \frac{2}{3} & 0 \\ \frac{2}{3} & 1 & \frac{2}{3} \\ 0 & \frac{2}{3} & 1 \end{pmatrix}.$$

A fuzzy relation S on X with matrix

$$S = \begin{pmatrix} 1 & p & q \\ p & 1 & r \\ q & r & 1 \end{pmatrix}$$

is a fuzzy equivalence relation smaller than or equal to R if and only if

$$\begin{array}{ccc} 0 \leq p \leq \dfrac{2}{3} & 0 \leq q \leq 0 & 0 \leq r \leq \dfrac{2}{3} \\ T(p,q) \leq r & T(p,r) \leq q & T(q,p) \leq r \\ T(q,r) \leq p & T(r,p) \leq q & T(r,q) \leq p. \end{array}$$

If L is the Łukasiewicz t-norm and we are looking for transitive openings, we can consider only solution with a 3 on the denominator (see (Boixader and Recasens 2011)), then there are 8 possible solutions in $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$:

$$\begin{array}{c} p = 0, \dfrac{1}{3}, \quad q = 0, \quad r = 0, \dfrac{1}{3}, \dfrac{2}{3} \\ p = \dfrac{2}{3}, \quad q = 0, \quad r = 0, \dfrac{1}{3}. \end{array}$$

Among them, there are 2 Ł-transitive openings of R. Namely

$$\begin{pmatrix} 1 & \frac{1}{3} & 0 \\ \frac{1}{3} & 1 & \frac{2}{3} \\ 0 & \frac{2}{3} & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \frac{2}{3} & 0 \\ \frac{2}{3} & 1 & \frac{1}{3} \\ 0 & \frac{1}{3} & 1 \end{pmatrix}.$$

The method is very greedy and in general needs many calculations. Fortunately, when combined with heuristic algorithms it can be very effective as it is shown in the next example.

Example 2.26. Consider the reflexive and symmetric fuzzy relation R on a set of cardinality 5 with matrix

$$R = \begin{pmatrix} 1.0 & 0.5 & 0.5 & 0.7 & 0.6 \\ 0.5 & 1.0 & 0.8 & 0.7 & 0.5 \\ 0.5 & 0.8 & 1.0 & 0.9 & 0.8 \\ 0.7 & 0.7 & 0.9 & 1.0 & 0.9 \\ 0.6 & 0.5 & 0.8 & 0.9 & 1.0 \end{pmatrix}.$$

In (De Baets and De Meyer 2003) an algorithm is used to calculate an $Ł$-fuzzy equivalence relation S which is not a transitive opening of R but smaller than R and close to a transitive opening:

$$S = \begin{pmatrix} 1.0 & 0.5 & 0.5 & 0.6 & 0.6 \\ 0.5 & 1.0 & 0.8 & 0.6 & 0.5 \\ 0.5 & 0.8 & 1.0 & 0.8 & 0.7 \\ 0.6 & 0.6 & 0.8 & 1.0 & 0.9 \\ 0.6 & 0.5 & 0.7 & 0.9 & 1.0 \end{pmatrix}.$$

There are only 2 T-fuzzy equivalence relations with entries in $\{0, 0.1, 0.2, ..., 1\}$ greater than or equal to S and smaller than R that can be obtained in the same way as in Example 2.25:

$$\begin{pmatrix} 1.0 & 0.5 & 0.5 & 0.6 & 0.6 \\ 0.5 & 1.0 & 0.8 & 0.6 & 0.5 \\ 0.5 & 0.8 & 1.0 & 0.8 & 0.7 \\ 0.6 & 0.6 & 0.8 & 1.0 & 0.9 \\ 0.6 & 0.5 & 0.7 & 0.9 & 1.0 \end{pmatrix}, \begin{pmatrix} 1.0 & 0.5 & 0.5 & 0.7 & 0.6 \\ 0.5 & 1.0 & 0.8 & 0.6 & 0.5 \\ 0.5 & 0.8 & 1.0 & 0.8 & 0.7 \\ 0.6 & 0.6 & 0.8 & 1.0 & 0.9 \\ 0.6 & 0.5 & 0.7 & 0.9 & 1.0 \end{pmatrix}.$$

The second matrix is a L-transitive opening of R.

2.4.2. Complete Linkage. The minimum t-norm

In the complete linkage, the entries of a proximity relation $R = (a_{ij})$ on a finite set $X = \{x_1, x_2, ...x_n\}$ are modified according to the next algorithm to obtain a min-transitive opening. Given two disjoint subsets C_1, C_2 of X their similarity degree S is defined by $S(C_1, C_2) = \min_{x_i \in C_1, x_j \in C_2}(a_{ij})$.

(1) Initially a cluster C_i is assigned to every element x_i of X (i.e. the clusters of the first partition are singletons).
(2) In each new step two clusters are merged in the following way.
    If $\{C_1, C_2, ..., C_k\}$ is the actual partition, then we must select the two clusters C_i and C_j for which the similarity degree $S(C_i, C_j)$ is maximal. (If there are several such maximal pairs, one pair is picked at random). The new cluster $C_i \cup C_j$ replaces the two clusters C_i and C_j, and all entries of a_{mn} and a_{nm} of R with $m \in C_i$ and $n \in C_j$ are lowered to $S(C_i, C_j)$.
(3) Step 2 is repeated until there remains one single cluster containing all the elements of X.

Example 2.27. Let us consider the proximity R on $X = \{x_1, x_2, x_3, x_4\}$ with matrix

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0.1 & 0.7 & 0.4 \\ 0.1 & 1 & 0.4 & 0.3 \\ 0.7 & 0.4 & 1 & 0.5 \\ 0.4 & 0.3 & 0.5 & 1 \end{pmatrix} \end{array}.$$

The first partition is $C_1 = \{x_1\}$, $C_2 = \{x_2\}$, $C_3 = \{x_3\}$, $C_4 = \{x_4\}$. The greatest similarity degree between clusters is $S(C_1, C_3) = 0.7$. These two clusters are merged to form $C_{13} = \{x_1, x_3\}$. The matrix does not change in this step.

The new partition is C_{13}, C_2, C_4. The similarity degrees are

$$\begin{aligned} S(C_{13}, C_2) &= \min(a_{12}, a_{32}) = \min(0.1, 0.4) = 0.1 \\ S(C_{13}, C_4) &= \min(a_{14}, a_{34}) = \min(0.4, 0.5) = 0.4 \\ S(C_2, C_4) &= a_{24} = 0.3. \end{aligned}$$

The greatest similarity degree is 0.4 and the new partition is therefore $C_{134} = \{x_1, x_3, x_4\}$, $C_2 = \{x_2\}$. The entries a_{14}, a_{41}, a_{34}, a_{43} of the matrix R are replaced by 0.4 obtaining

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0.1 & 0.7 & 0.4 \\ 0.1 & 1 & 0.4 & 0.3 \\ 0.7 & 0.4 & 1 & 0.4 \\ 0.4 & 0.3 & 0.4 & 1 \end{pmatrix} \end{array}.$$

In the last step, we merge the two clusters C_{134}, C_2. The similarity degree is

$$S(C_{134}, C_2) = \min(a_{12}, a_{32}, a_{42}) = \min(0.1, 0.4, 0.3) = 0.1.$$

The transitive opening of R obtained by complete linkage is then

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{pmatrix} 1 & 0.1 & 0.7 & 0.4 \\ 0.1 & 1 & 0.1 & 0.1 \\ 0.7 & 0.1 & 1 & 0.4 \\ 0.4 & 0.1 & 0.4 & 1 \end{pmatrix} \end{array}.$$

3. Extensional Fuzzy Subsets

Extensionality is the most important property that a fuzzy subset can satisfy with respect to a fuzzy equivalence relation because extensional fuzzy subsets are the only ones that can be observed when the fuzzy equivalence relation is considered.

Let us analyze for the moment what happens in the crisp case where there is a universe X and an equivalence relation $\sim$ defined on X. In this situation X is partitioned into equivalence classes that constitute the smallest parts of X that can be observed through $\sim$. Then all observable subsets of X are combinations (unions (and intersections if we want to obtain the empty set)) of equivalence classes of $\sim$. A subset A of X can be approximated by observable ones from above and from bellow and the result is a rough set (see figure 4).

If we have a fuzzy equivalence relation E on X, then the observable fuzzy subsets of X with respect to E are exactly the extensional ones. This section will start with the study of the structure of the set H_E of all extensional fuzzy subsets of X with respect to E. Then we will study two operators, ϕ_E and ψ_E, allowing us to obtain the optimal upper and lower approximations of a fuzzy subset of X by extensional ones. Finally, the concept of fuzzy point will be introduced.

We will see that the extensional fuzzy subsets can be obtained as combination of fuzzy equivalence classes and hence are the basis of the granularity of the system. Granularity, according to L.A. Zadeh, is one of the basic concepts that underlie human

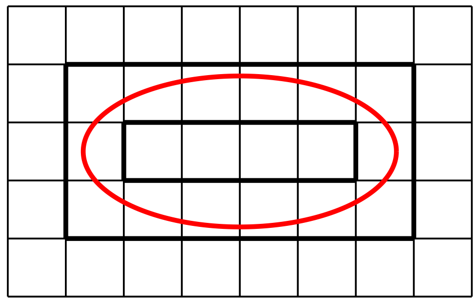


Figure 4. A subset in red and its upper and lower approximations.

cognition and the elements within a granule '*have to be dealt with as a whole rather than individually*'.

> Informally, granulation of an object A results in a collection of granules of A, with a granule being a clump of objects (or points) which are drawn together by indistinguishability, similarity, proximity or functionality. L.A. Zadeh

In this section all t-norms are assumed to be left-continuous.

3.1. The Set H_E of Extensional Fuzzy Subsets

Definition 3.1. Let E be a T-fuzzy equivalence relation on a set X. A fuzzy subset μ of X is extensional with respect to E (or simply extensional) if for all $x, y \in X$

$$T(E(x,y), \mu(y)) \leq \mu(x).$$

H_E will be the set of extensional fuzzy subsets of X with respect to E.

Remark. This definition fuzzifies the predicate

> If x and y are equivalent and $y \in \mu$, then $x \in \mu$.

The extensional sets of E coincide with its generators in the Representation Theorem.

Proposition 3.2. *Let E be a T-fuzzy equivalence relation on X, μ a fuzzy subset of X and E_μ the T-fuzzy equivalence relation generated by μ. $\mu \in H_E$ if and only if $E_\mu \geq E$.*

The columns of E (its fuzzy equivalence classes) are extensional.

Proposition 3.3. *Given a T-fuzzy equivalence relation E on a set X and an element $x \in X$, the column $\mu_x = E(x, \cdot)$ of x is extensional.*

Proposition 3.4. *(Klawonn and Castro 1995) Let E be a T-fuzzy equivalence relation on a set X. The following properties are satisfied for all $\mu \in H_E$, $(\mu_i)_{i \in I}$ a family of* *extensional fuzzy subsets and* $\alpha \in [0,1]$.

(1) $\sup_{i\in I}\{\mu_i\} \in H_E$.
(2) $\inf_{i\in I}\{\mu_i\} \in H_E$.
(3) $T(\alpha,\mu) \in H_E$.
(4) $\overrightarrow{T}(\mu|\alpha) \in H_E$.
(5) $\overrightarrow{T}(\alpha|\mu) \in H_E$.

where $T(\alpha,\mu)$, $\overrightarrow{T}(\mu|\alpha)$ *and* $\overrightarrow{T}(\alpha|\mu)$ *are the fuzzy subsets of* X *defined for all* $x \in X$ *by* $T(\alpha,\mu)(x) = T(\alpha,\mu(x))$, $\overrightarrow{T}(\mu|\alpha)(x) = \overrightarrow{T}(\mu(x)|\alpha)$ *and* $\overrightarrow{T}(\alpha|\mu)(x) = \overrightarrow{T}(\alpha|\mu(x))$.

The next important Proposition 3.5 states that there is a bijection between the sets satisfying the properties of the last proposition and the sets H_E.

Proposition 3.5. *(Klawonn and Castro 1995) Let* H *be a subset of* $[0,1]^X$ *satisfying the properties of the last proposition. Then there exists a* T*-fuzzy equivalence relation* E *on* X *such that* $H = H_E$. E *is uniquely determined and it is generated (using the Representation Theorem) by the family of elements of* H.

Reinterpreting the proof of this result, we obtain the important fact that the extensional fuzzy subsets of E can be obtained (are combinations) of the fuzzy equivalence classes of E, which generalizes the fact that the observable crisp subsets of a crisp equivalence relation are unions of its equivalence classes.

Corollary 3.6. *(Boixader and Recasens 2018b) Let* FC *be the set of fuzzy equivalence classes or columns of a* T*-fuzzy equivalence relation* E *on a set* X *(i.e.,* $FC = \{\mu_y \in [0,1]^X \mid \mu_y(x) = E(x,y)\ \forall x,y \in X\}$*). Then* H_E *is the smallest subset of* $[0,1]^X$ *closed by the operations of Proposition 3.4 and containing* FC.

In other words, for every extensional fuzzy subset μ of E and $x \in X$, $\mu(x) = \sup_{y\in X}\{T(\mu_y(x),\mu(y))\}$.

Remark. The obtained expression for μ: $\mu(x) = \sup_{y\in X}\{T(\mu_y(x),\mu(y))\}$ fuzzifies the predicate

> $x \in \mu$ if and only if there exists an equivalence class $[y]$ of E such that $x \in [y]$ and $[y] \subseteq \mu$.

In other words, in the crisp case μ is the union of equivalence classes of E.

In view of these results H_E can be interpreted as the set of fuzzy subsets of X/E.

To finish this subsection, let us point out that for local fuzzy equivalence relations (i.e.: non-necessarily reflexive) extensionality must be referred to couples of fuzzy subsets (Calvo and Recasens 2021).

Definition 3.7. Let E be a local T-fuzzy equivalence relation on a universe X. A couple (μ,ν) of fuzzy subsets of X is extensional with respect to E if $\mu \geq \nu$ and for all $x,y \in X$

$$T(E(x,y),\mu(y)) \leq \nu(x).$$

3.2. The Mappings ϕ_E and ψ_E

ϕ_E and ψ_E are operators associated to a fuzzy equivalence relation E that allow us to find the optimal upper and lower approximation of a fuzzy subset by extensional ones. They are a fuzzy consequence and fuzzy interior operator respectively.

Fuzzy consequence and fuzzy interior operators appear in many branches of Soft Computing. They have been used as logic operators in fuzzy logic (Elorza and Burillo 1999; Pavelka 1979) and in Approximate Reasoning (Klawonn and Castro 1995). They have also been proved useful for finding upper and lower approximations of fuzzy rough sets (Morsi and Yakout 1998; Boixader, Jacas and Recasens 2000b), for finding extensional fuzzy subsets of fuzzy relations (Jacas and Recasens 1994a), as closure and interior operators of a fuzzy topology (Jacas and Recasens 1994a) and in fuzzy mathematical morphology (Bloch and Maître 1995), among others.

Definition 3.8. Let E be a T-fuzzy equivalence relation on a set X. The map $\phi_E : [0,1]^X \to [0,1]^X$ is defined for all $x \in X$ by

$$\phi_E(\mu)(x) = \sup_{y \in X}\{T\left(E(x,y), \mu(y)\right)\}.$$

Proposition 3.9. *(Klawonn and Castro 1995; Morsi and Yakout 1998) Let E be a T-fuzzy equivalence relation on a universe X, μ, μ' fuzzy subsets of X, $(\mu_i)_{i \in I}$ a family of fuzzy subsets of X and $\alpha \in [0,1]$. Then*

(1) If $\mu \leq \mu'$ then $\phi_E(\mu) \leq \phi_E(\mu')$.
(2) $\mu \leq \phi_E(\mu)$.
(3) $\phi_E(\phi_E(\mu)) = \phi_E(\mu)$.
(4) $\phi_E(\sup_{i \in I}\{\mu_i\}) = \sup_{i \in I}\{\phi_E(\mu_i)\}$.
(5) $\phi_E(\{x\})(y) = \phi_E(\{y\})(x)$
(6) $\phi_E(T(\alpha, \mu)) = T(\alpha, \phi_E(\mu))$.

The extensional fuzzy subsets are characterized as the fixed points of ϕ_E.

Proposition 3.10. *Let E be a T-fuzzy equivalence relation on a universe X and μ a fuzzy subset of X. Then $\mu \in H_E$ if and only if $\phi_E(\mu) = \mu$.*

$\phi_E(\mu)$ is therefore an upper approximation of μ. The next proposition says that $\phi_E(\mu)$ is the most specific extensional set that contains μ and in this sense it is the optimal upper approximation of μ in H_E.

Proposition 3.11. *Let E be a T-fuzzy equivalence relation on a universe X and μ a fuzzy subset of X. Then*

$$\phi_E(\mu) = \inf_{\mu' \in H_E}\{\mu \leq \mu'\}.$$

ϕ_E is defined from the fuzzy equivalence relation E. Reciprocally, E can be recovered form ϕ_E.

Proposition 3.12. *Let $\phi : [0,1]^X \to [0,1]^X$ be a map satisfying the properties of Proposition 3.9. The fuzzy relation E_ϕ on X defined for all $x, y \in X$ by*

$$E_\phi(x,y) = \phi(\{x\})(y)$$

is a T-fuzzy equivalence relation on X.

This proposition provides a bijection between the operators satisfying the properties of Proposition 3.9 and fuzzy equivalence relations.

Dual to the operator ϕ_E providing upper approximations of fuzzy subsets by extensional ones, another operator ψ_E can be defined that gives lower approximations.

Definition 3.13. Let E be a T-fuzzy equivalence relation on a set X. The map $\psi_E : [0,1]^X \to [0,1]^X$ is defined for all $x \in X$ by

$$\psi_E(\mu)(x) = \inf_{y \in X} \{\overrightarrow{T}(E(x,y)|\mu(y))\}.$$

Proposition 3.14. *(Morsi and Yakout 1998) Let E be a T-fuzzy equivalence relation on a universe X, μ, μ' fuzzy subsets of X, $(\mu_i)_{i \in I}$ a family of fuzzy subsets of X and $\alpha \in [0,1]$. Then*

(1) $\mu \leq \mu' \Rightarrow \psi_E(\mu) \leq \psi_E(\mu')$.
(2) $\psi_E(\mu) \leq \mu$.
(3) $\psi_E(\psi_E(\mu)) = \psi_E(\mu)$.
(4) $\psi_E(\inf_{i \in I}\{\mu_i\}) = \inf_{i \in I}\{\psi_E(\mu_i)\}$.
(5) $\psi_E(\overrightarrow{T}(\{x\}|\alpha))(y) = \psi_E(\overrightarrow{T}(\{y\}|\alpha))(x)$.
(6) $\psi_E(\overrightarrow{T}(\alpha|\mu)) = \overrightarrow{T}(\alpha|\psi_E(\mu))$.

where the singleton $\{x\}$ is interpreted as the fuzzy subset defined for all $z \in X$ by $\{x\}(z) = \begin{cases} 1 & if\ z = x \\ 0 & otherwise. \end{cases}$ *(similarly with $\{y\}$).*

The extensional fuzzy subsets of E are characterized as the fixed point of ψ_E.

Proposition 3.15. *Let E be a T-fuzzy equivalence relation on a universe X and μ a fuzzy subset of X. Then $\mu \in H_E$ if and only if $\psi_E(\mu) = \mu$.*

The next proposition shows that $\psi_E(\mu)$ is the best lower approximation of μ by extensional fuzzy subsets.

Proposition 3.16. *Let E be a T-fuzzy equivalence relation on a universe X and μ a fuzzy subset of X. Then*

$$\psi_E(\mu) = \sup_{\mu' \in H_E} \{\mu' \leq \mu\}.$$

E can also be recovered from ψ_E.

Proposition 3.17. *Let $\psi : [0,1]^X \to [0,1]^X$ be an operator satisfying the properties of the Proposition 3.14. Then the fuzzy relation E_ψ on X defined for all $x, y \in X$ by*

$$E_\psi(x,y) = \inf_{\alpha \in [0,1]} \{\overrightarrow{T}(\psi(\overrightarrow{T}(\{x\}|\alpha)(y)|\alpha))\}.$$

is a T-fuzzy equivalence relation on X.

This proposition provides a bijection between the operators satisfying the properties of Proposition 3.14 and fuzzy equivalence relations.

3.3. Fuzzy Points

Zadeh fuzzified the concept of equivalence classes as the columns of a fuzzy equivalence relations. Fuzzy points (Boixader and Recasens 2008) are another generalization of equivalence classes with non-necessarily normal fuzzy subsets.

Definition 3.18. Let E be a T-fuzzy equivalence relation on a set X. $\mu \in H_E$ is a fuzzy point of X with respect to E if for all $x_1, x_2 \in X$

$$T\left(\mu(x_1), \mu(x_2)\right) \leq E(x_1, x_2).$$

P_X will denote the set of fuzzy points of X with respect to E.

The above definition fuzzifies the sentence

If x_1 and x_2 belong to μ, then x_1 and x_2 are equivalent.

The following map Λ_E allows us to characterize the columns, the fuzzy points and the maximal fuzzy points of a fuzzy equivalence relation E.

Definition 3.19. Let E be a T-fuzzy equivalence relation on a set X. The map $\Lambda_E : [0,1]^X \to [0,1]^X$ is defined by

$$\Lambda_E(\mu)(x) = \inf_{y \in X}\{\overrightarrow{T}\left(\mu(y)|E(y,x)\right)\} \ \forall x \in X.$$

Proposition 3.20. *Let E be a T-fuzzy equivalence relation on X and μ a normal fuzzy subset of X. Then $\Lambda_E(\mu) = \mu$ if and only if μ is a column of E.*

Proposition 3.21. *Let E be a T-fuzzy equivalence relation on X and $\mu \in H_E$. $\Lambda_E(\mu) \geq \mu$ if and only if $\mu \in P_X$.*

Proposition 3.22. *Let E be a T-fuzzy equivalence relation on X. The set of fixed point of Λ_E is the set of all maximal fuzzy points of X.*

The next two propositions relate fuzzy points, decomposable fuzzy equivalence relations and the Representation Theorem.

Proposition 3.23. *(Klawonn and Castro 1995) Let $(\mu_i)_{i \in I}$ be a family of normal fuzzy subsets of X and $(x_i)_{i \in I}$ a family of elements of X such that $\mu_i(x_i) = 1$ for all $i \in I$. Then the following two properties are equivalent.*

a) There exists a T-fuzzy equivalence relation E on X such that

$$\mu_i(x) = E(x, x_i) \ \forall i \in I \ \forall x \in X.$$

b) For all $i, j \in I$,

$$\sup_{x \in X}\{T(\mu_i(x), \mu_j(x))\} \leq \inf_{y \in X}\{\overleftrightarrow{T}\left(\mu_i(y), \mu_j(y)\right)\}.$$

Proposition 3.24. *(Klawonn and Castro 1995) Let $(\mu_i)_{i \in I}$ be a family of normal* *fuzzy subsets of X and $(x_i)_{i\in I}$ a family of elements of X such that $\mu_i(x_i) = 1$ for all $i \in I$ satisfying*

$$\sup_{x\in X}\{T(\mu_i(x), \mu_j(x))\} \leq \inf_{y\in X}\{\overleftrightarrow{T}(\mu_i(y), \mu_j(y))\}.$$

for all $i, j \in I$. Then the T-fuzzy equivalence relation $E = \sup_{i\in I}\{E^{\mu_i}\}$ is the smallest T-fuzzy equivalence relation on X satisfying

$$\mu_i(x) = E(x, x_i)\ \forall i \in I,\ \forall x \in X.$$

4. Similarity Relations

Similarity relations (min-fuzzy equivalence relations) are widely used in Taxonomy because they are closely related to hierarchical trees. Indeed, given a min-fuzzy equivalence relation on a set X and $\alpha \in [0, 1]$, the α-cuts of E are partitions of X and if $\alpha \geq \beta$, then the α-cut is a refinement of the partition corresponding to the β-cut. Therefore, E generates an indexed hierarchical tree. Reciprocally, from an indexed hierarchical tree a min-fuzzy equivalence relation can be generated. These results follow from the fact that $1-E$ is a pseudoultrametric. The topologies generated by ultrametrics are very peculiar, since if two balls are not disjoint, then one of them is included in the other one. The results of this section were already obtained in Zadeh's article.

4.1. Similarity Relations and Ultrametrics

Definition 4.1. A map $m : X \times X \to \mathbb{R}$ is a pseudoultrametric if and only if for all $x, y, z \in X$

(1) $m(x, x) = 0$.
(2) $m(x, y) = m(y, x)$.
(3) $\max(m(x, y), m(y, z)) \geq m(x, z)$.

If $m(x, y) = 0$ implies $x = y$, then it is called an ultrametric.

Pseudoultrametrics are pseudodistances where the triangular inequality has been strengthen by replacing the addition by the maximum.

Ultrametric spaces have a very special behavior.

Proposition 4.2. *Let m be an ultrametric on X. Then*

(1) If $B(x, r)$ denotes the ball of center x and radius r and $y \in B(x, r)$, then $B(x, r) = B(y, r)$. (All elements of a ball are its center).
(2) If two balls have non-empty intersection, then one of them is contained in the other one.

Proposition 4.3. *Let E be a fuzzy relation on a set X. E is a similarity relation on X if and only if $m = 1 - E$ is a pseudoultrametric.*

From this proposition interesting results follow. One of them is that the cardinality of $\mathrm{Im}(E) = \{E(x, y)\}$ is smaller than or equal to the cardinality of X. In particular, if X is finite of cardinality n and E is identified with its matrix, then the number of different entries of the matrix is less than or equal to n, which simplifies calculation and storage.

Another important result relates to the α-cuts.

Definition 4.4. For a fuzzy relation E on X and $\alpha \in [0,1]$, the α-cut of E is the set E_α of pairs $(x,y) \in X \times X$ such that $E(x,y) \geq \alpha$.

If $\alpha \geq \beta$, then $E_\alpha \leq E_\beta$ and as a consequence of Propositions 4.2 and 4.3 we have the following result.

Proposition 4.5. *Let E be a fuzzy relation on X. E is a similarity relation on X if and only if for each $\alpha \in [0,1]$, the α-cut of E is an equivalence relation on X.*

4.2. Similarity Relations and Hierarchical Trees

Definition 4.6. A hierarchical tree of a finite set X is a sequence of partitions $A_1, A_2, ..., A_k$ of X such that every partition refines the preceding one.

A hierarchical tree is indexed if every partition A_i has associated a non-negative number λ_i and $\lambda_i < \lambda_{i+1}$ for all $i = 1, 2, ..., k-1$.

Proposition 4.7. *Every similarity relation E on a finite set X generates an indexed hierarchical tree on X.*

Proof. We can consider the α-cuts of E with $\alpha \in \mathrm{Im}(E)$. □

Reciprocally,

Proposition 4.8. *Every indexed hierarchical tree $A_1, A_2, ..., A_k$ of a finite set X with $\lambda_k = 1$ generates a similarity relation E on X.*

Proof. If $A_1, A_2, ..., A_k$ is the set of partitions of the tree and $\lambda_1, \lambda_2, ..., \lambda_k$ the corresponding indexes, then it is easy to verify that the fuzzy relation E defined by

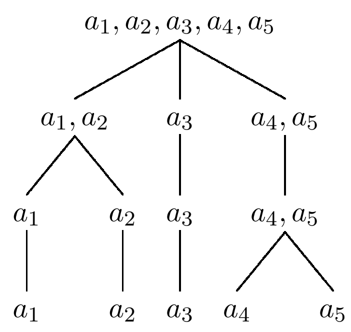
$$E(x,y) = \max_{i=1,2,...,k} \{\lambda_i \text{ such that } (x,y) \in A_i\}$$

for all $x, y \in X$ is a similarity relation. □

Example 4.9. Let $X = \{a_1, a_2, a_3, a_4, a_5\}$ and E the similarity relation on X with matrix

$$\begin{pmatrix} 1 & 0.5 & 0.2 & 0.2 & 0.2 \\ 0.5 & 1 & 0.2 & 0.2 & 0.2 \\ 0.2 & 0.2 & 1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.2 & 1 & 0.7 \\ 0.2 & 0.2 & 0.2 & 0.7 & 1 \end{pmatrix}.$$

The corresponding tree is



5. Other Aspects and Applications

Fuzzy equivalence relations, fuzzifying classical equivalence relations, appear in almost all branches of fuzzy logic. This section contains the most relevant aspects of such relations not treated until now in this article and the most relevant places where they are used.

5.1. Metric Aspects of Fuzzy Equivalence Relations

There are many relations between fuzzy equivalence relations and pseudometrics.

A first more or less straightforward result is that a fuzzy relation E on a universe X is a fuzzy $Ł$-equivalence relation if and only if $m = 1 - E$ is a pseudodistance on X. E separates points if and only if m is a distance. Though this result was implicitly obtained in different previous articles, the first one that formulated it explicitly was (Gottwald 1992). This is one of the main reasons why transitivity with respect to the Łukasiewicz t-norm is required in many places (Ruspini 1977).

Another important relation between fuzzy equivalence relations and pseudodistances was given in (Valverde 1985).

Proposition 5.1. *Let T be a continuous Archimedean t-norm T, t an additive generator of T and E a fuzzy relation on a universe X. Then E is a T-fuzzy equivalence relation if and only if $m = t \circ E$ is a pseudodistance of X.*
E separates points if and only if m is a distance on X.

Remark. If m is a pseudodistance on X, then $t^{[-1]} \circ m$ is a T-fuzzy equivalence relation on X.

Due to these results, many properties of metric spaces can be transferred to results on T-equivalence relations when T is continuous Archimedean. For example the Representation Theorem was first obtained by Ovchinnikov (1984) for the product t-norm, that is continuous Archimedean, by noticing that every T-fuzzy equivalence relation E with respect to the product t-norm is of the form e^{-m} with m a pseudodistance.

But the most important relation between fuzzy equivalence relations and pseudidistances is that both are generalized metric spaces. The idea of generalized metric space was developed in Trillas' PhD. thesis as he noticed the similarities between different structures like metric spaces, probabilistic metric spaces, Boolean premetric spaces, the positive part of a Riesz group, and so on. The main results of generalized metric spaces are compiled in (Trillas and Alsina 1978) and its definition is as follows.

Definition 5.2. Let X be a set, $(M, \circ, \leq)$ an ordered semigroup with identity element e and m a map $m : X \times X \to M$. (X, m) is called a generalized metric space and m a generalized metric on X if and only if for all $x, y, z \in X$

(1) $m(x, x) = e$
(2) $m(x, y) = m(y, x)$
(3) $m(x, y) \circ m(y, z) \geq m(x, z)$.

m separates points if and only if

(4) $m(x, y) = e$ implies $x = y$.

What is interesting for this article is that if T is a t-norm, then $([0, 1], T)$ is a (totally) ordered semigroup (1 being its smallest element and 0 its greatest one) and a set X with a T-fuzzy equivalence relation E is a generalized metric space valued on $([0, 1], T, \leq_T)$. This encapsulates the very intuitive idea that *two objects are similar, equivalent or indistinguishable when they are close* and allows us to look at T-fuzzy equivalence relations as similarities and distances at the same time. From a more technical point of view, it also allows us to relate diverse fuzzy equivalence relations and generate fuzzy equivalence relations from given ones as will be done in the next subsection.

Finally, let us recall that E is a min-fuzzy equivalence relation if and only if $1 - E$ is a pseudoultrametric as we have seen in Subsection 4.1.

5.2. Homomorphisms Between Fuzzy Equivalence Relations

This subsection will study the following three points.

- We will find the mappings transforming a T-fuzzy equivalence relation into another such relation with respect to the same continuous Archimedean t-norm.
- It will be shown that if T and T' are isomorphic t-norms, then there is a bijection between T-fuzzy equivalence relations and T'-fuzzy equivalence relations on the same universe as well as between their extensional fuzzy subsets.
- We will find when two fuzzy subsets μ and ν generate the same fuzzy equivalence relation (i.e.: when $E_\mu = E_\nu$). Proofs and comments of this subsection can be found in (Jacas and Recasens 2002).

Definition 5.3. A metric transform is a sub-additive and non-decreasing map $s : [0, \infty) \to [0, \infty)$ with $s(0) = 0$.

Proposition 5.4. *Let E be a T-fuzzy equivalence relation on a universe X with T a continuous Archimedean t-norm with an additive generator t and s a metric transform. Then $t^{[-1]} \circ s \circ t \circ E$ is a T-fuzzy equivalence relation on X.*

Example 5.5. The map $s : [0, \infty) \to [0, \infty)$ defined by $s(x) = x^\alpha$ for all $x \in [0, \infty)$ with $0 < \alpha \leq 1$ is a metric transform.

- If E is an $Ł$-fuzzy equivalence relation on a set X, $t(x) = 1 - x$ an additive generator, then

$$E'(x, y) = (t^{[-1]} \circ s \circ t)(E(x, y)) = 1 - (1 - E(x, y))^\alpha$$

is also an $Ł$-fuzzy equivalence relation on X.

- Let T_P be the product t-norm. If E is a T_P-fuzzy equivalence relation on a set X, then $E'(x,y) = e^{-(-\ln E(x,y))^{\alpha}}$ also is a T_P-fuzzy equivalence relation on X.

Let us recall the definition of isomorphism between t-norms.

Definition 5.6. Two continuous t-norms T, T' are isomorphic if and only if there exists a bijective map $f : [0,1] \to [0,1]$ such that $f \circ T = T' \circ (f \times f)$.

Isomorphisms f are continuous and increasing maps and we have the following result.

- All strict continuous Archimedean t-norms are isomorphic. In particular, they are isomorphic to the product t-norm.
- All nilpotent Archimedean t-norms are isomorphic. In particular, they are isomorphic to the Łukasiewicz t-norm.

Proposition 5.7. *If E is a T-fuzzy equivalence relation on a set X for a given t-norm T and f is a continuous, increasing and bijective map $f : [0,1] \to [0,1]$, then $f \circ E$ is a T'-fuzzy equivalence relation with $T' = f \circ T \circ (f^{-1} \times f^{-1})$.*

The next proposition relates the families of generators (2.14) of E and $f \circ E$.

Proposition 5.8. *Let T be a continuous t-norm and E a T-fuzzy equivalence relation on a set X. If $(\mu_i)_{i \in I}$ is a generating family of E and f a continuous, increasing and bijective map $f : [0,1] \to [0,1]$, then $(f \circ \mu_i)_{i \in I}$ is a generating family of the T'-fuzzy equivalence relation $f \circ E$.*

From this it follows directly that E and $f \circ E$ have the same dimension and that if $(\mu_i)_{i \in I}$ is a basis of E then $(f \circ \mu_i)_{i \in I}$ is a basis of $f \circ E$.

Moreover, since extensionality and generator are equivalent concepts, if H_E is the set of extensional fuzzy subsets of E, then $f(H_E)$ is the set of extensional fuzzy subsets of $f \circ E$.

Finally, it is interesting to know when two fuzzy subsets μ and ν generate the same T-fuzzy equivalence relation. The next two propositions answer the question for continuous Archimedean t-norms and for the minimum t-norm.

Proposition 5.9. *Let T be a continuous Archimedean t-norm, t a generator of T and μ, ν fuzzy subsets of X. $E_\mu = E_\nu$ if and only if $\forall x \in X$ one of the following conditions holds:*

a) $t(\mu(x)) = t(\nu(x)) + k_1$ *with* $k_1 \geq \sup\{-t(\nu(x)) | x \in X\}$
or
b) $t(\mu(x)) = -t(\nu(x)) + k_2$ *with* $k_2 \geq \sup\{t(\nu(x)) | x \in X\}$.

Moreover, if T is nilpotent, then $k_1 \leq \inf\{t(0) - t(\nu(x)) \mid x \in X\}$ and $k_2 \leq \inf\{t(0) + t(\nu(x)) \mid x \in X\}$.

Example 5.10.

- If L is the Łukasiewicz t-norm, with the previous notations

$$\mu(x) = \nu(x) + k \text{ with } \inf\{1 - \nu(x)\} \geq k \geq \sup_{x \in X}\{-\nu(x)\}$$

or

$$\mu(x) = -\nu(x) + k \text{ with } \inf_{x\in X}\{1+\nu(x)\} \geq k \geq \sup_{x\in X}\{\nu(x)\}.$$

- If T is the product t-norm, then

$$\mu(x) = \frac{\nu(x)}{k} \text{ with } k \geq \sup_{x\in X}\{\nu(x)\}$$

or

$$\mu(x) = \frac{k}{\nu(x)} \text{ with } k \leq \inf_{x\in X}\{\nu(x)\}.$$

Proposition 5.11. *Let T be the minimum t-norm and let μ be a fuzzy subset of X such that there exists an element x_M of X with $\mu(x_M) \geq \mu(x)\ \forall x \in X$. Let $Y \subseteq X$ be the set of elements x of X with $\mu(x) = \mu(x_M)$ and $s = \sup\{\mu(x) \text{ such that } x \in X-Y\}$. A fuzzy subset ν of X generates the same T-fuzzy equivalence relation as μ if and only if*

$$\forall x \in X - Y\ \ \mu(x) = \nu(x) \text{ and } \nu(y) = t \text{ with } s \leq t \leq 1\ \forall y \in Y.$$

5.3. Betweenness Relations

Metric betweenness relations were defined in (Menger 1928) in order to study convexity in metric spaces. If T is a continuous Archimedean t-norm, t an additive generator of T and E a T-fuzzy equivalence relation on a set X, then $t \circ E$ is a pseudometric. From this it follows that if E separates points, then it generates a betweenness relation. The nature of this betweenness relation is related to the complexity and structure of E. For example, E is one-dimensional if and only if it generates a linear betweenness relation on X and E is a decomposable fuzzy equivalence relation when it generates a radial betweenness relation on X.

Definition 5.12. A (metric) betweenness relation on a set X is a ternary relation B on X (i.e. $B \subseteq X^3$) satisfying for all $x, y, z \in X$

(1) $(x,y,z) \in B \Rightarrow x \neq y \neq z \neq x$
(2) $(x,y,z) \in B \Rightarrow (z,y,x) \in B$
(3) $(x,y,z) \in B \Rightarrow (y,z,x) \notin B, (z,x,y) \notin B$
(4) $(x,y,z) \in B$ and $(x,z,t) \in B \Rightarrow (x,y,t) \in B$ and $(y,z,t) \in B$.

If $(x,y,z) \in B$, then y is said to be between x and z.

If given any three elements of B, one of them is between the other two, then the betweenness relation is called linear or total.

If d is a distance defined on a set X, the relation "y is between x and z when $d(x,y) + d(y,z) = d(x,z)$" satisfies the axioms of a betweenness relation.

Proposition 5.13. *Let T be a continuous Archimedean t-norm and E a T-fuzzy equivalence relation separating points on a set X such that $E(x,y) \neq 0$ for all $x, y \in X$. The ternary relation B on X defined by $(x,y,z) \in B$ if and only if $x \neq y \neq z \neq x$ and*

$$T(E(x,y), E(y,z)) = E(x,z)$$

is a betweenness relation on X.

Remark. Given a T-fuzzy equivalence relation E on a set X, we can define on X the following crisp equivalence relation $\sim$: $x \sim y$ if and only $E(x, y) = 1$. In the quotient set $X/\sim$ the fuzzy relation $\overline{E}$ defined by $\overline{E}(\overline{x}, \overline{y}) = E(x, y)$ is a T-fuzzy equivalence relation separating points.

One-dimensional fuzzy equivalence relations (i.e.: fuzzy equivalence relations E_μ generated by a fuzzy subset μ) determine linear betweenness relations.

Proposition 5.14. *Let T be a continuous Archimedean t-norm and E a T-fuzzy equivalence relation on X such that for all $x, y \in X$, $\min\{E(x, y) \mid x, y \in X\} \neq 0$. E is one dimensional if and only if the betweenness relation B determined by $\overline{E}$ on $X/\sim$ is linear.*

Radial Betweenness Relations, defined below, characterize decomposable fuzzy equivalence relations.

Definition 5.15. A betweenness relation B on a set X is called radial if and only if there exists an element $a \in X$ such that a is between any other two elements of X and these are the only elements of B (i.e. $(x, y, z) \in B$ if and only if $y = a$). The element a is called the center of the betweenness relation.

If a distance on a set X generates a radial betweenness relation on X, then there is a node or hub a and the distance between two points x and y of X is the distance between x and a plus the distance between y and a.

Proposition 5.16. *Let T be a continuous Archimedean t-norm, E a T-fuzzy equivalence relation on a finite set X of cardinality n satisfying $E(x, y) \neq 0$ $\forall x, y \in X$ and B the betweenness relation generated on $X/\sim$ by $\overline{E}$. E is decomposable if and only if B is radial or E can be extended to a T-fuzzy equivalence relation E' on $X' = X \cup \{a\}$ with $a \notin X$ in such a way that the betweenness relation B' generated on $X'/\sim$ by $\overline{E'}$ is radial with center $\overline{a}$.*

This means that if E is decomposable on X, then for calculating the degree on similarity or equivalence between two elements x and y of X we can calculate $E(x, a)$ and $E(y, a)$ and then combine them with the corresponding t-norm.

5.4. Fuzzy Betweenness Relations

In real problems, the values of a T-fuzzy equivalence relation are distorted by inaccuracy, imprecision or some noise. If for example the values of a one-dimensional T-fuzzy equivalence relation are distorted, then the betweenness relation generated by this relation will probably be not linear and even can be empty. This means that the definition of betweenness relation cannot capture the possibility of a relation to be “almost” linear or -more generally speaking- is not capable of dealing with points being “more or less” between others. Fuzzy betweenness relations were studied in (Jacas and Recasens 1994b) in the case of strict continuous Archimedean t-norms to handle this problem. Later on, other researches have gone further and defined new kinds of fuzzy betweenness relations characterizing different types of fuzzy relations (see Zhang, Pérez-Fernández and De Baets 2020).

In this subsection, following the original ideas in (Jacas and Recasens 1994b), it will be shown that if T is strict continuous Archimedean, then fuzzy betweenness relations can be generated by T-fuzzy equivalence relations separating points and, reciprocally, that every fuzzy betweenness relation generates such a T-fuzzy equivalence relation.

Definition 5.17. A fuzzy betweenness relation with respect to a given strict Archimedean t-norm T on a set X is a fuzzy ternary relation, i.e. a map

$$\begin{aligned} X \times X \times X &\to [0,1] \\ (x,y,z) &\to xyz \end{aligned}$$

satisfying the following properties for all $x, y, z, t \in X$

(1) $xxy = 1$
(2) $xyz = zyx$
(3) (a) $T(xyz, xzt) \leq xyt$
    (b) $T(xyz, xzt) \leq yzt$
(4) If $x \neq y$, then $xyx < 1$.

Proposition 5.18. *The crisp part of a fuzzy betweenness relation in the set of triplets of different elements of X is a classical betweenness relation on X.*

Proposition 5.19. *Let T be a continuous strict Archimedean t-norm and E a T-fuzzy equivalence relation separating points on X with $E(x,y) \neq 0$ for all $x, y \in X$. Then the fuzzy ternary relation on X defined by*

$$xyz = \overrightarrow{T}(E(x,z)|T(E(x,y), E(y,z)))$$

is a fuzzy betweenness relation.

Reciprocally,

Proposition 5.20. *Let T be a continuous strict Archimedean t-norm. If xyz is a fuzzy betweenness relation on a set X, then the fuzzy relation E on X defined by $E(x,y) = xyx$ is T-fuzzy equivalence relations on X.*

The next example shows how a T-fuzzy equivalence relation can be replaced by a one-dimensional one close to it.

Example 5.21. Let T_P be the product t-norm and E the T_P-fuzzy equivalence relation on the set $X = \{1, 2, 3, 4, 5\}$ of cardinality 5 given by the following matrix

$$\begin{pmatrix} 1 & 0.74 & 0.67 & 0.50 & 0.41 \\ 0.74 & 1 & 0.87 & 0.65 & 0.53 \\ 0.67 & 0.87 & 1 & 0.74 & 0.60 \\ 0.50 & 0.65 & 0.74 & 1 & 0.80 \\ 0.41 & 0.53 & 0.60 & 0.80 & 1 \end{pmatrix}.$$

The associated fuzzy betweenness relation is given in Table 1.

The dimension of E is 3. Nevertheless, it is close to the one-dimensional T-fuzzy

Table 1. Fuzzy betweenness relation generated by E

x	y	z	xyz	x	y	z	xyz	x	y	z	xyz	x	y	z	xyz
1	2	3	0.96	2	3	1	0.79	3	4	1	0.55	4	5	1	0.66
1	2	4	0.96	2	3	4	0.99	3	4	2	0.55	4	5	2	0.65
1	2	5	0.96	2	3	5	0.98	3	4	5	0.99	4	5	3	0.65
1	3	2	0.79	2	4	1	0.44	3	5	1	0.37	5	1	2	0.57
1	3	4	0.99	2	4	3	0.55	3	5	2	0.37	5	1	3	0.46
1	3	5	0.98	2	4	5	0.98	3	5	4	0.65	5	1	4	0.26
1	4	2	0.44	2	5	1	0.29	4	1	2	0.57	5	2	1	0.96
1	4	3	0.55	2	5	3	0.37	4	1	3	0.45	5	2	3	0.77
1	4	5	0.98	2	5	4	0.65	4	1	5	0.26	5	2	4	0.46
1	5	2	0.29	3	1	2	0.57	4	2	1	0.96	5	3	1	0.98
1	5	3	0.37	3	1	4	0.45	4	2	3	0.76	5	3	2	0.98
1	5	4	0.66	3	1	5	0.46	4	2	5	0.43	5	3	4	0.56
2	1	3	0.57	3	2	1	0.96	4	3	1	0.99	5	4	1	0.98
2	1	4	0.57	3	2	4	0.76	4	3	2	0.99	5	4	2	0.98
2	1	5	0.57	3	2	5	0.77	4	3	5	0.56	5	4	3	0.99

equivalence relation E' with matrix

$$\begin{pmatrix} 1 & 0.76 & 0.67 & 0.50 & 0.40 \\ 0.76 & 1 & 0.88 & 0.68 & 0.53 \\ 0.67 & 0.88 & 1 & 0.75 & 0.60 \\ 0.50 & 0.66 & 0.75 & 1 & 0.80 \\ 0.40 & 0.53 & 0.60 & 0.80 & 1 \end{pmatrix}$$

whose associated fuzzy betweenness relation is shown in the Table 2.

Table 2. Fuzzy betweenness relation generated by E'.

x	y	z	xyz	x	y	z	xyz	x	y	z	xyz	x	y	z	xyz
1	2	3	1	2	3	1	0.78	3	4	1	0.56	4	5	1	0.64
1	2	4	1	2	3	4	1	3	4	2	0.56	4	5	2	0.64
1	2	5	1	2	3	5	1	3	4	5	1	4	5	3	0.64
1	3	2	0.78	2	4	1	0.43	3	5	1	0.36	5	1	2	0.58
1	3	4	1	2	4	3	0.56	3	5	2	0.36	5	1	3	0.45
1	3	5	1	2	4	5	1	3	5	4	0.64	5	1	4	0.25
1	4	2	0.43	2	5	1	0.28	4	1	2	0.58	5	2	1	1
1	4	3	0.56	2	5	3	0.36	4	1	3	0.45	5	2	3	0.78
1	4	5	1	2	5	4	0.64	4	1	5	0.25	5	2	4	0.43
1	5	2	0.28	3	1	2	0.58	4	2	1	1	5	3	1	1
1	5	3	0.36	3	1	4	0.45	4	2	3	0.78	5	3	2	1
1	5	4	0.64	3	1	5	0.45	4	2	5	0.43	5	3	4	0.56
2	1	3	0.58	3	2	1	1	4	3	1	1	5	4	1	1
2	1	4	0.58	3	2	4	0.78	4	3	2	1	5	4	2	1
2	1	5	0.58	3	2	5	0.78	4	3	5	0.56	5	4	3	1

Another way to approximate a fuzzy equivalence relation by a one-dimensional one is by relating it with Saaty's preference matrices as can be found in (Jacas and Recasens 1993a).

5.5. *Fuzzy Rough Sets and Modal Logic*

If $\sim$ is a crisp equivalence relation of X, then a crisp subset $A \subseteq X$ can be approximated from bellow by the union of all the equivalence classes of $\sim$ contained in A and from above by the union of all the equivalence classes with non-empty intersection with A. These two approximations of A by "observable" sets is called a rough set, a concept introduced by Pawlak (Pawlak 1982) (see Figure 4). In the fuzzy case, this upper and lower approximations are obtained with the operators ϕ_E and ψ_E (see Morsi and Yakout (1998); Boixader, Jacas and Recasens (2000b)).

Rough sets approximations have been interpreted as the necessity and possibility in modal logic and ϕ_E and ψ_E have been used to model the fuzzy possibility and the fuzzy necessity operators in fuzzy modal logic, where the accessibility relation between possible worlds is given by a T-fuzzy equivalence relation E or, more general, by a fuzzy relation (Bou et al. 2010).

A Kripke frame is a pair $F = (W, R)$ where W is a set called the set of possible words and R a (crisp) binary relation on W called the accessibility relation. When vRw we say that w is accessible from v. If R is replaced by a fuzzy relation on W, then we have a fuzzy Kripke frame.

A fuzzy Kripke model is a 3-tuple $M = (W, R, V)$ where (W, R) is a fuzzy Kripke frame and V is a map, called valuation, assigning to each element of the set Var of atomic formulas or variables and each world in W an element of $[0, 1]$ (i.e., $V : Var \times W \to [0, 1]$).

The valuation V can be extended to any formula φ and given a formula φ the map $V_\varphi : W \to [0, 1]$ can be defined as $V_\varphi(w) = V(\varphi, w)$.

The valuation of the necessity $\Box\varphi$ of φ in world w is then defined by

$$V(\Box\varphi, w) = \psi_R(V_\varphi)(w) = \inf_{w' \in W} \overrightarrow{T}\left(R(w, w')|V_\varphi(w')\right)$$

and the possibility $\Diamond\varphi$ of φ in world w is

$$V(\Diamond\varphi, w) = \phi_R(V_\varphi)(w) = \sup_{w' \in W} T(R(w, w'), V_\varphi(w')).$$

5.6. *Fuzzy Subsets, Fuzzification, Triangular and Trapezoidal Fuzzy Numbers*

For a fuzzy equivalence relation E on a set X the map ϕ_E provides the best upper approximation of a fuzzy set μ of X by an extensional one. If A is a crisp subset of X, then $\phi_E(A)(x) = \sup_{y \in A}\{E(x, y)\}$ and $\phi_E(A)$ is the best approximation of X with an extensional fuzzy subset containing A. It is therefore a natural fuzzification of A (Godo and Rodriguez 2008). In particular, for a singleton $\{x\}$, its fuzzification is its fuzzy equivalence class, i.e., the fuzzy subset μ_x of X defined by $\mu_x(y) = E(x, y)$ for all $y \in X$.

For an $\alpha \in [0, 1]$ and an element $x \in X$ the fuzzy singleton $\mu_{x,\alpha}$ is defined by

$$\mu_{x,\alpha}(y) = \begin{cases} \alpha & if y = x \\ 0 & otherwise. \end{cases}$$

The fuzzification of $\mu_{x,\alpha}$ is $T(\alpha, \mu_x)$.

If the universe is the real line and E is the L-fuzzy equivalence relation on $\mathbb{R}$ defined by $E(x, y) = \max(1 - \alpha|x - y|, 0)$ for some $\alpha > 0$, then the fuzzification of a point is a triangular number and the fuzzification of an interval is a trapezoidal number. This gives a sound theoretical interpretation to the usual fuzzification of points and intervals by these numbers: They are the smallest extensional fuzzy subsets containing the point or interval assuming the existence of the fuzzy equivalence relation E (Jacas and Recasens 1993b).

These results are related with the semantics of fuzzy sets based on similarity explained in (Dubois and Prade 1997). They present three semantics of fuzzy sets, one of them with similarity. For a fuzzy subset μ to be defined on X we can consider a set of prototypes A of μ. Then $\mu(x)$ is the degree of proximity of x to prototype elements of μ or, explicitly, $\mu(x) = \sup_{a \in A}\{E(x, a)\}$. This can be interpreted as μ being the fuzzification of the set A of prototypical elements.

5.7. *Fuzzy Topologies, Extensionality and Continuity*

The set H_E of extensional fuzzy subsets of a T-fuzzy equivalence relation E on a set X is a fuzzy topology on X and the maps ϕ_E and ψ_E are fuzzy closure and interior operators of it. E by its side also generates a classical topology T_E on X. We will show the close relationship between these classical and crisp topologies (see Jacas (1993); Jacas and Recasens (1994a); Demirci (2004)).

For a mapping $f : X \to Y$ the equivalence between being an extensional map and continuity with respect to the fuzzy topologies generated by fuzzy equivalence relations will also be established (Boixader and Recasens 2018b).

Let us recall the definition of fuzzy topology (Lowen 1976).

Definition 5.22. A subset τ of $[0, 1]^X$ is a fuzzy topology on X if and only if it satisfies the following properties.

(1) If $A \subseteq \tau$, then $\sup\{\mu \mid \mu \in A\} \in \tau$.
(2) If $\mu, \mu' \in \tau$, then $\min(\mu, \mu') \in \tau$.
(3) For any $k \in [0, 1]$, if we write μ_k as the constant k fuzzy subset of X (i.e. $\mu_k(x) = k \in [0, 1]$ for all $x \in X$), then $\mu_k \in \tau$.

The elements of a fuzzy topology τ are called open fuzzy subsets. Given a strong negation φ a fuzzy subset μ is called closed when $\varphi \circ \mu$ is open. The set $\tau^\varphi = \{\mu \in [0, 1]^X \mid \varphi \circ \mu \in \tau\}$ of closed fuzzy subsets of a fuzzy topology is called a fuzzy co-topology.

When there is a T-fuzzy equivalence relation E defined on a universe X, then H_E is both a fuzzy topology and a fuzzy co-topology. If T is nilpotent and φ_T is the strong negation defined for all $x \in [0, 1]$ by $\varphi_T(x) = \overrightarrow{T}(x|0)$, then there is this nice relationship.

Proposition 5.23. *Let T be a nilpotent t-norm and E a T-fuzzy equivalence relation on a universe X. Then $H_E^{\varphi_T} = H_E$.*

In particular, if T is nilpotent, the elements of H_E are open and closed (clopen).

On the other hand, a T-fuzzy equivalence relation E on a set X defines a crisp topology on X in a natural way.

Definition 5.24. Given a T-fuzzy equivalence relation E separating points on a set X and a strong negation φ, the open ball $B(x,r)$ of centre $x \in X$ and radius $r \in (0,1]$ is defined by

$$B(x,r) = \{y \in X \textit{ such that } E(x,y) > \varphi(r)\}.$$

The set of open balls is a basis of a topology denoted by T_E.

In order to relate the topology T_E and the fuzzy topology H_E generated by a T-fuzzy equivalence relation E, we recall the following definition (Lowen 1976).

Definition 5.25. Let $T(X)$ and $\tau(X)$ denote the sets of topologies and fuzzy topologies on a given set X respectively. The mappings $i : \tau(X) \to T(X)$ and $\omega : T(X) \to \tau(X)$ are defined in the following way.

$$\begin{aligned} i(\tau) &= \left\{\mu^{-1}(\alpha,1] \text{ such that } \alpha \in [0,1), \mu \in \tau\right\}. \\ \omega(T) &= \left\{\mu \in [0,1]^X \text{ such that } \mu \text{ is lower semicontinuous}\right\}. \end{aligned}$$

Proposition 5.26. *Let T be a nilpotent t-norm, E a T-fuzzy equivalence relation on a universe X and T_E and H_E the topology and the fuzzy topology generated by E. Then $i(H_E) = T_E$ and $\omega(T_E) = H_E$.*

Extensionality is a kind of Lipschitzian property and extensional maps have been widely used in areas such as Approximate Reasoning (Boixader 1998), vague algebras (Demirci 2002), logic (Hájek 1998) among others because it assures that the images $f(x)$ and $f(y)$ of two elements x and y are at least as similar as x and y are. Proposition 5.30 shows the equivalence between this property and continuity with respect to the fuzzy topologies generated by fuzzy equivalence relations.

Definition 5.27. Let E and F be T-fuzzy equivalence relations on X and Y respectively. A map $f : X \to Y$ is extensional with respect to E and F ((E,F)-extensional) if for $x, x' \in X$, $E(x,x') \le F(f(x), f(x'))$.

Definition 5.28. Let $f : X \to Y$ be a mapping and ν a fuzzy subset of Y. $f^{-1}(\nu)$ is the fuzzy subset of X defined by

$$f^{-1}(\nu)(x) = \nu(f(x))$$

for all $x \in X$.

Definition 5.29. Let σ be a fuzzy topology on X and τ a fuzzy topology on Y. A mapping $f : X \to Y$ is fuzzy-continuous with respect to σ and τ if for every $\nu \in \tau$, $f^{-1}(\nu) \in \sigma$.

Proposition 5.30. *(Boixader and Recasens 2018b) Let E and F be T-fuzzy equivalence relations on X and Y respectively. $f : X \to Y$ is fuzzy-continuous with respect to H_E and H_F if and only if f is (E,F)-extensional.*

5.8. Aggregation of Fuzzy Equivalence Relations

In many situations, there can be more than one fuzzy equivalence relation defined on a universe. Let us suppose, for example that we have a set of instances defined by some features. We can generate a fuzzy equivalence relation from each feature. In these cases, we may need to aggregate the obtained relations. The usual way to do it is calculating the minimum (or infimum) of them which is a fuzzy equivalence relation thanks to the Representation Theorem 2.14. Although this has a very clear interpretation in fuzzy logic since the infimum is used to model the universal fuzzy quantifier $\forall$, it leads many times to undesirable results in applications. The reason is that the minimum has a drastic effect. If for example two objects of our universe are very similar or indistinguishable for all but one fuzzy equivalence relation, but for this particular one very different, then the result applying the minimum will give this last measure and forget all the other ones. This can be reasonable and useful if we need a perfect matching with respect to all our relations, but this is not the case in many situations. When we need to take all the relations into account in a less dramatic way, other ways have to be found.

Definition 5.31. Let $t : [0,1] \to [-\infty, +\infty]$ be a monotonic map $x_1, x_2, ..., x_n \in [0,1]$ and $p_1, p_2, ..., p_n$ non-negative numbers with $\sum_{1=1}^{n} p_i = 1$. The weighted quasi-arithmetic mean $m_t^{p_1,p_2,...,p_n}$ of $x_1, x_2, ..., x_n$ generated by t is

$$m_t^{p_1,p_2,...,p_n}(x_1, x_2, ..., x_n) = t^{-1}\left(\frac{p_1 t(x_1) + p_2 t(x_2) + ... + p_n t(x_n)}{n}\right).$$

$p_1, p_2, ..., p_n$ are called the weights of $m_t^{p_1,p_2,...,p_n}$.

Example 5.32.

(1) If $t(x) = x$, then m_t is the weighted arithmetic mean.
(2) If $t(x) = \log x$, then m_t is the weighted geometric mean.

If T is an Archimedean t-norm with additive generator t, we can consider the weighted quasi-arithmetic mean generated by t. Then we have the following result.

Proposition 5.33. *Let $E_1, E_2, ..., E_n$ be T-fuzzy equivalence relations on X with T an Archimedean t-norm with additive generator t. Then $m_t^{p_1,p_2,...,p_n}(E_1, E_2, ..., E_n)$ is a T-fuzzy equivalence relation.*

Another way to aggregate fuzzy equivalence relations is using OWA operators.

Definition 5.34. Let $p_1, p_2, ..., p_n$ be numbers of the unit interval. The map $s : [0,1]^n \to [0,1]$ defined for all $x_1, x_2, ..., x_n \in [0,1]$ by $s(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} p_i x_{(i)}$ where $x_{(i)}$ is the i^{th} largest of the x_i is the OWA operator with weights $p_1, p_2, ..., p_n$.

Proposition 5.35. *Let $E_1, E_2, ..., E_n$ be T-fuzzy equivalence relations on X with T an Archimedean t-norm with additive generator t. Let $s : \mathbb{R}^n \to \mathbb{R}$ be the OWA operator with weights $(p_1, p_2, ..., p_n)$ and t a generator of T. The fuzzy relation E on X defined by*

$$E(x,y) = t^{[-1]}\left(s\left(t(E_{(1)}(x,y)) + t(E_{(2)}(x,y)) + ... + t(E_{(n)}(x,y))\right)\right)$$

is a T-fuzzy equivalence relation if and only if $p_i \geq p_j$ for $i < j$.

In some cases we have to aggregate a non-finite number of relations, for example if we need to compare two fuzzy sets of the real line. Let us suppose that we have a family of T-fuzzy equivalence relations $(E_i)_{i \in [a,b]}$ on a set X with the indices in the interval

$[a, b]$ of the real line and that for every couple (x, y) of X the map $f_{(x,y)} : [a, b] \to R$ defined by $f_{(x,y)}(i) = E_i(x, y)$ is integrable in some sense.

Definition 5.36. Let T be a continuous Archimedean t-norm, t an additive generator of T and $(E_i)_{i\in[a,b]}$ a family of T-fuzzy equivalence relations on a universe X. The aggregation of the family $(E_i)_{i\in[a,b]}$ with respect to T is the T-fuzzy equivalence relation E on X defined for all $x, y \in X$ by

$$E(x, y) = t^{[-1]}(\frac{1}{b-a} \int_a^b t(E_i(x, y))di)$$

More information on the aggregation of T-fuzzy equivalence relations is available in (Jacas and Recasens 2003; Pradera, Trillas and Castiñeira 2002; Pradera and Trillas 2002).

5.9. Approximating Proximity Relations and Extensional Fuzzy Subsets

The transitive closure of a proximity relation R provides its best upper approximation with a fuzzy equivalence relation. Its openings good lower approximations. But between the transitive closure and the openings there are other fuzzy equivalence relations closer to R.

Similarly, the mappings ϕ_E and ψ_E provide the best upper and lower approximations of a fuzzy subset μ with extensional ones. However, in general, there is no guarantee that there are no extensional fuzzy sub sets "in between" that approximate μ better in the sense that it is possible that the fuzzy extensional subset that best approximates μ is neither containing nor contained in it.

In this subsection we present three ways to obtain such intermediate good approximations for proximity relations (Garmendia and Recasens 2009) and for fuzzy subsets (Mattioli and Recasens 2015) when the t-norm is continuous Archimedean.

5.9.1. Approximation of Proximity Relations by T-fuzzy Equivalence Relations

The next proposition expresses explicitly the T-transitivity for continuous Archimedean t-norms.

Proposition 5.37. *Let $E = (e_{ij})_{i,j=1,\ldots,n}$ be a proximity matrix on a set X of cardinality n and T a continuous Archimedean t-norm with additive generator t. E is a T-fuzzy equivalence relation if and only if for all i, j, k $1 \le i < j < k \le n$*

$$t(e_{ij}) + t(e_{jk}) \ge t(e_{ik})$$
$$t(e_{ij}) + t(e_{ik}) \ge t(e_{jk})$$
$$t(e_{ik}) + t(e_{jk}) \ge t(e_{ij})$$

Given a proximity matrix R, we must then search for (one of) the closest matrices E to R satisfying the last $3\binom{n}{3}$ inequalities, which is a non-linear programming problem. This is a hard problem in general though for the Łukasiewicz and the product t-norms it turns out to be a quadratic programming case and well-known algorithms can be applied (see Paragraph 5.9.1.3).

In the following two paragraphs we propose alternative methods to obtain not the best but reasonably good approximations of proximity relations by a T-fuzzy equivalence relations.

5.9.1.1 Aggregating the Transitive Closure and a T-fuzzy Equivalence Relation smaller than R

Since the weighted quasi-arithmetic mean of T-fuzzy equivalence relations is also a T-fuzzy equivalence relation (Proposition 5.33), given a proximity matrix R, we can calculate its transitive closure $\overline{R}$ and a smaller T-fuzzy equivalence relation $\underline{R}$ than R and find the weights $p, 1-p$ to obtain the closest average of $\overline{R}$ and $\underline{R}$ to R.

To evaluate the closeness of two fuzzy relations we can use a similarity measure of the Euclidean distance. Here we will use the latter.

Proposition 5.38. *Let $R = (r_{ij})$ be a proximity matrix on a finite set X of cardinality n, T a continuous Archimedean t-norm with additive generator t, $\overline{R} = (\overline{r}_{i,j})$ its transitive closure, $\underline{R} = (\underline{r}_{ij})$ the T-fuzzy equivalence relation obtained from the columns of R with the Representation Theorem, $p \in [0,1]$ and $m_t^{p,1-p}(\overline{R}, \underline{R})$ the T-fuzzy equivalence relation quasi-arithmetic mean of $\overline{R}$ and $\underline{R}$ with weights p and $1-p$. Then*

$$D(R, m_t^{p,1-p}(\overline{R}, \underline{R})) = \Big(\sum_{1 \leq i,j \leq n} \left(t^{-1}\left(p \cdot t\left(\overline{r}_{ij}\right) + (1-p) \cdot t\left(\underline{r}_{ij}\right)\right) - t(r_{ij})\right)^2\Big)^{\frac{1}{2}}.$$

Proposition 5.39. *Let $Ł$ be the Łukasiewicz t-norm and R a proximity on a set X of cardinality n. The closest $m_t^{p,1-p}(\overline{R}, \underline{R})$ to R is attained by*

$$p = \frac{\sum_{1 \leq i < j \leq n} \left(\overline{r}_{ij} - \underline{r}_{ij}\right)\left(r_{ij} - \underline{r}_{ij}\right)}{\sum_{1 \leq i < j \leq n} \left(\overline{r}_{ij} - \underline{r}_{ij}\right)^2}.$$

Example 5.40. Let X be a set of cardinality 4 and L the Łukasiewicz t-norm. Below there is a proximity relation R on X and its closest $Ł$-fuzzy equivalence relation of the form $m_t^{p,1-p}(\overline{R}, \underline{R})$. The optimal p is $p = 0.6388889$.

$$R = \begin{pmatrix} 1 & 0.8 & 0.2 & 0.4 \\ 0.8 & 1 & 0.7 & 0.1 \\ 0.2 & 0.7 & 1 & 0.6 \\ 0.4 & 0.1 & 0.6 & 1 \end{pmatrix}, m_t^{0.639,0.361}(\overline{R}, \underline{R}) = \begin{pmatrix} 1 & 0.6917 & 0.3917 & 0.3639 \\ 0.6917 & 1 & 0.5917 & 0.2278 \\ 0.3917 & 0.5917 & 1 & 0.5278 \\ 0.3639 & 0.2278 & 0.5278 & 1 \end{pmatrix}$$

5.9.1.2 Applying a Homotecy to a T-fuzzy Equivalence Relation

In this paragraph, the fact that the power of a T-fuzzy equivalence relation in the sense of Definition 5.41 is again a T-fuzzy equivalence relation will be exploited to modify the entries of $\overline{R}$ or $\underline{R}$ to find an approximation of R.

Let us recall the definition of power with respect to a continuous t-norm T.

Definition 5.41. Let T be a continuous t-norm and n a natural number. We will call the n^{th} power $x^{(n)}$ of $x \in [0,1]$ with respect to T $x^{(n)} = T(\overbrace{x, x, ..., x}^{n})$. $x^{(\frac{1}{n})}$ is defined by $x^{(\frac{1}{n})} = \sup_{z \in [0,1]}\{T^n(z) \leq x\}$ and for p, q natural numbers, $x^{(\frac{p}{q})}(x) = (x^{(\frac{1}{q})})^{(p)}$. Passing to the limit it is possible to define x^r for all $r \in \mathbb{R}^+$.

Remark. If $r \leq s$, then $x^{(r)} \geq x^{(s)}$.

The following result allows us to calculate powers by using an additive generator t of T.

Proposition 5.42. *Let T be an Archimedean t-norm with additive generator t and $r \in \mathbb{R}^+$. Then:*

$$x^{(r)} = t^{[-1]}(r \cdot t(x))$$

Proposition 5.43. *Let T be a continuous t-norm, E a T-fuzzy equivalence relation on X and $p > 0$. Then $E^{(p)}$ is a T-fuzzy equivalence relation.*

If we consider the Euclidean distance D between R and the power $E^{(p)}$ of a T-fuzzy equivalence relation $E = (e_{ij})$, T a continuous Archimedean t-norm with t an additive generator, then we can prove the following result.

Proposition 5.44.

$$D(R, E^{(p)}) = \Big(\sum_{1 \le i,j \le n} \big(t^{-1}\left(p \cdot t\left(e_{ij}\right)\right) - r_{ij}\big)^2\Big)^{\frac{1}{2}}.$$

Example 5.45. Considering the proximity R of Example 5.40 and the Łukasiewicz t-norm, $D(R, \overline{R}^{(p)})$ attains its minimum for $p = 1.208633$ and $D(R, \underline{R}^{(p)})$ for $p = 0.821306$. Good approximations of R are $\overline{R}^{(1.208633)}$ and $\underline{R}^{(0.821306)}$.

$$\overline{R}^{(1.21)} = \begin{pmatrix} 1 & 0.7583 & 0.3957 & 0.2748 \\ 0.7583 & 1 & 0.6374 & 0.1540 \\ 0.3957 & 0.6374 & 1 & 0.5165 \\ 0.2748 & 0.1540 & 0.5165 & 1 \end{pmatrix}, \underline{R}^{(0.82)} = \begin{pmatrix} 1 & 0.8357 & 0.5893 & 0.5072 \\ 0.8357 & 1 & 0.7536 & 0.4251 \\ 0.5893 & 0.7536 & 1 & 0.6715 \\ 0.5072 & 0.4251 & 0.6715 & 1 \end{pmatrix}.$$

5.9.1.3 Using non-linear Programming Techniques

The inequalities of Proposition 5.37 are especially simple for the Łukasiewicz and the product t-norm. This will allow us to find the best approximation of a given proximity for these two t-norms with respect to the Euclidean distance.

Proposition 5.46. *Let $E = (e_{ij})_{i,j=1,\ldots,n}$ be a proximity matrix on a set X of cardinality n. E is a L-fuzzy equivalence relation where L is the Łukasiewicz t-norm if and only if for all i, j, k $1 \le i < j < k \le n$*

$$e_{ij} + e_{jk} - e_{ik} \le 1$$
$$e_{ij} + e_{ik} - e_{jk} \le 1$$
$$e_{ik} + e_{jk} - e_{ij} \le 1.$$

Proposition 5.47. *Let $E = (e_{ij})_{i,j=1,\ldots,n}$ be a proximity matrix on a set X of cardinality n. E is a T-fuzzy equivalence relation where T is the product t-norm if and only if for all i, j, k $1 \le i < j < k \le n$*

$$e_{ij} \cdot e_{jk} \le e_{ik}$$
$$e_{ij} \cdot e_{ik} \le e_{jk}$$
$$e_{ik} \cdot e_{jk} \le e_{ij}.$$

Let $R = (r_{ij})_{i,j=1,...,n}$ be a proximity relation on X of cardinality n. If we want to find the best approximation of R by a T-fuzzy equivalence relation $E = (e_{ij})_{i,j=1,...,n}$ with respect to the Euclidean distance D we must minimize $D^2(R,E) = \sum_{i<j}(e_{ij} - r_{ij})$ where the e_{ij} $i, j = 1, ..., n$ are subject to the conditions of Proposition 5.37.

Proposition 5.48. *Let $R = (r_{ij})_{i,j=1,...,n}$ be a proximity relation on X of cardinality n. The closest T-fuzzy equivalence relation to R with respect to the Euclidean distance D when T is the Łukasiewicz (product) t-norm is the solution of minimizing*

$$D^2(R,E) = \sum_{i<j}(e_{ij} - r_{ij})^2$$

where e_{ij} $i, j = 1, ..., n$ are subject to the linear inequalities of Proposition 5.46 (5.47).

This is a standard quadratic linear problem and there are some algorithms to solve it (Bazaraa, Sherali and Shetty 1993).

Example 5.49. Consider the matrix R of Example 5.40. The closest $Ł$-fuzzy equivalence relation $E = (e_{ij})_{i,j=1,...,4}$ must minimize

$$D^2(R,E) = (e_{ij} - 0.8)^2 + (e_{ij} - 0.2)^2 + (e_{ij} - 0.4)^2 + (e_{ij} - 0.7)^2 + (e_{ij} - 0.1)^2 + (e_{ij} - 0.6)^2$$

subject to the conditions of Proposition 5.46.

The matrix E is

$$E = \begin{pmatrix} 1 & 0.65 & 0.225 & 0.4 \\ 0.65 & 1 & 0.575 & 0.225 \\ 0.225 & 0.575 & 1 & 0.575 \\ 0.4 & 0.225 & 0.575 & 1 \end{pmatrix}.$$

5.9.2. Approximating Fuzzy Subsets by Extensional Ones

In this subsection similar methods to the ones of the last subsection will be used to find good approximations of a fuzzy subset by an extensional one.

5.9.2.1 Approximation using Weighted Means

We want to approximate μ by $m_t^{p,1-p}(\phi_E(\mu), \psi_E(\mu))$.

Proposition 5.50. *Let T be a continuous Archimedean t-norm, t an additive generator of T, E a T-fuzzy equivalence relation on a set X and μ, ν extensional fuzzy subsets of X with respect to E. Then:*

$$m_t^{p,1-p}(\mu, \nu) \in H_E.$$

Corollary 5.51. *Let T be a continuous Archimedean t-norm, t an additive generator of T, E a T-fuzzy equivalence relation on a set X and μ a fuzzy subset of X. Then:*

$$m_t^{p,1-p}(\phi_E(\mu), \psi_E(\mu)) \in H_E.$$

We search for the value of p that minimizes the Euclidean distance D between μ and $m_t^{p,1-p}(\phi_E(\mu), \psi_E(\mu))$.

For the Łukasiewicz t-norm the result below provides an explicit formula to find this optimal weight p.

Proposition 5.52. *Let μ be a fuzzy subset of a finite set $X = \{x_1, ..., x_n\}$ and $T =Ł$ the Łukasiewicz t-norm. Then $D(\mu, m_t^{p,1-p}(\phi_E(\mu), \psi_E(\mu)))$ is minimized when:*

$$r = \frac{\sum \mu(x_i)\phi_E(\mu)(x_i) - \sum \mu(x_i)\psi_E(\mu)(x_i) - \sum \phi_E(\mu)(x_i)\psi_E(\mu)(x_i) + \sum (\psi_E(\mu)(x_i))^2}{\sum (\phi_E(\mu)(x_i))^2 + \sum (\psi_E(\mu)(x_i))^2 - 2\sum \phi_E(\mu)(x_i)\psi(\mu)(x_i)}$$

5.9.2.2 Approximation using Powers

We will provide another method based on approximating μ by an adequate power $\psi_E(\mu)^{(p)}$.

Proposition 5.53. *Let E be an fuzzy equivalence relation on a set X, μ an extensional fuzzy subset of E and $p \leq 1$. Then $\mu^{(p)} \in H_E$.*

Corollary 5.54. *$\psi_E(\mu)^{(p)}$ is extensional for $p \leq 1$.*

We can find a good approximation of a fuzzy subset μ by an extensional one by adjusting the power p of $\psi_E(\mu)^{(p)}$ in order to minimize the Euclidean distance D between μ and $\psi_E(\mu)^{(p)}$.

For the Łukasiewicz t-norm we have the following result.

Proposition 5.55. *Let μ be a fuzzy subset of a finite set $X = \{x_1, ..., x_n\}$ and $T =Ł$ the Łukasiewicz t-norm. Then $D(\mu, \psi_E(\mu)^{(p)})$ is minimized when*

$$p = \frac{\sum \mu(x_i) + \sum \psi_E(\mu)(x_i) - \sum \mu(x_i) \cdot \psi_E(\mu)(x_i) - n}{2\sum \psi_E(\mu)(x_i) - \sum \psi_E^2(\mu)(x_i) - n}.$$

Given the duality between upper and lower approximations, ϕ_E and ψ_E, it would be expectable to find an analogous method to find an approximation based on powers of $\phi_E(\mu)$. But this is not possible. $\phi_E(\mu)$ is an upper approximation of μ, thus better approximations $\phi_E(\mu)^{(p)}$ would be found for values $r > 1$. However, if ν is an extensional fuzzy subset, for values of p greater than 1, $\nu^{(p)}$ is not extensional in general.

5.9.2.3 Approximation using Quadratic Programming

In this paragraph the aim will be to look for the extensional fuzzy subset that better approximates a given subset μ restricting the search to the region of extensional fuzzy subsets. For the Łukasiewicz and product t-norms this region is bounded by a system of linear inequalities and the problem turns out to be of Quadratic Programming.

Let μ be a fuzzy subset and σ an extensional fuzzy subset with respect to a T-fuzzy equivalence relation E on a set $X = \{x_1, x_2, ..., x_n\}$. The objective function to be minimized is $D^2(\mu, \sigma) = \sum_{i=1}^{n}(\mu(x_i) - \sigma(x_i))^2$.

The region of the space in which the solution has to be found is bounded by the conditions that ensure the extensionality of σ. Assuming T to be continuous Archimedean with an additive generator t, σ is extensional if and only if it verifies

$$t(E(x_i, x_j) + t(\sigma(x_j)) \geq t(\sigma(x_i)) \quad \forall i, j = 1, ..., n.$$

If $Ł$ is the Łukasiewicz t-norm, then these inequalities become

$$\sigma(x_i) - \sigma(x_j) \geq E(x_i, x_j) - 1 \quad \forall i, j = 1, ..., n.$$

Analogously, if T_P is the product t-norm, these conditions are

$$\sigma(x_i) \geq E(x_i, x_j) \cdot \sigma(x_j) \quad \forall i, j = 1, ..., n.$$

In both cases this is a linear system of n unknowns and $\binom{n}{2}$ inequalities (excluding the trivial cases $i = j$).

The best approximation of μ by an extensional fuzzy subset (i.e. the one minimizing F) is then the solution of a Quadratic Programming problem.

5.9.2.4 Example

Considering the $Ł$-fuzzy equivalence relation

$$E = \begin{pmatrix} 1 & 0.8 & 0.7 & 0.3 & 0.2 \\ 0.8 & 1 & 0.7 & 0.3 & 0.2 \\ 0.7 & 0.7 & 1 & 0.3 & 0.2 \\ 0.3 & 0.3 & 0.3 & 1 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 1 \end{pmatrix}$$

on $X = \{x_1, x_2, ..., x_5\}$ and the fuzzy subset $\mu(x) = (0.9, 0.5, 0.1, 0.8, 0.3)$, $\phi_E(\mu) = (0.9, 0.7, 0.6, 0.8, 0.3)$ and $\psi_E(\mu) = (0.4, 0.4, 0.1, 0.8, 0.3)$.

- The best p in $m_t^{p,1-p}(\phi_E(\mu), \psi_E(\mu))$ is $p = 0.116463$. Hence

$$(0.637288, 0.542373, 0.337288, 0.8, 0.3)$$

  is a good approximation of μ by an extensional fuzzy subset.
- The best p in $\psi_E^{(p)}(\mu)$ is $p = 0.825243$. Hence

$$\psi_E^{(0.825243)}(\mu) = (0.504854, 0.504854, 0.257282, 0.834951, 0.422330)$$

  is a good approximation of μ.
- The third method, based on solving a quadratic programming problem, gives the approximation $(0.65, 0.5, 0.35, 0.8, 0.3)$.

5.10. Observational Entropy. Fuzzy Observational Decision Trees

Shannon's measure of entropy on a finite set X is defined by

$$H(X) = -\sum_{x \in X} p(x) \log_2 p(x)$$

This measure was thought within the frame of communication theory, specifically for facing issues concerning channel reliability and reduction of transmission cost, but ignoring the semantic content of the messages involved. Let us suppose the existence of a T-fuzzy equivalence relation on X. Then the occurrence of two different events, but indistinguishable by the fuzzy equivalence relation defined, will count as the occurrence of the same event when measuring the "observational" entropy defined as follows (Yager 1992; Hernández and Recasens 2002).

Definition 5.56. Let E a T-fuzzy equivalence relation on a finite set X. The observation degree of $x_j \in X$ is defined by:

$$\pi(x_j) = \sum_{x \in X} p(x)E(x, x_j).$$

This definition has a clear interpretation: the possibility of observing x_j is given by the probability that x_j really happens plus the probability of occurrence of some element "very similar" to x_j, weighted by the similarity degree.

Definition 5.57. The quantity of information received by observing x_j is defined by:

$$C(x_j) = -\log_2 \pi(x_j).$$

Definition 5.58. Given a T-fuzzy equivalence relation E on X, and P a probability distribution on X, the observational entropy (HO) of the pair (E, P) is defined by:

$$HO(E, P) = \sum_{x \in X} p(x)C(x).$$

This measure has been applied to the generation of fuzzy decision trees when there is a fuzzy equivalence relation defined on the universe of discourse (Hernández and Recasens 2002).

5.11. Fuzzy Maps

Fuzzy maps (Demirci 1999) fuzzify the concept of map between two universes. They have been used in different fields like vague algebras (Demirci 2002) and have been proved useful in order to understand fuzzy reasoning, especially reasoning based on fuzzy rules (Klawonn 2000).

Definition 5.59. Let T be a t-norm and E and F two T-fuzzy equivalence relations on two universes X and Y respectively. A fuzzy subset $R : X \times Y \to [0, 1]$ of $X \times Y$ is a fuzzy map on $X \times Y$ (or from X to Y) if and only if for all $x, x' \in X$ and $y, y' \in Y$

(1) $T(E(x, x'), F(y, y'), R(x, y)) \leq R(x', y')$
(2) $T(R(x, y), R(x, y')) \leq F(y, y')$.

Definition 5.60. A fuzzy map R on $X \times Y$ is perfect if and only for any $x \in X$ there exists a unique $y_x \in Y$ with $R(x, y_x) = 1$.

Let us recall the definition of extensional map (see subsection 5.7).

Definition 5.61. Let E, F be two T-fuzzy equivalence relations on X and Y respectively. A map $f : X \to Y$ is extensional with respect to E and F if and only if

$$E(x,y) \leq F(f(x), f(y)).$$

There is an important relationship between extensional maps and perfect fuzzy maps.

Proposition 5.62. *Let T be a t-norm and E and F two T-fuzzy equivalence relations on X and Y respectively.*

- *If R is a fuzzy map from X to Y, then $f_R : X \to Y$ defined for all $x \in X$ by $f_R(x) = y_x$ is a (crisp) extensional map $f_R : X \to Y$.*
- *An extensional map $f : X \to Y$ defines a perfect fuzzy map R_f on $X \times Y$ by $R_f(x,y) = F(f(x), y)$ for all $(x,y) \in X \times Y$.*

5.12. Fuzzy Equivalence Relations and Approximate Reasoning

Fuzzy maps have been used in (Klawonn 2000) to interpret Mamdani's controllers and for approximate reasoning.

In a Mamdani's controller we start with a family of fuzzy rules of the form

$$\mathcal{R}_i : \text{ If } x \text{ is } A_i, \text{ then } y \text{ is } B_i \; i = 1, 2, ..., n$$

where the linguistic terms A_i and B_i are modelled by the fuzzy subsets μ_{A_i} of X and ν_{B_i} of Y.

Depending on the way the entailment is modelled, a fuzzy relation ρ_L or ρ_U is generated by

$$\rho_U(x,y) = \inf_{i=1,2,\ldots,n} \{\overrightarrow{T}(\mu_i(x)|\nu_i(y))\}$$

or

$$\rho_L(x,y) = \sup_{i=1,2,\ldots,n} \{T(\mu_i(x), \nu_i(y))\}.$$

In a control problem we want to find a map $f : X \to Y$. It is a common case to know only some values of f and from them we want to recover the whole map f.

Let $D = \{x_1, x_2, ..., x_n\}$ be the elements of X for which we know their images $f(x_i)$ and assume that there are two fuzzy equivalence relations E and F defined on X and Y respectively. Then we can consider the columns $\phi_E(\{x_i\})$ of E of the elements of D and the columns $\phi_F(\{f(x_i)\})$ of F of the elements $f(x_i)$. These are the less specific extensional fuzzy subsets containing these elements.

Assuming that f is an extensional map with respect to E and F, two fuzzy maps R_{f_D} and R_f from X to Y can be defined.

Definition 5.63. With the previous notations, the following two fuzzy maps R_{f_D} and R_f are defined for all $x \in X$ and $y \in Y$ by

$$R_{f_D}(x,y) = \sup_{i=1,2,\ldots,n} \{T(E(x,x_i), F(y, f(x_i)))\}$$

$$R_f(x,y) = F(y, f(x)).$$

The following result gives upper and lower bounds to f.

Proposition 5.64. *Let E and F be two T-fuzzy equivalence relations on X and Y respectively and $f : X \to Y$ an (unknown) extensional map with respect to E and F. Let $D = \{x_1, x_2, ..., x_n\}$ be some elements of X and let f_D denote the restriction of f to D (which is known). Let $\mu_i = \phi_E(x_i)$ and $\nu_i = \phi_F(f(x_i))$, $i = 1, 2, ..., n$. Then*

$$\rho_L = R_{f_D} \leq R_f \leq \rho_U.$$

This approach for recovering a map has also been used in the study of fuzzy transforms (Perfilieva 2006).

5.13. Vague Algebras

Demirci has introduced the concept of vague algebra that basically consists of operations compatible with given fuzzy equivalence relations (Demirci 2002).

As an example, let us see how a vague group is defined.

Definition 5.65. Let E_X and $E_{X\times X}$ be T-fuzzy equivalence relations on X and $X\times X$ respectively. A vague binary operation on X is a perfect fuzzy map $\tilde{\circ}$ from $X \times X$ to X.

$\tilde{\circ}(x, y, z)$ means the degree in which z is the result of operating x with y.

Definition 5.66. A vague group $(X, \tilde{\circ})$ is a set X with a T-fuzzy equivalence relation E_X and a vague binary operation $\tilde{\circ}$ satisfying

(1) $(T((\tilde{\circ}(b, c, d), \tilde{\circ}(a, d, m), \tilde{\circ}(a, b, q), \tilde{\circ}(q, c, w))) \leq E_X(m, w))$ $\forall a, b, c, d, m, q, w \in X$, (Associativity).
(2) There exists a (two sided) identity element $e \in X$ such that $\tilde{\circ}(e, a, a) = \tilde{\circ}(a, e, a) = 1$ for each $a \in X$.
(3) For each $a \in X$, there exists a (two-sided) inverse element $a^{-1} \in X$ such that $\tilde{\circ}(a^{-1}, a, e) = \tilde{\circ}(a, a^{-1}, e) = 1$.

The idea of vague operation has been used in the study of fuzzy actions (Boixader and Recasens 2018a) and of vague fuzzy t-norms (Boixader and Recasens 2021). Moreover, the relationship between vague groups and fuzzy subgroups have been analyzed in (Demirci and Recasens 2004) and will be presented at the end of the next subsection.

5.14. Fuzzy Subgroups

Fuzzy subgroups were introduced in (Rosenfeld 1971) as a natural generalization of the concept of subgroup and have been widely studied (Mordeson, Bhutani and Rosenfeld 2005).

To every fuzzy subgroup μ of a group $(G, \circ)$ two fuzzy equivalence relations E_μ and ${}_\mu E$ can be associated that are left and a right invariant under translations respectively. If the fuzzy subgroup μ is normal (Definition 5.73), then the left and right fuzzy equivalence relations E_μ and ${}_\mu E$ coincide and the operation of the group is compatible with them (Demirci and Recasens 2004; Fu-Gui 2000; Mordeson, Bhutani and Rosenfeld 2005).

All the results of these subsection can be found in the previous references and in (Formato, Gerla and Scarpati 1999).

Definition 5.67. Let $(G, \circ)$ be a group, e its identity element, μ a fuzzy subset of G and T a t-norm. Then μ is a T-fuzzy subgroup (or simply a fuzzy subgroup) of G if for all $x, y \in G$ the following properties hold

a) $\mu(e) = 1$
b) $\mu(x) = \mu(x^{-1})$
c) $T(\mu(x), \mu(y)) \leq \mu(x \circ y)$. (Closeness).

To every fuzzy subset μ of a group $(G, \circ)$ a pair of fuzzy relations can be associated that are fuzzy equivalence relations if and only if μ is a fuzzy subgroup of G.

Definition 5.68. Let $\circ$ be a binary operation on a set G and E a fuzzy relation on G. E is invariant under translations with respect to $\circ$ if for all $x, y, z \in G$,

a) $E(x, y) = E(z \circ x, z \circ y)$ (left invariant) and
b) $E(x, y) = E(x \circ z, y \circ z)$ (right invariant).

Invariance under translations for a fuzzy relation is essential in many applications as we will see in the next subsection on Fuzzy Morphology. Another example can be found in music (and in signal theory), where the relation between sounds (notes) is measured by the quotient of their frequencies, so that we obtain a fuzzy equivalence relation on $(\mathbb{R}^+, \cdot)$ invariant under products.

Definition 5.69. Let μ be a fuzzy subset of a group $(G, \circ)$. The fuzzy relations E_μ and ${}_\mu E$ on G defined for all $x, y \in G$ by

$$E_\mu(x, y) = \mu(x \circ y^{-1})$$
$${}_\mu E(x, y) = \mu(y^{-1} \circ x)$$

are the right and left fuzzy relations associated to μ respectively.

Proposition 5.70. *Let μ be a T-fuzzy subgroup of a group $(G, \circ)$. Then E_μ and ${}_\mu E$ are right and left invariant fuzzy equivalence relations on G respectively.*

Reciprocally, to every right (left) fuzzy equivalence relation on $(G, \circ)$ a fuzzy subgroup of G can be assigned.

Proposition 5.71. *Let E be a right (left) invariant T-fuzzy equivalence relation on a group $(G, \circ)$ with identity element e. Then the column μ_e of E (i.e., the fuzzy subset μ_e of G defined by $\mu_e(x) = E(e, x)\ \forall x \in G$) is a T-fuzzy subgroup of G and $E = E_{\mu_e}$ ($E = {}_{\mu_e}E$).*

Corollary 5.72. *Let $(G, \circ)$ be a group. There exist bijections between the set of fuzzy subgroups of G, the set of right invariant fuzzy equivalence relations on G and the set of left invariant fuzzy equivalence relations on G mapping every fuzzy subgroup μ of G into its associated fuzzy equivalence relations E_μ and ${}_\mu E$.*

The following definition fuzzifies the concept of normal subgroup.

Definition 5.73. A fuzzy subgroup μ of a group $(G, \circ)$ is a normal fuzzy subgroup if $\mu(x \circ y) = \mu(y \circ x)\ \forall x, y \in G$.

Proposition 5.74. *Let $(G, \circ)$ be a group and μ a normal T-fuzzy subgroup of G. The T-fuzzy equivalence relations E_μ and ${}_\mu E$ associated to μ coincide and are invariant* *under translations.*

Reciprocally,

Proposition 5.75. *Let $(G, \circ)$ be a group, μ a T-fuzzy subgroup of G and E_μ and ${}_\mu E$ its associated T-fuzzy equivalence relations. If E_μ and ${}_\mu E$ coincide, then μ is a normal fuzzy subgroup of G.*

Corollary 5.76. *Let $(G, \circ)$ be a group. There is a bijection between the set of normal T-fuzzy subgroups of G and the set of T-fuzzy equivalence relations on G invariant under translations mapping every normal T-fuzzy subgroup μ of G into its associated T-fuzzy equivalence relation E_μ.*

It is worth mentioning Theorems 11 and 13 of (Formato, Gerla and Scarpati 1999). Given a set S, consider the set Σ_S of permutations or bijective maps of S which is a group under composition. Then Theorem 11 of (Formato, Gerla and Scarpati 1999) states that if μ is a fuzzy subgroup of Σ_S, then the fuzzy relation E_μ on S defined for all $x, y \in S$ by $E_\mu(x, y) = \sup_{g \in \Sigma_S}\{\mu(g) \mid g(x) = y\}$ is a fuzzy equivalence relation. Reciprocally, Theorem 13 of (Formato, Gerla and Scarpati 1999) states that given a fuzzy equivalence relation E on S, the fuzzy subset μ of Σ_S defined for all $f \in \Sigma_S$ by $\mu(f) = \inf_{x \in S} E(x, f(x))$ is a fuzzy subgroup. If S is a group $(G, \cdot)$, then the mapping $< \cdot >: G \to \Sigma_G$ sending x to $< x >$ defined by $< x > (y) = x \cdot y$ is an embedding of g in Σ_G and to every fuzzy subgroup μ of G corresponds a fuzzy subgroup μ' of Σ_G defined for all $f \in \Sigma_G$ by $\mu'(f) = \mu(x)$ if $f =< x >$ and 0 otherwise. Then for all $x, y \in G$, $E_\mu(x, y) = E_{\mu'}(x, y) = \mu(x \cdot y^{-1}) = \mu'(< x > \circ < y >^{-1})$ and we recover Proposition 5.70.

Let us present a couple of examples on the real line and a third one on the plane.

Example 5.77. For the Łukasiewicz t-norm, the fuzzy set μ of $(\mathbb{R}, +)$ defined by all $x \in \mathbb{R}$ by $\mu(x) = \max(1 - |x|, 0)$ (a triangular fuzzy number) is a fuzzy subgroup (normal because $\mathbb{R}$ is an Abelian group). Its associated fuzzy equivalence relation E is defined for all $x, y \in \mathbb{R}$ by $E(x, y) = \mu(x - y) = \max(1 - |x - y|, 0)$.

Example 5.78. For the product t-norm, the fuzzy set μ of $(\mathbb{R}^+, \cdot)$ defined for all $x \in \mathbb{R}$ by

$$\mu(x) = \begin{cases} x & \text{if } x \leq 1 \\ \frac{1}{x} & \text{otherwise} \end{cases}$$

is a fuzzy subgroup (normal because $(\mathbb{R}^+, \cdot)$ is an Abelian group). Its associated fuzzy equivalence relation E is defined for all $x, y \in \mathbb{R}^+$ by

$$E_\mu(x, y) = \mu(\frac{x}{y}) = \begin{cases} \frac{x}{y} & \text{if } x \leq y \\ \frac{y}{x} & \text{otherwise.} \end{cases}$$

The following results will allow us to interpret a T-vague group as the quotient of a group modulo a normal T-fuzzy subgroup.

Proposition 5.79. *Let $(G, \circ)$ be a group and E a T-fuzzy equivalence relation on G invariant under translations with respect to $\circ$. Then $(G, \tilde{\circ})$ with $\tilde{\circ} : G \times G \times G \to [0, 1]$ defined by $\tilde{\circ}(x, y, z) = E(x \circ y, z)$ for all $x, y, z \in G$, is a T-vague group with respect* *to* E.

Corollary 5.80. *Let* $(G, \circ)$ *be a group with identity element* e, μ *a normal* T*-fuzzy subgroup of* $(G, \circ)$ *with* $\mu(e) = 1$ *and* E_μ *its associated* T*-fuzzy equivalence relation on* G $(E_\mu(x, y) = \mu(x \circ y^{-1}))$. *If* $\tilde{\circ} : G \times G \times G \to [0, 1]$ *is defined for all* $x, y, z \in G$ *by* $\tilde{\circ}(x, y, z) = \mu(x \circ y \circ z^{-1}) = E_\mu(x \circ y, z)$, *then* $(G, \tilde{\circ})$ *is a* T*-vague group.*

Proposition 5.81. *Let* $(G, \tilde{\circ})$ *be a* T*-vague group with respect to* E *and* $\circ : G \times G \to G$ *the map satisfying* $z = x \circ y$ *if* $\tilde{\circ}(x, y, z) = 1$. *Then,*

- $\tilde{\circ}(x, y, z) = E(x \circ y, z)$ $\forall x, y, z \in G$.
- E *is invariant under translations with respect to* $\circ$.

From these three results, given a group $(G, \circ)$ we have bijective maps between its T-vague groups, its T-fuzzy normal subgroups and its T-fuzzy equivalence relations invariant under translations and a simple way to generate two of them knowing the other one: For instance, $\tilde{\circ}(x, y, z) = E(x \circ y, z) = \mu(x \circ y \circ z^{-1})$.

Finally we will show that T-vague groups are in fact the fuzzy generalization of quotient groups and that if f is a homomorphism from $(G, \circ)$ to $(G, \tilde{\circ})$, then the kernel of f is precisely the normal fuzzy subgroup associated to $(G, \tilde{\circ})$.

Definition 5.82. Let $(G, \tilde{\circ})$ and $(H, \tilde{\star})$ be two T-vague groups. A map $f : G \to H$ is a homomorphism from G onto H if and only if

$$\tilde{\circ}(x, y, z) \leq \tilde{\star}(f(x), f(y), f(z)) \ \forall x, y, z \in G.$$

It is clear that a crisp group $(G, \circ)$ is a T-vague group defining $\circ(a, b, c) = 1$ if $a \circ b = c$ and 0 otherwise, and considering the crisp equality as the T-fuzzy equivalence relation.

Proposition 5.83. *Let* $(G, \tilde{\circ})$ *be a* T*-vague group with respect to a* T*-fuzzy equivalence relation* E *and* $\circ$ *an operation on* G *satisfying* $z = x \circ y$ *if* $\tilde{\circ}(x, y, z) = 1$. *Then the identity map* $\mathrm{id} : G \to G$ *is a homomorphism from* $(G, \circ)$ *onto* $(G, \tilde{\circ})$.

Definition 5.84. Let $f : G \to H$ be a homomorphism from $(G, \tilde{\circ})$ onto $(H, \tilde{\star})$ and F the T-fuzzy equivalence relation associated to H. The kernel of f is the fuzzy subset μ of G defined by $\mu(a) = F(f(a), e)$ $\forall a \in G$ where e is the identity element of $(H, \tilde{\star})$.

Proposition 5.85. *Let* $(G, \tilde{\circ})$ *be a* T*-vague group with respect to a* T*-fuzzy equivalence relation* E *and* $\circ$ *an operation on* G *satisfying* $z = x \circ y$ *if* $\tilde{\circ}(x, y, z) = 1$. *The kernel of the identity map* $\mathrm{id} : G \to G$ *is the normal* T*-fuzzy subgroup of* G *associated to* $(G, \tilde{\circ})$.

5.15. Fuzzy Mathematical Morphology

One field where invariance under translations is usually assumed and needed is fuzzy mathematical morphology (Bloch and Maître 1995). In the study of objects in the plane $\mathbb{R}^2$, isotropy is usually assumed and hence the same structural element is used throughout the plane. By using a fuzzy relation invariant under translations, we can build and model such a structural element. The most important operators in fuzzy mathematical morphology are dilation and erosion.

Definition 5.86. Let R be a fuzzy relation invariant under translations on $\mathbb{R}^2$.

- The dilation $D : [0,1]^{\mathbb{R}^2} \to [0,1]^{\mathbb{R}^2}$ is defined for all $\mu \in [0,1]^{\mathbb{R}^2}$ by

$$D(\mu)(\vec{x}) = \sup_{\vec{y}\in\mathbb{R}^2} T(R(\vec{y},\vec{x}),\mu(\vec{y})).$$

- The erosion $E : [0,1]^{\mathbb{R}^2} \to [0,1]^{\mathbb{R}^2}$ is defined for all $\mu \in [0,1]^{\mathbb{R}^2}$ by

$$E(\mu)(\vec{x}) = \inf_{\vec{y}\in\mathbb{R}^2} \overrightarrow{T}(\mu(\vec{y})|R(\vec{y},\vec{x})).$$

If R is an fuzzy equivalence relation, then the structural element is the column of the neutral element $\vec{0} = (0,0)$ of $(\mathbb{R}^2,+)$, which is a fuzzy subgroup of $\mathbb{R}^2$.

Example 5.87. In the plane $\mathbb{R}^2$ we can consider the L-fuzzy subgroup defined by $\mu(x,y) = \min(1-|x|, 1-|y|)$ (a square pyramid with vertex $(0,0,1)$). The associated invariant fuzzy equivalence relation E_μ is $E_\mu((x,y),(x',y')) = \mu((x,y)-(x',y')) = \min(1-|x-x'|, 1-|y-y'|)$. In the context of fuzzy mathematical morphology μ is a structural element.

5.16. Feature Selection

(Bezdek and Harris 1978) is the first paper dealing with the aggregation of T-fuzzy equivalence relations using the weighted arithmetic mean. They study L-fuzzy equivalence relations obtained as weighted arithmetic means of crisp equivalence relations. In (Armengol et al 2019) this study has been expanded and the so-called T-generable fuzzy equivalence relations have been characterized. Then the results have been applied to feature selection by introducing a new method called JADE.

It has been seen in Subsection 5.8 that the weighted quasi-arithmetic mean $m_t^{p_1,p_2,...,p_k}$ (where t is an additive generator of a continuous Archimedean t-norm T) of a finite family of T-fuzzy equivalence relations on a set X is also a T-fuzzy equivalence relation on X. Next proposition generalizes this result by imposing only that the numbers p_i are non-negative.

Proposition 5.88. *Let T be a continuous Archimedean t-norm, t an additive generator of T, $E_1, E_2, ..., E_k$ T-fuzzy equivalence relations on a set X and $p_1, p_2, ..., p_k$ non-negative real numbers. Then the fuzzy relation E on X defined for all $x, y \in X$ by*

$$E(x,y) = t^{[-1]}(p_1 \cdot t(E_1(x,y)) + p_2 \cdot t(E_2(x,y)) + ... + p_k \cdot t(E_k(x,y)))$$

is a T-fuzzy equivalence relation on X.

E will be denoted by $m_t^{p_1,p_2,...,p_k}(E_1, E_2, ..., E_k)$.

Next we define the concept of T-generability for a T-fuzzy equivalence relation.

Definition 5.89. Let T be a t-norm, X a set and E a T-fuzzy equivalence relation on X. E is T-generable if there exists a finite family $\mu_1, \mu_2, ..., \mu_m$ of fuzzy subsets of X such that $E = T(E_{\mu_1}, E_{\mu_2}, ..., E_{\mu_m})$.

Proposition 5.90. *Let T be a continuous Archimedean t-norm, t an additive generator of T, X a finite set, $\mu_1, \mu_2, ..., \mu_m$ fuzzy subsets of X and $p_1, p_2, ..., p_m \in [0,1]$.*

The T-fuzzy equivalence relation

$$E = m_t^{p_1,p_2,...,p_m}(E_{\mu_1}, E_{\mu_2}, ..., E_{\mu_m})$$

on X is T-generable.

The crisp equivalence relations E_A generated by a crisp subset A of X play a central role for characterizing T-generable fuzzy equivalence relations.

Definition 5.91. Let T be a continuous Archimedean t-norm, t an additive generator of T and X a finite set. The set $M(X)$ of T-fuzzy equivalence relations on X is defined by

$$M(X) = \{t^{[-1]}(\sum_{A \subseteq X} p_A \cdot t(E_A)) \mid p_A \geq 0\}.$$

Proposition 5.92. *Let T be a continuous Archimedean t-norm, t an additive generator of T and X a finite set.*

- *If T is strict, then $M(X)$ is the set of crisp equivalence relations on X.*
- *If T is nilpotent, then $M(X)$ is the set of T-generable fuzzy equivalence relations on X.*

Corollary 5.93. *Let T be a nilpotent t-norm and X a finite set. Then the crisp equivalence relations on X are T-generable.*

The assessment of weights to attributes in both, inductive learning methods and clustering, is closely related with feature selection methods. Feature selection and feature extraction methods' goal is to detect and eliminate redundant information and take only those features considered relevant for the task at hand. The selection of the appropriate features greatly influences the quality of the data used by the algorithms and, thus, the goodness of the final result. Below we present a method called JADE for feature selection with the use of T-generable fuzzy equivalence relations.

Let $X = \{x_1, x_2, ..., x_m\}$ be a set of labeled domain objects described by a set of attributes $A = \{a_1, a_2, \ldots, a_n\}$, where the attributes a_i are considered fuzzy subsets of X; and E_{a_i}, $i = 1, 2, ..., n$ the Ł-fuzzy equivalence relations generated by a_i. Each $x_i \in X$ belongs to one solution class $\{C_1, C_2, \ldots, C_k\}$, i.e., there are k classes where an unseen domain object could be classified. On the set X we can induce two kinds of partitions:

- The correct partition that is the one that separates the objects in X according the the solution classes $C = \{C_1 \ldots C_k\}$.
- The fuzzy partitions induced by each attribute of A. Given an attribute $a_j \in A$, the objects in X can be separated according to the value that they hold in the attribute a_j.

In terms of relations, the correct partition is a crisp equivalence relation R on X, and each partition induced by an attribute a_j is the T-fuzzy equivalence relation E_{a_j} on the set X. R and each E_{a_j} can be represented as matrices (R is a $m \times m$ matrix where each element r_{hl} is equal to 1 if the objects x_h and x_l belong to the same solution class and 0 otherwise).

The global similarity between two objects can be assessed by the weighted mean

$E = \sum_{i=1}^{n} E_{a_i} p_i$ of these T-fuzzy equivalence relations.

The goal of JADE is to assess the weights p_i for which E is closest to R. The weight p_i associated to a fuzzy equivalence relation E_{a_i} will denote the importance of feature a_i in the process of classifying the examples of X.

The attributes generating fuzzy equivalence relations with higher weights help more to the resemblance of E to R. In this way a ranking on the set of attributes is created.

The weights of the attributes give an idea of which of them are more relevant to describe a class. Also, the weights can be used to directly classify unseen objects since a new example y will be classified in the class of the example to which it is more similar.

Even in the case in which all attributes are categorical – they generate crisp equivalence relations – the obtained relation E from them would be fuzzy containing all the information of these equivalence relations and weighted by adequate weights. This would not be possible if we were trying to find a crisp relation E.

JADE has been applied to obtain a model supporting decision processes of a dairy farm (López-Suárez et al 2018).

5.17. Semipositive-definite Fuzzy Equivalence Relations and Kernels

Positive definite and semi-definite matrices appear in many branches of Mathematics such as in the study of quadratic forms, optimization problems and classification of quadric (hyper)-surfaces; but what is more important in Artificial Intelligence is that in linear algebra the matrix associated to an inner product of a vector space is positive definite. In this case, a Euclidean distance can be defined on this space. In many AI problems, in particular in machine learning, vision or cluster analysis, a distance on a universe X is needed. In these cases, the use of kernels on X allows us to define a Euclidean-like distance d on X with all its good properties (Moser 2006b; Pękalska and Duin 2005).

Kernels are based on semi-definite matrices.

Definition 5.94. A symmetric $n \times n$ real matrix A is positive definite if $\vec{u}^t A \vec{u} > 0$ for every non-zero vector $\vec{u} \in \mathbb{R}^n$ and positive semi-definite if $\vec{u}^t A \vec{u} \geq 0$.

Surprisingly, the study of the transitivity of a matrix A with respect to some t-norms has been proved a good tool for studying its semi-definiteness.

In (Moser 2006a) and (Tomás, Alsina and Rubio-Martinez 2006) it has been proved that if a tolerance relation (i.e.: a reflexive and symmetric fuzzy relation) on a set X is positive semi-definite, then it is transitive with respect to the continuous Archimedean t-norm T_{arccos} having the function arccos as additive generator. In (Tomás, Alsina and Rubio-Martinez 2006), it is also proved that it is also transitive with respect to the continuous Archimedean t-norm $T_{\sqrt{1-x}}$ having the function $t(x) = \sqrt{1-x}$ as additive generator. The reciprocal of both assertions is not true but in (Recasens and Tomás 2020) the following two characterizations in which an additional embedding property of metric spaces is involved can be found.

Proposition 5.95. *A tolerance relation $E = (x_{ij})$ on a set $X = \{x_1, x_2, ..., x_n\}$ of finite cardinality n is positive definite if and only if it is a $T_{\sqrt{1-x}}$-fuzzy equivalence relation and X with the distance $d(x_i, x_j) = \sqrt{2}\sqrt{1 - x_{ij}}$, $1 \leq i, j \leq n$, is isometrically embeddable into $\mathbb{R}^n$ in such a way that the images of the points of X lie on the hypersphere $\mathbb{S}^{n-1}$.*

Proposition 5.96. *A tolerance relation $E = (x_{ij})$ on a set $X = \{x_1, x_2, ..., x_n\}$ of finite cardinality is positive definite if and only if it is a T_{arccos}-fuzzy equivalence relation and X with the distance $d(x_i, x_j) = \arccos x_{ij}$, $1 \le i, j \le n$ can be mapped isometrically into $\mathbb{S}^{n-1}$ with the spherical metric.*

Another interesting result (Proposition 5.100) relates the positive semi-definiteness of a symmetric matrix A with non-negative entries and positive entries on the diagonal with its transitivity with respect to specific t-norms T_λ of the Yager's family where λ depends on the order of A. Due to the next lemma, we can restrict the study to tolerance relations.

Lemma 5.97. *Let $A = (a_{ij})$ $1 \le i, j \le n$ be a symmetric matrix with non-negative entries and $a_{ii} > 0$ for all $1 \le i \le n$ and $D = \text{diag}(\frac{1}{\sqrt{a_{11}}}, \frac{1}{\sqrt{a_{22}}} ..., \frac{1}{\sqrt{a_{nn}}})$ be the $n \times n$ matrix with diagonal $\frac{1}{\sqrt{a_{11}}}, \frac{1}{\sqrt{a_{22}}} ..., \frac{1}{\sqrt{a_{nn}}}$ and zeros otherwise. Then A is a positive semi-definite matrix if and only if $E = D \cdot A \cdot D$ is a positive semi-definite tolerance relation.*

Yager's family of t-norms is defined as follows.

Definition 5.98. (Klement, Mesiar and Pap 2000) The Yager's family of continuous Archimedean t-norms $(T_\lambda)_{\lambda \in (0,\infty)}$ is defined for all $x, y \in [0, 1]$ by

$$T_\lambda(x, y) = \max((1 - (1 - x)^\lambda + (1 - y)^\lambda)^{\frac{1}{\lambda}}, 0).$$

t_λ defined by $t_\lambda(x) = (1 - x)^\lambda$ for all $x \in [0, 1]$ is an additive generator of T_λ.

Properties 5.99.

- All t-norms of Yager's family are non-strict.
- If $\lambda > \mu$, then $T_\lambda(x, y) \ge T_\mu(x, y)$ for all $x, y \in [0, 1]$.
- If $\lambda = 1$, then we retrieve the Łukasiewicz t-norm Ł and $t_1(x) = 1 - x$ is an additive generator of Ł.
- $\lim_{\lambda \to \infty} T_\lambda(x, y) = \min(x, y)$ for all $x, y \in [0, 1]$.

It is a well-known and easy to prove result that if d is a distance on a set X and $0 < k \le 1$, then d^k defined for all $x, y \in X$ by $d^k(x, y) = (d(x, y))^k$ is also a distance on X. Moreover, if X is finite of cardinality n, then d^k is Euclidean for small enough k. More precise, in (Deza and Maehara 1990) they consider for every n the supremum c_n of all $k > 0$ such that d^k is Euclidean (i.e.: (X, d^k) can be isometrically embedded into an Euclidean space) for any distance d on a set of cardinality n. In particular, for sets of cardinality $n = 2, 3, 4, 6$, the corresponding c_n are $c_2 = \infty$, $c_3 = 1$, $c_4 = \frac{1}{2}$, $c_6 = \frac{1}{2} \log_2 \frac{3}{2} \sim 0.2924$. (The calculation of c_2 and c_3 is trivial, c_4 follows from (Blumenthal 1936) and c_6 is calculated in (Deza and Maehara 1990). As the cardinality n of the set increases, the corresponding greatest value c_n decreases.

Proposition 5.100. *For a given natural number n, let c_n be the greatest value satisfying that for every pseudodistance d on any finite set of cardinality n, d^{c_n} is a Euclidean pseudodistance. Then if $E : X \times X \to [0, 1]$ on a set $X = \{x_1, x_2, ..., x_n\}$ of cardinality n is a $T_{\frac{1}{2c_{n+1}}}$-fuzzy equivalence relation, then E is positive semi-definite.*

The constant c_n is not known except for small values of n but in (Deza and Maehara 1990) a lower bound k_n for c_n is given. Namely, $k_n = \frac{1}{2n} \log_2 e \sim \frac{0.7213}{n}$. Therefore, we have the following result.

Proposition 5.101. *If $E : X \times X \to [0,1]$ on a set X of cardinality n is a $T_{\frac{n+1}{\log_2 e}}$-fuzzy equivalence relation, then it is positive semi-definite.*

As an example of the relevance of Propositions 5.100 and 5.101 we will provide alternative and trivial proofs of the following two important well-known facts.

Proposition 5.102. *(Moser 2006b) Every* min*-fuzzy equivalence relation on a finite set is positive semi-definite.*

Proof. Since $\min(x,y) \geq T_\lambda(x,y)$ for all $\lambda \in (0,\infty)$ and $x,y \in [0,1]$, every min-fuzzy equivalence relation on a finite set is also a T_λ-fuzzy equivalence relation for all $\lambda \in (0,\infty)$ and the result follows from Proposition 5.100. □

Proposition 5.103. *(Vestfrid and Timan 1983) Every pseudoultrametric on a finite set is Euclidean.*

Proof. The power of a pseudoultrametric is also a pseudoultrametric. In particular, given a pseudoultrametric m, m^2 is also a pseudoultrametric. If the set has finite cardinality, then we can assume that m (and m^2) have their values in $[0,1]$.

On the other hand, E is a min-fuzzy equivalence relation on a set X if and only if $1-E$ is a pseudoultrametric. So, there exists a min-fuzzy equivalence relation E such that $m^2 = 1-E$.

By Proposition 5.102, E is positive semi-definite and therefore $\sqrt{1-E} = m$ is Euclidean (Schoenberg (1938)). □

5.18. Fuzzy Concept Lattices

In (fuzzy) concept analysis (Bělohlávek 2002) we start with a set X of objects and a set Y of attributes that are related by a (fuzzy) relation I from X to Y. $I(x,y)$ is interpreted as the degree to which object x has the attribute y. It is also assumed that there are fuzzy equivalence relations on X and Y.

All the results of these subsection are borrowed from (Bělohlávek 2002) although in this book the fuzzy subsets and fuzzy relations are not valued on the unit interval but, more general, on a complete residuated lattice.

Definition 5.104. Given a t-norm T, a triple (X,Y,I) where X and Y are universes with E, F T-fuzzy equivalence relations on X and Y respectively and I a fuzzy relation from X to Y is called a fuzzy context.

From a fuzzy context we try to find concepts where a concept will be a pair $(A,B) \subseteq X \times Y$ such that B is the set of all attributes that are shared by all objects of A and A is the set of all objects sharing all the attributes from B. Formally, we define two maps $^\uparrow$ and $^\downarrow$ relating the fuzzy subsets of X with the fuzzy subsets of Y.

Definition 5.105. Let (X,Y,I) be a fuzzy context. The maps $^\uparrow : [0,1]^X \to [0,1]^Y$ and $^\downarrow : [0,1]^Y \to [0,1]^X$ are defined for all $\mu \in [0,1]^X$ and $\nu \in [0,1]^Y$ by

$$\mu^\uparrow(y) = \inf_{x\in X}\{\overrightarrow{T}(\mu(x)|I(x,y)\}$$
$$\nu^\uparrow(x) = \inf_{y\in Y}\{\overrightarrow{T}(\nu(y)|I(x,y)\}.$$

Definition 5.106. A fuzzy concept in a fuzzy context (X, Y, I) is a pair (μ, ν) with $\mu \in [0,1]^X$ and $\nu \in [0,1]^Y$ such that

$$\mu^{\uparrow} = \nu \qquad \text{and} \qquad \nu^{\downarrow} = \mu.$$

The set of all fuzzy concepts of (X, Y, I) is denoted by $\mathcal{B}(X, Y, I)$.

Among all theoretical properties and applications of fuzzy concept analysis we will only focus on some aspects related to fuzzy equivalence relations. First fuzzy relations will be generated on X and Y from the fuzzy concepts of the context.

Definition 5.107. Let (X, Y, I) be a context. E^X and E^Y are the fuzzy equivalence relations on X and Y respectively defined for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ by

$$E^X(x_1, x_2) = \inf_{(\mu,\nu)\in\mathcal{B}(X,Y,I)} \{E_\mu(x_1, x_2)\}$$
$$E^Y(y_1, y_2) = \inf_{(\mu,\nu)\in\mathcal{B}(X,Y,I)} \{E_\nu(y_1, y_2)\}.$$

Proposition 5.108. *The fuzzy equivalence class $E^X(x, \cdot)$ of an object x coincides with $\{x\}^{\uparrow\downarrow}$ and the fuzzy equivalence class $E^Y(y, \cdot)$ of an attribute y coincides with $\{y\}^{\downarrow\uparrow}$.*

We can consider the similarity between concepts as well. We will define two fuzzy equivalence relations on the set of concepts of a context that will coincide.

Definition 5.109. Let (X, Y, I) be a context. On the set $\mathcal{B}(X, Y, I)$ of fuzzy concepts of the fuzzy context we define the following two fuzzy equivalence relations

$$E^{Ext}(\mu_1, \nu_1), (\mu_2, \nu_2)) = \inf_{x\in X}\{\overleftrightarrow{T}(\mu_1(x), \mu_2(x))\}$$
$$E^{Int}(\mu_1, \nu_1), (\mu_2, \nu_2)) = \inf_{y\in Y}\{\overleftrightarrow{T}(\nu_1(y), \nu_2(y))\}.$$

Proposition 5.110. *Let (X, Y, I) be a context. Then $E^{Ext} = E^{Int}$.*

6. Conclusions

Fuzzy equivalence relations are basic tools in many of the subfields of fuzzy logic as has been shown in this article, where their most relevant aspects and applications have been displayed. This is due to the need of an equality is essential in all branches of knowledge and because Zadeh had the intuition to provide a simple and robust fuzzification of classical equality. With the introduction of t-norms in their definition-something that was already suggested in Zadeh's original article and then popularized mainly to the effort of Trillas- they have increased in their adaptability to diverse situations.

The richness and applicability of their structure arises from many different reasons. On one side, they are fuzzy relations and hence the well-studied theory of fuzzy relations can be applied to them. Moreover, they have a clear metric behavior and this fact allows us to adapt the theory of metric spaces and topology to their study. Fuzzy biimplications (biresiduations) induce fuzzy equivalence relations and they can be analyzed from the point of view of mathematical logic. Finally, the close relationship between min-fuzzy equivalence relations (similarity relations) and hierarchical trees make the former useful in Taxonomy and Classification Theory.

The reader interested in deepening on the topics treated in this article or described below can have a look at the references. The monograph (Recasens 2011a) contains most of the known of fuzzy equivalence relations until 2011. A short monograph can be found online (Recasens 2011b).

The following are other areas where fuzzy equivalence relations have been used but not considered in this article.

- Many times there is a need for comparing fuzzy subsets of a given universe X like in ordering them or measuring their similarity. Fuzzy equivalence relations on fuzzy subsets have been introduced and related to fuzzy equivalence relations on X using the duality principle (Recasens (2011a, chap. 3)). This is a special case of similarity measures to compare both crisp and fuzzy subsets of a universe. There is a huge amount of work on similarity measures that goes beyond the scope of this survey (Li, Qin and He 2014; Yang and Li 2020).
- A number of articles on mathematical fuzzy logic with equality have been published where, apart from the usual axioms of fuzzy logic, a binary relation $\approx$ satisfying the axioms of a fuzzy equivalence relation is assumed. Then a n-ary relation A is compatible with $\approx$ when

$$\vdash (x_1 \approx y_1) \otimes (x_2 \approx y_2) \otimes ... \otimes (x_n \approx y_n) \to A(x_1, x_2, ..., x_n) \approx A(y_1, x_2, ..., x_n).$$

  (see Bělohlávek and Vychodil (2005, chap. 3)). This generalizes extensionality of fuzzy subsets and of fuzzy binary relations (Defintion 5.59.1).
- Going back to the Representation theorem 2.14, different families of fuzzy subsets can generate the same T-fuzzy equivalence relation E. This gives great interest to the theorem, since if we interpret the elements of the family as degrees of matching between the elements of X and a set of prototypes, we can use different features, giving different interpretations to E. Among the generating families of a relation, the ones with minimal cardinality (called basis) are of special interest, since they have an easy semantic interpretation and also because the information contained in the matrix can be packed in a few fuzzy subsets. The dimension of E is the cardinality of a basis of E. In (Jacas 1990) a couple of algorithms to find the dimension and a basis of a similarity relation is given. A geometric approach and an algorithm for calculating the dimension and a basis of T-fuzzy equivalence relations with T continuous Archimedean can be found in (Jacas and Recasens 1994a, 1995).
- Multiresolution is a general mathematical concept that allows us to study properties of a system by means of several changes of resolution. In the case of a family of nested fuzzy equivalence relations, the granularity of the system can be controlled by selecting the appropiate relation. In (De Soto and Recasens 2015) several such families are considered.
- Although fuzzy equivalence relations are usually valued on $[0, 1]$, a continuous domain, for practical purposes their values have to be truncated, rounded, replaced by intervals,... The preservation or loss of the transitivity of the relations after these changes has been studied in (Boixader and Recasens 2013).
- Two fuzzy T-equivalence relations E and F on a universe X are called per-

mutable when $E \circ F = F \circ E$ where $\circ$ is the sup $-T$ product of fuzzy relations. Permutability is an interesting property related to the compatibility of fuzzy equivalence relations with algebraic structures. It has been characterized in (Recasens 2012) in the following way: E and F are permutable if and only if $E \circ F$ is a T-fuzzy equivalence relation. Moreover, this occurs if and only if $E \circ F$ coincides with the T-transitive closure $\overline{\max(E, F)}$ of $\max(E, F)$.

- Fuzzy equivalence relations have been used in (Jacas and Recasens 2006) to give semantic interpretation to well-known aggregation functions and to generate new interesting ones. Using the biresiduation $\overleftrightarrow{T}$ of a continuous t-norm (which is a T-fuzzy equivalence relation on $[0, 1]$) it is reasonable to consider the aggregation of $x, y \in [0, 1]$ as the $a \in [0, 1]$ that is as close to x as to y. If T is continuous Archimedean and t an additive generator of T, then a is the quasi-arithmetic mean $m_t(x, y)$. If T is an ordinal sum, then new aggregation functions are obtained.

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