Some Results on the Laplacian Spectra of Token Graphs

Cristina Dalfó[1](✉), Frank Duque[2], Ruy Fabila-Monroy[3], Miquel Àngel Fiol[4], Clemens Huemer[5], Ana Laura Trujillo-Negrete[3], and Francisco J. Zaragoza Martínez[6]

[1] Universitat de Lleida, Igualada (Barcelona), Catalonia
cristina.dalfo@udl.cat

[2] Universidad de Antioquia, Antioquia, Colombia
rodrigo.duque@udea.edu.co

[3] Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Mexico City, Mexico
ruyfabila@math.cinvestav.edu.mx, ltrujillo@math.cinvestav.mx

[4] Barcelona Graduate School of Mathematics, Universitat Politècnica de Catalunya, Barcelona, Catalonia
miguel.angel.fiol@upc.edu

[5] Universitat Politècnica de Catalunya, Barcelona, Catalonia
clemens.huemer@upc.edu

[6] Universidad Autónoma Metropolitana Azcapotzalco, Mexico City, Mexico
franz@azc.uam.mx

Abstract. We study the Laplacian spectrum of token graphs, also called symmetric powers of graphs. The k-token graph $F_k(G)$ of a graph G is the graph whose vertices are the k-subsets of vertices from G, two of which being adjacent whenever their symmetric difference is a pair of adjacent vertices in G. In this work, we give a relationship between the Laplacian spectra of any two token graphs of a given graph. In particular, we show that, for any integers h and k such that $1 \leq h \leq k \leq \frac{n}{2}$, the Laplacian spectrum of $F_h(G)$ is contained in the Laplacian spectrum of $F_k(G)$. Besides, we obtain a relationship between the spectra of the k-token graph of G and the k-token graph of its complement $\overline{G}$. This generalizes a well-known property for Laplacian eigenvalues of graphs to token graphs.

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1 Introduction

Let G be a simple graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. For a given integer k such that $1 \le k \le n$, the *k-token graph* $F_k(G)$ of G is the graph whose vertex set $V(F_k(G))$ consists of the $\binom{n}{k}$ k-subsets of vertices of G, and two vertices A and B of $F_k(G)$ are adjacent whenever their symmetric difference $A \triangle B$ is a pair $\{a, b\}$ such that $a \in A$, $b \in B$, and $\{a, b\} \in E(G)$; see Fig. 1 for an example. Note that if $k = 1$, then $F_1(G) \cong G$; and if G is the complete graph K_n, then $F_k(K_n) \cong J(n, k)$, where $J(n, k)$ denotes a Johnson graph [5]. The naming token graph comes from an observation by Fabila-Monroy, Flores-Peñaloza, Huemer, Hurtado, Urrutia, and Wood [5], that vertices of $F_k(G)$ correspond to configurations of k indistinguishable tokens placed at distinct vertices of G, where two configurations are adjacent whenever one configuration can be reached from the other by moving one token along an edge from its current position to an unoccupied vertex. Such graphs are also called *symmetric k-th power of a graph* by Audenaert, Godsil, Royle, and Rudolph [2]; and *n-tuple vertex graphs* in Alavi, Lick, and Liu [1]. The token graphs have some applications in physics. For instance, a connection between symmetric powers of graphs and the exchange of Hamiltonian operators in quantum mechanics is given in [2]. They have also been considered in relation to the graph isomorphism problem, see Rudolph [8].

In this work, we focus on the Laplacian spectrum of $F_k(G)$ for any value of k. Recall that the Laplacian matrix $\boldsymbol{L}(G)$ of a graph G is $\boldsymbol{L}(G) = \boldsymbol{D}(G) - \boldsymbol{A}(G)$, where $\boldsymbol{A}(G)$ is the adjacency matrix of G, and $\boldsymbol{D}(G)$ is the diagonal matrix whose non-zero entries are the vertex degrees of G. For a d-regular graph G, each eigenvalue λ of $\boldsymbol{L}(G)$ corresponds to an eigenvalue μ of $\boldsymbol{A}(G)$ via the relation $\lambda = d - \mu$. In [3], Carballosa, Fabila-Monroy, Leaños, and Rivera proved that, for $1 < k < n-1$, the k-token graph $F_k(G)$ is regular only if G is the complete graph K_n or its complement, or if $k = n/2$ and G is the star graph $K_{1,n-1}$ or its complement. Then, for most graphs, we cannot directly infer the Laplacian spectrum of $F_k(G)$ from the adjacency spectrum of $F_k(G)$. In fact, when considering the adjacency spectrum, we find graphs G whose spectrum is not contained in the spectrum of $F_k(G)$; see Rudolph [8]. Surprisingly, for the Laplacian spectrum, this holds and it is our first result.

This extended abstract is organized as follows. In Sect. 2, we show that the Laplacian spectrum of a graph G is contained in the Laplacian spectrum of its k-token graph $F_k(G)$. Besides, with the use of a new $(n;k)$-binomial matrix, we give the relationship between the Laplacian spectrum of a graph G and that of its k-token graph. In Sect. 3, we show that an eigenvalue of a k-token graph is also an eigenvalue of the $(k+1)$-token graph for $1 \le k < n/2$. Besides, we define another matrix, called $(n;h,k)$-binomial matrix. With the use of this matrix, it

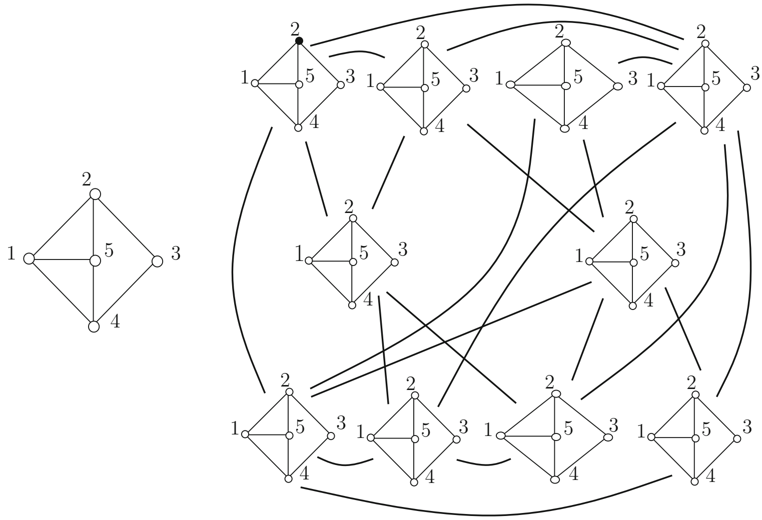

Fig. 1. A graph G (left) and its 2-token graph $F_2(G)$ (right). The Laplacian spectrum of G is $\{0, 2, 3, 4, 5\}$. The Laplacian spectrum of $F_2(G)$ is $\{0, 2, 3^2, 4, 5^3, 7, 8\}$.

is shown that, for any integers h and k such that $1 \leq h \leq k \leq \frac{n}{2}$, the Laplacian spectrum of $F_h(G)$ is contained in the spectrum of $F_k(G)$. Finally, in Sect. 4, we obtain a relationship between the Laplacian spectra of the k-token graph of G and the k-token graph of its complement $\overline{G}$. This generalizes a well-known property for Laplacian eigenvalues of graphs to token graphs.

For more information, not included in this extended abstract, see [4].

2 The Laplacian Spectra of Token Graphs

Let us first introduce some notation used throughout the paper. Given a graph $G = (V, E)$, we indicate with $a \sim b$ that a and b are adjacent in G. As usual, the transpose of a matrix $\boldsymbol{M}$ is denoted by $\boldsymbol{M}^\top$, the identity matrix by $\boldsymbol{I}$, the all-1 vector $(1, \ldots, 1)^\top$ by $\mathbf{1}$, the all-1 (universal) matrix by $\boldsymbol{J}$, and the all-0 vector and all-0 matrix by $\mathbf{0}$ and $\boldsymbol{O}$, respectively. Let $[n] := \{1, \ldots, n\}$. Let $\binom{[n]}{k}$ denote the set of k-subsets of $[n]$, the set of vertices of the k-token graph.

Our first theorem deals with the Laplacian spectrum of a graph G and its k-token graph $F_k(G)$.

Theorem 1. *Let G be a graph and $F_k(G)$ its k-token graph. Then, the Laplacian spectrum (eigenvalues and their multiplicities) of G is contained in the Laplacian spectrum of $F_k(G)$.*

Theorem 1 has a direct proof using token movements, and it can also be obtained using the $(n;k)$ binomial matrix B, defined in the following. Given some integers n and k (with $k \in [n]$), we define the $(n;k)$-*binomial matrix* $\boldsymbol{B}$.

This is a $\binom{n}{k} \times n$ matrix whose rows are the characteristic vectors of the k-subsets of $[n]$ in a given order. Thus, if the i-th k-subset is A, then

$$(\boldsymbol{B})_{ij} = \begin{cases} 1 \text{ if } j \in A, \\ 0 \text{ otherwise.} \end{cases}$$

Lemma 1. *The $(n;k)$-binomial matrix $\boldsymbol{B}$ satisfies*

$$\boldsymbol{B}^\top \boldsymbol{B} = \binom{n-2}{k-1}\boldsymbol{I} + \binom{n-2}{k-2}\boldsymbol{J}.$$

Let G be a graph with n vertices and, for $k \leq \frac{n}{2}$, let $F_k = F_k(G)$ be its k-token graph. The following result gives the relationship between the corresponding Laplacian matrices, $\boldsymbol{L}_1$ and $\boldsymbol{L}_k$.

Theorem 2. *Given a graph G and its k-token graph F_k, with corresponding Laplacian matrices $\boldsymbol{L}_1$ and $\boldsymbol{L}_k$, and $(n;k)$-binomial matrix $\boldsymbol{B}$, the following holds:*

$$\boldsymbol{B}^\top \boldsymbol{L}_k \boldsymbol{B} = \binom{n-2}{k-1}\boldsymbol{L}_1. \tag{1}$$

Corollary 1. *Given a graph G, with $G \cong F_1$, and its k-token graph F_k, with corresponding Laplacian matrices $\boldsymbol{L}_1$ and $\boldsymbol{L}_k$, and $(n;k)$-binomial matrix $\boldsymbol{B}$, the following implications hold:*

(i) If $\boldsymbol{v}$ is a λ-eigenvector of $\boldsymbol{L}_1$, then $\boldsymbol{Bv}$ is a λ-eigenvector of $\boldsymbol{L}_k$.
(ii) If $\boldsymbol{w}$ is a λ-eigenvector of $\boldsymbol{L}_k$ and $\boldsymbol{B}^\top \boldsymbol{w} \neq \boldsymbol{0}$, then $\boldsymbol{B}^\top \boldsymbol{w}$ is a λ-eigenvector of $\boldsymbol{L}_1$.

Corollary 2. *(i) The Laplacian spectrum of $\boldsymbol{L}_1$ is contained in the Laplacian spectrum of $\boldsymbol{L}_k$.*
(ii) Every eigenvalue λ of $\boldsymbol{L}_k$, having eigenvector $\boldsymbol{w}$ such that $\boldsymbol{B}^\top \boldsymbol{w} \neq \boldsymbol{0}$, is a λ-eigenvector of $\boldsymbol{L}_1$.

3 A More General Result

In this section, we show a stronger result. Namely, for any $1 \leq k < n/2$, the Laplacian spectrum of the k-token graph $F_k(G)$ of a graph G is contained in the Laplacian spectrum of its $(k+1)$-token graph $F_{k+1}(G)$.

A 'local analysis', based on token movements, is used to show that every eigenvalue of $F_k(G)$ is also an eigenvalue of $F_{k+1}(G)$.

Theorem 3. *Let G be a graph on n vertices. Let h, k be integers such that $1 \leq h \leq k \leq n/2$. If λ is an eigenvalue of $F_h(G)$, then λ is an eigenvalue of $F_k(G)$.*

All previous results can be seen as consequences of the following matricial formulation. First, we define, for some integers n, k_1, and k_2 (with $1 \leq k_1 < k_2 < n$), the $(n; k_2, k_1)$-*binomial matrix* $\boldsymbol{B} = \boldsymbol{B}(n; k_2, k_1)$. This is a $\binom{n}{k_2} \times \binom{n}{k_1}$ $(0,1)$-matrix, whose rows are indexed by the k_2-subsets $A \subset [n]$, and its columns are indexed by the k_1-subsets $X \subset [n]$. The entries of $\boldsymbol{B}$ are

$$(\boldsymbol{B})_{AX} = \begin{cases} 1 \text{ if } X \subset A, \\ 0 \text{ otherwise.} \end{cases}$$

The transpose of $\boldsymbol{B} = \boldsymbol{B}(n; k_2, k_1)$ is known as the *set-inclusion matrix*, denoted by $W_{k_1,k_2}(n)$ (see, for instance, Godsil [7]).

Lemma 2. *The matrix $\boldsymbol{B}$ satisfies the following simple properties.*

(i) The number of 1's of each column of $\boldsymbol{B}$ is $\binom{n-k_1}{k_2-k_1}$.
(ii) The number of common 1's of any two columns of $\boldsymbol{B}$, corresponding to k_2-subsets of $[n]$ whose intersection has $k_1 - 1$ elements, is $\binom{n-k_1-1}{k_2-k_1-1}$.

The new matrix $\boldsymbol{B}$ allows us to give the following result that can be seen as a generalization of Theorem 2 (see also Corollary 3).

Theorem 4. *Let G be a graph on $n = |V|$ vertices, with k_1- and k_2-token graphs $F_{k_1}(G)$ and $F_{k_2}(G)$, where $1 \leq k_1 \leq k_2 \leq n$. Let $\boldsymbol{L}_{k_1}$ and $\boldsymbol{L}_{k_2}$ be the respective Laplacian matrices, and $\boldsymbol{B}$ the $(n; k_2, k_1)$-binomial matrix. Then, the following holds:*

$$\boldsymbol{B}\boldsymbol{L}_{k_1} = \boldsymbol{L}_{k_2}\boldsymbol{B}. \tag{2}$$

Let us now see some consequences of this theorem. First, we get again Theorem 2.

Corollary 3. *With the same notation as before, the following holds.*

(i) For every k_1, k_2 with $1 \leq k_1 \leq k_2 \leq n$,

$$\boldsymbol{B}^{\top}\boldsymbol{L}_{k_2}\boldsymbol{B} = \boldsymbol{B}^{\top}\boldsymbol{B}\boldsymbol{L}_{k_1}. \tag{3}$$

(ii) For $k_1 = 1$ and $k_2 = k$,

$$\boldsymbol{B}^{\top}\boldsymbol{L}_{k}\boldsymbol{B} = \binom{n-2}{k-1}\boldsymbol{L}_1.$$

Since $F_k(G) \cong F_{n-k}(G)$ assume, without loss of generality, that $1 \leq k_1 \leq k_2 \leq \frac{n}{2}$. Then, we have a generalization of Corollary 3.

Corollary 4. *For any integers h, k such that $1 \leq h \leq k \leq \frac{n}{2}$, let $\boldsymbol{B}$ be the $(n; k, h)$-binomial matrix. Then, the eigenvalues and eigenvectors of the Laplacian matrices of the token graphs F_h and F_k are related in the following way.*

(i) If $\boldsymbol{v}$ is a λ-eigenvector of $\boldsymbol{L}_h$, then $\boldsymbol{B}\boldsymbol{v}$ is a λ-eigenvector of $\boldsymbol{L}_k$. Moreover, the linear independence of the different eigenvectors is preserved. (That is, the spectrum of $\boldsymbol{L}_h$ is contained in the spectrum of $\boldsymbol{L}_k$.)

(ii) If $\boldsymbol{w}$ is a λ-eigenvector of $\boldsymbol{L}_k$ and $\boldsymbol{B}^\top \boldsymbol{w} \neq \boldsymbol{0}$, then $\boldsymbol{B}^\top \boldsymbol{w}$ is a λ-eigenvector of $\boldsymbol{L}_h$. Moreover, all the eigenvalues, including multiplicities, of $\boldsymbol{L}_h$ are obtained (that is, one eigenvalue each time that the above non-zero condition is fulfilled).

In our context, Theorem 4 allows us to obtain the Laplacian matrix of F_h in terms of the Laplacian matrix of F_k, provided that we know the binomial matrix $\boldsymbol{B}(n; k, h)$ with its rows and columns in the right order (that is, the same order as the columns of $\boldsymbol{L}_h$ and $\boldsymbol{L}_k$, respectively). Indeed, in this case, (3) with $k_1 = h$ and $k_2 = k$ leads to

$$\boldsymbol{L}_h = (\boldsymbol{B}^\top \boldsymbol{B})^{-1} \boldsymbol{B}^\top \boldsymbol{L}_k \boldsymbol{B}. \tag{4}$$

Notice that $\boldsymbol{B}^\top \boldsymbol{B}$ is a Gram matrix of the columns of $\boldsymbol{B}$, which are linearly independent vectors and, hence, $\boldsymbol{B}^\top \boldsymbol{B}$ is invertible.

Following with the simplified notation $k_1 = h$ and $k_2 = k$, the result of Theorem 4 can also be written in terms of the adjacency matrices $\boldsymbol{A}_h$ and $\boldsymbol{A}_k$ of F_h and F_k, respectively. Then, we get

$$\boldsymbol{A}_k \boldsymbol{B} - \boldsymbol{B} \boldsymbol{A}_h = \boldsymbol{D}_k \boldsymbol{B} - \boldsymbol{B} \boldsymbol{D}_h, \tag{5}$$

where $\boldsymbol{D}_h$ and $\boldsymbol{D}_k$ are the diagonal matrices with non-zero entries the degrees of the vertices of F_h and F_k, respectively. Some consequences of this are obtained when both F_h and F_k are regular.

Corollary 5. *Assume that a graph $G \equiv F_1$ and its k-token graph F_k are d_1-regular and d_k-regular graphs, respectively. Let $\boldsymbol{B}$ be the $(n; k, 1)$-binomial matrix. Let $\boldsymbol{A}$ and $\boldsymbol{A}_k$ be the respective adjacency matrices of G and F_k. If $\boldsymbol{v}$ is a λ-eigenvector of $\boldsymbol{A}$, then $\boldsymbol{B}\boldsymbol{v}$ is a μ-eigenvector of $\boldsymbol{A}_k$, where $\mu = (d_k - d_1 + \lambda)$.*

4 A Graph, Its Complement, and Their Token Graphs

Let us consider a graph G and its complement $\overline{G}$, with respective Laplacian matrices $\boldsymbol{L}$ and $\overline{\boldsymbol{L}}$. We already know that the eigenvalues of G are closely related to the eigenvalues of $\overline{G}$, since $\boldsymbol{L} + \overline{\boldsymbol{L}} = n\boldsymbol{I} - \boldsymbol{J}$. Hence, the same relationship holds for the algebraic connectivity, see Fiedler [6].

Observe that the k-token graph of $\overline{G}$ is the complement of the k-token graph of G with respect to the Johnson graph $J(n, k)$ (the k-token graph of K_n), see Carballosa, Fabila-Monroy, Leaños, and Rivera [3, Prop. 3]. Then, it is natural to ask whether a similar relationship holds between the Laplacian spectrum of the k-token graph of G and the Laplacian spectrum of the k-token graph of $\overline{G} = K_n - G$. In this section, we show that, indeed, this is the case.

Our result is a consequence of the following property.

Lemma 3. *Given a graph G and its complement $\overline{G}$, the Laplacian matrices $\boldsymbol{L} = \boldsymbol{L}(F_k(G))$ and $\overline{\boldsymbol{L}} = \boldsymbol{L}(F_k(\overline{G}))$ of their k-token graphs commute: $\boldsymbol{L}\overline{\boldsymbol{L}} = \overline{\boldsymbol{L}}\boldsymbol{L}$.*

Theorem 5. *Let $G = (V, E)$ be a graph on $n = |V|$ vertices, and let $\overline{G}$ be its complement. For a given k, with $1 \leq k < n-1$, let us consider the token graphs $F_k(G)$ and $F_k(\overline{G})$. Then, the Laplacian spectrum of $F_k(\overline{G})$ is the complement of the Laplacian spectrum of $F_k(G)$ with respect to the Laplacian spectrum of the Johnson graph $J(n,k) = F_k(K_n)$. That is, every eigenvalue λ_J of $J(n,k)$ is the sum of one eigenvalue $\lambda_{F_k(G)}$ of $F_k(G)$ and one eigenvalue $\lambda_{F_k(\overline{G})}$ of $F_k(\overline{G})$, where each $\lambda_{F_k(G)}$ and each $\lambda_{F_k(\overline{G})}$ is used once:*

$$\lambda_{F_k(G)} + \lambda_{F_k(\overline{G})} = \lambda_J.$$

Theorem 5 leads to the following consequence.

Corollary 6. *Let G be a graph such that its complement $\overline{G}$ has c connected components. Then, for $1 \leq k \leq n-1$, the k-token graph $F_k(G)$ has at least c integer eigenvalues. If each of the c components of $\overline{G}$ has at least k vertices, then $F_k(G)$ has at least $\binom{c+k-1}{k}$ integer eigenvalues.*

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