INTEGRABLE SYSTEMS ON SINGULAR SYMPLECTIC MANIFOLDS: FROM LOCAL TO GLOBAL

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ABSTRACT. In this article, we consider integrable systems on manifolds endowed with symplectic structures with singularities of order one. These structures are symplectic away from a hypersurface where the symplectic volume goes either to infinity or to zero transversally, yielding either a b-symplectic form or a folded symplectic form. The hypersurface where the form degenerates is called critical set. We give a new impulse to the investigation of the existence of action-angle coordinates for these structures initiated in [36] and [37] by proving an action-angle theorem for folded symplectic integrable systems. Contrary to expectations, the action-angle coordinate theorem for folded symplectic manifolds cannot be presented as a cotangent lift as done for symplectic and b-symplectic forms in [36]. Global constructions of integrable systems are provided and obstructions for the global existence of action-angle coordinates are investigated in both scenarios. The new topological obstructions found emanate from the topology of the critical set Z of the singular symplectic manifold. The existence of these obstructions in turn implies the existence of singularities for the integrable system on Z.

1. INTRODUCTION

In this article, we investigate the integrability of Hamiltonian systems on manifolds endowed with a smooth 2-form which is symplectic away from a hypersurface Z (called the *critical set*) and which degenerates in a controlled way (of order one) along Z. Either this form lowers its rank at Z and it induces a form on Z with maximal rank or its associated symplectic volume blows up with a singularity of order one. The manifolds endowed with the first type of singular structure are called *folded symplectic manifolds* and the ones endowed with the second one are called b-symplectic forms. Folded symplectic manifolds can be visualized as symplectic manifolds *with a fold, Z* that "mirrors" the symplectic structure on both sides. The study of *folded symplectic manifolds* complements that of their *"duals"* called b-symplectic manifolds which have been largely investigated since [18] and [16] and are better described as Poisson manifolds whose Poisson bracket drops rank along a hypersurface keeping some transversality properties. This article is also an invitation to consider more degenerate cases (such as higher-order singularities) which will be studied elsewhere. The models provided here can be considered as a *toy model* for more complicated singularities. Integrable systems on singular manifolds show up naturally, for instance, in the study of the Toda systems when the particles in interaction collide or are far-away. Singular symplectic manifolds naturally model symplectic manifolds with boundary and, as such, the notion of integrable system is naturally extended to manifolds with boundary.

The research of the integrability of Hamiltonian systems on these manifolds is of interest both from a Poisson and symplectic point of view. In the symplectic scenario, the study of action-angle coordinates for integrable systems initially motivated by integration purposes has been of major

importance for qualitative aspects of the associated Hamiltonian systems. For instance, action-angle coordinates provide normal forms which are fundamental to analyze the perturbation of the system (KAM theory).

Action-angle coordinates are also useful in the field of geometric quantization as already observed by Einstein when reformulating the Bohr-Sommerfeld quantization conditions [13]. Integrable systems singular symplectic manifolds define natural Lagrangian foliations yielding natural examples of real polarizations on these manifolds. In particular, they are of interest to study geometric quantization of symplectic manifolds with boundary as one of the sources of examples for these singular structures. On symplectic manifolds with boundary deformation quantization is already well-understood [47] and formal geometric quantization has been object of recent study in [53] for non-compact manifolds and in [23, 24] and [5] for b^m-symplectic manifolds. More specifically, the existence of action-angle coordinates for these structures provides a *primitive* first model for geometric quantization by counting the integral fibers of the integrable system. As proved in [26, 25, 27, 46, 43] this model has been tested to be successful in geometric quantization of toric symplectic manifolds and refines the idea of Bohr-Sommerfeld quantization. Understanding action-angle coordinates for integrable systems on singular symplectic manifolds represents a first step towards understanding geometric quantization of singular symplectic manifolds. Action-angle manifolds on singular symplectic manifolds also provide natural cotangent-type models that can be useful in understanding the notion of quantum integrable systems ([50], [3]) in the singular set-up.

The study of *folded symplectic manifolds* includes the case of origami manifolds [7] where additional conditions are imposed on the critical set. Origami manifolds are associated to global toric actions. For symplectic manifolds, Delzant theorem [10] puts the investigation of $2n$-dimensional toric manifolds on the same page as the study of a class of polytopes (Delzant's polytopes) on $\mathbb{R}^n$. Indeed origami manifolds inherit their name from origami paper templates where a superposition of Delzant polytopes gives rise to a toric action on this class of folded symplectic manifolds. Symplectic origami provide examples of integrable systems on folded symplectic manifolds but there are other examples motivated by physical systems such as the folded spherical pendulum or the Toda system when the interacting particles are far-away.

In this article, we construct b-integrable systems on b-symplectic manifolds of dimension 4 having a critical set induced by a periodic symplectomorphism, and via the desingularization technique, this yields folded integrable systems on the associated desingularized folded symplectic manifold. We prove the existence of action-angle coordinates à la Liouville-Mineur-Arnold exploring the Hamiltonian actions by tori on folded symplectic manifolds. The action variables are not exactly coordinates, since these variables can degenerate in a certain way. We show that this action-angle theorem cannot always be interpreted in terms of a cotangent model as in the case of symplectic and b-symplectic manifolds [36].

We end up this article investigating the obstruction theory for global existence of action-angle coordinates, exhibiting a new topological obstruction for singular symplectic manifolds that is localized on the critical set of the singular symplectic form. This yields examples of integrable systems on b-symplectic manifolds and folded symplectic manifolds with critical set non-diffeomorphic to a product of a symplectic leaf with a circle. For those systems the toric action does not even extend to a neighborhood of the critical set. We end up this article observing that the existence of finite isotropy for the transverse S^1-action given by the modular vector field obstructs the uniformization of periods of the associated torus action on the b-symplectic manifold. This automatically yields the existence of singularities of the integrable system on the critical locus of the b-symplectic structure.

Organization of this article: In Section 2 we introduce the basic tools in b-symplectic and folded symplectic geometry. In Section 3 we investigate Hamiltonian dynamics on folded symplectic manifolds and introduce folded integrable systems. In Section 4 we provide a list of motivating examples for integrable systems on folded symplectic manifolds. In Section 5 we prove an action-angle theorem (Theorem 6) for folded symplectic manifolds and discuss folded cotangent models. We end up this section by discussing why this theorem does not always yield equivalence with a cotangent model. Section 6 contains constructions of integrable systems on 4-dimensional b-symplectic manifolds having with a prescribed critical set (Theorem 21) and on any folded symplectic manifold which desingularizes it. Section 7 investigates the existence of global action-angle coordinates and highlights the non-triviality of the mapping torus as a topological obstruction to the global existence of action-angle coordinates (Theorems 26 and 27).

Conventions: All the manifolds in this paper are smooth. In particular, singular symplectic manifolds are smooth manifolds with singular geometric structures.

2. Preliminaries

Through this work we consider forms ω on even-dimensional manifolds M^{2n} which are symplectic away from a hypersurface Z and such that ω^n either cuts the zero section of the bundle $\Lambda^n(T^*M)$ transversally or goes to infinity in a controlled way along Z. So, in particular, ω^n defines a volume form away from Z.

For the class of forms for which ω^n cuts the zero section of the bundle $\Lambda^n(T^*M)$ transversally, we require an extra condition to guarantee maximal rank (see below). These forms are called *folded symplectic forms* and a manifold equipped with such a form is called a folded symplectic manifold. Near a point of the hypersurface Z, a folded symplectic form may be regarded as the pullback of a symplectic form by a map that folds along that hypersurface.

2.1. **Basics on folded symplectic manifolds.** We recall here some basic facts of folded symplectic manifolds.

Definition 1. *Let M be a $2n$-dimensional manifold. We say that $\omega \in \Omega^2(M)$ is folded-symplectic if*

(1) $d\omega = 0$,
(2) $\omega^n \pitchfork \mathcal{O}$, *where $\mathcal{O}$ is the zero section of $\bigwedge^{2n}(T^*M)$, hence $Z = \{p \in M, \omega^n(p) = 0\}$ is a codimension 1 submanifold,*
(3) $i : Z \to M$ *is the inclusion map, $i^*\omega$ has maximal rank $2n - 2$.*

*We say that (M, ω) is a **folded symplectic manifold** and we call $Z \subset M$ the folding hypersurface.*

The following theorem is an analog of Darboux's theorem for folded symplectic forms [41]:

Theorem 1 (Martinet). *For any point p on the folding hypersurface Z of a folded symplectic manifold (M^{2n}, ω) there is a local system of coordinates $(x_1, y_1, \ldots, x_n, y_n)$ centered at p such that Z is locally given by $x_1 = 0$ and*

$$\omega = x_1 dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \ldots + dx_n \wedge dy_n.$$

Let (M, ω) be a $2n$-dimensional folded symplectic manifold. Let $i : Z \hookrightarrow M$ be the inclusion of the folding hypersurface Z. The induced form $i^*\omega$ has a one-dimensional kernel at each point. We denote by V the bundle $\ker \omega|_Z$, and $L = V \cap TZ$ the null line bundle, which will always be assumed to be trivializable.

The following theorem [8, Theorem 1] is a Moser type theorem for folded symplectic manifolds which extends the local normal form above to a neighborhood of Z. For the null line bundle we consider α a one-form such that $\alpha(v) = 1$ for a non-vanishing section v of L.

Theorem 2. *Assume that Z is compact, then there exists a neighborhood U of Z and an orientation preserving diffeomorphism,*

$$\varphi : Z \times (-\varepsilon, \varepsilon) \to U$$

with $\varepsilon > 0$ such that $\varphi(x, 0) = x$ for all $x \in Z$ and

$$\varphi^* \omega = p^* i^* \omega + d\left(t^2 p^* \alpha\right) \ ,$$

where $p : Z \times (-\varepsilon, \varepsilon) \to Z$ is the projection onto the first factor, the map $i : Z \hookrightarrow M$ the inclusion and t the real coordinate on the interval $(-\varepsilon, \varepsilon)$.

A special class of folded symplectic manifolds is given by origami symplectic manifolds, folded symplectic manifolds for which the null foliation L defines a circle fibration.

2.1.1. *Origami manifolds.*

Definition 2. *An **origami manifold** is a folded symplectic manifold (M, ω) whose null foliation is fibrating with oriented circle fibers, π, over a compact base, B.*

$$S^1 \lhook\joinrel\longrightarrow Z$$
$$\downarrow \pi$$
$$B$$

*The form ω is called an **origami form** and the null foliation is called the **null fibration**.*

Remark 1. On an origami manifold, the base B is naturally symplectic with symplectic form ω_B on B satisfying $i^* \omega = \pi^* \omega_B$.

Example. *Consider the unit sphere $S^{2n} \subset \mathbb{R}^{2n+1}$ given by equation $\sum_{i1}^{n}(x_i^2 + y_i^2) + z^2 = 1$ with global coordinates $x_1, y_1, \ldots, x_n, y_n, z$ on $\mathbb{R}^{2n+1}$. Let ω_0 be the restriction to S^{2n} of the form $dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$. Then ω_0 is a folded symplectic form. The folding hypersurface is the sphere given by the intersection with the plane $z = 0$. It is easy to check that the null foliation is the Hopf foliation (see for instance [7]).*

2.1.2. *Toric origami manifolds.* An important class of origami manifolds is associated to toric actions. The study of toric folded symplectic manifolds was initiated in [7] in the origami case (see [29] for the general case). The classical theory of toric symplectic manifolds is closely related to a theorem by Delzant [10] which gives a one-to-one correspondence between toric symplectic manifolds and a special type of convex polytopes (called Delzant polytopes) up to equivalence. Grosso modo, toric symplectic manifolds can be classified by their moment polytope, and their topology can be read directly from the polytope in terms of equivariant cohomology. In [31, 32] the authors examine the toric origami case and describe how toric origami manifolds can also be classified by their combinatorial moment data.

Toric origami manifolds can be recovered from a *collection of Delzant polytopes* intersecting each other on a certain manner. Namely, given be a toric origami manifold. the image of the moment map is the superimposition of certain Delzant polytopes [10]. Such Delzant polytopes are associated to the connected components of its symplectic cut space. As explained in [7] the moment map sends the folding hypersurface to certain facets which can be shared by two polytopes which coincide near those facets.

These are called *origami templates*. More concretely,

Definition 3. [7] *An n-dimensional **origami template** is a pair $(\mathcal{P}, \mathcal{F})$, where $\mathcal{P}$ stands for a (non-empty and finite) collection of n-dimensional Delzant polytopes Δ_i and $\mathcal{F}$ is a collection of facets and pairs of facets of polytopes Δ_i in $\mathcal{P}$ satisfying the following properties:*

(1) *for each pair of facets of polytopes $\{F_1, F_2\} \in \mathcal{F}$, the associated polytopes in $\mathcal{P}$ agree near those facets;*

(2) *if a facet F occurs in $\mathcal{F}$, either by itself or as a member of a pair, then neither F nor any of its neighboring facets occur elsewhere in $\mathcal{F}$;*

(3) *Let $\mathcal{W}$ be the topological space constructed from the disjoint union $\sqcup \Delta_i$, with $\Delta_i \in \mathcal{P}$ by identifying facet pairs in $\mathcal{F}$. Then $\mathcal{W}$ is connected.*

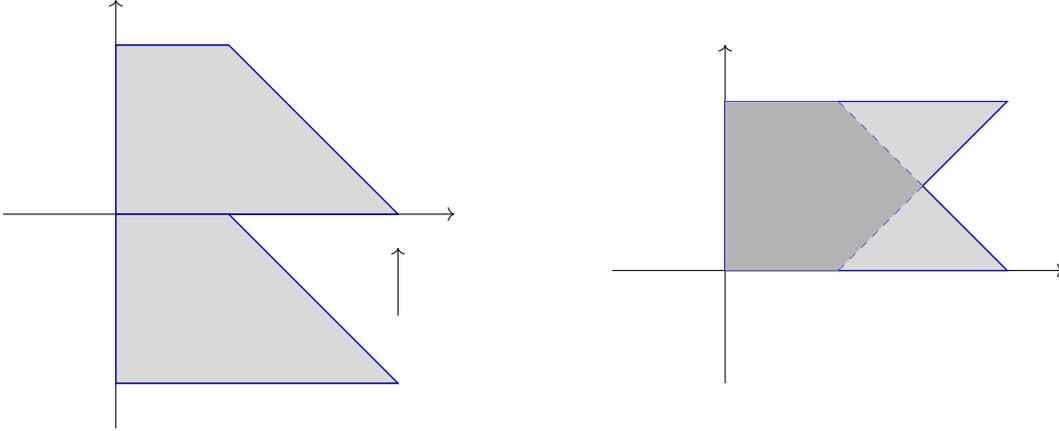See Figure 1 for an example of a template obtained by folding two trapezoids.

FIGURE 1. Folding two Delzant polytopes to obtain an origami template.

For example, each trapezoid above corresponds to a Hirzebruch surface in the standard Delzant dictionary. Similarly, it is possible to associate to the origami template the folded symplectic manifold obtained by a radial blow-up of two Hirzebruch surfaces [7].

This works in full generality, as proved in [7], and gives a correspondence between toric origami manifolds and origami templates taking the ideas of Delzant [10] to this new singular scenario.

Theorem 3. [7] *Toric origami manifolds are classified by origami templates up to equivariant symplecto-morphism preserving the moment map. More specifically, there is a one-to-one correspondence*

$$\{2n\text{-dimensional toric origami manifolds}\} \longrightarrow \{n\text{-dimensional origami templates}\}$$
$$(M^{2n}, \omega, \mathbb{T}^n, \mu) \longmapsto \mu(M).$$

2.2. **Basics on b-symplectic manifolds.** In this section, we give a crash course on b-**symplectic/Poisson manifolds**.

The study of b-symplectic manifolds starts with a similar definition to that of folded symplectic manifolds but in the context of Poisson geometry. Given a symplectic form ω we can naturally associate a Poisson bracket to any pair of smooth functions $f, g \in C^\infty(M^{2n})$ from the symplectic structure as follows

$$\{f, g\} = \omega(X_f, X_g)$$

where the vector fields X_f and X_g stand for the Hamiltonian vector fields with respect to ω.

From the equation above it is simple to check that $\{f,g\} = X_f(g)$ so the bracket defines a biderivation (Leibniz rule), it is antisymmetric and because $X_{\{f,g\}} = [X_f, X_g]$, it also satisfies the Jacobi identity (i.e., $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$ for any triple of smooth functions f, g and h.

A general Poisson structure is defined as a general antisymmetric bracket on any manifold (not necessarily even dimensional) $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$ satisfying Leibniz rules and Jacobi identity.

Because a Poisson bracket defines a biderivation, we can write it as a bivector field $\Pi \in \Gamma(\Lambda^2(TM))$. The correspondence between Poisson brackets and Poisson bivector fields is clarified by the equation

$$\Pi(df, dg) = \{f, g\}.$$

The Jacobi identify defines an additional constraint and not every bivector field defines a Poisson structure. Bivector fields which are Poisson satisfy the integrability equation $[\Pi, \Pi] = 0$ where the bracket is the Schouten bracket, the natural extension of the Lie bracket to bivector fields.

In total analogy with the symplectic case, given a function we may define the Hamiltonian vector field via the equation: $X_f := \Pi(df, \cdot)$. Observe that, in particular the equation $\{f, g\} = X_f(g)$ also holds in the general Poisson context.

Let us now consider Poisson bivector fields on even-dimensional manifolds that are symplectic away from a hypersurface. When these Poisson bivector fields fulfill transversality conditions along the hypersurface, many techniques from the symplectic realm can be exported to study them. These are called b-Poisson manifolds and have been studied and analyzed in detail starting in [18].

Definition 4. *Let (M^{2n}, Π) be a Poisson manifold. If the map*

$$p \in M \mapsto (\Pi(p))^n \in \bigwedge^{2n}(T(M^{2n}))$$

*is transverse to the zero section of $\bigwedge^{2n}(T(M^{2n}))$, then Π is called a b-**Poisson structure** on M. The pair (M^{2n}, Π) is called a b-**Poisson manifold**. The vanishing set of Π^n is a hypersurface denoted by Z and called the **critical hypersurface** of (M^{2n}, Π).*

A list of examples can be found and analyzed in detail in [44]. In the particular case of surfaces, these structures coincide with the stable Poisson structures classified by Radko in [48]. This is why, from now on, we refer to a compact oriented surface endowed with a b-Poisson structure as a Radko surface.. The next example is the prototypical *Radko sphere*.

Example. *We endow the 2-sphere $\mathbb{S}^2$ with the coordinates (h, θ), where h denotes the height function and θ is the angle. The Poisson structure written as $\Pi = h \frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}$ vanishes transversally along the equator $Z = \{h = 0\}$ and thus it defines a b-Poisson structure on the pair $(\mathbb{S}^2, Z)$.*

The product of two b-Poisson manifolds is not a b-Poisson manifold but the product of a b-Poisson surface with a symplectic manifold is a b-Poisson manifold as described in the example below.

Example. *For higher dimensions we may consider the following product structures: let $(\mathbb{S}^2, Z)$ be the sphere in the example above and (S^{2n}, π_S) be a symplectic manifold, then $(\mathbb{S}^2 \times S, \pi_{\mathbb{S}^2} + \pi_S)$ is a b-Poisson manifold of dimension $2n + 2$. We may replace $(\mathbb{S}^2, Z)$ by any compact Radko surface (R, π_R) (see for instance [18]).*

Other examples come from foliation theory and from the theory of cosymplectic manifolds:

Example. *Let (N^{2n+1}, π) be a Poisson manifold of constant corank 1. Let us assume that there exists a vector field X which is a Poisson vector field and let $f : \mathbb{S}^1 \to \mathbb{R}$ be a smooth function. The bivector field given by*

$$\Pi = f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi$$

defines a b-Poisson structure on the product $S^1 \times N$ whenever the function f vanishes linearly and the vector field X is transverse to the symplectic leaves of N^{2n+1}. In this case, the critical hypersurface is formed by the union of as many copies of N as zeros of f.

The example above is generic in the sense that any b-Poisson structure can be described in this way in a neighborhood of a critical hypersurface N. The critical hypersurface N has a natural cosymplectic structure (see Definition 7) associated with it. In particular, this example realizes a given cosymplectic manifold as a connected component of a cosymplectic manifold which is a critical set of a b-Poisson manifold. This is the content of example 19 in [18].

Around any point in Z, the b-Darboux theorem (see [18] and [47]) guarantees that it is always possible to find local coordinates with respect to which the b-Poisson structure as stated below:

Theorem 4 (b-Darboux). *For any point $p \in Z$ on the critical hypersurface of a b-Poisson manifold we may find local coordinates centered at p for which the b-Poisson structure Π can be written as:*

$$\Pi = \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + t\frac{\partial}{\partial t} \wedge \frac{\partial}{\partial z}.$$

Thus b-Poisson manifolds and symplectic manifolds have many things in common. Indeed it is possible to work with the language of forms by admitting $\frac{df}{f}$ where f is the defining function of Z as a *legal* form of an extended complex. This function f can be considered well-defined semi-locally or near each connected component of Z or globally. See [49] for a discussion of this fact. This is the complex of b-forms originally introduced by Richard Melrose [42] in the context of b-calculus to study the index theorem on manifolds with boundary.

In order to introduce this language properly, we briefly recall the construction of b-forms.

Definition 5. *Let (M, Z) be a pair where M is a manifold and Z is a hypersurface. The b-tangent bundle ^bTM is the bundle whose sections are vector fields in M tangent to Z. A vector field tangent to Z is called a b-vector field. Similarly, the b-cotangent bundle $^bT^*M$ is the dual bundle to ^bTM and its sections are called b-forms.*

Any b-form of degree k can be written near Z (if f exists globally, this decomposition is global) as $\omega = \frac{df}{f} \wedge \alpha + \beta$ where α and β are $k-1$ and k smooth De Rham forms respectively and f is a defining function for Z.

Definition 6. *(b-functions) A b-function on a b-manifold (M, Z) is a function which is smooth away from the critical set Z, and near Z has the form*

$$c \log |t| + g,$$

where $c \in \mathbb{R}, g \in C^\infty$, and t is a local defining function of Z. The sheaf of b-functions is denoted $^bC^\infty$.

A closed b-form of degree 2 which is nondegenerate as a section of the bundle $\Lambda^2(^bT^*M)$ is called a b-**symplectic form**. As it is proved in [18], there is a one-to-one correspondence between b-symplectic forms and b-Poisson forms. In particular, we may re-state the Darboux normal form in the language of b-forms as done below

Theorem 5. *(b-Darboux theorem) Let ω be a b-symplectic form on (M, Z) and $p \in Z$. Then we can find a coordinate chart $(U, x_1, y_1, \ldots, x_n, y_n)$ centered at p such that on U the hypersurface Z is locally defined by $y_1 = 0$ and*

$$\omega = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

For any b-function f on a b-symplectic manifold (M, ω) the b-**Hamiltonian** vector field is the one X_f defined by $\iota_{X_f}\omega = -df$.

A $\mathbb{T}^k$-action on a b-symplectic manifold (M^{2n}, ω) is called b-**Hamiltonian** if the fundamental vector fields are the b-Hamiltonian vector fields of functions which Poisson commute. Such an action is called **toric** if $k = n$.

The critical hypersurface Z of a b-symplectic structure has an induced regular Poisson structure which can also be visualized as a cosymplectic manifold (see [18, 20]).

Definition 7. *A cosymplectic manifold is a manifold M of odd dimension $2n + 1$ equipped with a pair (α, ω), where α is a closed one-form and ω is a closed two-form such that $\alpha \wedge \omega^n$ is a volume-form.*

In [20] it was shown that if Z is compact and connected, and one of the symplectic leaves L is compact, then the critical set Z is the mapping torus of one of its symplectic leaves L by the flow at time k of any modular vector field u:

$$Z = (L \times [0, k])/_{(x,0)\sim(\phi(x),k)},$$

where k is a fixed positive real number and ϕ is the time-k flow of u. In particular, all the symplectic leaves inside Z are symplectomorphic. As in [18], we refer to a fixed symplectomorphism inducing the mapping torus as the **monodromy** of Z.

This yields the following definition:

Definition 8 (Modular period). *Taking any modular vector field u_{mod}^{Ω}, the* **modular period** *of Z is the number k such that Z is the mapping torus*

$$Z = (L \times [0, k])/_{(x,0)\sim(\phi(x),k)},$$

and the time-t flow of u_{mod}^{Ω} is translation by t in the $[0, k]$ factor above.

2.2.1. *The (twisted) b-cotangent lift.* The cotangent lift can also be defined on the b-cotangent bundle of a smooth manifold. In this case there are two different 1-forms that provide the same geometrical structure on the b-cotangent bundle (a b-symplectic form). These are the canonical (Liouville) b-form and the twisted b-form. Both forms of degree 1 have the same differential (a smooth b-symplectic form) but are indeed non-smooth forms. The b-cotangent lift in each of the cases is defined differently. These were studied in detail in [36]. In this article, we focus on the twisted b-cotangent lift as it gives the right model for the structure of a b-integrable system.

Definition 9. *Let $T^*\mathbb{T}^n$ be endowed with the standard coordinates (θ, a), $\theta \in \mathbb{T}^n$, $a \in \mathbb{R}^n$ and consider again the action on $T^*\mathbb{T}^n$ induced by lifting translations of the torus $\mathbb{T}^n$. Define the following non-smooth one-form away from the hypersurface $Z = \{a_1 = 0\}$:*

$$\lambda_{tw,c} = c\log|a_1|d\theta_1 + \sum_{i=2}^{n} a_i d\theta_i,$$

where c is a constant. Then, the form $\omega := -d\lambda_{tw,c}$ is a b-symplectic form on $T^*\mathbb{T}^n$, called the twisted b-symplectic form on $T^*\mathbb{T}^n$. In coordinates:

$$\omega_{tw,c} := \frac{c}{a_1}d\theta_1 \wedge da_1 + \sum_{i=2}^{n} d\theta_i \wedge da_i. \tag{1}$$

Observe that this twisted forms comes endowed with a local invariant: The constant c. The interpretation of this invariant is that this gives the period of the modular vector field, which depends on each connected component of Z.

We call the lift together with the b-symplectic form (1) the **twisted b-cotangent lift** with modular period c on the cotangent space of a torus.

As it was deeply studied in [36] the lifted action can be extended to groups of type $S^1 \times H$ which turns out to be b-Hamiltonian in general.

2.2.2. *b-Integrable systems.* A b-symplectic manifold/b-Poisson manifold can be seen as a standard Poisson manifold. Along the critical set Z the Liouville tori determined by a standard integrable system have dimension $n-1$ which is not convenient to model integrable systems on b-symplectic manifolds where the critical set Z represent the direction of *infinity* in celestial mechanics and we would expect to have n-dimensional tori.

This is why in this context it is more natural to talk about *b-integrable system* as follows:

Definition 10. *A b-integrable system on a $2n$-dimensional b-symplectic manifold (M^{2n}, ω) is a set of n b-functions which are pairwise Poisson commuting $F = (f_1, \ldots, f_{n-1}, f_n)$ with $df_1 \wedge \cdots \wedge df_n \neq 0$ as a section of $\wedge^n({}^bT^*(M))$ on a dense subset of M and on a dense subset of Z. A point in M is **regular** if the vector fields $X_{f_1}, \ldots, X_{f_n}$ are linearly independent (as* smooth *vector fields) at it.*

For these systems an action-angle coordinate theorem, proved in [37], shows the existence of a semilocal invariant in the neighbourhood of Z (the modular period):

Theorem 6. *Let $(M, \omega, F = (f_1, \ldots, f_{n-1}, f_n = \log|t|))$ be a b-integrable system, and let $m \in Z$ be a regular point for which the integral manifold containing m is compact, i.e. a Liouville torus F_m. Then there exists an open neighborhood U of the torus F_m and coordinates $(\theta_1, \ldots, \theta_n, \sigma_1, \ldots, \sigma_n) : U \to \mathbb{T}^n \times B^n$ such that*

$$\omega|_U = \sum_{i=1}^{n-1} d\sigma_i \wedge d\theta_i + \frac{c}{\sigma_n}d\sigma_n \wedge d\theta_n, \tag{2}$$

where the coordinates $\sigma_1, \ldots, \sigma_n$ depend only on F and the number c is the modular period of the component of Z containing m.

In [36] this normal form was identified as a cotangent model:

Theorem 7. *Let $F = (f_1, \ldots, f_n)$ be a b-integrable system on the b-symplectic manifold (M, ω). Then semilocally around a regular Liouville torus $\mathbb{T}$, which lies inside the exceptional hypersurface Z of M, the system is equivalent to the cotangent model $(T^*\mathbb{T}^n)_{tw,c}$ restricted to a neighbourhood of $(T^*\mathbb{T}^n)_0$. Here c is the modular period of the connected component of Z containing $\mathbb{T}$.*

2.2.3. *A bridge between b-Symplectic manifolds and folded symplectic manifolds.* In [21] the following theorem is proved in the more general setting of b^m-symplectic structures with singularities of higher order. For $m = 1$, it associates a family of folded-symplectic forms to a given b-symplectic form. This process is know as *desingularization*.

Theorem 8. *Let ω be a b-symplectic structure on a compact manifold M and let Z be its critical hyper-surface. There exists a family of folded symplectic forms ω_ϵ which coincide with the b-symplectic form ω outside an ϵ-neighborhood of Z.*

As a consequence of this result any b-symplectic manifold admits a folded symplectic structure. However, it is well-known that the converse statement does not hold as not every folded symplectic form can be presented as a desingularization of a b-symplectic structures. In particular, any compact orientable 4-dimensional manifold admits a folded symplectic form [2] but not every 4-dimensional compact manifold admits a b-symplectic manifold. For instance the 4-sphere S^4 does not admit a b-symplectic structure as it was proven in [18] that the class determined by the b-symplectic form is non-vanishing.

3. HAMILTONIAN DYNAMICS ON FOLDED SYMPLECTIC MANIFOLDS

Let (M, ω) be a folded symplectic manifold, with folding hypersurface Z. Consider p a point in Z, applying Theorem 1 the folded-symplectic form ω can be written in a neighborhood of p as:

$$\omega = tdt \wedge dq + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

with $t = 0$ defining the folding hypersurface. The singularity in ω prevents the Hamiltonian equation $\iota_X \omega = -df$ from having a solution for every possible function f. So not every function $f \in C^\infty(U)$ defines locally a Hamiltonian vector field.

Example. Let $(U; t, q, ..., x_n, y_n)$ be a chart where ω is written as the folded-Darboux form mentioned above. Take for example the function $f = t$. The associated Hamiltonian vector field is $X = \frac{1}{t}\frac{\partial}{\partial q}$, which is not a well-defined smooth vector field.

Fortunately, we can characterize the set of functions which define smooth Hamiltonian vector fields as follows:

Lemma 9. *A function $f : M \to \mathbb{R}$ on a folded symplectic manifold (M, ω) defines a smooth Hamiltonian vector field X_f if and only if $df|_p(v) = 0$ for every $p \in Z$ and $v \in V| = \ker \omega|_p$. Furthermore X_f is tangent to Z.*

Proof. Assume that f defines a smooth Hamiltonian vector field at a point p in Z. This means that the equation $\iota_X \omega = -df$ has a solution. Assume that there is some point $p \in Z$ and some tangent vector $v \in V|_p \subset TM|_p$ such that $df_p(v) \neq 0$. Then $\omega_p(X_p, v_p) \neq 0$ by the Hamiltonian equation, which contradicts the fact that $v \in \ker \omega|_Z$.

Conversely, assume that $df|_p(v) = 0$ for every $p \in Z$ and $v \in V_p = \ker \omega|_p$. Take Martinet's coordinates $(t, q, ..., x_n, y_n)$ at a neighborhood U of p. The form has the following expression:

$$\omega = tdt \wedge dq + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Any vector field can be written as $X = a_1 \frac{\partial}{\partial t} + b_1 \frac{\partial}{\partial q} + ... + a_n \frac{\partial}{\partial x_n} + b_n \frac{\partial}{\partial y_n}$. The Hamiltonian equation $\iota_X \omega = -df$ can be written component-wise as

$$\begin{cases} a_1 & = -\frac{\partial f}{\partial q}\frac{1}{t} \\ b_1 & = \frac{\partial f}{\partial t}\frac{1}{t} \\ a_i & = -\frac{\partial f}{\partial q}, \quad i = 2, ..., n \\ b_i & = \frac{\partial f}{\partial t}, \quad i = 2, ..., n. \end{cases}$$

In these coordinates, we can write $V = \langle \frac{\partial}{\partial q}, \frac{\partial}{\partial t} \rangle$. Since $df(v) = 0$ along Z for each $v \in V$, we deduce that,

$$\frac{\partial f}{\partial q} = tH \tag{3}$$

$$\frac{\partial f}{\partial t} = tF \tag{4}$$

for some smooth function H and G. Then the coefficients a_1 and b_1 are well-defined and so is X as claimed.

From Equation (4) we get that $f = t^2 f_1 + f_2(q, x_2, ..., y_n)$ for some smooth functions f_1, f_2. Equation (3) implies that $\frac{\partial f_2}{\partial q} = 0$. Hence f has the form

$$f = t^2 f_1 + f_2(x_2, ..., y_n). \tag{5}$$

From this equation, we deduce that that $\frac{\partial f}{\partial q} = t^2 \frac{\partial f_1}{\partial q}$, which implies that

$$a_1|_{\{t=0\}} = -t \frac{\partial f_1}{\partial q}|_{\{t=0\}} = 0,$$

where a_i are the components of the Hamiltonian vector field defined above. We deduce that X is tangent to Z. □

This motivates the following definition.

Definition 11. *A function $f : M \to \mathbb{R}$ in a folded symplectic manifold (M, ω) is a **folded function** if $df|_Z(v) = 0$ for every $v \in V = \ker \omega|_Z$.*

Remark 2. The space of folded functions corresponds to the space of admissible functions $C^\infty_{\mathrm{adm}}(M)$ defining Hamiltonian vector fields on a Dirac manifold (confer [6, Section 1.4.3]), where we interpret our folded symplectic manifold as a Dirac manifold.

Note that even if a Hamiltonian vector field X_f is always tangent to Z, one can obtain non-vanishing components of X_f in the null line bundle L. If one takes n folded functions, we will always have $df_1 \wedge ... \wedge df_n|_Z = 0$ when we look at it as a section of $\Lambda^n T^* M$. However, the n functions can define n linearly independent Hamiltonian vector fields even at points in Z. This justifies the following natural definition:

Definition 12. *A folded integrable system is a set of n folded functions $F = (f_1, ..., f_n)$ on a folded symplectic manifold (M, ω) of dimension $2n$ with critical surface Z, which define Hamiltonian vector fields that are independent on a dense set of Z and M, and commute with respect to ω (i.e. $\omega(X_{f_i}, X_{f_j}) = 0$ for all i, j).*

We say that a point $p \in Z$ is *regular* if the Hamiltonian vector fields of f_i are all independent at p. Around the regular points of the integrable system, the expression of the functions can be simplified and as a consequence the Poisson bracket of the functions is well-defined:

Lemma 10. *Let F be a folded integrable system on a folded symplectic manifold of dimension $2n$. Near a regular point of an integrable system, there exist coordinates $(t, q, x_2, ..., y_n)$ such that $\omega = tdt \wedge dq +$*

$\sum_{i=2}^{n} dx_i \wedge dy_i$, *and the integrable system has the form*

$$f_1 = t^2/2$$
$$f_2 = g_2(t, q, x_2, ..., y_n)t^{k_2} + h_2(x_2, y_2, ..., x_n, y_n)$$
$$\vdots$$
$$f_n = g_n(t, q, x_2, ..., y_n)t^{k_n} + h_n(x_2, y_2, ..., x_n, y_n),$$

for $k_2, \ldots, k_n \in \mathbb{N}$ all of them ≥ 2 and t is a defining function of Z.

Proof. Denote the inclusion of Z in M by $i : Z \hookrightarrow M$. Since the pullback to Z of the folded symplectic form $i^*\omega$ has rank $2n - 2$, there are at most $n - 1$ independent Hamiltonian vector fields tangent to Z such that $\langle X_1, ..., X_{n-1} \rangle$ has no component in $\ker i^*\omega$. This implies that at any regular point $p \in Z$ of an integrable system, one of the n independent Hamiltonian vector fields $X_1, ..., X_n$ has a component in $\ker i^*\omega$. We might assume it is the first one X_1.

Let us show that in the points close to p in Z, this vector field X_1 can be written as $X_1 = v + X'$, where $v \in \ker i^*w$ and $X' \in \langle X_2, ..., X_n \rangle$. Decompose the tangent two-space $V_p = \ker \omega_p$ as $V_p = L_p \oplus H_p$ where $L_p = \ker i^*\omega|_p \subset T_pZ$. In particular H_p is transverse to T_pZ. Observe that the tangent space of M at p decomposes as $T_pM = W \oplus L_p \oplus H_p$, where W is a symplectic subspace of T_pZ of dimension $2n-2$. This symplectic space decomposes as $W = \langle X_2, ..., X_n \rangle|_p \oplus \langle X_2, ..., X_n \rangle|_p^{\perp}$, where the orthogonal is taken with respect to the symplectic bilinear form on W. However we know that Hamiltonian vector fields are tangent to Z so X_1 cannot have a component in H_p. If it has a component in $\langle X_2, ..., X_n \rangle|_p^{\perp}$, then we would not have $\omega(X_1, X_j)|_p = 0$ for every $j = 2, ..., n$. This proves that X_1 admits the decomposition $X_1 = v + X'$.

From the previous discussion we deduce that $\langle X_1, ..., X_n \rangle = \langle v, X_2, ..., X_n \rangle$. Take local coordinates in a neighborhood U of p such that $v = \frac{\partial}{\partial q}$. Take symplectic coordinates $(x_2, y_2, ..., x_n, y_n)$ of $i^*\omega$, the existence of such coordinates follows from the Darboux theorem for closed two forms of constant rank [39, Proposition 13.7]. Since v is a section of the null line bundle, we can apply Theorem 2 with $\alpha = dq$ to conclude that

$$\omega = tdt \wedge dq + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

In these coordinates the vector field X_1 is the Hamiltonian vector field of $t^2/2$, hence $f_1 = t^2/2$. The remaining functions $f_2, ..., f_n$ are folded functions and can be expressed as in Equation (5). This concludes the proof of the lemma. $\qquad \square$

3.1. **Folded cotangent bundle.** In this subsection, we recall the construction of folded cotangent bundle described in [29]. It is a vector bundle whose construction is analogous to that of the b-cotangent bundle (originally introduced by Melrose [42]). The b-cotangent bundle is defined as the dual bundle to the bundle defined by vector fields tangent to a given hypersurface. Contrarily, the folded cotangent bundle is defined as differential forms which vanish (in some specific sense) transversally along a prescribed hypersurface.

Definition 13. *Let M a manifold and Z a closed hypersurface. Let V a rank 1 subbundle of i_Z^*TM so that for all $p \in Z$ the fiber V_p is transverse to T_pZ. We define for each open subset $U \subset M$*

$$\Omega_V^1(U) := \{\alpha \in \Omega^1(U)| \ \alpha|_V = 0\},$$

the space of 1-forms on U vanishing on V. If $U \cap Z = \emptyset$ then it is just $\Omega^1(U)$.

Following [29] there exists a vector bundle $T_V^* M$ called the **folded cotangent bundle**, of rank n whose global sections are isomorphic to $\Omega_V^1(M)$. This vector bundle is unique up to isomorphism, independently of the chosen V. For a small open neighborhood U of a point in Z, there exist suitable coordinates $(x_1, ..., x_{n-1})$ in $U \cap Z$ and a coordinate t such that $(x_1, ..., x_{n-1}, t)$ are coordinates in U and $T_V^*(U)$ is generated by $dx_1, ..., dx_{n-1}, tdt$. The dual bundle to $T_V^* M$ is denoted by $T_V M$ and called the folded tangent bundle.

In this bundle there is a canonical folded symplectic form which is obtained by taking a Liouville form λ_f which is canonical as it satisfies the *Liouville-type* equation $\langle \lambda_f|_p, v \rangle = \langle p, (\pi_p)_*(v) \rangle$ for every $v \in T_V(T_V^* M)$ and $p \in T_V^* M$. In coordinates $(x_1, ..., x_n, p_1, ..., p_n)$ we can write

$$\lambda = p_1 x_1 dx_1 + \sum_{i=2}^n p_i dx_i.$$

Its derivative gives rise to a folded symplectic structure

$$\omega_f = d\lambda = x_1 dp_1 \wedge dx_1 + \sum_{i=2}^n dp_i \wedge dx_i$$

which looks like the Darboux-type folded symplectic structure. The introduction of this bundle allows restating the definition of a folded integrable system in terms of the folded cotangent bundle.

Definition 14. *An integrable system on a $2n$-dimensional folded symplectic manifold (M, ω) is a set of folded functions $F = (f_1, ..., f_n)$ for which $df_1 \wedge ... \wedge df_n \neq 0$ as sections of $\Lambda^n T_V^* M$ on a dense set of M and Z, and whose Hamiltonian vector fields commute with respect to ω, i.e. $\omega(X_{f_i}, X_{f_j}) = 0, \forall i, j$.*

Even if ω does not define a Poisson bracket in Z because the Hamiltonian vector fields are not defined for non-folded functions, the bracket is well-defined for folded functions and the commutation condition $\omega(X_{f_i}, X_{f_j}) = 0$ for two Hamiltonian vector fields is still well-defined (see also Remark 2 and [6, Section 1.4.3]).

4. EXAMPLES OF FOLDED INTEGRABLE SYSTEMS

In this section, we present a series of examples of folded integrable systems. In particular, we exhibit examples of folded integrable systems whose dynamics cannot possibly be modeled by b-integrable systems. This motivates the development of the theory of folded integrable systems, and in particular the study of action-angle coordinates.

4.1. **Double collision in two particles system.** In the literature of celestial mechanics concerning the restricted 3-body problem or the n-body problem, several regularization transformations associated with ad-hoc changes (like time reparametrization) endow the symplectic structure with natural singularities. Below we describe a model of double collision in two-particle systems where McGehee-type changes are implemented. We model a system of two particles under the influence of a potential energy function of the form $U(x) = -|x|^{-\alpha}$, with $\alpha > 0$. In the phase space $(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2$ it is a Hamiltonian system with Hamiltonian function $F = \frac{1}{2}|y|^2 - |x|^{-\alpha}$. Let us introduce a notation for two constants: denote $\beta = \alpha/2$ and $\gamma = \frac{1}{\beta+1}$. By performing the change of coordinates:

$$\begin{cases} x &= r^\gamma e^{i\theta} \\ y &= r^{-\beta\gamma}(v + iw)e^{i\theta} \end{cases}$$

and scaling with a new time parameter τ such that $dt = rd\tau$ we obtain the equations of motion

$$\begin{cases} r' &= (\beta + 1)rv \\ v' &= w^2 + \beta(v^2 - 2) \\ \theta' &= w \\ w' &= (\beta - 1)\omega v \end{cases}.$$

We will model the collision set $\{r = 0\}$ in the case $\beta = 1$ as the folding hypersurface of a folded symplectic manifold endowed with a folded integrable system. Let us consider the folded symplectic form $\omega = rdr \wedge dv + d\theta \wedge dw$ in the manifold $T^*(\mathbb{R} \times S^1) \cong \mathbb{R}^2 \times S^1 \times \mathbb{R}$ with coordinates (r, v, θ, w). We take the folded Hamiltonian function

$$H = -\frac{r^2}{2}(w^2 + (v^2 - 2)) + \frac{w^2}{2}.$$

Observe that $dH = -r^2v dv + (w^2 + v^2 - 2)2rdr + (w - r^2w)dw$, and the Hamiltonian vector field is

$$X_H = -rv\frac{\partial}{\partial r} + (w^2 + v^2 - 2)\frac{\partial}{\partial v} + (w + r^2w)\frac{\partial}{\partial \theta}$$

The equations of motion in the critical hypersurface $\{r = 0\}$ coincide with the equations of motion in the collision manifold of the original problem, hence providing a folded Hamiltonian model for it. Even the linear asymptotic behavior close to collision is captured by the model. Observe that X commutes with $\frac{\partial}{\partial \theta}$, which is a Hamiltonian vector field for the function $f_2 = w$. Hence the dynamics are modelled by a folded integrable system given by $F = (f_1 = H, f_2)$ in $T^*(\mathbb{R} \times S^1)$ with folded symplectic structure $rdr \wedge dv + d\theta \wedge dw$.

4.2. Folded integrable systems on toric origami manifolds.
Not all integrable systems on folded symplectic manifolds come from standard systems on symplectic manifolds after singularization transformations or regularization techniques as in the example above. Take for instance $\mathbb{R}^4$ with the standard symplectic structure $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. The function $f_1 = x_1^2 + y_1^2 + x_2^2 + y_2^2$ and $f_2 = x_1y_2 - x_2y_1$ commute with respect to ω. There is a natural folding map from the sphere S^4 to $\bar{D}^4$, that we denote π. It is a standard fact that $\pi^*\omega$ is a folded symplectic structure in S^4, which is in addition of origami type. Taking $F = (\pi^*f_1, \pi^*f_2)$ yields an example of a folded integrable system in S^4 with its induced folded symplectic structure. Note that this is an example of an integrable system on a singular symplectic manifold which is not b-symplectic, as shown by the obstructions in [18] and [40].

4.3. Symplectic manifolds with fibrating boundary.
Consider a symplectic manifold with boundary such that close to the boundary the symplectic form tends to degenerate and admits adapted Martinet-Darboux charts such that the boundary has local equation $x_1 = 0$ and the symplectic form degenerates on the boundary with the following local normal form:

$$\omega = x_1 dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \ldots + dx_n \wedge dy_n.$$

Let us take as a starting point some integrable system naturally defined on a manifold with boundary. Assume the folding hypersurface fibrates by circles over a compact symplectic base (origami type). It would be enough to consider an integrable system on the $(2n - 2)$-symplectic base $f_2, \ldots f_n$ and add t^2 as f_1. The set $(t^2, f_2, \ldots, f_n)$ defines a folded integrable system. Observe that complete integrability comes as a consequence of Theorem 2.

4.4. Product of folded surfaces with symplectic manifolds endowed with integrable systems.

Take an orientable surface Σ, and ω a non-vanishing two form. Denote t any function in Σ which is transverse to the zero section. The critical set is a finite number of closed curves $\gamma_j, \ j = 1, ..., k$. Then the function t^2 defines a folded integrable system in $(\Sigma, t\omega)$, where $t\omega$ is a folded symplectic structure. Let $F = (f_1, ..., f_n)$ be an integrable system in a symplectic manifold (M^{2n}, ω_1). Then $(t^2, f_1, ..., f_n)$ defines a folded integrable system in the manifold $M^{2n} \times \Sigma$ endowed with the folded symplectic form $\omega_f = t\omega + \omega_1$. In fact, taking any $(n+1)$-tuple of the form $(t^2 \sum_{i=1}^{n} \lambda_i f_i, f_1, ..., f_n)$ for some non-trivial n-tuple of constants λ_i yields a folded integrable system. The critical set is of the form $Z = \sqcup_{j=1}^{k} \gamma_j \times M^{2n}$.

4.5. Origami templates.

In subsection 2.1.2 we introduced the classification of toric action on origami symplectic manifolds. The moment map of these toric actions are examples of folded integrable systems which are described by a global Hamiltonian action of a torus. Indeed any integrable system can be semi-locally described in these terms (as we will see in the next section).

Origami templates form a visual way to describe toric origami manifolds and thus a particular case of integrable systems on folded symplectic manifolds. In Figure 2 we depict the origami template associated to Example 4.2 in this section.
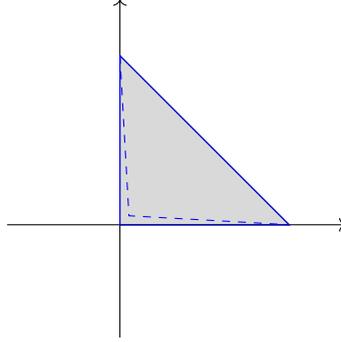

FIGURE 2. Origami template associated with Example 4.2.

4.6. The folded spherical pendulum.

Consider the spherical pendulum on S^2 defined as follows: Take spherical coordinates (θ, ϕ) with $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$ if we denote each momentum as P_θ and P_ϕ respectively, the Hamiltonian function is

$$H = \frac{1}{2}(P_\theta^2 + \frac{1}{\sin^2 \theta} P_\phi^2) + \cos \theta.$$

Instead of taking the standard symplectic form in T^*S^2 we consider the folded symplectic form

$$\omega = P_\phi dP_\phi \wedge d\phi + dP_\theta \wedge d\theta.$$

Computing the Hamiltonian vector field associated to H we get

$$X_H = \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \phi} + P_\theta \frac{\partial}{\partial \theta} + (\sin \theta + \frac{\cos \theta}{\sin^3 \theta} P_\phi^2) \frac{\partial}{\partial P_\theta}.$$

This vector field clearly commutes with $\frac{\partial}{\partial \phi}$, which is the Hamiltonian vector field of $f = P_\phi^2$. Observe furthermore that

$$dH \wedge dP_\phi^2 = -(\sin \theta + \frac{\cos \theta}{\sin^3 \theta} P_\phi^2) 2 P_\phi d\theta \wedge dP_\phi,$$

which is nondegenerate on a dense set of M and on a dense set of Z when seen as a section of the second exterior product of the folded cotangent bundle. The manifold is $M = T^*(S^2 \setminus \{N, S\})$, i.e. we are taking out the poles of the sphere. In this sense M is equipped with an origami symplectic form: the critical set is $T^*(S^2 \setminus \{N, S\})$ and the null line bundle is an S^1 fibration generated by $\frac{\partial}{\partial \varphi}$.

Observe that dynamically this system is different from the standard spherical pendulum. When $P_\phi = 0$, the vector field can have a non vanishing $\frac{\partial}{\partial \phi}$ component.

4.7. A folded integrable system which cannot be modelled as a b-integrable system.
Consider S^2 with the folded symplectic form $\omega = h\, dh \wedge d\theta$. A folded function whose exterior derivative is a non-vanishing one-form (when considered as a section of the folded cotangent bundle) on a dense set of M and of Z defines a folded integrable system. Take for instance $f = \cos\theta h^2$, which satisfies this condition. Computing its Hamiltonian vector field we obtain

$$X_f = h\sin\theta \frac{\partial}{\partial h} + 2\cos\theta \frac{\partial}{\partial \theta}.$$

This vector field vanishes at some points in the critical locus $Z = \{h = 0\}$. A b-integrable system on a surface Σ is defined by a function $f = c\log|h| + g$ with $g \in C^\infty(\Sigma)$. In particular, its Hamiltonian vector field cannot vanish at any point on the critical hypersurface, as it happens in this example of folded integrable system. Thus, even if the structure $h\, dh \wedge d\theta$ can be seen as the desingularization of $\frac{1}{h} dh \wedge d\theta$, the dynamics of this folded integrable system cannot be modeled using the b-symplectic structure.

4.8. Cotangent lifts for folded symplectic manifolds.
In this section, we describe the cotangent lift in the set-up of folded symplectic manifolds.

When the group acting on the base is a torus this procedure provides examples of folded integrable systems.

Consider a Lie group G acting on M by an action $\phi : G \times M \longrightarrow M$.

Definition 15. *The cotangent lift of ϕ is the action on T^*M given by $\hat{\phi}_g := \phi_{g^{-1}}^*$, where $g \in G$.*

The following commuting diagram holds:

$$
\begin{array}{ccc}
T^*M & \xrightarrow{\ \hat{\phi}_g\ } & T^*M \\
\pi \downarrow & & \downarrow \pi \\
M & \xrightarrow{\ \phi_g\ } & M
\end{array}
$$

where π is the projection from T^*M to M. The cotangent bundle has the symplectic form $\omega = -d\lambda$ where λ is the Liouville form. This form is defined by the property $\langle \lambda_p, v \rangle = \langle p, (\pi_p)_*(v) \rangle$, where $v \in T(T^*M)$ and $p \in T^*M$. It can be shown easily that the cotangent lift is a Hamiltonian action with momentum map $\mu : T^*M \to \mathfrak{g}^*$ given by

$$\langle \mu(p), X \rangle := \langle \lambda_p, X^\#|_p \rangle = \langle p, X^\#|_{\pi(p)} \rangle.$$

Here $X^\#$ denotes the fundamental vector field of X associated to the action. The Liouville form is invariant by the action which implies the invariance of the moment map. In particular, the map is Poisson.

The construction called b-symplectic cotangent lift for b-symplectic manifolds done in [36] can be similarly done in the folded symplectic case which we will do below.

For the standard Liouville form in the folded cotangent bundle, the singularity is in the base space, and we would like to have it on the fiber. A different form, that we call **twisted folded Liouville form** can be defined on $T_V^* S^1$ with coordinates (θ, p):

$$\lambda_{tw} = \frac{p^2}{2} d\theta_1.$$

this way the singularity is in the fiber, and we can apply it to define a folded cotangent lift on the torus. Let $\mathbb{T}^n$ be the manifold and the group acting by translations, and take the coordinates $(\theta_1, ..., \theta_n, a_1, ..., a_n)$ on $T^* M$. The standard symplectic Liouville form in these coordinates is

$$\lambda = \sum_{i=1}^{n} p_i d\theta_i.$$

The moment map $\mu_{can} : T^* \mathbb{T}^n \to \mathfrak{t}^*$ of the lifted action with respect to the canonical symplectic form is

$$\mu_{can}(\theta, p) = \sum_i p_i d\theta_i$$

where the θ_i are seen as elements of $\mathfrak{t}^*$. In fact, one can identify the moment map as just the projection of $T^* \mathbb{T}^n$ into the second component since $T^* \mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$.

This torus action in the cotangent bundle of the torus can be seen as a folded-Hamiltonian action with respect to a folded symplectic form. Similarly to the Liouville one-form we define the following singular form away from the hypersurface $Z = \{p_1 = 0\}$:

$$p_1^2 d\theta_1 + \sum_{i=2}^{n} p_i d\theta_i.$$

The negative differential gives rise to a folded symplectic form called twisted folded symplectic form on $T^* \mathbb{T}^n$:

$$\omega_{tw,f} := p_1 d\theta_1 \wedge dp_1 + \sum_{i=2}^{n} d\theta_i \wedge dp_i.$$

The moment map is then

$$\mu_{tw,f} = (p_1^2, p_2, \dots, p_n),$$

where we identify $\mathfrak{t}^*$ with $\mathbb{R}^n$ as before.

We call this lift the **folded cotangent lift**. Note that, in analogy to the symplectic case, the components of the moment map define a folded integrable system on $(T^* \mathbb{T}^n, \omega_{tw,f})$.

Remark 3. As we will see in the next section, the folded cotangent lift does not always serve as the semilocal model for an integrable system in the neighborhood of a Liouville torus, in contrast to what happens in symplectic and b-symplectic geometry.

5. CONSTRUCTION OF ACTION-ANGLE COORDINATES

In this section, we prove the existence of action-angle coordinates for folded integrable systems at regular fibers defined on Z. Away from Z the manifold is symplectic and the standard Arnold-Liouville-Mineur theorem applies. One may have the temptation to use a desingularization procedure as in [21] and the action-angle coordinate theorem proved in [37] to conclude. However, as we saw in previous sections, there are manifolds that admit folded symplectic structures but no b-symplectic structure. Furthermore, we say that there are dynamics induced by folded integrable

systems which cannot be obtained by b-integrable systems. Thus, we provide below a proof of the action-angle coordinates theorem for the general folded symplectic case. Action-angle coordinates for general Dirac manifolds under some additional hypotheses near the regular fiber are studied in [54].

5.1. **Topology of the integrable system.** We first show that for a folded integrable system there is a foliation by Liouville tori in the neighborhood of a regular fiber in Z of the integrable system. In Figure 3 we schematically depict a trivial fibration by tori near a regular fiber.

Proposition 11. *Let $p \in Z$ be a regular point of a folded-integrable system (M, ω, F). Assume that the integral manifold $\mathcal{F}_p$ is compact. Then there is neighborhood U of $\mathcal{F}_p$ and a diffeomorphism*

$$\varphi : U \cong \mathbb{T}^n \times B^n$$

which takes the foliation $\mathcal{F}$ to the trivial foliation $\{\mathbb{T}^n \times \{b\}\}_{b \in B^n}$.

Proof. Arguing as in the proof of Lemma 10, the kernel of $i^*\omega$ is generated by the joint distribution of the Hamiltonian vector fields of $f_1, ..., f_n$ at every point p of a neighborhood of a regular fiber. Hence, we can assume that $f_1 = t^2/2$ for some semi-local coordinate t defining Z. The foliation given by the Hamiltonian vector fields of $F = (t^2/2, ..., f_n)$ is the foliation $\mathcal{F}$ integrating the distribution $\langle X_{f_1}, ..., X_{f_n} \rangle$. Each Hamiltonian vector field is tangent to the level sets of F, and hence also tangent to the level sets of $\bar{F} = (t, f_2, ..., f_n)$.

The set of functions $\bar{F}$ are functionally independent near Z. This can be easily deduced from the normal form of folded integrable systems Lemma 10 near any point $p \in Z$. The first function is always $t^2/2$, and the other functions are of the form $f_i = g_i t^2 + h_i(x_2, y_2, ..., x_n, y_n)$ in some Darboux coordinates where $\omega = t dt \wedge dq + \sum_{i=2}^n dx_i \wedge dy_i$. The independence of the Hamiltonian vector fields along the regular fiber $\mathcal{F}_p$ containing $p \in Z$ implies that the functions f_i satisfy $df_2 \wedge ... \wedge df_n|_{\mathcal{F}_p} \neq 0$. Furthermore, since $df_i|_p = dg_i$ and the functions g_i do not depend on t, we deduce that $dt \wedge df_2 \wedge ... df_n \neq 0$ at any point in the regular fiber containing p. Hence this holds at all points in a neighborhood U of the regular fiber because it is an open condition. We deduce that

$$\bar{F} = (t, f_2, ..., f_n) : U \longrightarrow \mathbb{R}^n$$

defines a submersion. The neighboring fibers are all tori. The proof of this fact is standard as there are n commuting non-vanishing vector fields on a leaf of dimension n (confer [9]). We now finish the proof as in [38, Proposition 3.2] to show that the fibration is trivial in U.

At any point $q \in \mathcal{F}_p$, there is a neighborhood $U_q \subset M$ which is diffeomorphic to the product $V_q \times B_q$, where V_q is a small neighborhood of q inside $\mathcal{F}_p$ and B_q is simply an open ball of dimension n. The diffeomorphism ϕ_p is obtained by the implicit function theorem as a lifting of $\bar{F}$. We obtain the following commuting diagram:

$$
\begin{array}{ccc}
U_q & \xrightarrow{\phi_q} & V_q \times B_q \\
 & \searrow & \downarrow{p_B} \\
 & & B_q
\end{array}
$$

By covering $\mathcal{F}_p$ by finitely many open neighborhoods V_i and balls B_i, we obtain some diffeomorphisms ϕ_i and we need to check that they coincide on each pair-wise intersection of the domains. This would be enough to find a diffeomorphism of a neighborhood of $\mathcal{F}_p$, by considering the smallest ball of the B_i.
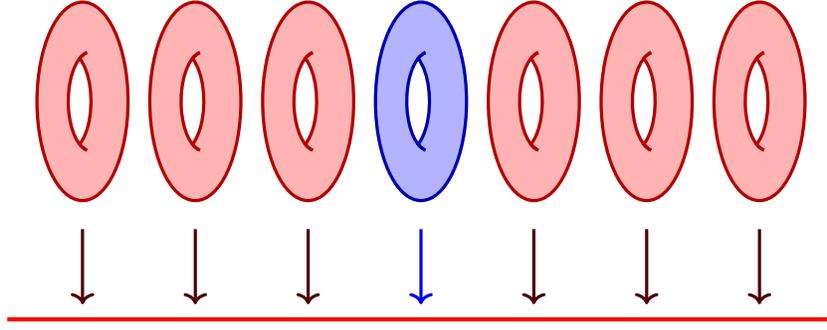
FIGURE 3. Fibration by Liouville tori: The middle fiber of the point $p \in Z$ in blue, the neighboring Liouville tori in red.

By means of an auxiliary Riemannian metric g in M, we can identify the normal bundle of $\mathcal{F}_p$ with a neighborhood of it inside M using the exponential map. For each V_i, we find smooth maps $\varphi_i : U_i = V_i \times B_i \to V_i$ that agree on the intersection of their domains. By possibly shrinking the domains U_i, we can choose the diffeomorphisms $\tilde{\phi}_i = \varphi_i \times (t, ..., f_n)$ which are now defined on a neighborhood of $\mathcal{F}_p$ and have the required matching condition. This yields a diffeomorphism

$$\varphi : U \longrightarrow \mathbb{T}^n \times B^n$$

taking the foliation $\mathcal{F}$ to the trivial foliation by tori.   □

We now prove a Darboux-Carathéodory theorem for folded symplectic manifolds to (locally) complete a set of folded functions that commute with respect to ω. We do this applying the arguments of the proof of Darboux theorem provided in [1]. The Darboux-Carathéodory theorem will be a key point in the proof of the existence of action-angle coordinates.

Theorem 12 (Folded Darboux-Carathéodory theorem). *Let $p \in Z$ be a point of the folding hypersurface of a folded symplectic manifold (M, ω) and let t be the function defining Z. Consider $f_1, ..., f_n$ to be n folded functions whose Hamiltonian vector fields are smooth, independent at p and commute pairwise with respect to ω. Then in a neighborhood U of p there exists n functions $q_1, ..., q_n$ such that near p the folded symplectic form is written as*

$$\omega = \sum_{i=1}^{n} dq_i \wedge df_i. \tag{6}$$

A system of coordinates is given by $q_1, ..., q_n$ and some coordinates $t, y_2, .., y_n$ such that the f_i only depend on the latter.

Remark 4. Note that the functions $f_1, ..., f_n$ do not define a set of n independent coordinate functions in U and, thus, Equation (6) does not correspond to a symplectic form but a folded symplectic form.

Proof. In order to construct this decomposition, we first construct the folded symplectic conjugate of the function f_1 following the classical recipe which we can find in [1].

In U the foliation $\mathcal{F}$ induced by the level sets of $(t, f_2, ..., f_n)$ coincides with the one generated by $\mathcal{D} = \langle X_1, ..., X_n \rangle$, where X_i denotes the Hamiltonian vector field of f_i. Take B a submanifold of dimension n containing p and transverse to this foliation. We will now construct a function q_1 such that $X_1 = \frac{\partial}{\partial q_1}$ and $dq_1(X_j) = \delta_{1j}$.

For each point $m \in B$, there is a leaf $L_m \subset U$ of the foliation $\mathcal{F}$ containing p. Each L_m is foliated by $n-1$ dimensional leaves, induced by the foliation integrating $\mathcal{D}' = \langle X_2, ..., X_n \rangle$. Denote by

$L'_m \subset L_m$ the $n - 1$ dimensional leaf containing M. For some ε, the flow ϕ_1^t of X_1 is defined for a smaller neighborhood U' for $|t| < \varepsilon$ and with its image contained in U. For a point $x \in L_m \cap U'$, there is some $x' \in L'_m \cap U'$ and a t' such that $|t'| < \varepsilon$ and $\phi_1^{t'}(x') = x$. Define the function:

$$q_1 : U' \longrightarrow \mathbb{R}$$
$$x \longmapsto t'(x).$$

We claim that $dq_1(X_j) = \delta_{ij}$. First, observe that the flow ϕ_1^t preserves the foliations induced by D and D' because of the commuting conditions given by the integrable system. This implies that q_1 is constant along such foliation and hence $dq_1(X_j) = 0$ if $j \neq 1$. The definition of t' yields $dq_1(X_1) = 1$.

We now fix both q_1 and f_1, and apply iteratively this construction for the flow of the Hamiltonian vector field of each of the remaining functions $f_2, ..., f_n$. We get smooth functions $q_2, ..., q_n$ such that $dq_i(X_j) = \delta_{ij}$. At the points in $U \setminus Z$, the functions $f_1, ..., f_n, q_1, ..., q_n$ do form a set of coordinates. This implies that in $U \setminus Z$, where ω is symplectic, we have

$$\omega = \sum_{i=1}^{n} dq_i \wedge df_i.$$

But both ω and the functions q_i, f_i are smooth and defined along U, hence this expression extends to U. Extend the functions q_i to a set of coordinates $(q_1, ..., q_n, y_1, y_2, ..., y_n)$. We can assume that $y_1 = t$ is a defining function of Z, since the q_i are coordinates along the level sets of F: the vector fields $\frac{\partial}{\partial q_i}$ are the Hamiltonian vector fields of the f_i. The fact that $\omega(X_i, X_j) = 0$ implies that $df_i(X_j) = \frac{\partial}{\partial q_j}(f_i) = 0$. This proves that the f_i depend only on $(t, y_2, .., y_n)$. $\qquad\square$

In contrast with the Darboux-Carathéodory in symplectic and b-symplectic geometry, one can not obtain a canonical normal form as in Martinet's theorem. This is a consequence of the fact that when you fix several commuting folded functions, various of those functions can have a Hamiltonian vector field with a non-vanishing component in the null line bundle.

Remark 5. The classical Darboux-Carathéodory theorem considers a set of $k < n$ commuting independent functions $f_1, ..., f_k$. The same proof can be adapted in this situation and the same theorem applies for a set of $k < n$ commuting functions which are independent $f_1, ..., f_k$. We can find then a set of coordinates $(q_1, ..., q_k, y_1, ..., y_k, x_{k+1}, y_{k+1}, ..., x_n, y_n)$ such that $\omega = \sum_{i=1}^{k} dq_i \wedge df_i + \sum_{i=k+1}^{n} dx_i \wedge dy_i$. This form of Darboux-Carathéodory theorem is convenient for the study of non-commutative integrable systems (see for instance [35]).

5.2. Equivariant relative Poincaré's lemma for folded symplectic forms.
We start this section with some lemmas which we will need for the proof of the action-angle theorem. They concern Relative Poincaré's lemma for folded symplectic forms and their equivariant versions. Recall from [52, page 25].

Theorem 13 (Relative Poincaré lemma). *Let $N \subset M$ a closed submanifold and $i : N \hookrightarrow M$ the inclusion map. Let ω a closed k-form on M such that $i^*\omega = 0$. Then there is a $(k-1)$-form α on a neighborhood of N in M such that $\omega = d\alpha$.*

This Relative Poincaré lemma can be used in the particular case in which the form is folded and the submanifold is a Liouville torus.

Proposition 14. *In a neighborhood $U(L)$ of a Liouville torus the folded symplectic form can be written*

$$\omega = d\alpha.$$

If ω is invariant under a compact group action, α can be assumed to be invariant by the same compact group action.

Proof. Let $i : L \hookrightarrow M$ be the natural inclusion of the Liouville torus on the folded symplectic manifold, since $i^*\omega = 0$ we may apply the following relative Poincaré theorem.

Let us check that the hypotheses of the Relative Poincaré lemma are met. The form is closed and we only need to check $i^*\omega = 0$. Since every Y_i is Hamiltonian with Hamiltonian function σ_i, we obtain that $\iota_{Y_i}\omega = d\sigma_i$. And therefore the tangent space to L is generated by $Y_1, ..., Y_n$. However, we know that $i^*d\sigma_i = 0$, since L is the level set of the integrable system. This implies that $\iota_{Y_i}i^*\omega = 0$ and hence $i^*\omega = 0$.

Now define the averaged $\bar{\alpha}$ as

$$\bar{\alpha} = \int_G \rho_g^*\alpha d\mu,$$

where μ is a Haar measure and ρ_g is the group action. This 1-form is G-invariant, and as ρ preserves ω, we can write,

$$\omega = \int_G \rho_g^*\omega d\mu$$
$$= \int_G d\rho_g^*\alpha d\mu$$

Thus $\omega = d(\int_G \rho_g^*(\alpha)d\mu)$. In particular this proves that the primitive $\bar{\alpha}$ is invariant by the action. i.e, for any Y_i fundamental vector field of the torus action one obtains, $\mathcal{L}_{Y_i}\bar{\alpha} = 0$. Thus finishing the proof of the proposition. $\square$

5.3. **Statement and proof of the action-angle coordinate theorem.** We proceed now with the statement and the proof of the action-angle theorem.

Theorem 15. *Let $F = (f_1, ..., f_n)$ be a folded integrable system on (M, ω) and $p \in Z$ a regular point in the folding hypersurface. We assume the integral manifold $\mathcal{F}_p$ containing p is compact. Then there exist an open neighborhood U of the torus $\mathcal{F}_p$ and a diffeomorphism*

$$(\theta_1, ..., \theta_n, t, b_2, ..., b_n) : U \to \mathbb{T}^n \times B^n,$$

where t is a defining function of Z and such that

$$\omega_U = \sum_{i=1}^n d\theta_i \wedge dp_i.$$

where the p_i are folded functions which depend only on $(t, b_2, ..., b_n)$ (and so do the f_i).

The S^1-valued functions

$$\theta_1, ..., \theta_n$$

are called angle coordinates and the $\mathbb{R}$-valued folded functions

$$p_1, p_2, ..., p_n$$

are called folded action coordinates.

Remark 6. Comparing this theorem with the analog in [37] observe that unlike the b-symplectic case, the expression of ω in a neighborhood of the Liouville torus is not in a folded Darboux-type form.

Besides the lemmas in the former subsection, we will need the following technical lemma.

In [38] (see Claim 2 in page 1856) it is shown that given a complete vector field Y of period 1 and a bivector field P such that $\mathcal{L}_Y \mathcal{L}_Y P = 0$ then $\mathcal{L}_Y P = 0$. If instead of a bivector field we take a 2-form, the proof can be easily adapted as follows.

Lemma 16. *If Y is a complete vector field of period 1 and ω is a 2-form such that $\mathcal{L}_Y \mathcal{L}_Y \omega = 0$ then $\mathcal{L}_Y \omega = 0$.*

Proof. Denote $v = \mathcal{L}_Y \omega$. Denote ϕ_t the flow of Y.

$$\frac{d}{dt}\left(\phi_t^* \omega\right) = \phi_t^*(\mathcal{L}_Y \omega)$$
$$= \phi_t^* v$$
$$= v$$

In the last equality we used that $\mathcal{L}_Y v = 0$. By simple integration we obtain,

$$(\phi_t)^* \omega = \omega + tv.$$

At time $t = 1$ the flow is the identity because Y has period 1 and hence $v = 0$. $\square$

We now proceed to the action-angle theorem proof.

Proof. The vector fields $X_{f_1}, ..., X_{f_n}$ define a torus action on each Liouville torus $\mathbb{T}^n \times \{b\}_{b \in B^n}$. We would like an action defined in a neighborhood of the type $\mathbb{T}^n \times B^n$. For the first part of the proof, we follow the proofs in [38] and [37] and construct a toric action. For this, we consider the classical action of the joint-flow (which is an $\mathbb{R}^n$-action) and prove uniformization of periods to induce a $\mathbb{T}^n$-action.

We denote by φ_i^t the time-t-flow of the Hamiltonian vector fields X_{f_i}. Consider the joint flow of these Hamiltonian vector fields.

$$\varphi : \mathbb{R}^n \times (\mathbb{T}^n \times B^n) \longrightarrow \mathbb{T}^n \times B^n$$
$$\left((t_1, \ldots, t_n), (x, y)\right) \longmapsto \varphi_1^{t_1} \circ \cdots \circ \varphi_n^{t_n}(x, y).$$

The vector fields X_{f_i} are complete and commute with one another so this defines an $\mathbb{R}^n$-action on $\mathbb{T}^n \times B^n$. When restricted to a single orbit $\mathbb{T}^n \times \{b\}$ for some $b \in B^n$, the kernel of this action is a discrete subgroup of $\mathbb{R}^n$, a lattice Λ_b. We call Λ_b the period lattice of the orbit. The rank of Λ_b is n because the orbit is assumed to be compact.

The lattice Λ_b will in general depend on b. The idea of uniformization of periods is to modify the action to get constant isotropy groups such that $\Lambda_b = \mathbb{Z}^n$ for all b. For any $b \in B^{n-1} \times \{0\}$ and any $a_i \in \mathbb{R}$ the vector field $\sum a_i X_{f_i}$ on $\mathbb{T}_n \times \{b\}$ is the Hamiltonian vector field of the function

$$\sum_{i=1}^{n} a_i f_i.$$

To perform the uniformization we pick smooth functions

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) : B^n \to \mathbb{R}^n$$

such that $(\lambda_1(b), \lambda_2(b), \ldots, \lambda_n(b))$ is a basis for the period lattice Λ_b for all $b \in B^n$. Such functions λ_i exist such that they satisfy this condition (perhaps after shrinking B^n) by the implicit function theorem, using the fact that the Jacobian of the equation $\Phi(\lambda, m) = m$ is regular with respect to the s variables.

We define a uniformized flow using the functions λ_i as

$$\tilde{\Phi} : \mathbb{R}^n \times (\mathbb{T}^n \times B^n) \to \mathbb{T}^n \times B^n$$

$$((s_1, \ldots, s_n), (x, b)) \mapsto \Phi\left(\sum_{i=1}^n s_i \lambda_i(b), (x, b)\right).$$

The period lattice of this $\mathbb{R}^n$ action is $\mathbb{Z}^n$, and therefore constant hence the initial action descends to the quotient to define a new action of the group $\mathbb{T}^n$.

We want to find now functions $\sigma_1, \ldots, \sigma_n$ such that their Hamiltonian vector fields are precisely the ones constructed above $Y_i = \sum_{j=1}^n \lambda_i^j X_{f_j}$. We compute the Lie derivative of the vector fields Y_i using Cartan's formula:

$$\mathcal{L}_{Y_i}\omega = d\iota_{Y_i}\omega + \iota_{Y_i}d\omega$$

$$= d(-\sum_{j=1}^n \lambda_i^j df_j)$$

$$= -\sum_{j=1}^n d\lambda_i^j \wedge df_j$$

We deduce that

$$\mathcal{L}_{Y_i}\mathcal{L}_{Y_i}\omega = \mathcal{L}_{Y_i}(-\sum_{j=1}^n d\lambda_i^j \wedge df_j).$$

In the last equality we have used the fact that λ_i^j are constant on the level sets of F. Lemma 16 applied to the vector fields Y_i yields $\mathcal{L}_{Y_i}\omega = 0$ and the folded-symplectic structure is preserved.

The next step is to prove that the collection of 1-forms $\iota_{Y_i}\omega$ are exact in the neighbourhood of a Liouville torus. So the new action is indeed Hamiltonian. We apply proposition 14 in a neighbourhood of a Liouville torus and the symplectic form ω can be written as $\omega = d\alpha$. Now since $\mathcal{L}_{Y_i}\omega = 0$, consider the toric action generated by the vector fields Y_i. Applying the equivariant version of Proposition 14 with the group $G = \mathbb{T}^n$ the form ω is G-invariant and we can find a new $\bar{\alpha}$ which is at the same time a primitive for the folded symplectic structure ω and $\mathbb{T}^n$-invariant.

Cartan's formula yields:

$$\iota_{Y_i}\omega = -\iota_{Y_i}d\bar{\alpha}$$
$$= -d\iota_{Y_i}\bar{\alpha}.$$

Thus we deduce that the fundamental vector fields Y_i are indeed Hamiltonian with Hamiltonian folded functions $\iota_{Y_i}\bar{\alpha}$. Denoting by $\sigma_1, \ldots, \sigma_n$ these Hamiltonian functions, they are now the natural candidates for "action" coordinates. Each of these functions defines a smooth Hamiltonian vector field, so by definition they are all folded functions.

We are under the hypotheses of Theorem 12 (Darboux-Carathéodory theorem), so we can find a coordinate system

$$(t, y_2, \ldots, y_n, q_1, \ldots, q_n)$$

and some folded functions σ_i such that

$$\omega = \sum_{i=1}^n d\sigma_i \wedge dq_i,$$

and the σ_i depend only on $(t, \ldots, y_n)$. The functions σ_i were defined using an equivariant form α_i which is defined in a neighborhood of the whole regular fiber. Hence the σ_i extend to all

$U' = \sigma^{-1}(\sigma(U))$. For the sake of simplicity we denote these extensions using the same notation. The Hamiltonian vector fields of σ_i have period one, so the functions q_i can be viewed as angle variables θ_i. It remains to check that, in the extended functions, ω can be written in the desired *Darboux-type* form.

Observe that $\omega(\frac{\partial}{\partial \sigma_i}, \frac{\partial}{\partial \theta_i}) = \delta_{ij}$ in U' by the definition of θ_i: these are defined locally as variables conjugated to σ_i. By abuse of notation we denote by X_{θ_i} the vector fields which solve the equations: $\iota_{X_{\theta_i}}\omega = -d\theta_i$. By construction, the equality $\omega(Y_i, Y_j) = 0$ holds in U'. This follows from the fact that ω is symplectic away from Z, and since $[Y_i, Y_j] = 0$ we get that $\omega(Y_i, Y_j) = 0$ in $U' \setminus Z$ and hence the equality extends to all U'.

We know, by the Darboux-Carathéodory coordinates, that $\omega(\frac{\partial}{\partial \sigma_i}, \frac{\partial}{\partial \sigma_j}) = \omega(X_{\theta_i}, X_{\theta_j}) = 0$ in the neighborhood U of the regular point. Applying the definition of exterior derivative, using that ω is closed and that the vector fields commute we obtain:

$$d\omega(X_{\theta_i}, X_{\theta_j}, X_{\sigma_k}) = X_{\theta_i}(\omega(X_{\theta_j}, X_{\sigma_k})) - X_{\theta_j}(\omega(X_{\theta_i}, X_{\sigma_k}))$$
$$+ X_{\sigma_k}(\omega(X_{\theta_i}, X_{\theta_j}))$$
$$= 0$$

Using that $\omega(X_{\sigma_i}, X_{\theta_j}) = \delta_{ij}$ for all i and j, we obtain

$$X_{\sigma_k}(\omega(X_{\theta_i}, X_{\theta_j})) = 0.$$

In particular, by using the joint flow $\tilde{\Phi}$ of the vector fields X_{σ_k} we prove that the relation $\omega(\frac{\partial}{\partial \sigma_i}, \frac{\partial}{\partial \sigma_j}) = \{\theta_i, \theta_j\} = 0$ holds in the whole neighborhood U'. We conclude that ω has the desired form

$$\omega = \sum_{i=1}^{n} d\sigma_i \wedge d\theta_i.$$

We consider the change $p_i := -\sigma_i$ so that we can write ω in the form

$$\omega = \sum_{i=1}^{n} d\theta_i \wedge dp_i.$$

Taking some coordinates $(t, b_2, ..., b_n)$ in B^n, it is clear that the functions $(t, b_2, ..., b_n, \theta_1, ..., \theta_n)$ form a coordinate system and the f_i only depend on $(t, b_2, .., b_n)$. This concludes the proof. $\quad\square$

5.4. Desingularization and equivalence with cotangent models.
In [36] several examples of b-integrable systems are provided using the b-cotangent lift. In fact, the construction can be generalized to the context of b^m-symplectic manifolds. From the definition of cotangent lift and the results in [36] which we recalled in subsection 2.2.1 we obtain:

Proposition 17. *The twisted b-cotangent lift of the action of an abelian group G of rank n on M^{2n} yields a b-Hamiltonian action in T^*M. If the action is free or locally free, the twisted cotangent lift yields a b-integrable system.*

In [22] the desingularization of torus actions was explored in detail. As a consequence of theorem 6.1 in [22] where an equivariant desingularization procedure is established for effective torus actions, we obtain the following *desingularized models*.

Proposition 18. *The equivariant desingularization takes the twisted b-cotangent lift of an action of a torus $\mathbb{T}^n$ on M to a twisted folded cotangent lift model.*

Remark 7. In [45] explicit desingularization formulae are given for action-angle coordinates of desingularized systems. These are convenient for the refinement of KAM theory for singular symplectic manifolds.

Proof. Denote t the defining function of the critical hypersurface Z. The moment map of the action in the S^1 coordinate is a function of the form $f = c \log(|p|)$ for some constant c, where p denotes the momentum coordinate in $T^* S^1$. Its Hamiltonian vector field is $X_f = c \frac{\partial}{\partial \theta}$. Take $f' = \frac{cp^2}{2}$ as new momentum map component for the folded symplectic structure in $T^* M$. $\qquad \square$

This construction provides a way to produce examples of folded integrable systems via desingularization of b-integrable systems which are given by toric actions. However, we know that not every integrable system on a folded symplectic manifold comes from desingularization. Indeed a folded integrable system may not admit a folded cotangent model as we show in the next section.

5.5. **About equivalence with cotangent models: a case study.** For integrable systems in symplectic and b-symplectic geometry, the action-angle coordinates has a Darboux-type expression for the associated geometrical structure in a semi-local neighborhood. This does not always apply in the case of folded integrable systems, since we do not obtain an expression as in Martinet theorem:

$$\omega = t dt \wedge dq + \sum_{i=2}^{n} dx_i \wedge dy_i$$

for some coordinates $(t, q, x_1, y_1, ..., x_n, y_n)$. In general, the previous theorem cannot be simplified to obtain such an expression.

Indeed, assume that we can find some action-angle type coordinates of the form, say, $\omega = t dt \wedge d\theta_1 + \sum_{i=2}^{n} dp_i \wedge d\theta_i$ for some $\mathbb{R}$-valued coordinates p_i and S^1-valued coordinates θ_i. In these coordinates, the null line line bundle $\ker i^* \omega|_Z$ is generated by $\langle \frac{\partial}{\partial \theta_1} \rangle$. This would imply that the null line bundle is a fibration by circles near the regular torus but we know this is not the case in general. We will now show an example of folded integrable system on a folded symplectic manifold whose null bundle has only two closed orbits.

Example. *Consider the mapping torus of S^2 generated by a smooth irrational rotation ϕ around a fixed axis, which is a symplectomorphism of S^2 equipped with the symplectic form $dh \wedge d\varphi$ which has only two periodic points. We get $S^2 \times S^1$, and a cosymplectic manifold $(\alpha, \tilde{\omega})$ where the form $\tilde{\omega}$ is obtained by gluing $dh \wedge d\varphi$ with $\phi^*(dh \wedge d\theta)$. It satisfies that $\ker \tilde{\omega}$ is a suspended vector field, which is an irrational rotation on each torus given by $h = c$ where h is the height function on S^2.*

By multiplying by S^1, we get $S^2 \times S^1 \times S^1$, which can be endowed with the folded symplectic form

$$\omega = \sin \theta d\theta \wedge \alpha + \tilde{\omega}.$$

The critical hypersurface is given by two copies of $S^2 \times S^1$, at $\theta = 0, \pi$, where $\ker i^* \omega|_Z = \ker \tilde{\omega}$.

The pair $(f_1, f_2) = (\cos \theta, h)$ defines a folded integrable system in $(S^2 \times S^1 \times S^1, \omega)$. Indeed, we have $df_1 \wedge df_2 = -\sin \theta d\theta \wedge dh$, which is non-vanishing in a dense set of M and Z as a section of $\Lambda^2(T_V^* M)$. The null line bundle of of ω is $\ker \tilde{\omega}|_Z$, which generates a vector field which has only two closed orbits at $h = 1$ and $h = 0$. We deduce that this folded integrable system does not admit a cotangent model. In particular, in the normal form obtained in Theorem 15, none of the functions p_i is of the form $p_i = t^2$ for some defining function t of Z.

This proves the following proposition.

Proposition 19. *Folded integrable systems do not admit, in general, cotangent models near a regular point.*

Typically, folded symplectic structures exhibited more flexibility (in the geometrical sense) than b-symplectic structures. This is captured by the fact that they adhere to an existence h-principle as proved Cannas [2] and in particular, all 4-dimensional compact orientable manifolds admit a

folded structure. On the other hand, the previous proposition can be seen as a rigidity phenomenon, which arises from considering dynamical aspects rather than geometrical ones. This rigidity arises from the existence of a canonical null foliation on the folding hypersurface. For b-symplectic manifolds, this null foliation is not canonical: it is defined up to Hamiltonian vector fields tangent to the leaves. This explains why from this dynamical point of view, this flexibility allows to obtain canonical normal forms for b-integrable systems.

6. CONSTRUCTIONS OF b-INTEGRABLE SYSTEMS

In this section, we study the existence of integrable systems on b-symplectic manifolds and their possible desingularization into folded integrable systems. We construct ad-hoc integrable systems on any 4-dimensional b-symplectic manifold whose critical locus admits a transverse Poisson S^1-action, starting from integrable systems defined on the leaf of a cosymplectic manifolds. This provides the first general construction of b-integrable systems on b-symplectic 4-manifolds. The proof of the existence of integrable systems on any symplectic manifold, due to Brailov (see [14, Proposition 5.1]), does not readily generalize to b-symplectic manifolds. In particular, it is interesting to understand which b-symplectic manifold do admit b-integrable systems. Our constructions yield b-integrable systems with non-trivial structures on a neighborhood of Z. This is not the case in Brailov's construction (see also Remark 10), where the integrable systems on standard symplectic manifolds are constructed with a trivial structure via Darboux balls. We will use these ad-hoc constructions in the last section to study topological obstructions to the existence of global (in fact, semi-local) action-angle coordinates already near the singular locus Z. In what follows we will always assume that the symplectic foliation on the critical set Z contains a compact leaf, and thus Z is a symplectic mapping torus by [20, Theorem 19]. Furthermore, we will assume that the first singular integral induces an S^1-action in a neighborhood of Z which is transverse to the symplectic foliation on Z. In particular, the monodromy obtained by the first return map of the Hamiltonian vector field of the first integral induces a finite group action on the symplectic leaf of Z. The finite group action detects the points where the initial circle action is not free.

6.1. **Structure of a b-integrable system in Z.** We start analyzing how a b-integrable system behaves on Z, the critical hypersurface of a b-symplectic manifold (M, ω).

Claim 1. *Let F stand for a b-integrable system on a b-symplectic manifold (M, ω). Then for a fixed symplectic leaf L of Z there is a dense set of points in L that are regular points of F.*

Proof. Assume that the set of regular points in a fixed leaf L is not dense. Then we can find an open neighbourhood U in L which does not contain any regular point, i.e. $df_1 \wedge \cdots \wedge df_n = 0$ (when seen as a section of $\Lambda^n(^bT^*M)$). However, in order for F to define a b-integrable system, one of the functions has to be a genuine (i.e., non-smooth) b-function in a neighborhood of Z. In other words, $f = c \log|t| + g$ with $c \neq 0$ and g a smooth function. We can assume that $f_1 = f$ is a genuine b-function in a neighborhood U' in Z containing U. Since $c \neq 0$, it defines a Hamiltonian vector field whose flow is transverse to the symplectic leaf L. The function f_1 Poisson commutes with all the other integrals, and so the the flow of f_1 preserves $df_1 \wedge \cdots \wedge df_n$.

Denote by φ_t the flow of X_{f_1}. Then the set $V = \{\varphi_t(U') \mid t \in (0, \varepsilon)\}$ is an open subset of Z where $df_1 \wedge \cdots \wedge df_n = 0$. This is a contradiction with the fact that $F = (f_1, \ldots, f_n)$ defines a b-integrable system. $\qquad\square$

Once we take into account that the first integral $f_1 = c \log|t|$ induces an S^1-action, we can deduce the semi-local structure of the system.

Proposition 20. *Let (M, ω) be a b-symplectic manifold admitting a b-integrable system such that $f_1 = c \log |t|$ defines an S^1-action in the neighborhood of Z. Then $(f_2, ..., f_n)$ induces an integrable system on each symplectic leaf L on Z which is invariant by the monodromy of the S^1-action.*

Proof. The fact that we may always assume that in a neighborhood of Z the first integral is $f_1 = c \log |t|$ follows from remark 16 in [37](see also Proposition 3.5.3 in [34]), where c is the modular period of that connected component, and $f_2, ..., f_n$ are smooth. Observe that because f_1 is regular everywhere in a neighborhood of Z, the induced S^1-action has no fixed points.

By hypothesis, the Hamiltonian vector field $X_{c \log |t|}$ commutes with the Hamiltonian vector fields $X_{f_2}, ..., X_{f_n}$ which implies that the flow φ_t of the S^1-action preserves each of the functions $f_2, ..., f_n$. The flow also preserves the symplectic foliation in Z. Thus, fixing a symplectic leaf L, the flow φ_t satisfies $\varphi_t(L) \cong L$ and $\varphi_t^*(f_2, ..., f_n) = (f_2, ..., f_n)$. This shows that on each leaf the functions $f_2, ..., f_n$ induce the same integrable system. In particular, this integrable system in L is preserved by the first return map of the monodromy in that fixed leaf, implying that the system is invariant by that finite group action. $\square$

Remark 8. In the jargon of three-dimensional topology, the connected components of the critical set Z of a 4-dimensional b-symplectic manifold admitting such systems are Seifert manifolds with orientable base and vanishing Euler number. This follows from the fact that Z is a mapping torus and that the first Hamiltonian vector field induces an S^1-action without fixed points. This natural assumption of admitting a transverse S^1-action is key to our construction in the next section.

6.2. **Construction of b-integrable systems.** Taking into account the last remark, to construct b-integrable systems, we will assume that Z is the mapping torus of a periodic symplectomorphism of a compact leaf L on Z. This symplectomorphism defines a finite group action on L. This is why to construct b-integrable systems on 4 dimensional b-symplectic manifolds, we start by proving that we can always find a non-constant analytic function that is invariant under a symplectic finite group action on a surface.

Claim 2. *Let $\mathbb{Z}_k$ be a finite group acting of a symplectic surface Σ. Then there exists a non-constant analytic function F invariant by the group action.*

Proof. Let f be a generic analytic function in Σ. Consider the averaged function given by the *averaging trick*:

$$F(x) := \sum_{i=1}^{k-1} f(i.x)$$

. By construction this analytic function is invariant by the action of $\mathbb{Z}_k$. Given a point p in Σ, the differential of F vanishes at p if and only if $df_p + df_{2.p} + ...df_{(k-1).p} = 0$. Observe that for a generic f, there exists a point where this condition is not fulfilled. In particular, we deduce that $dF_p \neq 0$ at some point p, and hence F is not a constant function. $\square$

In the claim above we can replace the analytic condition with a Morse condition. See for instance [51] for a proof of the existence and density of invariant Morse functions by the action of a compact Lie group.

Theorem 21. *Let (M, ω) be a b-symplectic manifold of dimension 4 with critical set Z which is a mapping torus associated with a periodic symplectomorphism. Then (M, ω) admits a b-integrable system.*

Proof. In this case, a leaf of the critical set is a surface L. Take a neighborhood of Z of the form $U = Z \times (-\varepsilon, \varepsilon)$. Denote by X the Hamiltonian vector field of the function $\log |t|$ for some defining

function of Z. By hypothesis, X defines a Poisson S^1-action in Z transverse to the leaves as studied in [4]. This S^1-action may admit exceptional orbits (hence with non-trivial monodromy). Denote by α and β the defining one and two forms of ω at Z. That is, in U we can assume that ω has the form $\omega = \alpha \wedge \frac{dt}{t} + \beta$ with $\alpha \in \Omega^1(Z), \beta \in \Omega^2(Z)$. Recall that both forms are closed and $i_L^* \beta$ is a symplectic form in a leaf L of Z.

The critical set can be described as follows: There is an equivariant cover $L \times S^1 \times (-\varepsilon, \varepsilon)$ of U, and we denote by p the projection to U. This equivariant cover can be equipped with the b-Poisson structure

$$\omega = \pi_{Z_0}^* \tilde{\alpha} \wedge \frac{dt}{t} + \pi_{Z_0}^* \tilde{\beta},$$

where $\tilde{\alpha} = p^* \alpha$ and $\tilde{\beta} = p^* \beta$. Then U is Poisson isomorphic by [4, Corollary 17] to the quotient of the equivariant by the action of a finite group $\mathbb{Z}_k$ in the leaf given by the return time flow of the S^1-action and extended trivially to the neighborhood $L \times S^1 \times (-\varepsilon, \varepsilon)$.

The action of $\mathbb{Z}_k$ acts by symplectomorphisms on L. By Lemma 2, there is an analytic function F in L which is invariant by the action. In particular, F can be extended to $\tilde{F}$ in all Z by the S^1-action. If π is the projection in $U = Z \times (-\varepsilon, \varepsilon)$ to the first component, then we extend $\tilde{F}$ to U by considering $\pi^* \tilde{F}$ and denote it again $\tilde{F}$.

We construct in the neighborhood U the pair of functions $(f_1, f_2) = (\varphi(t) c \log |t|, \varphi(t) \tilde{F})$ in U. The function $\varphi(t)$ denotes a bump function which is constantly equal to 1 for $t \in (-\delta, \delta)$ and constantly equal to 0 for $|t| > \delta'$, with $\delta < \delta' < \varepsilon$. Observe the functions f_1 and f_2 are linearly independent in $^bT^*M$ in a dense set of $Z \times (-\delta', \delta')$. The Hamiltonian vector field of $\varphi(t) f_1$ generates the transverse S^1 action extended to U, and the Hamiltonian vector field of $\tilde{F}$ is tangent to the symplectic leaves in each $Z \times \{t_0\}$. Hence $\{f_1, \tilde{F}\} = 0$. Now using the properties of the Poisson bracket we obtain

$$\begin{aligned} \{f_1, \varphi(t)\tilde{F}\} &= -\{\varphi(t)\tilde{F}, f_1\} \\ &= \{\varphi(t), f_1\}\tilde{F} + \{\tilde{F}, f_1\} \\ &= 0 + 0, \end{aligned}$$

where we used that f_1 only depends on the coordinate t. We obtain an integrable system in the neighborhood of the critical locus U. In order to extend the integrable system to M, we do it as in the proof of existence of integrable systems in symplectic manifolds as shown by Brailov (cf. [14]). That is, cover $M \setminus U$ by Darboux balls, each of them equipped with a local integrable system of the form $f_i' = x_i^2 + y_i^2$. By cutting off this system using a function $\varphi(\sum_{i=1}^2 (x_i^2 + y_i^2))$, we can obtain for each Darboux ball a globally defined pair of functions $f_i = \varphi \cdot f_i'$. We can now cover $M \setminus U$ by a finite amount of balls B_i whose intersection is only the union of their boundaries. We choose φ in each ball such that the locally defined integrable systems vanish in all derivatives exactly at these boundaries. The closed set of zero measure where the globally constructed n-tuple of functions are not independent is composed of the boundaries of the balls, and includes $Z \times \{-\varepsilon, \varepsilon\}$. This is illustrated in Figure 4, where only some balls are depicted close to the boundary of $Z \times [-\varepsilon, \varepsilon]$. The closed set where the functions vanish are represented by the black-colored boundaries.

This allows us to glue the system in each ball and with the system we constructed in U, yielding a pair of commuting functions F_1, F_2 such that $dF_1 \wedge dF_2 \neq 0$ in a dense set of M and Z.   □

Remark 9. The proof generalizes to higher dimensions as long as one can construct an integrable system in the symplectic leaf invariant by the finite group action. This is the content of Claim 6.2 for the case of a symplectic surface.
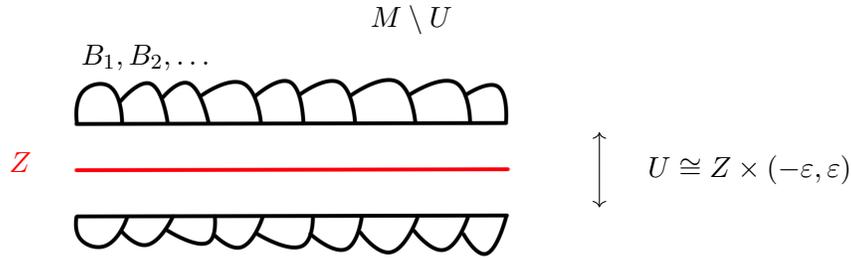
$$M \setminus U$$

$$B_1, B_2, \ldots$$

$$Z$$

$$U \cong Z \times (-\varepsilon, \varepsilon)$$

FIGURE 4. Some Darboux balls filling $M \setminus U$.

Remark 10. The original construction of integrable systems in regular symplectic manifolds via the covering of Darboux balls yields an integrable system without any interesting property. However, the construction in Theorem 21 gives rise to a lot of examples of b-integrable systems that near the singular set Z can be very rich from a semi-global point of view.

The following theorem is Theorem B in [15]:

Theorem 22. *Any cosymplectic manifold of dimension 3 is the singular locus of orientable, closed, b-symplectic manifolds.*

In particular, whenever the cosymplectic manifold has periodic monodromy, it can be realized as the critical locus of a b-symplectic manifold with a b-integrable system. Theorem 22 requires specifically that Z is connected. If we drop that requirement, there is a direct construction (Example 19 in [18]) to realize any cosymplectic manifold as the singular locus of a b-symplectic manifold that we will introduce later.

The proof of Theorem 21 can be adapted to obtain folded integrable systems in the desingularized folded symplectic manifold resulting from applying Theorem 8.

Corollary 23. *Let (M, ω) be a b-symplectic manifold in the hypotheses of Theorem 21. Then the desingularized folded symplectic manifold (M, ω_ε) admits a folded integrable system.*

Proof. The desingularization given by Theorem 8 sends ω to ω_ε, which is a folded symplectic structure in M with critical hypersurface Z. The induced structure on Z remains unchanged: it is a cosymplectic manifold with compact leaves whose monodromy is periodic. The S^1-action generated by the modular vector field becomes the null line bundle of ω_ε. Such line bundle is generated in a neighborhood of Z by the Hamiltonian vector field of t^2, where t is defining function of Z. By Claim 6.2, there is an analytic function invariant by the first return map X_{t^2}. One can do exactly the same construction as in the proof of Theorem 21, taking as first function $f_1 = \varphi(t)t^2$ instead of $\varphi(t) \log |t|$ in the neighborhood U of Z. $\square$

7. GLOBAL ACTION-ANGLE COORDINATES: CONSTRUCTIONS AND EXISTENCE

In this section, we extend toric actions on the symplectic leaf on the critical set of a b-symplectic manifold to a toric action in the neighborhood of the critical set Z. Thus obtaining global action-angle coordinates. For certain compact extensions of this neighborhood, we obtain global action-angle coordinates on the compact manifold. The existence of global coordinates requires an associated global toric action. However, the topology of the critical set Z may hinder such an action. Thus the topology of Z can be an obstruction to the existence of global action-angle coordinates for

any b-integrable system defined on such b-symplectic manifold. In other words, our construction admits global action-angle coordinates if and only if the toric structure of the symplectic leaf of the critical set Z extends to a toric action of the b-symplectic/folded symplectic manifolds. Toric symplectic manifolds are well-understood thanks to [19] and [17].

In this section we will need to following lemma (which is Corollary 16 in [19]):

Lemma 24. *Let (M^{2n}, Z, ω) be a b-symplectic manifold with a toric action and L a symplectic leaf of Z. Then $Z \cong L \times S^1$.*

Let L be a toric symplectic manifold of dimension $2n - 2$ and let $F = (f_2, \ldots, f_n)$ be its moment map.

We know from Delzant's theorem [10] that the image of F is a Delzant polytope. From the definition of moment map the components of F Poisson commute and are functionally independent so they form an integrable system on L. Consider now ϕ be a symplectomorphism of L which is equivariant with respect to the toric action and let $Z = L \times [0, 1] / \sim$ be the cosymplectic manifold associated to it. Extend the integrable system on Z to an integrable system on Z just by observing that by hypothesis the toric action commutes with the symplectomorphism defining the cosymplectic manifold. Observe that the integrable system F on the leaf extends to Z only if Z is a product or F is invariant by the monodromy. Denote by (α, ω) the pair of 1 and 2-forms associated to the cosymplectic structure i.e, ω restricted to the symplectic leaves defines the symplectic structure on Z and α is a closed form defining the codimension one symplectic foliation.

Following the extension theorem (Theorem 50 in [18]) we consider now the open b-symplectic manifold $U = Z \times (-\epsilon, \epsilon)$ with b-symplectic form,

$$\omega = \frac{df}{f} \wedge \pi^*(\alpha) + \pi^*(\omega)$$

where $\pi : U \to Z$ stands for the projection in the first component of U, Z and f is the defining function for the critical set Z.

Consider the map $\hat{F} = (c \log |t|, \pi^*(f_2), \ldots, \pi^*(f_n))$ on with c the modular period of Z where we abuse notation and we write the components on the covering $L \times [0, 1]$ of the mapping torus Z.

In this section we prove,

Theorem 25. *The mapping $\hat{F} = (c \log |t|, \pi^*(f_2), \ldots, \pi^*(f_n))$ defines a b-integrable system on the open b-symplectic manifold $Z \times (-\epsilon, \epsilon)$ thus extending the integrable system defined by the toric structure of L. The toric structure of L extends to a toric structure on the b-symplectic manifold $Z \times (-\epsilon, \epsilon)$ if and only if the cosymplectic structure of Z is trivial, i.e., $Z = L \times [0, 1]$.*

Proof. Observe that the functions $f_2, \ldots, f_n$ define an integrable system on the cosymplectic manifold Z as the gluing symplectomorphism that defines the mapping torus commutes with the torus action defined by F. So this torus action descends to the quotient Z and the functions f_i are well-defined on the mapping torus Z. From the definition of b-symplectic form the projection π is a Poisson map and thus $\{\pi^*(f_i), \pi^*(f_j)\} = \{f_i, f_j\} = 0$ for all $i, j \geq 2$. Observe also that functional independence on a dense set W of L, of the functions $f_2, \ldots, f_n$ on L (a factor of U) together with the functional independence of the pure b-function $c \log |t|$ from the functions $\pi^*(f_2), \ldots, \pi^*(f_n)$ implies the functional independence on the dense open set $W \times I$ with the product topology.

Furthermore, the Poisson bracket $\{c \log |t|, \pi^*(f_j)\} = 0$ from the expression of b-symplectic structure. Thus the system $\hat{F}$ defines an integrable system on $Z \times (-\epsilon, \epsilon)$.

To conclude observe that the action-angle coordinates associated with the global toric action on L extend to Z (and thus to a neighborhood $Z \times (-\epsilon, \epsilon)$ if and only if the action extends to a toric action. We now use Lemma 24 above to conclude that the toric structure extends to Z if and only if the mapping torus is trivial, i.e., $Z = L \times [0,1]$. This ends the proof of the theorem. $\square$

Given any cosymplectic compact manifold Z, following the construction from Example 19 in [18], $Z \times S^1$ admits a b-symplectic structure simply by considering the dual b-Poisson structure (where π is the corank regular Poisson structure associated to the cosymplectic structure and X is a Poisson vector field transverse to the symplectic foliation in Z as it was proved in [20]). The function f is a function vanishing linearly. The critical locus of this b-Poisson structure has as many copies of the original Z as zeros of the function f.

$$\Pi = f(\theta) X \wedge \frac{\partial}{\partial \theta} + \pi$$

The theorem above admits its compact version:

Theorem 26. *The mapping $\hat{F} = (c \log |f(\theta)|, \pi^*(f_2), \ldots, \pi^*(f_n))$ defines a b-integrable system on the b-symplectic manifold $Z \times S^1$ thus extending the integrable system defined by the toric structure of L. The toric structure of L extends to a toric structure on the b-symplectic manifold $Z \times S^1$ if and only if the cosymplectic structure of Z is trivial, i.e., $Z = L \times S^1$.*

As a corollary, we can detect situations in which the topological obstruction to the global existence of action-angle coordinates lies in the non-triviality of the mapping torus defined by the critical set Z.

Theorem 27. *Any b-integrable system on b-symplectic manifold extending a toric system on a symplectic leaf of Z does not admit global action-angle coordinates whenever the critical set Z is not a trivial mapping torus $Z = L \times S^1$.*

Below we show an example of a b-symplectic manifold M of dimension 6 that admits some b-integrable system which is not toric even though the leaves on the critical hypersurface are toric. Observe that the b-integrable system cannot define a toric action (and thus admit global action-angle coordinates) because of the topological structure of Z.

Example (Topological obstructions to semi-local action-angle coordinates). *Consider a product of spheres $S^2 \times S^2$ with coordinates $(h_1, \theta_2, h_2, \theta_2)$ and standard product symplectic form $\omega = dh_1 \wedge d\theta_1 + dh_2 \wedge d\theta_2$. The map*

$$\varphi : S^2 \times S^2 \longrightarrow S^2 \times S^2$$
$$(p, q) \longmapsto (q, p)$$

is a symplectomorphism satisfying that $\varphi^2 = \text{Id}$. The induced map in homology swaps the generators of $H_2(S^2 \times S^2) \cong H_2(S^2) \oplus H_2(S^2)$. This shows that φ is not in the connected component of the identity, as this would imply that induced map in homology would act trivially [28, Theorem 2.10]. Thus, the mapping torus with gluing diffeomorphism φ cannot be a trivial product $S^2 \times S^2 \times S^1$.

The pair of functions $F = (f_1, f_2) = (h_1 + h_2, h_1 h_2)$ are invariant with respect to φ and hence descend to the mapping torus. Furthermore, they define an integrable system (and in fact a toric action) on $S^2 \times S^2$, since they clearly Poisson commute and satisfy that $df_1 \wedge df_2 = (h_2 - h_2) dh_1 \wedge dh_2 \neq 0$ almost everywhere. In particular, by Remark 9, any b-symplectic manifold with critical set Z admits a b-integrable system. However since the critical hypersurface is not a trivial product, any b-integrable system will not be toric in a neighborhood of Z.

By the discussion before the statement of Theorem 26, the cosymplectic manifold N can be realized as a connected component of a critical hypersurface of a compact b-symplectic manifold diffeomorphic to $M = N \times S^1$. Thus any b-integrable system in M will not be toric even in a neighborhood of Z.

Observe that with the magic trick of the desingularization we obtain examples of folded-integrable systems without global action-angle coordinates. This is done by arguing as in Corollary 23.

Theorem 28. *Let F be a folded integrable system obtained by desingularization of a b-integrable system, if the critical set Z of the original b-symplectic structure is not a trivial mapping torus, then the folded integrable system F does not admit global action-angle coordinates.*

Let us finish this article with a couple of concluding remarks:

- For symplectic manifolds the obstructions to global action-angle coordinates started with Duistermaat in his seminal paper [12] where Duistermaat related the existence of obstructions to the existence of monodromy which in its turn was naturally associated with the existence of singularities.

  In this article we have concluded that for a singular symplectic manifold there are topological obstructions for the existence of global action-angle coordinates that are detectable at first sight: The critical set Z has to be a trivial mapping torus $Z = L \times [0, 1]$ thus the existence of monodromy associated to this mapping torus is also an obstruction.

- Furthermore, the existence of action-angle coordinates yields a free action of a torus in the neighborhood of a regular torus action thus the existence of isotropy groups for the candidate of torus action defining the system, automatically implies that the locus with non-trivial isotropy groups is singular for the integrable system. The same holds for a sub-circle. In particular:

Corollary 29. *Let F be a b-integrable system as in Proposition 20 on a b-symplectic manifold and denote by T the union of the exceptional orbits of the S^1-action defined by $c \log |t|$. Then the system has singularities at the set T.*

Thus not only the topology of the critical set Z yields an obstruction to the existence of global action-angle coordinates but it also detects singularities of integrable systems. In particular along the exceptional orbits for the transverse S^1-action given by Proposition 20. This motivates us to investigate singularities of integrable systems on singular symplectic manifolds, a study which we will pursue in a different article.

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