PERCOLATION ON RANDOM GRAPHS WITH A FIXED DEGREE SEQUENCE

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Abstract. We consider bond percolation on random graphs with given degrees and bounded average degree. In particular, we consider the order of the largest component after the random deletion of the edges of such a random graph. We give a rough characterisation of those degree distributions for which bond percolation with high probability leaves a component of linear order, known usually as a *giant component*. We show that essentially the critical condition has to do with the tail of the degree distribution. Our proof makes use of recent technique which is based on *the switching method* and avoids the use of the classic configuration model on degree sequences that have a limiting distribution. Thus our results hold for sparse degree sequences without the usual restrictions that accompany the configuration model.

`keywords`: random graphs with given degrees, bond percolation, giant component, power law
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1. Introduction

Random graphs with a given degree sequence have become an integral part of the theory of random graphs. Let $n \geq 2$ and let $\mathcal{D} = (d_1, \dots, d_n)$ be a degree sequence of length n; that is, a vector of non-negative integers which represent the degrees of the set of vertices $[n] := \{1, \dots, n\}$. In other words, vertex i has degree d_i for each $i \in [n]$. Without loss of generality, we assume that $d_1 \leq \dots \leq d_n$. In fact, our results deal with properties that are closed under automorphisms and remain valid when relabelling the vertex set. If not stated otherwise, we will assume that $d_1 \geq 1$. The results for degree sequences containing vertices of degree 0 can be easily deduced from the analysis of degree sequences without them. We will also assume that $\mathcal{D}$ is feasible; that is, there exists at least one graph with degree sequence $\mathcal{D}$. The main object of our study is $G^{\mathcal{D}}$, which is a graph chosen uniformly at random among all *simple* graphs on $[n]$ having degree sequence $\mathcal{D}$.

Random graphs with a given degree distribution appear also in the context of graph enumeration. Bender and Canfield [4], as well as Bollobás [6] and Wormald [26], came up with the now well-known *configuration model*, which has become a standard tool in the analysis of random graphs that are sampled uniformly from the set of all simple graphs with a given degree sequence. However, the study of such random graphs through the configuration model has some limitations, as it often requires bounds on the growth of the maximum degree of the degree sequence. Typically, these are implicitly imposed by bounds on the second (or higher) moment of the degree sequence.

In 1995, Molloy and Reed [18] investigated the component structure of $G^{\mathcal{D}}$ and, more specifically, the emergence of the *giant component* (a component containing at least a constant fraction of the vertices). This is one of the central questions in the theory of random graphs. They provided a condition on $\mathcal{D}$ that characterises the emergence of a giant component in $G^{\mathcal{D}}$ given that $\mathcal{D}$ satisfies a number of technical conditions. This result has been widely applied to the analysis of a variety of complex networks [1, 2, 5, 22] and there are several refinements of it [8, 12, 14, 16, 19]. The technical restrictions on $\mathcal{D}$ in [18] result from the use of the configuration model. These restrictions have been weakened in subsequent papers [8, 14]. Recently, Joos, Perarnau, Rautenbach, and Reed [15] managed to completely remove all restrictions on $\mathcal{D}$ by using an analysis based on the switching method. This provides a new criterion for the existence of a giant component in $G^{\mathcal{D}}$ that can be applied to every degree sequence.

In this paper, we follow this novel approach and consider the *component sizes* of a random graph with a given degree sequence under the random deletion of its edges. For a graph G and

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a real number $p \in [0,1]$, we denote by G_p the random subgraph of G in which every edge of G is retained independently with probability p. This is commonly known as *bond percolation* on G. The theme of this paper is the component structure of $G_p^{\mathcal{D}}$. Since $G^{\mathcal{D}}$ itself is a random graph, $G_p^{\mathcal{D}}$ should be understood as follows: first, we choose a graph $G^{\mathcal{D}}$ uniformly at random from the set of all simple graphs with degree sequence $\mathcal{D}$ and thereafter each edge of $G^{\mathcal{D}}$ is retained independently with probability p.

The structure of $G^{\mathcal{D}}$ has been studied in great detail for the case $d_i = d$ for all $i \in [n]$ and some $d \in \mathbb{N}$ (in this case we also write $G(n,d)$ for $G^{\mathcal{D}}$). The bond percolation of $G(n,d)$ was first studied by Goerdt [11]. He proved that there exists a critical value $p_{crit} = 1/(d-1)$ such that the existence of a giant component depends on whether $p < p_{crit}$ or $p > p_{crit}$. Bond percolation of $G(n,d)$ near the critical probability p_{crit} has been extensively studied [21, 23, 24]. The first author [10] and Janson [13] considered the bond percolation of $G^{\mathcal{D}}$, proving the existence of a critical probability, provided that $\mathcal{D}$ satisfies some technical conditions, similar to those required in [18]. These results have been extended to a more general setting by Bollobás and Riordan [8].

In the present work, we determine those conditions on $\mathcal{D}$ which ensure that p_{crit} is bounded away from 0. We consider arbitrary degree sequences without restrictions as in [8, 10, 13, 18] and we only insist that the total number of edges grows linearly with the number of vertices n. We call those sequences *sparse*. (We briefly discuss the non-sparse case at the end of the paper.) Besides the mathematical motivation, sparse graphs are also the main focus in the theory of complex networks as this is a property that is observed in several networks that arise in applications [2].

Consider a sequence of degree sequences $\mathfrak{D} = (\mathcal{D}_n)_{n\geq 2}$, where $\mathcal{D}_n = (d_1^{(n)}, \ldots, d_n^{(n)})$. Let D_n be the random variable that is the degree in $\mathcal{D}_n$ of a vertex selected uniformly at random. For $c \in \mathbb{N}$, we define $W(c,\mathcal{D}) := \{i \ : \ d_i \geq c\}$; that is, $W(c,\mathcal{D})$ is the set of vertices of degree at least c and set $W_n(c) := W(c,\mathcal{D}_n)$. The sequence $(D_n)_{n\geq 2}$ is *uniformly integrable* if for every $\varepsilon > 0$, there exist c, n_0 such that for every $n \geq n_0$, we have

$$\sum_{i\in W_n(c)} d_i^{(n)} \leq \varepsilon n, \tag{1}$$

and *strongly uniformly integrable* if for every $\varepsilon > 0$, there exists c_0 such that for any $c \geq c_0$ there exists n_0 such that for every $n \geq n_0$, we have

$$\sum_{i\in W_n(c)} d_i^{(n)} \leq \frac{\varepsilon}{c} \cdot n. \tag{2}$$

Strong uniform integrability can be seen as a weaker version of bounded second moment conditions.

For a graph G, we denote by $L_1(G)$ the number of vertices in the largest component (ties are resolved following the lexicographic ordering of the vertices).

We now state our main result in the context of sequences of degree sequences that satisfy a *mild* convergence condition.

Theorem 1. *Suppose that $\bar{d} \geq 1$ and let $\mathfrak{D} = (\mathcal{D}_n)_{n\geq 2}$ be a sequence of degree sequences such that for all $n \geq 2$ the average degree of $\mathcal{D}_n$ is at most $\bar{d}$. Suppose that*

$$d = d(\mathfrak{D}) := \sup_{c\geq 1} \lim_{n\to\infty} \max\left\{ \frac{\sum_{i\in V\setminus W_n(c)} d_i^{(n)}(d_i^{(n)}-1)}{\sum_{i\in V\setminus W_n(c)} d_i^{(n)}}, 1 \right\}$$

exists. Let D_n be the degree in $\mathcal{D}_n$ of a vertex chosen uniformly at random. Then

(i) if the sequence $(D_n)_{n\geq 2}$ is strongly uniformly integrable and $d < \infty$, then for every $\epsilon > 0$ the following hold:
  - *if $0 \leq p \leq (1-\epsilon)\frac{1}{d}$, then*
  $$\mathbb{P}[L_1(G_p^{\mathcal{D}_n}) = o(n)] = 1 - o(1) ;$$
  - *if $(1+\epsilon)\frac{1}{d} \leq p \leq 1$, then there exists $\rho = \rho(\epsilon)$ such that*
  $$\mathbb{P}[L_1(G_p^{\mathcal{D}_n}) > \rho n] = 1 - o(1) .$$

(ii) otherwise, for all $p, \delta \in (0,1)$, there exist $\rho > 0$ and n_0 such that for all $n \geq n_0$, we have

$$\mathbb{P}[L_1(G_p^{\mathcal{D}_n}) > \rho n] \geq 1 - \delta \ .$$

The proof of this theorem relies on a dichotomy within the class of all sparse degree sequences $\mathcal{D}$ of length n. This dichotomy is expressed through Theorems 2 and 3 below, which are much stronger, albeit somewhat technical, results. Roughly speaking, if the tail of the degree sequence $\mathcal{D}$ is sufficiently thin (conditions A_1 and A_2 below are satisfied), then there exists a critical probability p_{crit} bounded away from 0 (essentially determined by $\mathcal{D}$) such that when p crosses p_{crit} the fraction of vertices that belong to the largest component undergoes a rapid increase with high probability. On the other hand, if the tail is sufficiently heavy (either A_1 or A_2 are not satisfied), then for every $p \in (0,1]$, there is a giant component with high probability.

Let us now make these statements precise. For every $x > 0$ and every $c \in \mathbb{N}$, we say that $\mathcal{D}$ satisfies $A_1(x, c)$ if

$$\sum_{i \in W(c,\mathcal{D})} d_i \leq \frac{x}{c} \cdot n \ . \tag{3}$$

For all $x > 0$ and $c_1, c_2 \in \mathbb{N}$, we say that $\mathcal{D}$ satisfies $A_2(x, c_1, c_2)$ if

$$\sum_{i \in W(c_1,\mathcal{D}) \setminus W(c_2,\mathcal{D})} d_i^2 \leq \frac{x}{4} \cdot n \ . \tag{4}$$

Note that being strongly uniformly integrable is equivalent to satisfying condition $A_1(\varepsilon, c_0)$ for every $\varepsilon > 0$ and $c_0 = c_0(\varepsilon)$. Also observe that these integrability notions naturally extend to D_n, even if the sequence D_n does not converge in distribution.

The first part of this dichotomy describes which degree sequences have a percolation threshold.

Theorem 2. *For all $\epsilon, \gamma \in (0,1)$, all $c_1, c_2 \in \mathbb{N}$ and $\bar{d} \geq 1$, there exist $\rho = \rho(\epsilon, c_1)$, $\eta = \eta(\gamma, \epsilon, c_1)$ and n_0 such that for every $n \geq n_0$ and every degree sequence $\mathcal{D} = (d_1, \ldots, d_n)$ with average degree at most $\bar{d}$ that satisfies $A_1(\eta, c_2)$, then for*

$$p_{crit} := p_{crit}(c_2, \mathcal{D}) = \min \left\{ \frac{\sum_{i \in V \setminus W(c_2)} d_i}{\sum_{i \in V \setminus W(c_2)} d_i(d_i - 1)}, 1 \right\} \tag{5}$$

we have

(i) if $0 \leq p \leq (1-\epsilon)p_{crit}$, then

$$\mathbb{P}[L_1(G_p^{\mathcal{D}}) > \gamma n] = o_n(1) \ ;$$

(ii) if $\mathcal{D}$ satisfies $A_2(\epsilon, c_1, c_2)$ and $(1+\epsilon)p_{crit} \leq p \leq 1$, then

$$\mathbb{P}[L_1(G_p^{\mathcal{D}}) > \rho n] = 1 - o_n(1) \ .$$

In order to obtain a meaningful statement we need to apply Theorem 2 as follows: first, choose ε (width of the transition window) and $c_1, c_2, \bar{d}$. This fixes the value of p_{crit} and ρ. Now, choose γ which might be arbitrarily smaller than ρ. This fixes η and n_0. After these choices, the theorem then gives a sufficient criterion for degree sequences whose size of the largest component jumps from at most γn to at least ρn in a window of width 2ε around p_{crit}.

Interestingly, this theorem gives a criterion for the existence of "sudden" jumps in $L_1(G_p^{\mathcal{D}})$ that do not necessarily correspond to the phase transition of the appearance of a linear order component. In particular, it applies to cases where the degree sequence is not uniformly integrable. In Section 1.3 we will see an example of this behaviour.

The other part of the dichotomy settles the case of robust degree sequences.

Theorem 3. *For all $p, \delta \in (0,1)$ and all $\bar{d} \geq 1$, there exist $K, c_0 \in \mathbb{N}$ such that for every $c \geq c_0$, there exist $\rho > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every degree sequence $\mathcal{D} = (d_1, \ldots, d_n)$ with average degree at most $\bar{d}$ that does not satisfy either $A_1(K, c)$ or $A_2(K, 0, c)$, then*

$$\mathbb{P}[L_1(G_p^{\mathcal{D}}) > \rho n] \geq 1 - \delta \ .$$

As defined in (5), we have $p_{crit} > 0$ (assuming that $0/0 = 1$). Intuitively speaking, Theorem 3 classifies the degree sequences that have a "critical percolation threshold" at $p = 0$. It is also worth noticing that if $p = p(n) \to 0$ and $\bar{d} = O(1)$, then $L_1(G_p^{\mathcal{D}})$ is at most the number of edges in $G_p^{\mathcal{D}}$, which is $O(p\bar{d}n) = o(n)$ with probability $1 - o(1)$.

Theorems 2 and 3 show that the existence of a critical p for the emergence of a giant component is determined by the shape of the degree sequence for degrees that are bounded by a constant as well as by the degree sum of vertices of larger degree. For example, whether a degree sequence contains one vertex of degree $n/2$ or $n/(2\log n)$ vertices of degree $\log n$ does not make any difference.

1.1. **Approach to the proof of Theorems 2 and 3.** Previous work on bond percolation in $G^{\mathcal{D}}$ relies on the study of the configuration model. Given $\mathcal{D} = (d_1, \dots, d_n)$, let $\hat{G}^{\mathcal{D}}$ denote the random (multi)graph obtained using the configuration model (see e.g. [27]). The first author observed that the percolated random graph $\hat{G}_p^{\mathcal{D}}$ has the same distribution as $\hat{G}^{\mathcal{D}_p}$ where $\mathcal{D}_p = (d_1^p, \dots, d_n^p)$ is the random sequence obtained by choosing d_i^p distributed as a binomial random variable with parameters d_i and p, conditional on $\Sigma_{i\in[n]} d_i^p$ being even (see Lemma 3.1 in [10]). Loosely speaking, this result states that one could interchange the two random processes, percolating first the degree sequence conditional on its sum being even) and then choosing a random graph with the percolated degree sequence. Using this observation one can transfer results for the configuration model to its percolated instances [8, 10, 13]. These results can be transferred to the simple random graph $G_p^{\mathcal{D}}$ provided $\mathcal{D}$ satisfies certain technical conditions (see Section 1.2).

Joos et al. [15] established a criterion for the existence of a linear order component for any degree sequence. Following the previous observation of interchanging the two random processes, one could hope that the largest component of $G_p^{\mathcal{D}}$ could be studied directly, applying this criterion. However, the following example discusses a degree sequence for which the random graphs $G_p^{\mathcal{D}}$ and $G^{\mathcal{D}_p}$ are drastically different.

Example 1. *Let n be an even integer, let $p = 1/2$ and consider the degree sequence $\mathcal{D} = (n-1, n-1, 3, 3, \dots, 3)$. By standard concentration inequalities, with high probability, the degree sequence $\mathcal{D}_p$ satisfies $d_1^p(v_1), d_1^p(v_2) \approx n/2$ and $\Sigma_{i\in[n]} d_i^p \approx 5n/2$. An easy switching argument (see Section 3.1) shows that, with probability $1-o(1)$, we have $v_1v_2 \in E(G^{\mathcal{D}_p})$. However, by definition, the probability that $v_1v_2 \in E(G_p^{\mathcal{D}})$ is at most $1/2$. Moreover, the order of the largest component will strongly depend on the existence of v_1v_2. So the component structure of $G_p^{\mathcal{D}}$ and $G^{\mathcal{D}_p}$ is different.*

1.2. **Comparison of Theorem 1 to previous results.** The strongest statements in our paper are Theorems 2 and 3. However, as previous results deal with sequences of degree sequences $\mathfrak{D} = (\mathcal{D}_n)_{n\geq 2}$, it is more convenient to compare them to Theorem 1. Additionally, we will assume that D_n converges in distribution to the random variable D, denoted by $D_n \to D$, where D has finite and positive mean $\mathbb{E}[D]$; that is, there exists a probability distribution $(r_k)_{k\geq 0}$ such that for every $k \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{\{i \in [n] : d_i^{(n)} = k\}}{n} = r_k \ . \tag{6}$$

and $\mathbb{E}[D] = \sum_{k\geq 0} k r_k \in (0, \infty)$. Observe that (6) implies that $d(\mathfrak{D})$ exists.

Note that Theorem 1 only requires the existence of d. This is a slightly weaker condition than the convergence of D_n in distribution, but it is similar in spirit.

In this context, consider the following condition on the convergence of means

$$\lim_{n\to\infty} \mathbb{E}[D_n] = \mathbb{E}[D] \ . \tag{7}$$

Condition (7) can be easily replaced by the slightly weaker condition that D is uniformly integrable (see e.g. Remark 2.2 in [14]). Moreover, given that $D_n \to D$, the condition that D_n is (strongly) uniformly integrable as in (1) (and (2)) is equivalent to D being (strongly) uniformly integrable.

Additionally, we say that the sequence $(D_n)_{n\geq 2}$ has *bounded second moment* if

$$\mathbb{E}[D_n^2] = O(1) \ . \tag{8}$$

Under the assumption $D_n \to D$, we have following implications:

$$d(\mathfrak{D}) < \infty \quad \Leftarrow \quad \textit{Bounded second moment} \quad \Rightarrow \quad \textit{(Strong) uniform integrability}$$

All the implications are straightforward to check, so we omit their proofs.

We now state the two central results from the literature on bond percolation.

Theorem 4 (Proposition 3.1 in [13]). *Suppose $\mathfrak{D} = (\mathcal{D}_n)_{n\geq 2}$ is a sequence of degree sequences such that $D_n \to D$ and $(D_n)_{n\geq 2}$ has bounded second moment. If $p > 1/d(\mathfrak{D})$, there exists $\xi > 0$ such that*

$$\frac{L_1(G_p^{\mathcal{D}_n})}{n} \xrightarrow{p} \xi \ ,$$

and if $p < 1/d(\mathfrak{D})$, then

$$\frac{L_1(G_p^{\mathcal{D}_n})}{n} \xrightarrow{p} 0 \ .$$

Now, let $\rho(D)$ be the survival probability of a Galton-Watson tree with offspring distribution given by D. Let D_p be the *p-thinned* version of D, defined by

$$\mathbb{P}[D_p = i] = \sum_{j\geq i} \mathbb{P}[D = j] \binom{j}{i} p^i (1-p)^{j-i} \ .$$

Theorem 5 (Theorem 22 in [8]). *Suppose $\mathfrak{D} = (\mathcal{D}_n)_{n\geq 2}$ is a sequence of degree sequences such that $D_n \to D$ and $(D_n)_{n\geq 2}$ is uniformly integrable. Then for every $p \in (0,1)$*

$$\frac{L_1(G_p^{\mathcal{D}_n})}{n} \xrightarrow{p} \rho(D_p) \ .$$

The aim of this paper is to obtain results on the existence of a linear order component in $G_p^{\mathcal{D}}$ that are as widely applicable as possible, even if some precision is lost due to their generality. While Theorems 4 and 5 are more precise in their conclusions, they require conditions on $\mathfrak{D}$ (bounded second moment and uniform integrability, respectively) that are not necessary in Theorem 1. For instance, the results in Theorem 1 (ii) when $\mathfrak{D}$ is not uniformly integrable are not implied by any of the previous results in the literature. As we will show in the next sub-section, the case of non-uniformly integrable sequences is particularly interesting, and one cannot hope for very strong results in this setting. In conclusion, Theorem 1 and the previous results complement each other, and their use is a compromise between precision and generality.

1.3. Non-uniformly integrable sequences.

Roughly speaking, a degree sequence is not uniformly integrable if the vertices of unbounded degree have non-negligible contribution to the average degree. Here, we include two illustrative examples.

Example 2. *The order of the largest component may not be concentrated. Consider the degree sequence $\mathcal{D}_n = (n/4, n/4, 1, 1 \dots, 1)$ and $p \in (0,1)$. Using switchings, one can show that v_1 and v_2 are adjacent in $G^{\mathcal{D}_n}$ with probability $1 - o(1)$. Hence, for any fixed $p \in (0,1)$ the probability that u, v are adjacent in $G_p^{\mathcal{D}_n}$ is $(1 - o(1))p$, bounded away from 0 and 1. If v_1, v_2 are adjacent in $G_p^{\mathcal{D}_n}$, the order of the largest component will be distributed as $Bin(n/2 - 2, p)$; otherwise, it will be distributed as the maximum over two independent copies of $Bin(n/4, p)$.*

This example can be generalised to produce degree sequences $\mathcal{D}_n$ satisfying the following: for every $\rho > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} \mathbb{P}[L_1(G_p^{\mathcal{D}_n}) < \rho n] &> \delta \\ \mathbb{P}[L_1(G_p^{\mathcal{D}_n}) > (1-\rho) n] &> \delta \ . \end{aligned} \tag{9}$$

Thus, one cannot expect that $n^{-1} L_1(G_p^{\mathcal{D}_n})$ converges in probability to a constant as in Theorems 4 and 5. Moreover, (9) *shows that the statement of Theorem 1 (ii) cannot hold with probability $1 - o(1)$.*

Example 3. *As mentioned above, Theorem 2 can be used to detect sudden changes of the order of the largest component that do not coincide with the birth of the giant component. Consider the following degree sequence* $\mathcal{D}_n = (\xi n, 3, 3, \dots, 3)$*, for some* $\xi > 0$*. Theorem 3 shows that the size of the largest component of* $G_p^{\mathcal{D}_n}$ *is linear for every* $p > 0$*, in fact, it stochastically dominates a* $Bin(\xi n, p)$*. However, we can also apply Theorem 2 with* $p_{crit} = 1/2$*.In particular, given* $\varepsilon, \gamma, \rho$*, we can choose* ξ *sufficiently small so Theorem 2 gives a jump in* $L_1(G_p^{\mathcal{D}_n})$ *from at most* γn *to at least* ρn *in a window of width* 2ε *around* p_{crit}*. Hence the birth of the giant component is at* $p = 0$ *(by Theorem 3) and there is a boost at* $p = 1/2$ *(by Theorem 2).*

Intuitively, if a sequence is not uniformly integrable, then it has linearly many edges in vertices of unbounded degree. Under some weak conditions, these vertices will typically form a connected core, even after percolation. This core contains linearly many edges and it typically creates a linear order component. However, if $p_{crit} > 0$*, then for* $p > p_{crit}$*, the vertices of bounded degree will percolate even without the help of unbounded degree vertices, and thus, the growth of the giant component changes at* p_{crit}*. To our knowledge, this critical point has not been studied in the literature. It would be interesting to get a better understanding of the size of the largest component around this point.*

Structure of the paper: The paper is structured as follows. In Section 2, we provide the basic notation and some technical estimates that will be used throughout the proof. Section 3 presents the main combinatorial tool we will use, the switching method, and provides an overview of the proof of Theorem 2 and 3. In Section 4, we present three important technical propositions. Assuming them, in Section 5 and 6 we prove Theorem 2 and 3, respectively. In Section 7, 8 and 9 we prove these three propositions. Section 10 is devoted to the proof of Theorem 1. We provide an application of the results obtained to power-law degree sequences in Section 11. Finally, in Section 12, we state some remarks of our results and discuss a number of open questions.

2. Notation and some probabilistic tools

We consider labelled graphs G with vertex set $V = V(G) = [n] := \{1, \dots, n\}$ and edge set $E(G)$. If we refer to a graph or degree sequence on the set V, we always implicitly assume that $V = [n]$ and thus $|V| = n$. We say that a graph G on V has degree sequence $\mathcal{D} = (d_1, \dots, d_n)$ if for every $i \in [n]$, the degree of i is d_i. We let $\Sigma^{\mathcal{D}}$ denote the sum of these degrees; that is, $\Sigma^{\mathcal{D}} := \sum_{i=1}^n d_i$. We denote by $|\mathcal{D}|$ the length of the degree sequence.

For an arbitrary vertex $v \in V$, we will often write $d(v) = d_G(v)$ for its degree. Let H be a subgraph of G; if $v \in V(H)$, then $d_H(v)$ denotes the degree of v in H; if $v \in V \setminus V(H)$, then $d_H(v) = 0$. For a graph G, a subset of vertices $U \subseteq V$ and $v \in V$, we occasionally use the notation $d_G(v, U)$ to denote the number of neighbours of v in G that are in the subset U. Given a degree sequence $\mathcal{D}$ and a graph G, we denote by $\Delta(\mathcal{D})$ (and $\delta(\mathcal{D})$) and by $\Delta(G)$ (and $\delta(G)$) the maximum (and minimum) degree of the sequence $\mathcal{D}$ and of the graph G, respectively.

We denote by $N(v) = N_G(v)$ the set of neighbours of v in G. For $S \subseteq V$, we use $N(S) = N_G(S)$ for the set of vertices in $V \setminus S$ that have a neighbour in S. We also use $N[S] = N_G[S]$ for the set of vertices that are either in S or in $N(S)$. For $S \subseteq V$, we denote by $G[S]$ the (sub)graph of G induced by S. For disjoint $S, T \subseteq V$, we denote by $G[S, T]$ the bipartite graph induced between S and T.

We will make use of some classical concentration inequalities that can be found in [20].

Lemma 6 (Chernoff's inequality)**.** *Let* $X_1, \dots, X_N$ *be a set of independent Bernoulli random variables with expected value* p *and let* $X = \sum_{i=1}^N X_i$*. Then, for every* $0 < t < Np$

$$\mathbb{P}[|X - \mathbb{E}[X]| > t] \le 2e^{-\frac{t^2}{3Np}} .$$

Lemma 7 (McDiarmid's inequality)**.** *Let* $X_1, \dots, X_s$ *be a set of independent random variables taking values in* $[0, 1]$*. Let* $f : [0, 1]^s \to \mathbb{R}$ *be a function of* $X_1, \dots, X_s$ *that satisfies for every* $1 \le i \le s$*, every* $x_1, \dots, x_s \in [0, 1]$ *and every* $x_i' \in [0, 1]$*,*

$$|f(x_1, \dots, x_i, \dots, x_s) - f(x_1, \dots, x_i', \dots, x_s)| \le c_i ,$$

for some $c_i > 0$*. Then, for every* $t > 0$

$$\mathbb{P}[|f(X_1,\dots,X_s) - \mathbb{E}[f(X_1,\dots,X_s)]| > t] \le 2e^{-\frac{2t^2}{\sum_{i=1}^s c_i^2}} .$$

Many of our results have complicated hierarchies of constants. To be precise, if we say that a statement holds whenever $a \ll b \ll c \le 1$, then this means that there are non-decreasing functions $f, g : (0,1] \to (0,1]$ such that the result holds for all $0 < a, b, c \le 1$ with $a \le f(b)$ and $b \le g(c)$. In particular, such hierarchies need to be read from right to left. We will not calculate these functions explicitly in order to simplify the presentation. Hierarchies with more terms are defined in a similar way. Finally, we write $x = a \pm b$ to denote that $x \in [a-b, a+b]$, for real numbers x, a, b.

3. Overview of the proofs

In this section we present an overview of the proofs of Theorems 2 and 3 as well as of the main method used in them.

3.1. **The switching method.**

Let $\mathcal{G}^{\mathcal{D}}$ be the set of simple graphs with degree sequence $\mathcal{D}$. Throughout our proofs we want to consider the probability that $G^{\mathcal{D}}$, a uniformly chosen element of $\mathcal{G}^{\mathcal{D}}$, satisfies a certain property. We will slightly abuse notation by indistinctly referring to $\mathcal{G}^{\mathcal{D}}$ as a set of graphs and as a probability space equipped with the uniform distribution. Similarly, we will consider $\mathcal{F} \subseteq \mathcal{G}^{\mathcal{D}}$ and talk about the probability of $\mathcal{F}$, thought as an event in the probability space $\mathcal{G}^{\mathcal{D}}$.

The main combinatorial tool that we use to estimate different probabilities in $\mathcal{G}^{\mathcal{D}}$ is the *switching method*. Given a graph G with degree sequence $\mathcal{D}$ and two ordered edges $uv, xy \in E(G)$ we can perform the following graph operation, called a $\{uv, xy\}$-switch, or simply a *switch*: obtain G' by deleting the edges uv and xy from G and adding the edges ux and vy in G. Observe that the $\{uv, xy\}$-switch is different from the $\{vu, xy\}$-switch, but equal to the $\{vu, yx\}$-switch. If either ux or vy are edges of G, the graph G' will have multiple edges, and if either $u = x$ or $v = y$, the graph G' will have loops. Since we restrict here to simple graphs, we say that a switch is *valid* if none of these occur.

The basic idea of the switching method is the following. In order to determine the probability of $\mathcal{F} \subseteq \mathcal{G}^{\mathcal{D}}$ in terms of the probability of $\mathcal{F}' \subseteq \mathcal{G}^{\mathcal{D}}$, we use the average number of valid switches between a graph in $\mathcal{F}$ and a graph in $\mathcal{F}'$, denoted by $d(\mathcal{F} \to \mathcal{F}')$, and vice versa. A simple double-counting of such switches gives

$$\mathbb{P}[\mathcal{F}] = \frac{d(\mathcal{F}' \to \mathcal{F})}{d(\mathcal{F} \to \mathcal{F}')} \cdot \mathbb{P}[\mathcal{F}'] . \tag{10}$$

Although this relation is very simple, the switching method is very powerful. In particular, we avoid the use of the configuration model and all the technicalities that come with it.

3.2. **The emergence of the giant component.**

In [15], a characterisation is given of those degree sequences $\mathcal{D}$ for which the random graph $G^{\mathcal{D}}$ has a giant component with high probability. Let $j^{\mathcal{D}}$ be the smallest $j \in \mathbb{N}$ such that

$$\sum_{i=1}^{j} d_i(d_i - 2) > 0$$

if such j exists and else $j^{\mathcal{D}} = n$. Also, they set $R^{\mathcal{D}} := \sum_{i=j^{\mathcal{D}}}^{n} d_i$ and $M^{\mathcal{D}} := \sum_{i:d_i \neq 2} d_i$. Effectively, $G^{\mathcal{D}}$ has a giant component with high probability if and only if $R^{\mathcal{D}}$ grows linearly in $M^{\mathcal{D}}$.

As we deal with bond percolation on $G_p^{\mathcal{D}}$, we need to consider generalizations of these quantities.

(i) Let $j_p^{\mathcal{D}}$ be the minimum between n and the smallest natural number j such that

$$\sum_{i=1}^{j} d_i(p(d_i - 1) - 1) > 0 .$$

(ii) Let $R_p^{\mathcal{D}} := \sum_{i=j_p^{\mathcal{D}}}^{n} (p(d_i - 1) - 1)$.

Observe that if $\sum_{i=1}^{n} d_i(p(d_i - 1) - 1) \leq 0$, we have $R_p^{\mathcal{D}} = p(d_n - 1) - 1$.

Note that in the case $p = 1$, the definition of $j_p^{\mathcal{D}}$ coincides with the definition of $j^{\mathcal{D}}$. Moreover, $R_1^{\mathcal{D}} \leq R^{\mathcal{D}} \leq 3R_1^{\mathcal{D}}$. In our setting we do not need to define an analogue of $M^{\mathcal{D}}$ since vertices of degree 2 play no special role here (see Section 12 for a detailed discussion). Also note that vertices of degree 0 have no contribution to these parameters. So, it is useful to assume, and we will do so if not stated otherwise, that $d_i \geq 1$ for every $i \in [n]$. The result for degree sequences containing vertices of degree 0 can be easily deduced from the analysis of degree sequences without them.

Theorem 2 and 3 essentially distinguish between two cases on the tail of the degree sequence. In Theorem 2, we show that a critical percolation threshold p_{crit} exists if the two conditions $A_1(\cdot, c_1, c_2)$ and $A_2(\cdot, c_2)$ hold. These conditions bound the number of edges incident to vertices of large degree. One should note that the two conditions are only required for particular values of the degrees, namely c_1 and c_2, and thus, they are much weaker than a domination condition on the whole tail of the degree sequence. The heart of the proof of Theorem 2 is the analysis of an *exploration process*, in which we reveal the components of $G_p^{\mathcal{D}}$ by exposing the neighbours of each vertex sequentially (cf. Proposition 9). To avoid technical difficulties that arise due to high degree vertices, we include them together with their neighbours in the initial set of explored vertices. So during the process, we only reveal the connections of vertices of low or moderately-growing degree. Let S denote the set of high degree vertices (we will specify the exact magnitude during the proof). We show that if $\sum_{v \notin N[S]} d(v)(p(d(v) - 1) - 1)$ is negative and actually decays linearly with n, then the exploration process is subcritical in the sense that all components it reveals are sub-linear. If $\sum_{v \notin N[S]} d(v)(p(d(v) - 1) - 1)$ is positive and grows linearly, then one can use condition A_2 to ensure that $R_p^{\mathcal{D}}$ grows linearly. In that case, it turns out that the exploration process will reveal a component of linear order with high probability. In Sections 5 and 6, we show how these two conditions give a critical value p_{crit} such that when p goes from less than $(1 - \epsilon)p_{crit}$, to at least $(1 + \epsilon)p_{crit}$, the fraction of vertices in the largest component undergoes an abrupt increase.

Regarding Theorem 3, recall that its premises cover degree sequences that have a quite heavy tail, that is, either Condition A_1 or Condition A_2 fails. Here, we distinguish between two sub-cases. The first is that a set S_1 of very high (growing) degree vertices have linear total degree. The boundedness of the average degree implies that S_1 is small (of sub-linear size). Moreover, we show that with high probability $G_p^{\mathcal{D}}[S_1]$ is connected and that more than half of the edges incident to S_1 in $G^{\mathcal{D}}$, have their other endpoint in $V \setminus S_1$. Deleting each such edge with probability that is bounded away from 0 leaves with high probability a giant component. This is stated in Proposition 10. Now, if S_1 does not have linear total degree, then we show that its removal leaves a degree sequence $\mathcal{D}'$ that is *super-critical*: the quantity $R_p^{\mathcal{D}'}$ grows linearly in n. One can then apply again Proposition 9 to find a giant component, concluding the proof of Theorem 3.

The transition from $R_p^{\mathcal{D}}$ to $R_p^{\mathcal{D}'}$ requires a result (Proposition 8) which shows that if $R_p^{\mathcal{D}}$ grows linearly in n and we remove a set of vertices of small total degree, then the resulting degree sequence $\mathcal{D}'$ is such that $R_p^{\mathcal{D}'}$ still grows linearly, albeit with a smaller coefficient.

Finally, the proof of Theorem 1 is a relatively straightforward application of both Theorem 2 and 3 and it is proved in Section 10.

4. Three technical propositions

In this section we introduce three important propositions that will allow us to prove Theorem 2 and 3. We defer their proofs to Sections 7, 8 and 9, respectively.

The first one is a *deterministic* proposition which proves the following. Suppose G is a graph with degree sequence $\mathcal{D}$ and $S \subseteq V(G)$ such that $\sum_{u \in S} d(u)$ is small. Suppose $\mathcal{D}'$ is a possible degree sequence of the graph $G - S$, then $R_p^{\mathcal{D}'}$ is bounded from below by $R_p^{\mathcal{D}}/50$.

Proposition 8. *Suppose $1/n \ll \nu$, $400\nu \leq \mu \leq 1$, and $p \in (0, 1]$. Suppose $\mathcal{D}$ is a degree sequence on V with $R_p^{\mathcal{D}} \geq \mu n$ and let $S \subseteq V$ be such that $\sum_{v \in S} d(v) \leq \nu n$. Assume that G' is a graph obtained from a graph G with degree sequence $\mathcal{D}$ by deleting all vertices in S and afterwards by* *deleting all vertices of degree* 0*. Let* $\mathcal{D}'$ *be the degree sequence of* G' *and assume it has length* n'*. Then* $n' \geq (1-2\nu)n$ *and* $R_p^{\mathcal{D}'} \geq \frac{\mu}{50}n'$*.*

Moreover, if $G = G^{\mathcal{D}}$ *and* $\mathcal{D}'$ *is the degree sequence of* G'*, then* G' *is a uniformly random graph with degree sequence* $\mathcal{D}'$*, that is* $G' = G^{\mathcal{D}'}$*.*

The key ingredient for the proof of both Theorem 2 and 3 is the following proposition that gives us the component structure of $G_p^{\mathcal{D}}$. This proposition can be viewed as the extension of the main theorem in [15]. It states that if the drift on the bulk of the vertices (that is, excluding vertices of very high degree and their neighbours) is negative, then the fraction of vertices in the largest component is a.a.s. bounded by some small constant. On the other hand, if $\Delta(\mathcal{D}) \leq n^{1/4}$ and $R_p^{\mathcal{D}} \geq \mu n$, then we have a.a.s. a component containing a constant fraction of all vertices.

Proposition 9. *Suppose* $n \in \mathbb{N}$ *and* $1/n \ll \alpha \ll \gamma \ll \mu, 1/\bar{d}, p \leq 1$*. Let* $\mathcal{D}$ *be a degree sequence on* V *with* $\Sigma^{\mathcal{D}} \leq \bar{d}n$*.*

(i) If there exists a set $S \subseteq V$ *such that* $d(v) \leq n^{1/4}$ *for every* $v \notin S$*, and for every graph* G *with degree sequence* $\mathcal{D}$ *one has* $\sum_{u\in N_G[S]} d(u) \leq \alpha n$*, and* $\sum_{u \in V\setminus N_G[S]} d(u)(p(d(u)-1)-1) \leq -\mu n$*, then*
$$\mathbb{P}[L_1(G_p^{\mathcal{D}}) \leq \gamma n] = 1 - o_n(1)\ .$$
(ii) If $\Delta(\mathcal{D}) \leq n^{1/4}$ *and* $R_p^{\mathcal{D}} \geq \mu n$*, then*
$$\mathbb{P}[L_1(G_p^{\mathcal{D}}) \geq \gamma n] = 1 - o_n(1)\ .$$

Our last result is a version of Theorem 3 for degree sequences that have many edges incident to vertices of unbounded degree. Recall that for $c \in \mathbb{N}$, $W(c,\mathcal{D})$ denotes the set of vertices of degree at least c.

Proposition 10. *For all* $p, \delta, \epsilon \in (0,1)$ *and all* $\bar{d} \geq 1$*, there exist* $\gamma > 0, n_0 \geq 1$ *such that for every* $n \geq n_0$ *and every degree sequence* $\mathcal{D}$ *on* V *with* $\Sigma^{\mathcal{D}} \leq \bar{d}n$ *that satisfies*
$$\sum_{i\in W(\log^2 n, \mathcal{D})} d_i \geq \epsilon n\ ,$$
we have
$$\mathbb{P}[L_1(G_p^{\mathcal{D}}) > \gamma n] \geq 1 - \delta\ .$$

In the next two sections we proceed with the proof of Theorem 2 and 3 assuming these three propositions.

5. Degree sequences with thin tails: proof of Theorem 2

Proof of Theorem 2. Let $\mathcal{D}$ be a degree sequence on V. For convenience, we set $W(c) := W(c,\mathcal{D}) = \{i \in V : d_i \geq c\}$. We choose η such that $\eta \ll \epsilon, \gamma, 1/c_1$. Next, let $p_{crit} := p_{crit}(c_2, \mathcal{D})$ be as in (5). Note that the definition of p_{crit} excludes the contribution of all the vertices of degree at least c_2. Moreover, with this definition we have

$$\sum_{i\in V\setminus W(c_2)} d_i(p_{crit}(d_i-1)-1) \leq 0 \tag{11}$$

and equality holds if $p_{crit} < 1$.

We first prove Theorem 2 (i). Suppose that $p \leq (1-\epsilon)p_{crit}$. Our strategy is to apply Proposition 9 (i) with $W(c_2)$, 2η and $\epsilon/3$ playing the role of S, α and μ, respectively. In order to do so, we need to give an upper bound on $\sum_{i\in N[W(c_2)]} d_i$ and on $\sum_{i\in V\setminus N[W(c_2)]} d_i(p(d_i-1)-1)$ for every graph G with degree sequence $\mathcal{D}$.

By assumption, $A_1(\eta, c_2)$ holds; that is,

$$|N(W(c_2))| \leq \sum_{i\in W(c_2)} d_i \leq \frac{\eta}{c_2}\cdot n, \tag{12}$$

and therefore

$$\sum_{i\in N[W(c_2)]} d_i \le \sum_{i\in W(c_2)} d_i + \sum_{i\in N(W(c_2))} d_i \le \frac{\eta}{c_2}\cdot n + c_2|N(W(c_2))| \le (1+c_2)\frac{\eta}{c_2}\cdot n \le 2\eta n\ . \tag{13}$$

We next bound $\sum_{i\in V\setminus N[W(c_2)]} d_i(p(d_i-1)-1)$ from above. Since $d_i(p(d_i-1)-1) \ge -d_i$ for every $i \in V$, we obtain

$$\begin{aligned}\sum_{i\in V\setminus N[W(c_2)]} d_i(p(d_i-1)-1) &= \sum_{i\in V\setminus W(c_2)} d_i(p(d_i-1)-1) - \sum_{i\in N(W(c_2))} d_i(p(d_i-1)-1)\\ &\le \sum_{i\in V\setminus W(c_2)} d_i(p(d_i-1)-1) + c_2|N(W(c_2))|\\ &\overset{(12)}{\le} \sum_{i\in V\setminus W(c_2)} d_i(p(d_i-1)-1) + \eta n\ . \end{aligned}\tag{14}$$

It follows that

$$\begin{aligned}\sum_{i\in V\setminus W(c_2)} d_i(p(d_i-1)-1) &\le (1-\epsilon)\sum_{i\in V\setminus W(c_2)} d_i(p_{crit}(d_i-1)-1) - \epsilon\sum_{i\in V\setminus W(c_2)} d_i\\ &\overset{(11)}{\le} -\epsilon\sum_{i\in V\setminus W(c_2)} d_i\ .\end{aligned}$$

Using that $d_i \ge 1$, we obtain

$$\sum_{i\in V\setminus W(c_2)} d_i = \sum_{i\in V} d_i - \sum_{i\in W(c_2)} d_i \ge n - \sum_{i\in W(c_2)} d_i \overset{(12)}{\ge} \frac{n}{2}. \tag{15}$$

Therefore,

$$\sum_{i\in V\setminus W(c_2)} d_i(p(d_i-1)-1) \le -\frac{\epsilon}{2}\cdot n.$$

Using (14), it follows that

$$\sum_{i\in V\setminus N[W(c_2)]} d_i(p(d_i-1)-1) \le -\frac{\epsilon}{2}\cdot n + \eta n \le -\frac{\epsilon}{3}\cdot n. \tag{16}$$

As (13) and (16) hold, Proposition 9 (i) completes the proof of Theorem 2 (i).

We proceed with the proof of Theorem 2 (ii). Suppose now that $p \ge (1+\epsilon)p_{crit}$. Here we may assume that $p_{crit} < 1$ as otherwise $p > 1$ does not satisfy the assumption of part (ii). We will first show that there exists $\mu = \mu(\epsilon, \bar{d}, c_1)$ such that $R_p^{\mathcal{D}} \ge \mu n$.

For $k \in \{1,2\}$, we define $j_k := \min\{n+1, i \in [n] : d_i \ge c_k\}$. Since $p_{crit} < 1$, (11) holds with equality. Using the definition of $j_p^{\mathcal{D}}$, we obtain

$$\begin{aligned}\sum_{i=j_p^{\mathcal{D}}}^{j_2-1} d_i(p(d_i-1)-1) &= \sum_{i=1}^{j_2-1} d_i(p(d_i-1)-1) - \sum_{i=1}^{j_p^{\mathcal{D}}-1} d_i(p(d_i-1)-1)\\ &\ge \sum_{i=1}^{j_2-1} d_i(p(d_i-1)-1)\\ &\ge (1+\epsilon)\sum_{i=1}^{j_2-1} d_i(p_{crit}(d_i-1)-1) + \epsilon\sum_{i=1}^{j_2-1} d_i\\ &\overset{(11)}{=} 0 + \epsilon\sum_{i=1}^{j_2-1} d_i \overset{(15)}{\ge} \frac{\epsilon}{2}\cdot n\ .\end{aligned}\tag{17}$$

It follows from $A_2(\epsilon, c_1, c_2)$ that

$$\sum_{i=j_p^{\mathcal{D}}}^{j_1-1} d_i(p(d_i-1)-1) \overset{(17)}{\geq} \frac{\epsilon}{2}\cdot n - \sum_{i=j_1}^{j_2-1} d_i(p(d_i-1)-1) \geq \frac{\epsilon}{2}\cdot n - \sum_{i=j_1}^{j_2-1} d_i^2 \geq \frac{\epsilon}{4}\cdot n.$$

We conclude that

$$R_p^{\mathcal{D}} = \sum_{i=j_p^{\mathcal{D}}}^{n}(p(d_i-1)-1) \geq \sum_{i=j_p^{\mathcal{D}}}^{j_1-1}(p(d_i-1)-1) \geq \frac{1}{c_1}\sum_{i=j_p^{\mathcal{D}}}^{j_1-1} d_i(p(d_i-1)-1) \geq \frac{\epsilon}{4c_1}\cdot n =: \mu n.$$

Recall that, by assumption, we have $\sum_{i\in W(c_2)} d_i \leq \frac{\eta}{c_2}\cdot n \leq \eta n$. Since $\eta \ll \mu$, we can apply Proposition 8 with η and $W(c_2)$ playing the role of ν and S. Let $\mathcal{D}'$ be the random degree sequence of the subgraph $G^{\mathcal{D}}[V\setminus W(c_2)]$. Let $\mathfrak{D}_0$ be the set of degree sequences $\mathcal{D}_0$ that satisfy $\mathbb{P}[\mathcal{D}'=\mathcal{D}_0]>0$. By Proposition 8, we deduce that if $\mathcal{D}_0\in\mathfrak{D}_0$, then $R_p^{\mathcal{D}_0}\geq \mu n/50$ and $|\mathcal{D}_0|\geq \frac{n}{2}$. Moreover, conditional on $\mathcal{D}'=\mathcal{D}_0$, we have that $G^{\mathcal{D}}[V\setminus W(c_2)]$ and $G^{\mathcal{D}_0}$ have the same probability distribution. Now we select ρ such that $\eta \ll \rho \ll \mu, 1/\bar{d}, p$. We can apply Proposition 9 (ii) to the degree sequence $\mathcal{D}_0$ with 2ρ playing the role of γ, from which Theorem 2 (ii) follows:

$$\begin{aligned}\mathbb{P}[L_1(G_p^{\mathcal{D}})>\rho n] &\geq \mathbb{P}[L_1(G_p^{\mathcal{D}}[V\setminus W(c_2)])>\rho n]\\ &\geq \min_{\mathcal{D}_0\in\mathfrak{D}_0}\mathbb{P}[L_1(G_p^{\mathcal{D}_0}))>2\rho|\mathcal{D}_0|]\\ &= 1-o_n(1)\ .\end{aligned}$$

□

6. Robust degree sequences: proof of Theorem 3

Proof of Theorem 3. We choose c_0 and K such that $1/c_0 \ll 1/K \ll \delta, p, \epsilon, 1/\bar{d}$. For any given $c \geq c_0$, we choose n_0 such that $1/n_0 \ll 1/c$. Let $n \geq n_0$.

Suppose first that $\sum_{i\in W(\log^2 n)} d_i \geq Kn/c^3$. We apply Proposition 10 with K/c^3 playing the role of ϵ and obtain a γ_1 such that $\mathbb{P}[L_1(G_p^{\mathcal{D}})>\gamma_1 n]\geq 1-\delta$.

Hence we may assume that $\sum_{i\in W(\log^2 n)} d_i \leq Kn/c^3$. Let $S := W(\log^2 n)$. We will show that the (random) subgraph $G_p^{\mathcal{D}}[V\setminus S]$ has a giant component, and thus also $G_p^{\mathcal{D}}$ has a giant component. Let us first show that $R_p^{\mathcal{D}} \geq \frac{Kp}{16c}\cdot n$. We consider two cases:

Case 1: $A_1(K,c)$ does not hold; that is, $\sum_{i\in W(c)} d_i \geq Kn/c$.

We define $j_1 := \min\{j\in[n] : d_j \geq c\}$ and let j_2 be the smallest integer j such that $\sum_{i=j_1}^{j} d_i \geq Kn/(2c)$. Since $A_1(K,c)$ does not hold, j_1 and j_2 are well-defined. We have

$$\begin{aligned}\sum_{i=1}^{j_2} d_i(p(d_i-1)-1) &= p\sum_{i=1}^{j_2} d_i^2 - (1+p)\sum_{i=1}^{j_2} d_i \geq p\sum_{i=j_1}^{j_2} d_i^2 - (1+p)\bar{d}n\\ &\geq cp\sum_{i=j_1}^{j_2} d_i - 2\bar{d}n \geq \left(\frac{Kp}{2}-2\bar{d}\right)n\ .\end{aligned}$$

Therefore, $j_p^{\mathcal{D}} \leq j_2$. Since $1/d_{j_2} \leq 1/c \leq 1/c_0 \ll p$, we conclude that $p(d_j-1)-1 \geq pd_j/4$ for all $j\geq j_2$. By the definition of j_2 and using $A_1(K,c)$ (in fact, its negation), it follows that

$$R_p^{\mathcal{D}} \geq \sum_{i=j_2}^{n}(p(d_i-1)-1) \geq \frac{p}{4}\sum_{i=j_2}^{n} d_i \geq \frac{p}{4}\left(\sum_{i\in W(c)} d_i - \sum_{i=j_1}^{j_2-1} d_i\right) \geq \frac{Kp}{16c}\cdot n\ .$$

Case 2: $A_2(K,0,c)$ does not hold; that is, $\sum_{i\in V\setminus W(c)} d_i^2 \geq Kn/4$.

Now let j_3 be the smallest integer j such that $\sum_{i=1}^{j} d_i^2 \geq Kn/8$. Since $A_2(K, 0, c)$ does not hold, j_3 is well-defined and $d_{j_3} < c$. Using the definition of $j_p^{\mathcal{D}}$, similarly as before

$$\begin{aligned}\sum_{j=j_p^{\mathcal{D}}}^{j_3} d_j(p(d_j-1)-1) &\geq \sum_{j=1}^{j_3} d_j(p(d_j-1)-1) \\ &\geq p\sum_{j=1}^{j_3} d_j^2 - (1+p)\bar{d}n \\ &\geq \left(\frac{Kp}{8} - 2\bar{d}\right) n > \frac{Kp}{16}\cdot n\,.\end{aligned}$$

Thus

$$R_p^{\mathcal{D}} \geq \sum_{j=j_p^{\mathcal{D}}}^{j_3} (p(d_j-1)-1) > \frac{1}{c}\sum_{j=j_p^{\mathcal{D}}}^{j_3} d_j(p(d_j-1)-1) \geq \frac{Kp}{16c}\cdot n\,.$$

Let $\mathcal{D}'$ be the random degree sequence of the subgraph $G^{\mathcal{D}}[V\setminus S]$. Let $\mathfrak{D}_0$ be the set of degree sequences $\mathcal{D}_0$ that satisfy $\mathbb{P}[\mathcal{D}' = \mathcal{D}_0] > 0$. By Proposition 8 applied to S with K/c^3 and $Kpn/(16c)$ playing the role of ν and μ (note that $400\nu \leq \mu$), we obtain that, for every $\mathcal{D}_0 \in \mathfrak{D}_0$, one has $R_p^{\mathcal{D}_0} \geq Kpn/(800c)$ and $|\mathcal{D}_0| \geq \frac{n}{2}$. Moreover, conditional on $\mathcal{D}' = \mathcal{D}_0$, we have that $G^{\mathcal{D}}[V\setminus S]$ and $G^{\mathcal{D}_0}$ have the same probability distribution. Using that for every $\mathcal{D}_0 \in \mathfrak{D}_0$, $\Delta(\mathcal{D}_0) \leq \log^2 n$ and $R_p^{\mathcal{D}_0} \geq Kpn/(800c)$, we can apply Proposition 9 (ii) and obtain $\gamma_2 > 0$ such that

$$\begin{aligned}\mathbb{P}[L_1(G_p^{\mathcal{D}}) > \gamma_2 n] &\geq \mathbb{P}[L_1(G_p^{\mathcal{D}}[V\setminus S]) > \gamma_2 n] \\ &\geq \min_{\mathcal{D}_0\in\mathfrak{D}_0} \mathbb{P}[L_1(G_p^{\mathcal{D}_0})) > 2\gamma_2|\mathcal{D}_0|] \\ &\geq 1 - o_n(1)\,.\end{aligned}$$

We conclude the proof of the theorem by setting $\gamma := \min\{\gamma_1, \gamma_2\}$. □

7. Proof of Proposition 8

Although the statement of Proposition 8 may sound very natural and also easy to prove, the fact that some edge deletions may cause significant reordering in ordered degree sequences makes the proof technical and complex.

Proof of Proposition 8. For every $k \in [n]$, let d_k' be the degree of the vertex k in G' and define $r_k := d_k - d_k'$. Note that $r_k = d_k$ for every $k \in S$ whereby

$$\sum_{k\in V} r_k \leq 2\nu n\,.$$

Clearly, by deleting at most $\sum_{k\in S} d_k \leq \nu n$ edges, we have created at most $2\nu n$ vertices of degree 0, which we do not consider in $\mathcal{D}'$. Let n' be the number of vertices with positive degree after the deletion of S. Thus $n' \geq (1-2\nu)n$.

Note that the statement follows if $\Delta(G) \geq \mu n/(40p)$, since then a vertex of maximum degree is not contained in S and has degree at least $\Delta(G) - \nu n$ in $\mathcal{D}'$, whereby

$$R_p^{\mathcal{D}'} \geq p((\Delta(G) - \nu n) - 1) - 1 \geq \frac{\mu}{50}\cdot n \geq \frac{\mu n'}{50}\,,$$

where we used that $400\nu \leq \mu$. Thus, we may now assume that $\Delta(G) < \mu n/(40p)$.

We define $f_k := p(d_k - 1) - 1$ and $f_k' := p(d_k' - 1) - 1$ for all $i \in [n]$. Recall that $d_1 \leq \cdots \leq d_n$ and that $j_p^{\mathcal{D}}$ is the smallest integer j such that

$$\sum_{k=1}^{j} d_k(p(d_k-1)-1) = \sum_{k=1}^{j} d_k f_k > 0\,.$$

Let $A := \{k \in [n] : k < j_p^{\mathcal{D}}\}$ and $B := [n] \setminus A$. Let σ be a permutation such that $d'_{\sigma_1} \leq \ldots \leq d'_{\sigma_n}$ where we set $d'_k := 0$ if vertex k is deleted. Note that $d'_k f'_k = 0$ if $d'_k = 0$. Thus $R_p^{\mathcal{D}'}$ does not change if we add isolated vertices to $\mathcal{D}'$. For the sake of simplicity, we consider $\mathcal{D}'$ to be a degree sequence on $[n]$ with isolated vertices. Let $A' := \{\sigma_k : k \in [n], \sigma_k < j_{\mathcal{D}'}^p\}$ and $B' := [n] \setminus A'$.

Let $T := \{k \in B : f'_k \geq f_k/2\}$. Observe that if we delete an edge ij from G, then f_i and f_j decrease by p, respectively. Thus

$$\sum_{k\in B\setminus T} f_k \leq 2 \sum_{k\in B\setminus T} (f_k - f'_k) = 2p \sum_{k\in B\setminus T} r_k \leq 4p\nu n.$$

Hence

$$\sum_{k\in T} f_k = R_p^{\mathcal{D}} - \sum_{k\in B\setminus T} f_k \geq \mu n - 4p\nu n \geq \frac{4\mu}{5} n \ . \tag{18}$$

Suppose $k \in T$. Then,

$$d'_k \geq \frac{d_k}{2} + \frac{1}{2p} + \frac{1}{2} \geq \frac{d_k}{2} \ .$$

We define $c := d_{j_p^{\mathcal{D}}}$. It follows that,

$$\sum_{k\in T} d'_k f'_k \geq \frac{1}{4} \sum_{k\in T} d_k f_k \geq \frac{c}{4} \sum_{k\in T} f_k \overset{(18)}{\geq} \frac{c\mu}{5} n \ . \tag{19}$$

Let $T_1 \subseteq T$ be such that

$$\sum_{k\in T_1} d'_k f'_k > \frac{c\mu}{20} n \ ,$$

and $\max\{\sigma_k : k \in T_1\}$ is minimized (choose the set of consecutive vertices in T smallest with respect to the order $\sigma_1, \ldots, \sigma_n$). By (19) such a set exists. Let $k_{max} = \arg\max\{\sigma_k : k \in T_1\}$. So the above definition implies that

$$\sum_{k\in T_1\setminus\{k_{max}\}} d'_k f'_k \leq \frac{c\mu}{20} n \ .$$

Observe also that $f'_{k_{max}} \leq p\Delta(G) \leq \mu n/40$ and that $d'_k \geq d_k/2 \geq c/2$ for each $k \in T_1$ (whereby $\frac{2}{c} d'_k \geq 1$). We conclude that

$$\begin{aligned}
\sum_{k\in T\setminus T_1} f'_k &= \sum_{k\in T} f'_k - \sum_{k\in T_1\setminus\{k_{max}\}} f'_k - f'_{k_{max}} \\
&\geq \frac{1}{2} \sum_{k\in T} f_k - \frac{2}{c} \sum_{k\in T_1\setminus\{k_{max}\}} d'_k f'_k - f'_{k_{max}} \\
&\overset{(18)}{\geq} \frac{2\mu}{5} n - \frac{\mu}{10} n - \frac{\mu}{40} n \geq \frac{\mu}{4} n \ .
\end{aligned}$$

Let $B_1 := \{k \in B : f'_k < 0\}$ and note that $B_1 \subseteq B \setminus T$ and $B_1 \subseteq A'$.

Recall that $d'_k = d_k - r_k$ and that $f'_k = f_k - pr_k$. By the definition of A and c, we observe that $\sum_{k\in A} d_k f_k \geq -c(p(c-1)-1)$. Thus

$$\begin{aligned}
\sum_{k\in A} d'_k f'_k &\geq \sum_{k\in A} d_k f_k - \sum_{k\in A} r_k (f_k + p d_k) \\
&\geq -c(p(c-1)-1) - \sum_{k\in A} r_k (p(2c-1)-1) \\
&\geq -pc^2 - 4cp\nu n \ .
\end{aligned}$$

If $k \in B_1$, then $f_k > 0$ and from $f'_k < 0$, we obtain $d_k < 1/p + r_k + 1$. Thus

$$\begin{aligned}\sum_{k\in B_1} d'_k f'_k &= \sum_{k\in B_1} ((d_k - r_k)f_k - pr_k(d_k - r_k)) \\ &\geq -\sum_{k\in B_1} pr_k(d_k - r_k) \\ &> -\sum_{k\in B_1} r_k(1+p) \\ &\geq -2(1+p)\nu n \geq -4cp\nu n \ ,\end{aligned}$$

where we used that $1/c \leq p$ in the last line. Since A and B_1 are disjoint, we deduce that

$$\sum_{k\in A\cup B_1} d'_k f'_k \geq -pc^2 - 8c\nu pn \ .$$

Let $T_2 \subseteq T$ be such that

$$\sum_{k\in T_2} d'_k f'_k > pc^2 + 8c\nu pn \ ,$$

and $\max\{\sigma_k : k \in T_2\}$ is minimized (choose the set of consecutive vertices in T smallest with respect to the order $\sigma_1, \ldots, \sigma_n$). As $\Delta(G) \leq \mu n/(40p)$, we conclude that $pc^2 \leq c\mu n/40$. Since $8c\nu pn \leq c\mu n/40$, this implies that $T_2 \subseteq T_1$. Therefore, by using (20), we obtain

$$\sum_{k\in T\setminus T_2} f'_k \geq \frac{\mu n}{4} \ . \tag{20}$$

Since $A \cup B_1$ and T_2 are disjoint, we conclude that

$$\sum_{k\in A\cup B_1\cup T_2} d'_k f'_k > 0 \ . \tag{21}$$

The previous inequality suggests that the vertices in $T\setminus T_2$ might belong to B' and thus contribute to $R_p^{\mathcal{D}'}$. However, in the new ordering σ, there might be vertices in A with larger degree than some of the vertices in $T \setminus T_2$. Let $P := (T \setminus T_2) \cap A'$. If $\sum_{k\in P} f'_k \leq \frac{\mu n}{8}$, then (20) implies

$$R_p^{\mathcal{D}'} \geq \sum_{k\in(T\setminus T_2)\cap B'} f'_k \geq \frac{\mu n}{4} - \frac{\mu n}{8} = \frac{\mu n}{8} \geq \frac{\mu n'}{8} \ ,$$

and we are done.

Hence, we may assume $\sum_{k\in P} f'_k > \frac{\mu n}{8}$. Recall that, since $P \subseteq T$, we have $d'_k \geq d_k/2 \geq c/2$ for every $k \in P$ and hence, by (21),

$$\sum_{k\in A\cup B_1\cup T_2\cup P} d'_k f'_k > \frac{c\mu n}{16} \ . \tag{22}$$

Thus a significant amount of vertices in $A\cup B_1\cup T_2\cup P$ need to be in B'. Note that $B_1\cup P \subseteq A'$. Let $Q := (A \cup T_2) \cap B'$. By our choice of T_2, the degree of a vertex in T_2 in G' is at most the degree of a vertex in P in G'; that is, vertices in T_2 are smaller than vertices in P with respect to the ordering σ. Thus if a vertex of T_2 is contained in Q, then $P = \emptyset$ and this a contradiction to our assumption. Therefore, $Q = A \cap B'$ and hence $d'_k \leq d_k \leq c$ for each $k \in Q$. Using (22) we obtain,

$$R_p^{\mathcal{D}'} \geq \frac{1}{c}\sum_{k\in Q} d'_k f'_k \geq \frac{1}{c} \cdot \frac{c\mu}{16} n \geq \frac{\mu}{16} n \ .$$

This completes the proof. □

8. Proof of Proposition 9

Let $G^{\mathcal{D}}$ be a graph chosen according to the uniform distribution on $\mathcal{G}^{\mathcal{D}}$. The proof of Proposition 9 will consist of a careful analysis of an exploration of the different components of $G_p^{\mathcal{D}}$ and will heavily rely on the switching method. Our proofs are similar in spirit to those in [15], but the additional level of randomness that is due to bond percolation makes the arguments more involved and additional arguments are needed.

In order to bound the number of switches it is more convenient to denote by $d(u)$ the degree of a vertex $u \in V$, instead of using d_i for $i \in V$. We will use this notation in this and in the next section.

8.1. **Connection probabilities via the switching method.**

The following three technical lemmas provide the necessary tools needed for proving that the random exploration process follows closely to what we expect it to do. To prove these lemmas we make extensive use of the switching method and, in particular, of inequality (10).

Lemma 11. *Suppose $n \in \mathbb{N}$ and $1/n \ll 1$ and let $Z' \subseteq V$. Suppose H' is a graph with vertex set Z' and F' is a bipartite graph with vertex partition $(Z', V \setminus Z')$. Suppose $u \in Z'$ and $v \in V \setminus Z'$ such that $uv \notin E(F')$. Suppose $\mathcal{D}$ is a degree sequence on V such that*

- (E1) $d(u) \leq n^{1/4}$,
- (E2) *if $w \in V$ and $d(w) > n^{1/4}$, then $w \in Z'$ and $d(w) = d_{H'}(w)$, and*
- (E3) $\sum_{w \in V \setminus Z'} (d(w) - d_{F'}(w)) \geq n/20$.

Then,

$$\mathbb{P}[uv \in E(G^{\mathcal{D}}) \mid G^{\mathcal{D}}[Z'] = H',\, F' \subseteq G^{\mathcal{D}}] \leq 40n^{-1/2}.$$

Proof. Let $\mathcal{F}^+$ be the set of graphs with degree sequence $\mathcal{D}$ such that $G[Z'] = H'$, $F' \subseteq G$ and $uv \in E(G)$, and let $\mathcal{F}^-$ be the set of graphs with degree sequence $\mathcal{D}$ such that $G[Z'] = H'$, $F' \subseteq G$ and $uv \notin E(G)$. We will only perform switches that involve edges that are not contained in $E(H') \cup E(F')$. This ensures that the graph G_0 obtained from a switch also satisfies $G_0[Z'] = H'$ and $F' \subseteq G_0$. As this is the first proof that involves the switching method, we will provide an extra level of detail.

For every $G \in \mathcal{F}^+$, let $s^+(G)$ be the number of switches that transform G into a graph in $\mathcal{F}^-$. We seek for a lower bound on $s^+(G)$. Indeed, we will find many edges xy such that the $\{uv, xy\}$-switch leads to a graph in $\mathcal{F}^-$. For this, it suffices to select an edge xy such that xy is at distance at least 2 from uv, we have $xy \notin E(F')$, and $x \in V \setminus Z'$. By (E3), there are at least $n/20$ edges that have one endpoint in $V \setminus Z'$ and are not contained in $E(F')$. Therefore, it suffices to count how many of them lie at distance at most 1 from uv. Note that $d(u), d(v) \leq n^{1/4}$. Moreover, v has no neighbour with degree larger than $n^{1/4}$. While u can have neighbours $w \in Z'$ with degree larger than $n^{1/4}$, all the edges incident to w have both endpoints in Z' (by (E2)). It follows, that there are at most $2n^{1/2}$ edges at distance at most 1 from uv with at least one endpoint in $V \setminus Z'$. Note that for any such xy, the $\{uv, xy\}$-switch transforms G into a simple graph G_0 with degree sequence $\mathcal{D}$, $G_0[Z'] = H'$, $F' \subseteq G_0$, and $uv \notin E(G_0)$. Therefore,

$$s^+(G) \geq \frac{n}{20} - 2n^{1/2} \geq \frac{n}{30} .$$

For every $G \in \mathcal{F}^-$, let $s^-(G)$ be the number of switches that transform G into a graph in $\mathcal{F}^+$. We bound $s^-(G)$ from above. Clearly, any such switch is of the form $\{ux, vy\}$ for some $x, y \in V$. Since $d(u), d(v) \leq n^{1/4}$, there are at most $d(u)d(v) \leq n^{1/2}$ choices for the edges ux and vy. Therefore,

$$s^-(G) \leq n^{1/2} .$$

Using (10) we obtain

$$\begin{aligned}\mathbb{P}[uv \in E(G^{\mathcal{D}}) \mid G^{\mathcal{D}}[Z'] = H',\, F' \subseteq G^{\mathcal{D}}] &\le \frac{s^-(G)}{s^+(G)} \cdot \mathbb{P}[uv \notin E(G^{\mathcal{D}}) \mid G^{\mathcal{D}}[Z'] = H',\, F' \subseteq G^{\mathcal{D}}] \\ &\le \frac{30n^{1/2}}{n} \cdot \mathbb{P}[uv \notin E(G^{\mathcal{D}}) \mid G^{\mathcal{D}}[Z'] = H',\, F' \subseteq G^{\mathcal{D}}] \\ &\le 30n^{-1/2} .\end{aligned}$$

□

Lemma 12. *Suppose $n \in \mathbb{N}$ and $1/n \ll \nu \ll 1$. Suppose $Z' \subseteq V$. Suppose H' is a graph with vertex set Z', and F' is a bipartite graph with vertex partition $(Z', V \setminus Z')$. Suppose $x \in V \setminus Z'$ and $z \in Z'$. Suppose $\mathcal{D}$ is a degree sequence on V such that*

- *if $w \in V$ and $d(w) > n^{1/4}$, then $w \in Z'$ and $d(w) = d_{H'}(w)$,*
- *$\sum_{w\in Z'}(d(w) - d_{H'}(w) - d_{F'}(w)) \le \nu n$,*
- *$\sum_{w\in V\setminus Z'}(d(w) - d_{F'}(w)) \ge n/20$, and*
- *$xz \notin E(F')$,*

Then, for every $i \ge 0$ and $Z'' := Z' \setminus \{z\}$,

$$\mathbb{P}[d_{G^{\mathcal{D}}}(x, Z'') - d_{F'}(x) > \lfloor\sqrt{\nu}(d(x) - d_{F'}(x))\rfloor + i \mid G^{\mathcal{D}}[Z'] = H',\ F' \subseteq G^{\mathcal{D}},\ \mathcal{E}] \le (22\sqrt{\nu})^{i+1} ,$$

where $\mathcal{E} \in \{\{xz \in E(G^{\mathcal{D}})\}, \{xz \notin E(G^{\mathcal{D}})\}\}$. Therefore, by averaging, we also have

$$\mathbb{P}[d_{G^{\mathcal{D}}}(x, Z'') - d_{F'}(x) > \lfloor\sqrt{\nu}(d(x) - d_{F'}(x))\rfloor + i \mid G^{\mathcal{D}}[Z'] = H',\ F' \subseteq G^{\mathcal{D}}] \le (22\sqrt{\nu})^{i+1} .$$

Proof. Let $K := \lfloor\sqrt{\nu}(d(x) - d_{F'}(x))\rfloor$. For every $k \ge K$, let $\mathcal{F}_k = \mathcal{F}_k(\mathcal{E})$ be the set of graphs G with degree sequence $\mathcal{D}$ such that $G[Z'] = H'$, $F' \subseteq G$, $d_G(x, Z'') - d_{F'}(x) = k$, and $\mathcal{E}$ is satisfied. As before, we will only perform switches using edges that are not contained in $E(H') \cup E(F')$.

Consider a graph in $\mathcal{F}_k$. Then in any of the two possibilities for $\mathcal{E}$, there are at most $(d(x) - d_{F'}(x))\nu n$ switches that lead to a graph in $\mathcal{F}_{k+1}$.

For every graph in $\mathcal{F}_{k+1}$, arguing similar as in Lemma 11, there are at least $(k+1)(n/20 - \nu n - 2n^{1/2}) \ge (k+1)n/21$ switches that lead to a graph in $\mathcal{F}_k$. This is the number of pairs of edges where one element is among the $k+1$ edges between x and Z'' and which are not contained in $E(F')$, and the other element is among the edges with both endpoints in $V \setminus Z'$ (at least $n/20 - \nu n - 2n^{1/2}$) which are at distance at least 2 from the endpoints of the first element.

Thus, for $k \ge K$, we obtain

$$\mathbb{P}[\mathcal{F}_{k+1}] \le \frac{21(d(x) - d_{F'}(x))\nu n}{(k+1)n}\mathbb{P}[\mathcal{F}_k] \le 22\sqrt{\nu}\mathbb{P}[\mathcal{F}_k] ,$$

which implies that

$$\mathbb{P}[d_{G^{\mathcal{D}}}(x, Z'') - d_{F'}(x) > K + i \mid G^{\mathcal{D}}[Z'] = H',\ F' \subseteq G^{\mathcal{D}}, \mathcal{E}] \le (22\sqrt{\nu})^{i+1} .$$

□

Lemma 13. *Suppose $n \in \mathbb{N}$ and $1/n \ll \nu \ll 1$. Suppose $Z \subseteq V$. Suppose H is a graph with vertex set Z and F is a bipartite graph with vertex partition $(Z, V \setminus Z)$. Suppose $z \in Z$ and $x \in V \setminus Z$ such that $xz \notin E(F)$. Suppose $\mathcal{D}$ is a degree sequence on V (write $\hat{d}(u) = d(u) - d_H(u) - d_F(u)$ for all $u \in V$) such that*

- *if $w \in V$ and $d(w) > n^{1/4}$, then $w \in Z$ and $d(w) = d_H(w)$,*
- *$\sum_{w\in Z} \hat{d}(w) \le \nu n$, and*
- *$M := \sum_{w\in V\setminus Z} \hat{d}(w) \ge n/10$.*

Then,

$$\mathbb{P}[xz \in E(G^{\mathcal{D}}) \mid G^{\mathcal{D}}[Z] = H,\, F \subseteq G^{\mathcal{D}}] = \frac{\hat{d}(x)\hat{d}(z)}{M}(1 \pm 25\sqrt{\nu}).$$

Proof. Let $\mathcal{F}^+_{xz}$ be the set of graphs G with degree sequence $\mathcal{D}$ such that $G[Z] = H$, $F \subseteq G$ and $xz \in E(G)$ and $\mathcal{F}^-_{xz}$ the set of graphs with degree sequence $\mathcal{D}$ such that $G[Z] = H$, $F \subseteq G$

but $xz \notin E(G)$. As before, we consider only switches using edges that are not contained in $E(H) \cup E(F)$.

First, note that if $\min\{\hat{d}(x), \hat{d}(z)\} = 0$, then the statement holds trivially. Therefore, we may assume that $\hat{d}(x), \hat{d}(z) \geq 1$. Suppose $G \in \mathcal{F}^-_{xz}$. Applying Lemma 12 with $Z' = Z$, $H' = H$, $F' = F$ and $i = 0$, we deduce that

$$\mathbb{P}[d_G(x, Z) - d_F(x) \geq \sqrt{\nu}\hat{d}(x) \mid G[Z] = H, F \subseteq G, xz \notin E(G)] \leq 22\sqrt{\nu}\ .$$

Let $\hat{\mathcal{F}}^-_{xz}$ denote the subset of $\mathcal{F}^-_{xz}$ where $d_G(x, Z) < d_F(x) + \sqrt{\nu}\hat{d}(x)$ holds. Then the above implies that

$$|\hat{\mathcal{F}}^-_{xz}| \geq (1 - 22\sqrt{\nu})|\mathcal{F}^-_{xz}|. \tag{23}$$

In other words, for at least $(1-22\sqrt{\nu})|\mathcal{F}^-_{xz}|$ of the graphs in $\mathcal{F}^-_{xz}$, the vertex x has at most $\sqrt{\nu}\hat{d}(x)$ neighbours $z' \in Z \setminus \{z\}$ with $xz' \notin E(F)$.

Since $\hat{d}(z) \geq 1$, the vertex z has at least one neighbour $V \setminus Z$ through an edge not in $E(F)$. We now partition the set $\hat{\mathcal{F}}^-_{xz}$ into sets according to the neighbours of z in $V \setminus Z$ and the neighbours of x in Z (through edges that do not belong to $E(F)$). We will use $\bar{y}$ to denote sets of vertices in $\{y_1, \ldots, y_r\} \subseteq V \setminus (Z \cup \{x\})$ and $\bar{z}$ to denote sets of vertices in $\{z_1, \ldots, z_m\} \subseteq Z \setminus \{z\}$. We define $\hat{\mathcal{F}}^-_{xz}(\bar{y}, \bar{z})$ to be the subset of graphs in $\hat{\mathcal{F}}^-_{xz}$ such that the vertices in $\bar{y}$ are the neighbours of z in $V \setminus Z$ and the vertices in $\bar{z}$ are the neighbours of x in Z. In both cases, we only consider the neighbours that are connected to either z or x by an edge not in $E(F)$.

Thus, $\hat{\mathcal{F}}^-_{xz}$ is the disjoint union of all subsets $\hat{\mathcal{F}}^-_{xz}(\bar{y}, \bar{z})$, ranging over all $\bar{y}$ and $\bar{z}$ as specified above; that is, in particular, $|\bar{y}| = \hat{d}(z)$ and $|\bar{z}| \leq \sqrt{\nu}\hat{d}(x)$. We will now use Lemma 11 to show that for most members of $\hat{\mathcal{F}}^-_{xz}(\bar{y}, \bar{z})$, the vertex x is not adjacent to any vertex in $\bar{y}$.

To apply Lemma 11, we set $Z' := Z \cup \{x\}$, $V(H') = Z'$, and $E(H')$ consists of $E(H)$, the edges that join x and $\bar{z}$, and the edges in F that are incident to x. The graph F' is the bipartite graph with vertex set $(Z', V \setminus Z')$ and edge set $E(F) \setminus \{x\hat{z} : \hat{z} \in Z\}$. Also observe that (E1), (E2), and (E3) are satisfied; in particular, $\sum_{w \in V \setminus Z'}(d(w) - d_{F'}(w)) \geq M - d(x) \geq n/20$ holds. Let $\hat{\mathcal{F}}^{--}_{xz}(\bar{y}, \bar{z})$ be the subset of $\hat{\mathcal{F}}^-_{xz}(\bar{y}, \bar{z})$ in which x is not adjacent to a vertex in $\bar{y}$. Since $xy \notin E(F')$ for each $y \in \bar{y}$, Lemma 11 implies that

$$|\hat{\mathcal{F}}^{--}_{xz}(\bar{y}, \bar{z})| \geq (1 - 30n^{-1/2}n^{1/4})|\hat{\mathcal{F}}^-_{xz}(\bar{y}, \bar{z})| = (1 - 30n^{-1/4})|\hat{\mathcal{F}}^-_{xz}(\bar{y}, \bar{z})|, \tag{24}$$

because $\bar{y}$ contains at most $\hat{d}(z) \leq n^{1/4}$ vertices.

Next, we partition the set $\hat{\mathcal{F}}^{--}_{xz}(\bar{y}, \bar{z})$ according to the neighbours of x in $V \setminus Z$. We will use $\bar{w}$ to denote the set of neighbours of x in $V \setminus Z$. Thus $\bar{w}$ does not contain any member of $\bar{y} \cup \{x\}$ and $(1 - \sqrt{\nu})\hat{d}(x) \leq |\bar{w}| \leq \hat{d}(x)$. For such a $\bar{w}$, we let $\hat{\mathcal{F}}^{--}_{xz}(\bar{y}, \bar{z}, \bar{w})$ be the subset of $\hat{\mathcal{F}}^{--}_{xz}(\bar{y}, \bar{z})$ where $\bar{w}$ are the neighbours of x in $V \setminus Z$.

Assume now that $\bar{y} = \{y_1, \ldots, y_r\}$ and $\bar{w} = \{w_1, \ldots, w_\ell\}$, with $r = \hat{d}(z)$ and $(1 - \sqrt{\nu})\hat{d}(x) \leq \ell \leq \hat{d}(x)$. We fix some $i \in [r]$ and $j \in [\ell]$. An straightforward switching argument as for example performed in Lemma 11 shows that for at least $(1-n^{-1/10})|\hat{\mathcal{F}}^{--}_{xz}(\bar{y}, \bar{z}, \bar{w})|$ graphs in $\hat{\mathcal{F}}^{--}_{xz}(\bar{y}, \bar{z}, \bar{w})$, the edge $y_i w_j$ is not present. In this case, we apply the switch $\{zy_i, xw_j\}$. Thus, in total, the number of switches from graphs in $\hat{\mathcal{F}}^{--}_{xz}(\bar{y}, \bar{z}, \bar{w})$ to graphs in $\mathcal{F}^+_{xz}$ is at least

$$(1 - n^{-1/10})\ell r|\hat{\mathcal{F}}^{--}_{xz}(\bar{y}, \bar{z}, \bar{w})| \geq (1 - n^{-1/10})(1 - \sqrt{\nu})\hat{d}(x)\hat{d}(z)|\hat{\mathcal{F}}^{--}_{xz}(\bar{y}, \bar{z}, \bar{w})|.$$

Hence the number of switches from graphs in $\hat{\mathcal{F}}^-_{xz}(\bar{y}, \bar{z})$ to graphs in $\mathcal{F}^+_{xz}$ is at least

$$(1 - n^{-1/10})(1 - \sqrt{\nu})\hat{d}(x)\hat{d}(z)|\hat{\mathcal{F}}^{--}_{xz}(\bar{y}, \bar{z})| \overset{(24)}{\geq} (1 - 2\sqrt{\nu})\hat{d}(x)\hat{d}(z)|\hat{\mathcal{F}}^-_{xz}(\bar{y}, \bar{z})|.$$

This in turn implies that the number of switches from graphs in $\mathcal{F}^-_{xz}$ to graphs in $\mathcal{F}^+_{xz}$ is at least

$$(1 - 2\sqrt{\nu})\hat{d}(x)\hat{d}(z)|\hat{\mathcal{F}}^-_{xz}| \overset{(23)}{\geq} (1 - 24\sqrt{\nu})\hat{d}(x)\hat{d}(z)|\mathcal{F}^-_{xz}|.$$

Furthermore, since the edges of F are not involved in such switches, there are at most $\hat{d}(x)\hat{d}(z)$ switches transforming a graph in $\mathcal{F}^-_{xz}$ into a graph in $\mathcal{F}^+_{xz}$.

Consider now a graph in $\mathcal{F}_{xz}^+$. Any switch that transforms it into a graph in $\mathcal{F}_{xz}^-$ must use the edge xz. It suffices to bound the number of choices for the other edge. On the one hand, it is easy to see that there are at most M switches leading to a graph in $\mathcal{F}_{xz}^-$. On the other hand, since $d(x), d(z) \leq n^{1/4}$, there are at least $M - \nu n - 2n^{1/2}$ edges in distance at least 2 from xz which belong to $G[V \setminus Z]$. Thus there are at least $M - 2\nu n$ switches leading to a graph in $\mathcal{F}_{xz}^-$.

Combining all four bounds, leads to the desired statement. □

8.2. **The exploration process.** In order to bound the order of the largest component in $G_p^{\mathcal{D}}$ we will perform an exploration process on $G^{\mathcal{D}}$ that reveals the components of $G_p^{\mathcal{D}}$. An *input* is a pair $(G, \mathfrak{S})$ with the following properties. For a given degree sequence $\mathcal{D}$ on the vertex set V, we let G be a graph on V with degree sequence $\mathcal{D}$ and for every vertex v, we arbitrarily assign the labels $1, \ldots, d(v)$ to its incident edges. In this way, each edge obtains two labels. Since each label is associated with one of the endpoints of the corresponding edge, it is convenient to understand this labelling as a labelling of the *semi-edges* of the graph in such a way that the semi-edges incident to v are given the labels $1, \ldots, d(v)$. Thus, during the exploration process, G is equipped with an arbitrary labelling of the semi-edges incident to each vertex. The semi-edge labelling fits well with the switching method: if G' is obtained from G by switching two edges, then the semi-edges of G' naturally inherit the labelling on the semi-edges of G. The set $\mathfrak{S} = \{\sigma_v : v \in V\}$ is a collection of permutations, one for each vertex $v \in V$, where σ_v is a permutation of length $d(v)$. For technical reasons that will become apparent soon, we will need to consider the exploration process on an input. The labelling on the semi-edges together with $\mathfrak{S}$, will determine the order in which the vertices are explored during the process.

Given an input $(G, \mathfrak{S})$ and a subset of vertices $S_0 \subseteq V$, we proceed to describe the exploration of G from S_0. First, for every vertex in $v \in V$, we permute the labels of its incident semi-edges according to σ_v. Observe that a uniformly selected set of permutations $\mathfrak{S}$ leads to a uniformly selected labelling of the semi-edges incident to each vertex of G. First, we expose the graph $G[S_0]$. For every $t \geq 0$, let S_t be the set of vertices that have been explored up to time t, let $H_t := G[S_t]$ and let F_t be the bipartite subgraph with vertex partition $(S_t, V \setminus S_t)$ that contains those edges of $E(G)$ that have been exposed but have not survived the random deletion – we will be referring to these edges as the edges that have *failed to percolate.* For a vertex $u \in V$, we define its *free degree at time t* as

$$\hat{d}_t(u) := d(u) - d_{H_t}(u) - d_{F_t}(u) \ .$$

We may assume that V has some fixed ordering. If at time t there exists at least one vertex $v \in S_t$ with $\hat{d}_t(v) \geq 1$, we select[1] the smallest vertex $v_{t+1} \in S_t$ such that $\hat{d}_t(v_{t+1}) \geq 1$. Let w_{t+1} be the vertex $w \in V \setminus S_t$ with $v_{t+1}w \in E(G) \setminus E(F_t)$ that minimizes $\sigma_{v_{t+1}}(\ell(w))$, where $\ell(w)$ is the label of the semi-edge incident to v_{t+1} that corresponds to $v_{t+1}w$. After that, with probability p, we retain the edge $v_{t+1}w_{t+1}$ in G_p. If the edge survives percolation, we proceed as follows:

1. we set $S_{t+1} := S_t \cup \{w_{t+1}\}$;
2. we expose all the edges (*back edges*) from w_{t+1} to $S_t \setminus \{v_{t+1}\}$ that are not in F_t; we define the *backward degree*[2] of w_{t+1} as
$$d'_t(w_{t+1}) := d(w_{t+1}, S_t \setminus \{v_{t+1}\}) - d_{F_t}(w_{t+1}) \ ;$$
3. we retain each of the back edges in G_p independently with probability p; and
4. we define $H_{t+1} := G[S_{t+1}]$ and let F_{t+1} be the bipartite subgraph with vertex partition $(S_{t+1}, V \setminus S_{t+1})$ that contains all the edges between S_{t+1} and $V \setminus S_{t+1}$ that have failed to percolate so far.

If $v_{t+1}w_{t+1}$ fails to percolate, we set $S_{t+1} := S_t$, $H_{t+1} := H_t$, $V(F_{t+1}) := V(F_t) \cup \{w_{t+1}\}$, and $E(F_{t+1}) := E(F_t) \cup \{v_{t+1}w_{t+1}\}$.

[1]To be precise, the selection of a new vertex and the updates of the considered parameters happen between time t and $t + 1$.

[2]Note that the backward degree does not include the contribution of v_{t+1}.

Finally, if there is no $v \in S_t$ with $\hat{d}_t(v) \geq 1$, we let $w_{t+1} = u$, where $u \in V \setminus S_t$ is chosen with probability proportional to $\hat{d}_t(u)$, and we set $S_{t+1} := S_t \cup \{w_{t+1}\}$, $H_{t+1} := G[S_{t+1}]$ and $F_{t+1} := F_t$. This marks the beginning of a new component.

Note that at time t we have explored at most t new vertices and that $G[S_t]$ is fully exposed (as well as $G_p[S_t]$). Moreover, there is a set of edges $E(F_t)$ joining S_t and $V \setminus S_t$ that have also been exposed but failed to percolate.

Let $\mathcal{H}_t$ denote the history of the exploration process after t rounds (at time t). More precisely, this is the random object composed of the collection of all the choices that have been made in the exploration process up to time t, and include the choice of S_t, $H_t = G[S_t]$ and F_t. Observe that for a fixed input $(G, \mathfrak{S})$, the only randomness in this exploration process stems from the percolation process and the random selection of a new vertex if H_t is a union of components in the percolated graph.

The next two variables will be crucial to control our exploration process at time t:

- $M_t := \sum_{u \in V \setminus S_t} \hat{d}_t(u)$, which equals the number of ordered edges uv with $u \in V \setminus S_t$ and $uv \notin E(F_t)$.
- $X_t := \sum_{u \in S_t} \hat{d}_t(u)$, which equals the number of edges uv with $u \in S_t$, $v \in V \setminus S_t$ and $uv \notin E(F_t)$.

The variable X_t counts the number of edges that are suitable to be used in the step $t+1$ to continue the exploration process. If $X_t = 0$, then we have completed the exploration of a component of G_p.

In order to deduce Proposition 9, we will analyse the exploration procedure on the input $(G^{\mathcal{D}}, \mathfrak{S})$, where each permutation in $\mathfrak{S}$ is chosen uniformly at random among all permutations of the appropriate length. In order to show that the largest component in $G_p^{\mathcal{D}}$ is large or small, we will consider the evolution of the random process $\{X_t\}_{t \geq 0}$ conditional on its history $\mathcal{H}_t$, that is, the set of all decisions taken up to step t. More formally, $\mathcal{H}_t$ is the σ-algebra generated by all random decision taken up to step t. Note that now $\mathcal{H}_t$ does not only depend on the indicator random variables associated with whether the edges survive percolation, but also on the random graph $G^{\mathcal{D}}$. Using the method of the deferred decisions, we can generate each random permutation while we perform the exploration process. This ensures that, at step t, any choice of w_{t+1} satisfying the desired properties is equally possible (see Section 2.2 in [15] for a more details).

8.3. **The expected increase of X_t.** For every $uv \in E(G)$, let $I(uv)$ be the indicator random variable that the edge uv percolates.

If $X_t > 0$, then the increase of X_t can be written as

$$X_{t+1} - X_t = -(1 - I(v_{t+1}w_{t+1})) + I(v_{t+1}w_{t+1})((\hat{d}_t(w_{t+1}) - 2) - 2d'_t(w_{t+1})) \ , \tag{25}$$

and if $X_t = 0$, as

$$X_{t+1} - X_t = \hat{d}_t(w_{t+1}) \ . \tag{26}$$

The next three lemmas use Lemmas 12 and 13, (25), (26), and $\mathbb{E}(I(v_{t+1}w_{t+1})) = p$ to provide bounds on $\mathbb{E}[X_{t+1} - X_t \mid \mathcal{H}_t]$ assuming that t is small, M_t is large, and X_t is small and for the first lemma also positive.

Lemma 14. *Suppose $n \in \mathbb{N}$ and $1/n \ll \beta, \rho, \eta \ll \lambda \ll \mu, 1/\bar{d}, p \leq 1$. Let $S_0 \subseteq V$ and let $\mathcal{D}$ be a degree sequence on V such that $\Sigma^{\mathcal{D}} \leq \bar{d}n$ and $\sum_{u \in V \setminus S_0} d(u)(p(d(u)-1)-1) \leq -\mu n$.*

Consider the exploration process described above on $(G^{\mathcal{D}}, \mathfrak{S})$ with initial set S_0 and suppose $t \leq \rho n$. Conditional on $\mathcal{H}_t$ satisfying $d_{H_t}(w) = d(w)$ for every $w \in V$ with $d(w) > n^{1/4}$, $M_t \geq (1-\eta)\Sigma^{\mathcal{D}}$, and $0 < X_t \leq \beta n$, we have

$$\mathbb{E}[X_{t+1} - X_t \mid \mathcal{H}_t] \leq -\lambda.$$

Proof. At time t, there are at most t vertices $u \in V \setminus S_t$ such that $\hat{d}_t(u) = 0$. This is the case since $d(u) = \hat{d}_t(u) + d_{F_t}(u)$ for all $u \in V \setminus S_t$ and at each step $s \leq t$ there is at most one edge added to F_s. Observe also that the function $h(x) = x(x-2)$ is monotone increasing for $x \geq 1$

and $h(0) = h(2) = 0$, $h(1) = -1$. This implies that $\hat{d}_t(u)(\hat{d}_t(u)-2) > d(u)(d(u)-2)$ only if $d(u) = 1$ and $\hat{d}_t(u) = 0$. It follows that

$$\sum_{u \in V \setminus S_t} \hat{d}_t(u)(\hat{d}_t(u)-2) \leq t + \sum_{u \in V \setminus S_0} d(u)(d(u)-2) \,. \tag{27}$$

The fact that S_0 contains all the neighbours in $G^{\mathcal{D}}$ of vertices of degree larger than $n^{1/4}$ and that $S_0 \subseteq S_t$, ensures that for every $v \in S_t$ such that $\hat{d}_t(v) \geq 1$, we have $d(v) \leq n^{1/4}$. Choose $u \in V \setminus S_t$. Since $M_t \geq n/10$ and $X_t \leq \beta n$, and provided $v_{t+1}u \notin E(F_t)$, we can apply Lemma 13 with $\nu = \beta$, $Z = S_t$, $H = H_t$, $F = F_t$, $z = v_{t+1}$, and $x = u$ to conclude that

$$\mathbb{P}[v_{t+1}u \in E(G^{\mathcal{D}}) \mid \mathcal{H}_t] = \frac{\hat{d}_t(v_{t+1})\hat{d}_t(u)}{M_t}(1 \pm 25\sqrt{\beta}) \,.$$

Observe that every edge incident to v_{t+1} that is not contained in $E(F_t) \cup E(H_t)$ is chosen with the same probability to continue the exploration process. Thus the probability that u is the vertex w that minimizes $\sigma_{v_{t+1}}(\ell(w))$, where $\ell(w)$ is the label of the semi-edge incident to v_{t+1} and corresponding to $v_{t+1}w$, among all $w \in V \setminus S_t$ with $v_{t+1}w \in E(G^{\mathcal{D}}) \setminus E(F_t)$, is precisely $1/\hat{d}_t(v_{t+1})$. Therefore,

$$\mathbb{P}[u = w_{t+1} \mid \mathcal{H}_t] = \frac{\hat{d}_t(u)}{M_t}(1 \pm 25\sqrt{\beta}) \,.$$

Note that if $v_{t+1}u \in E(F_t)$, then $\mathbb{P}[u = w_{t+1} \mid \mathcal{H}_t] = 0$.

Let n_1 denote the number of vertices $v \in V \setminus S_t$ with $\hat{d}_t(v) = 1$ and let $A_t \subseteq V \setminus S_t$ denote the set of vertices u such that $v_{t+1}u \in E(F_t)$. Since $d(v_{t+1}) \leq n^{1/4}$, we have $|A_t| \leq n^{1/4}$. Also $\hat{d}_t(u) < d(u) \leq n^{1/4}$ for all $u \in A_t$. Therefore, $|\sum_{u \in A_t} \hat{d}_t(u)(\hat{d}_t(u)-2)| \leq n^{3/4}$.

Using (25) and the fact that an edge percolates independently from the underlying graph, we conclude that

$$\begin{aligned}
&\mathbb{E}[X_{t+1} - X_t \mid \mathcal{H}_t] \leq -(1-p) + p \sum_{u \in V \setminus S_t} \mathbb{P}[u = w_{t+1}](\hat{d}_t(u)-2) \\
\leq\quad & -(1-p) + \frac{p}{M_t}\left((1+25\sqrt{\beta}) \sum_{u \in V \setminus S_t : \hat{d}_t(u) \geq 2} \hat{d}_t(u)(\hat{d}_t(u)-2) - n_1(1-25\sqrt{\beta}) + 2n^{3/4}\right) \\
\leq\quad & -(1-p) + (1+25\sqrt{\beta})\frac{p}{M_t}\left(\sum_{u \in V \setminus S_t} \hat{d}_t(u)(\hat{d}_t(u)-2)\right) + 100\sqrt{\beta} + \frac{2pn^{3/4}}{M_t} \\
\overset{(27),\beta,\eta \ll 1}{\leq}\quad & -(1-p) + (1+25\sqrt{\beta})\frac{p}{M_t}\left(\sum_{u \in V \setminus S_0} d(u)(d(u)-2)\right) + 2\rho + 101\sqrt{\beta}.
\end{aligned}$$

Now, we write

$$\begin{aligned}
&\frac{p}{M_t}\left(\sum_{u \in V \setminus S_0} d(u)(d(u)-2)\right) = \frac{p}{M_t}\left(\sum_{u \in V \setminus S_0} d(u)(d(u)-1) - \sum_{u \in V \setminus S_0} d(u)\right) \\
&= \frac{p}{M_t}\left(\sum_{u \in V \setminus S_0} d(u)(d(u)-1) - \sum_{u \in V \setminus S_0} d(u)\right) - \frac{1}{M_t}\sum_{u \in V \setminus S_0} d(u) + \frac{1}{M_t}\sum_{u \in V \setminus S_0} d(u) \\
&= \frac{1-p}{M_t}\sum_{u \in V \setminus S_0} d(u) + \frac{1}{M_t}\left(\sum_{u \in V \setminus S_0} d(u)(p(d(u)-1)-1)\right) \,.
\end{aligned} \tag{28}$$

Hence,

$$\mathbb{E}[X_{t+1} - X_t \mid \mathcal{H}_t] \leq \frac{1-p}{M_t}\left(-M_t + (1+25\sqrt{\beta})\sum_{u\in V\setminus S_0} d(u)\right) + \frac{1+25\sqrt{\beta}}{M_t}\left(\sum_{u\in V\setminus S_0} d(u)(p(d(u)-1)-1)\right) + 2\rho + 101\sqrt{\beta}.$$

But since $M_t \geq (1-\eta)\Sigma^{\mathcal{D}} \geq (1-\eta)\sum_{v\in V\setminus S_0} d(v)$, we have

$$\frac{1-p}{M_t}\left(-M_t + (1+25\sqrt{\beta})\sum_{u\in V\setminus S_0} d(u)\right) \leq \frac{1-p}{1-\eta}\left(-(1-\eta)+1+25\sqrt{\beta}\right) \leq \eta + \beta^{1/3},$$

where the previous inequality follows as $\eta \leq p$ and $\beta \ll 1$.

Thereby, using that $\beta, \rho, \eta \ll \lambda \ll \mu, 1/\bar{d}$,

$$\begin{aligned}\mathbb{E}[X_{t+1} - X_t \mid \mathcal{H}_t] &\leq \frac{1+25\sqrt{\beta}}{M_t}\left(\sum_{u\in V\setminus S_0} d(u)(p(d(u)-1)-1)\right) + 2\rho + 101\sqrt{\beta} + \eta + \beta^{1/3}\\ &\leq -\frac{\mu}{\bar{d}} + 2\rho + 101\sqrt{\beta} + \eta + \beta^{1/3}\\ &\leq -\lambda\,,\end{aligned}$$

which completes the proof. □

For the following two lemmas we do not require the condition $X_t > 0$.

Lemma 15. *Suppose $n \in \mathbb{N}$ and $1/n \ll \beta, \rho, \eta \ll \lambda \ll \mu, 1/\bar{d}, p \leq 1$. Let $S_0 \subseteq V$ and let $\mathcal{D}$ be a degree sequence on V such that $\Sigma^{\mathcal{D}} \leq \bar{d}n$ and $R_p^{\mathcal{D}} \geq \mu n$.*

Consider the exploration process described above on $(G^{\mathcal{D}}, \mathfrak{S})$ with initial set S_0 and suppose $t \leq \rho n$. Conditional on $\mathcal{H}_t$ satisfying $d_{H_t}(w) = d(w)$ for every $w \in V$ with $d(w) > n^{1/4}$, $M_t \geq (1-\eta)\Sigma^{\mathcal{D}}$ and $X_t \leq \beta n$, we have

$$\mathbb{E}[\hat{d}_t(w_{t+1}) - 2 \mid \mathcal{H}_t] \geq \frac{2\lambda + 1 - p}{p}\,.$$

Proof. We will first provide a lower bound on $\sum_{u\in V\setminus S_t} \hat{d}_t(u)(p(\hat{d}_t(u)-1)-1)$. Consider a realisation of the degree sequence of $G^{\mathcal{D}}[V \setminus S_t]$ which satisfies the conditions on $\mathcal{H}_t$, which we denote by $\mathcal{D}'_t = (d'_1, \ldots, d'_{n'})$ with $d'_1 \leq \cdots \leq d_{n'}$ and $n' = |\mathcal{D}'_t|$. Since $M_t \geq (1-\eta)\Sigma^{\mathcal{D}}$, we have that $\sum_{v\in S_t} d(v) \leq \eta\bar{d}n$. By Proposition 8, with $S = S_t$ and $\nu = \eta\bar{d}$, and since $R_p^{\mathcal{D}} \geq \mu n$ (observe that $\nu \ll \mu$), we have $R_p^{\mathcal{D}'_t} \geq \frac{\mu n}{50}$. (At this point we want to stress that the previous bound is not a with-high-probability statement; it holds for every possible realisation of $\mathcal{D}'_t$.) Recall that for

$j \geq j^p_{\mathcal{D}'_t}$, we have $p(d'_j - 1) - 1 > 0$. It follows that

$$\begin{aligned}
&\sum_{u \in V \setminus S_t} \hat{d}_t(u)(p(\hat{d}_t(u) - 1) - 1) \\
\geq\ & \sum_{u \in V \setminus S_t} d_{G^{\mathcal{D}}[V \setminus S_t]}(u)(p(d_{G^{\mathcal{D}}[V \setminus S_t]}(u) - 1) - 1) - X_t \\
\geq\ & \sum_{j=1}^{j^p_{\mathcal{D}'_t}} d'_j(p(d'_j - 1) - 1) + \sum_{j=j^p_{\mathcal{D}'_t}}^{n'} d'_j(p(d'_j - 1) - 1) - d_{j^p_{\mathcal{D}'_t}}(p(d_{j^p_{\mathcal{D}'_t}} - 1) - 1) - \beta n \\
\geq\ & \sum_{j=j^p_{\mathcal{D}'_t}}^{n'} d'_j(p(d'_j - 1) - 1) - 2\beta n \\
\geq\ & \sum_{j=j^p_{\mathcal{D}'_t}}^{n} (p(d'_j - 1) - 1) - 2\beta n \\
=\ & R_p^{\mathcal{D}'_t} - 2\beta n \overset{\beta \ll \mu}{\geq} \frac{\mu n}{60}. \qquad (29)
\end{aligned}$$

Let n_1 denote the number of vertices $v \in V \setminus S_t$ with $\hat{d}_t(v) = 1$ and let $A_t \subseteq V \setminus S_t$ denote the set of vertices u such that $v_{t+1}u \in E(F_t)$. As in Lemma 14, we have $|\sum_{u \in A_t} \hat{d}_t(u)(\hat{d}_t(u) - 2)| \leq n^{3/4}$.

If $X_t > 0$ holds[3], we can use Lemma 13 to show that

$$\begin{aligned}
p\mathbb{E}[\hat{d}_t(w_{t+1}) - 2 \mid \mathcal{H}_t] &= p \sum_{u \in V \setminus S_t} \mathbb{P}[w_{t+1} = u \mid \mathcal{H}_t](\hat{d}_t(u) - 2) \\
&\geq \frac{p}{M_t}\left((1 - 25\sqrt{\beta}) \sum_{u \in V \setminus S_t : \hat{d}(u) \geq 2} \hat{d}_t(u)(\hat{d}_t(u) - 2) - n_1(1 + 25\sqrt{\beta}) - n^{3/4}\right) \\
&\geq (1 - 25\sqrt{\beta})\frac{p}{M_t}\left(\sum_{u \in V \setminus S_t} \hat{d}_t(u)(\hat{d}_t(u) - 2)\right) - 101\sqrt{\beta}\ .
\end{aligned}$$

Therefore, using a similar calculation as in (28), we conclude

$$\begin{aligned}
-(1 - p) + p\mathbb{E}[\hat{d}_t(w_{t+1}) - 2 \mid \mathcal{H}_t] \geq\ & \frac{1 - p}{M_t}\left(-M_t + (1 - 25\sqrt{\beta}) \sum_{u \in V \setminus S_t} \hat{d}_t(u)\right) \\
& + \frac{1 - 25\sqrt{\beta}}{M_t}\left(\sum_{u \in V \setminus S_t} \hat{d}_t(u)(p(\hat{d}_t(u) - 1) - 1)\right) - 101\sqrt{\beta}. \qquad (30)
\end{aligned}$$

Note that $\sum_{u \in V \setminus S_t} \hat{d}_t(u) \geq M_t - t \geq (1 - \rho)M_t$. Similarly as before,

$$\begin{aligned}
\frac{1 - p}{M_t}\left(-M_t + (1 - 25\sqrt{\beta}) \sum_{u \in V \setminus S_t} \hat{d}_t(u)\right) &\geq (1 - p)\left(-1 + (1 - 25\sqrt{\beta})(1 - \rho)\right) \\
&\geq -(\rho + 25\sqrt{\beta}). \qquad (31)
\end{aligned}$$

Using (29), (30), (31), and that $\beta, \rho \ll \lambda \ll \mu, 1/\bar{d}$, we conclude the proof of the lemma as

$$-(1 - p) + p\mathbb{E}[\hat{d}_t(w_{t+1}) - 2 \mid \mathcal{H}_t] \geq \frac{(1 - 25\sqrt{\beta})\mu}{60\bar{d}} - \rho - 126\sqrt{\beta} \geq 2\lambda\ .$$

□

[3] Observe that this calculation is also correct if $X_t = 0$.

Lemma 16. *Suppose $n \in \mathbb{N}$ and $1/n \ll \beta, \rho, \eta \ll \lambda \ll \mu, 1/\bar{d}, p \leq 1$. Suppose $S_0 \subseteq V$ and let $\mathcal{D}$ be a degree sequence on V such that $\Sigma^{\mathcal{D}} \leq \bar{d}n$ and $R_p^{\mathcal{D}} \geq \mu n$.*

Consider the exploration process described above on $(G^{\mathcal{D}}, \mathfrak{S})$ with initial set S_0 and suppose $t \leq \rho n$. Conditional on $\mathcal{H}_t$ satisfying $d_{H_t}(w) = d(w)$ for every $w \in V$ with $d(w) > n^{1/4}$, $M_t \geq (1-\eta)\Sigma^{\mathcal{D}}$ and $X_t \leq \beta n$, we have

$$\mathbb{E}[d'_t(w_{t+1}) \mid \mathcal{H}_t] \leq \frac{1}{10}\mathbb{E}[\hat{d}_t(w_{t+1}) - 2 \mid \mathcal{H}_t],$$

and

$$\mathbb{E}[X_{t+1} - X_t \mid \mathcal{H}_t] \geq \lambda\ .$$

Proof. Suppose first that $X_t = 0$. Recall that in this case, we start the exploration of a new component and we select a vertex in $V \setminus S_t$ with probability proportional to its free degree. As this is at least 1, we deduce $\mathbb{E}[X_{t+1} - X_t \mid \mathcal{H}_t] \geq p \geq \lambda$. By Lemma 15, we have $\mathbb{E}[\hat{d}_t(w_{t+1}) - 2 \mid \mathcal{H}_t] > 0$. As $d'_t(w_{t+1}) = 0$, the first bound also follows.

Suppose now that $X_t > 0$. In order to bound the expectation of $d'(w_{t+1})$ it is clear from Lemma 12 with $\nu = \beta$, $Z = S_t$, $H = H_t$, $F' = F_t$, $x = w_{t+1}$, $z = v_{t+1}$ and $\mathcal{E} = \{xz \in E(G^{\mathcal{D}})\}$ that

$$\begin{aligned}
\mathbb{E}[d'_t(w_{t+1}) \mid \mathcal{H}_t] &\leq 2\sqrt{\beta}\mathbb{E}[\hat{d}_t(w_{t+1}) \mid \mathcal{H}_t] \\
&= 2\sqrt{\beta}\mathbb{E}[\hat{d}_t(w_{t+1}) - 2 \mid \mathcal{H}_t] + 4\sqrt{\beta} \\
&\leq \frac{1}{10}\mathbb{E}[\hat{d}_t(w_{t+1}) - 2 \mid \mathcal{H}_t]\ ,
\end{aligned} \tag{32}$$

where the previous inequality follows from the fact that $\mathbb{E}[\hat{d}(w_{t+1}) - 2 \mid \mathcal{H}_t] \geq 2\lambda$ by Lemma 15 and $\beta \ll \lambda$.

Using (25), (32), Lemma 15 and $\beta \ll \lambda$, we obtain

$$\begin{aligned}
\mathbb{E}[X_{t+1} - X_t \mid \mathcal{H}_t] &= -(1-p) + p(\mathbb{E}[\hat{d}_t(w_{t+1}) - 2 \mid \mathcal{H}_t] - 2\mathbb{E}[d'_t(w_{t+1}) \mid \mathcal{H}_t]) \\
&\geq -(1-p) + p \cdot (1 - 4\sqrt{\beta})\mathbb{E}[\hat{d}_t(w_{t+1}) - 2 \mid \mathcal{H}_t] - 8p\sqrt{\beta} \\
&\geq 2(1 - 4\sqrt{\beta})\lambda - (1-p)4\sqrt{\beta} - 8p\sqrt{\beta} \geq \lambda\ ,
\end{aligned}$$

which completes the proof. □

8.4. Another concentration inequality.

The following lemma will be used to show that several parameters of our process do not deviate much from their expected value.

Lemma 17. *Suppose $a < 0, b > 0$, $m \in \{0,1\}$, $t \in \mathbb{N}$, and $y \in [a, 0)$. Suppose τ is a stopping time with respect to a filtration $(\mathcal{F}_s)_{s=0}^t$. Suppose $Y_0, Y_1, \dots, Y_t$ are random variables such that Y_s is measurable at time s and $Y_s - Y_{s-1} \in [a, b]$. Suppose that for any $s \in [t]$, we have*

$$\mathbb{E}[\mathbf{1}_{\{s \leq \tau\}}(-1)^m(Y_s - Y_{s-1}) \mid \mathcal{F}_{s-1}] \leq y\mathbf{1}_{\{s \leq \tau\}}.$$

Then

$$\mathbb{P}\left[(-1)^m(Y_{\tau \wedge t} - Y_0) + \mathbf{1}_{\{t > \tau\}}(t - \tau)y > \frac{yt}{2}\right] < e^{-\frac{y^2}{12(b-a)^2} \cdot t}.$$

To this end, we shall use the following lemma which was proved in [25] and is a corollary of a martingale concentration theorem (Theorem 3.12) from [17].

Proposition 18. *Let $W_1, \dots, W_t$ be random variables taking values in $[0,1]$ such that*

$$\mathbb{E}[W_s \mid W_1, \dots, W_{s-1}] \leq w_s$$

for each $s \in [t]$. Let $\lambda_t := \sum_{s=1}^t w_s$. Then for any $0 < \delta \leq 1$, we have

$$\mathbb{P}\left[\sum_{s=1}^t W_s \geq (1+\delta)\lambda_t\right] \leq e^{-\frac{\delta^2 \lambda_t}{3}}.$$

Proof of Lemma 17. The assumption of the lemma implies that for every $s \in [t]$

$$\mathbb{E}[\mathbf{1}_{\{s\le\tau\}}(-1)^m(Y_s - Y_{s-1}) + y\mathbf{1}_{\{s>\tau\}} \mid \mathcal{F}_{s-1}] \le y.$$

We set $Z_s := \mathbf{1}_{\{s\le\tau\}}(-1)^m(Y_s - Y_{s-1}) + y\mathbf{1}_{\{s>\tau\}}$. The history $\mathcal{F}_{s-1}$ completely determines $Z_1, \dots, Z_{s-1}$, whereby

$$\mathbb{E}[Z_s \mid Z_1, \dots, Z_{s-1}] \le y. \tag{33}$$

We now rescale the variables Z_s to obtain random variables in $[0,1]$. To this end, we set

$$W_s := \frac{Z_s - a}{b-a}. \tag{34}$$

It follows directly from (33) that

$$\mathbb{E}[W_s \mid W_1, \dots, W_{s-1}] \le \frac{y-a}{b-a} =: w\ .$$

By Proposition 18 with $w_s := w$ for each $s \in [t]$ and $\lambda_t := wt$, for any $\delta \in (0,1]$, it follows that

$$\mathbb{P}\left[\sum_{s=1}^{t} W_s \ge (1+\delta)tw\right] < e^{-\delta^2 wt/3}\ .$$

Using the definition of W_s in (34), we obtain

$$\mathbb{P}\left[\sum_{s=1}^{t} Z_s \ge (1+\delta)(y-a)t + at\right] \le e^{-\frac{\delta^2 t(y-a)}{3(b-a)}}\ .$$

Recall that $a < y < 0$. Choosing $\delta = -\frac{y}{2(y-a)} \in (0,1]$, we have $(1+\delta)(y-a)t+at = yt+\delta(y-a)t = \frac{yt}{2}$. Using that $b-a \ge y-a$, we obtain

$$\mathbb{P}\left[\sum_{s=1}^{t} Z_s \ge \frac{ty}{2}\right] < e^{-\frac{y^2}{12(b-a)^2}\cdot t}\ .$$

Finally, note that

$$\begin{aligned}\sum_{s=1}^{t} Z_s &= \mathbf{1}_{\{t>\tau\}}((-1)^m(Y_\tau - Y_0) + (t-\tau)y) + \mathbf{1}_{\{t\le\tau\}}(-1)^m(Y_t - Y_0)\\ &= (-1)^m(Y_{\tau\wedge t} - Y_0) + \mathbf{1}_{\{t>\tau\}}(t-\tau)y\end{aligned}$$

and the lemma follows. □

8.5. **Proof of Proposition 9.** In this subsection we will prove the Proposition 9. We first need three more technical statements.

Lemma 19. *Suppose $n \in \mathbb{N}$ and $1/n \ll \alpha, \beta \ll \xi \ll \eta, \rho \ll \mu, 1/\bar{d}, p \le 1$. Suppose $S_0 \subseteq V$ and let $\mathcal{D}$ be a degree sequence on V such that $\Sigma^{\mathcal{D}} \le \bar{d}n$ and $w \in S_0$ for every $w \in V$ with $d(w) > n^{1/4}$ and $\sum_{v\in S_0} d(v) \le \alpha n$. Let τ be the smallest $t \le n$ such that either $X_t > \beta n$ or $M_t < (1-\eta)\Sigma^{\mathcal{D}}$ - if this does not exist, we set $\tau = n+1$. Conditional on the event that $d_{H_0}(w) = d(w)$ for every $w \in V$ with $d(w) > n^{1/4}$, then*

$$\mathbb{P}[\tau \le \xi n, X_\tau \le \beta n] = o(n^{-2}).$$

Proof. Observe that $\mathbb{P}[\tau \le \xi n, X_\tau \le \beta n, M_\tau \ge (1-\eta)\Sigma^{\mathcal{D}}] = 0$. Thus

$$\mathbb{P}[\tau \le \xi n, X_\tau \le \beta n] = \mathbb{P}[\tau \le \xi n, X_\tau \le \beta n, M_\tau < (1-\eta)\Sigma^{\mathcal{D}}]\ . \tag{35}$$

Note first that if $M_\tau < (1-\eta)\Sigma^{\mathcal{D}}$, then $\sum_{v\in S_\tau} d(v) \ge \eta\Sigma^{\mathcal{D}} \ge \eta n$. Let R_t be the set of times $s \in \{0,\dots,t\}$ where the edge $v_{s+1}w_{s+1}$ has percolated and let R'_t be the set of times where $s \in \{0,\dots,t\}$ where $X_s = 0$. Therefore, we have

$$\sum_{t\in R_\tau} (\hat{d}_t(w_{t+1}) + d_{F_t}(w_{t+1})) + \sum_{t\in R'_\tau} \hat{d}_t(w_{t+1}) \ge \sum_{v\in S_\tau} d(v) - \sum_{v\in S_0} d(v) \ge (\eta-\alpha)n\ . \tag{36}$$

At each step $s \in \{0, \dots, t\}$ of the process at most one edge is added to F_s. For every $1 \le t \le \tau$, it follows that

$$\sum_{s \in R_t} d_{F_s}(w_{s+1}) \le t + 1 \le \xi n + 1 \,.$$

Since $\alpha \ll \xi \ll \eta$, using (36) one concludes

$$\sum_{t \in R_\tau \cup R'_\tau} \hat{d}_t(w_{t+1}) \ge \left(\sum_{t \in R_\tau} \hat{d}_t(w_{t+1}) + d_{F_t}(w_{t+1}) \right) - \xi n - 1 + \sum_{t \in R'_\tau} \hat{d}_t(w_{t+1}) \ge \frac{\eta n}{2} \,. \tag{37}$$

From (25) and (26), and using again that $\xi \ll \eta, p$, it follows that

$$\begin{aligned} X_\tau &= X_0 - (\tau - |R_\tau|) + \sum_{t \in R_\tau} (\hat{d}_t(w_{t+1}) - 2 - 2d'_t(w_{t+1})) + \sum_{t \in R'_\tau} \hat{d}_t(w_{t+1}) \\ &\ge -3\tau + \frac{1}{2} \sum_{t \in R_\tau \cup R'_\tau} \hat{d}_t(w_{t+1}) + \frac{1}{2} \left(\sum_{t \in R_\tau} \hat{d}_t(w_{t+1}) - 4d'_t(w_{t+1}) \right) \\ &\overset{(37)}{\ge} \frac{\eta n}{4} - 3\xi n + \frac{1}{2} \left(\sum_{t \in R_\tau} \hat{d}_t(w_{t+1}) - 4d'_t(w_{t+1}) \right) \\ &\ge \frac{\eta n}{8} + \frac{1}{2} \left(\sum_{t \in R_\tau} \hat{d}_t(w_{t+1}) - 4d'_t(w_{t+1}) \right) \,. \end{aligned} \tag{38}$$

Therefore, we deduce that

$$\begin{aligned} &\mathbb{P}[\tau \le \xi n, X_\tau \le \beta n, M_\tau < (1-\eta)\Sigma^{\mathcal{D}}] \le \\ &\quad \mathbb{P}\left[\tau \le \xi n, X_\tau \le \beta n, X_\tau \ge \frac{\eta n}{8} + \frac{1}{2} \left(\sum_{t \in R_\tau} \hat{d}_t(w_{t+1}) - 4d'_t(w_{t+1}) \right) \right] . \end{aligned} \tag{39}$$

Observe that $\beta \ll \eta$. So in order that $X_\tau \le \beta n$, it suffices to prove that $-\sum_{t \in R_\tau} (\hat{d}_t(w_{t+1}) - 4d'_t(w_{t+1}))$ is not too large.

We define the following sequence $Y_1, \dots, Y_{\xi n}$ of random variables. Let $Y_0 := 0$. Suppose t is the s-th smallest entry in R_τ. We set

$$Y_s := Y_{s-1} - (\hat{d}_t(w_{t+1}) - 4d'_t(w_{t+1}));$$

in the case where $|R_\tau| < s$ and $s \le \xi n$, we set

$$Y_s := Y_{s-1} - 1.$$

Observe that $|Y_s - Y_{s-1}| \le 4n^{1/4}$. Let $\{\mathcal{F}_s\}_{s=0}^{\xi n}$ be the filtration induced by the sequence $\{Y_s\}_{s=0}^{\xi n}$.

Suppose again that t is the s-th smallest entry in R_τ. We apply Lemma 12 with $\nu = \beta$, $Z' = S_t$, $H' = G^{\mathcal{D}}[S_t]$, $F' = F_t$, $x = w_{t+1}$, $z = v_{t+1}$ and $\mathcal{E} = \{xz \in E(G^{\mathcal{D}})\}$. The first three conditions of the lemma are satisfied: the first one is immediate from our hypothesis, the second one follows from $X_t \le \beta n$ and the third one from the fact that $M_t - \sum_{w \in V \setminus S_t} d_{F_t}(w) \ge (1-\eta)\Sigma^{\mathcal{D}} - t \ge n/20$. Moreover, $xz \notin E(F')$ holds. Similarly as in (32), we obtain,

$$\mathbb{E}[d'(w_{t+1}) \mid \mathcal{H}_t] \le \frac{1}{10} \mathbb{E}[\hat{d}(w_{t+1}) \mid \mathcal{H}_t] \,.$$

Let t_{-1} be defined such that $t_{-1} - 1$ is the $(s-1)$-th smallest entry in R_τ or -1 if $s = 1$. Observe that given $\mathcal{H}_{t_{-1}}$ we still can apply Lemma 12 to any possible input with history $\mathcal{H}_t$. This implies that

$$\mathbb{E}[d'(w_{t+1}) \mid \mathcal{H}_{t_{-1}}] \le \frac{1}{10} \mathbb{E}[\hat{d}(w_{t+1}) \mid \mathcal{H}_{t_{-1}}] \,.$$

Clearly $\mathbb{E}[\hat{d}_t(w_{t+1}) | \mathcal{H}_{t_{-1}}] \ge 1$, so

$$\mathbb{E}[Y_{s+1} - Y_s | \mathcal{F}_s] \le -\frac{1}{2} \,.$$

We can apply Lemma 17 to the collection $Y_0, \dots Y_{\xi n}$ with $-a = b = 4n^{1/4}, y = -1/2, m = 0$ and $t = \xi n$, where τ is the stopping time $\tau = \xi n$, to conclude that

$$\mathbb{P}\left[-Y_{\xi n} < \frac{\xi n}{4}\right] = o(n^{-2}).$$

Observe that by construction of Y_s we have $\sum_{t \in R_{\tau \wedge \xi n}} \left(\hat{d}_t(w_{t+1}) - 4d'_t(w_{t+1})\right) \geq -Y_{\xi n} - \xi n$. Hence

$$\mathbb{P}\left[\sum_{t \in R_{\tau \wedge \xi n}} \left(\hat{d}_t(w_{t+1}) - 4d'_t(w_{t+1})\right) < -\frac{3\xi n}{4}\right] = o(n^{-2}) \; . \tag{40}$$

So by (39) we can further bound

$$\begin{aligned} &\mathbb{P}[\tau \leq \xi n, X_\tau \leq \beta n, M_\tau < (1-\eta)\Sigma^{\mathcal{D}}] \leq \\ &\qquad \mathbb{P}\left[\tau \leq \xi n, X_\tau \leq \beta n, X_\tau \geq \frac{\eta n}{8} - \frac{3\xi n}{8}\right] + \mathbb{P}\left[\sum_{t \in R_{\tau \wedge \xi n}} \left(\hat{d}_t(w_{t+1}) - 4d'_t(w_{t+1})\right) < -\frac{3\xi n}{4}\right] \\ &\overset{(40)}{\leq} \quad \mathbb{P}\left[\tau \leq \xi n, X_\tau \leq \beta n, X_\tau \geq \frac{\eta n}{8} - \frac{3\xi n}{8}\right] + o(n^{-2}). \end{aligned}$$

We will show that the first term is 0. Indeed,

$$X_\tau > \left(\frac{\eta}{8} - \frac{3\xi}{8}\right) n \geq \frac{\eta}{16} \cdot n > \beta n \; ;$$

so this cannot hold simultaneously with $X_\tau \leq \beta n$. Finally by (35), we conclude that the probability that $\tau \leq \xi$ and $X_\tau \leq \beta n$ is $o(n^{-2})$. ☐

Lemma 20. *Suppose $n \in \mathbb{N}$ and $1/n \ll \alpha \ll \beta \ll \eta, \rho \ll \mu, 1/\bar{d}, p \leq 1$. Let $S_0 \subseteq V$ and let $\mathcal{D}$ be a degree sequence on V such that $\Sigma^{\mathcal{D}} \leq \bar{d}n$ and $w \in S_0$ for every $w \in V$ with $d(w) > n^{1/4}$, and $\sum_{v \in S_0} d(v) \leq \alpha n$. Let τ be the smallest $t \leq n$ such that either $X_t > \beta n$ or $M_t < (1-\eta)\Sigma^{\mathcal{D}}$ - if this does not exist, we set $\tau = n+1$. Conditional on the event that $d_{H_0}(w) = d(w)$ for every $w \in V$ with $d(w) > n^{1/4}$, the following holds:*

(i) *If $\sum_{u \in V \setminus S_0} d(u)(p(d(u)-1)-1) \leq -\mu n$ and τ_1 is the smallest t such that $X_t = 0$, then the probability that $\tau_1 > \rho n$ is $o(1/n)$.*
(ii) *If $R_p^{\mathcal{D}} \geq \mu n$, then the probability that $\tau > \rho n$ or that $X_\tau \leq \beta n$ is $o(1/n)$.*

Proof. Recall that $I(uv)$ is the indicator random variable that is equal to 1 if and only if $uv \in E(G^{\mathcal{D}})$ survives percolation when it is exposed. Also, recall (25): if $X_t > 0$, then

$$X_{t+1} - X_t = -(1 - I(v_{t+1}w_{t+1})) + I(v_{t+1}w_{t+1})((\hat{d}_t(w_{t+1}) - 2) - 2d'_t(w_{t+1})) \; .$$

We first prove (i). Consider the sequence $Y_0, Y_1, \dots$ of random variables such that $Y_0 := X_0$ and

$$Y_s := Y_{s-1} + \mathbf{1}_{\{s \leq \tau \wedge \tau_1\}}(X_s - X_{s-1}) \, .$$

Thus $|Y_s - Y_{s-1}| \leq 2n^{1/4}$. Let λ be such that $\eta, \rho \ll \lambda \ll \mu, 1/\bar{d}, p$. Since $X_{s-1} > 0$ if $s \leq \tau \wedge \tau_1$, by Lemma 14, we have

$$\mathbb{E}[Y_s - Y_{s-1} | \mathcal{H}_{s-1}] \leq -\lambda \mathbf{1}_{\{s \leq \tau \wedge \tau_1\}} =: y\mathbf{1}_{\{s \leq \tau \wedge \tau_1\}} \; .$$

Let ν and ξ be such that $\alpha \ll \nu \ll \beta \ll \xi \ll \eta, \rho$. We now apply Lemma 17 to Y_s with $-a = b = 2n^{1/4}$, $m = 0$ and $t = cn$, for some c such that $1/n \ll c < 1$, to conclude that

$$\begin{aligned} \mathbb{P}\left[(X_{\tau \wedge \tau_1 \wedge cn} - X_0) - \mathbf{1}_{\{cn > \tau \wedge \tau_1\}}(cn - \tau \wedge \tau_1)\lambda \leq -\frac{\lambda}{2} \cdot cn\right] &\geq 1 - e^{-\frac{\lambda^2}{48} \cdot cn^{1/2}} \\ &\geq 1 - n^{-2}. \end{aligned} \tag{41}$$

Let $\mathcal{E}_1(t)$ denote the event $(X_{\tau\wedge\tau_1\wedge t} - X_0) - \mathbf{1}_{\{t>\tau\wedge\tau_1\}}(t-\tau\wedge\tau_1)\lambda \leq -\frac{\lambda}{2}\cdot t$. We use Lemma 19 (for the second inequality below) and we write

$$\begin{aligned}
\mathbb{P}[\tau_1 > \rho n] &= \mathbb{P}[\tau_1 > \rho n, \tau \leq \xi n] + \mathbb{P}[\tau_1 > \rho n, \tau > \xi n]\\
&\leq \mathbb{P}[\tau_1 > \rho n, \tau \leq \xi n, X_\tau > \beta n] + n^{-2} + \mathbb{P}[\tau_1 > \rho n, \tau > \xi n]\\
&\leq \mathbb{P}[\tau_1 > \rho n, \tau \leq \xi n, X_\tau > \beta n, \mathcal{E}_1(\nu n)] + \mathbb{P}[\overline{\mathcal{E}_1(\nu n)}] + n^{-2} + \mathbb{P}[\tau_1 > \rho n, \tau > \xi n]\\
&\overset{(41)}{=} \mathbb{P}[\tau_1 > \rho n, \tau \leq \xi n, X_\tau > \beta n, \mathcal{E}_1(\nu n)] + 2n^{-2} + \mathbb{P}[\tau_1 > \rho n, \tau > \xi n].
\end{aligned}$$

Suppose that the events $\tau_1 > \rho n$, $\frac{3\nu}{4}n \leq \tau \leq \xi n, X_\tau > \beta n$ and $\mathcal{E}_1(\nu n)$ are realised simultaneously. Recall that $\alpha \ll \nu \ll \beta \ll \xi \ll \rho \ll \lambda$. Since $\tau_1 > \rho n$, we have $X_{\tau\wedge\tau_1\wedge\nu n} > 0$. Then, we reach a contradiction in the following way

$$0 < X_{\tau\wedge\tau_1\wedge\nu n} \leq -\frac{\lambda}{2}\nu n + X_0 + \mathbf{1}_{\{\nu n > \tau\wedge\tau_1\}}(\nu n - \tau\wedge\tau_1)\lambda \leq -\frac{\lambda\nu}{8}n < 0\ .$$

Suppose that the events $\tau_1 > \rho n$, $\tau \leq \frac{3\nu}{4}n, X_\tau > \beta n$ and $\mathcal{E}_1(\nu n)$ are realised simultaneously. Again, we reach a contradiction as follows

$$\beta n < X_{\tau\wedge\tau_1\wedge\nu n} = X_\tau \leq -\frac{\lambda}{2}\nu n + X_0 + (\nu n - \tau)\lambda \leq 2\lambda\nu n < \beta n\ .$$

Hence, $\mathbb{P}[\tau_1 > \rho n, \tau \leq \xi n, X_\tau > \beta n, \mathcal{E}_1(\nu n)] = 0$.

Thereby,

$$\begin{aligned}
\mathbb{P}[\tau_1 > \rho n] &\leq \mathbb{P}[\tau_1 > \rho n, \tau > \xi n] + 2n^{-2}\\
&\leq \mathbb{P}[\tau_1 > \rho n, \tau > \xi n, \mathcal{E}_1(\xi n)] + \mathbb{P}[\overline{\mathcal{E}_1(\xi n)}] + 2n^{-2}\\
&\overset{(41)}{=} \mathbb{P}[\tau_1 > \rho n, \tau > \xi n, \mathcal{E}_1(\xi n)] + 3n^{-2}.
\end{aligned}$$

But again the event $\tau_1 > \rho n, \tau > \xi n$ cannot occur simultaneously with $\mathcal{E}_1(\xi n)$, since otherwise,

$$X_{\xi n} = X_{\tau\wedge\tau_1\wedge\xi n} \leq -\frac{\lambda}{2}\xi n + X_0 < 0\ .$$

We conclude that $\mathbb{P}[\tau_1 > \rho n] \leq 3n^{-2}$.

We proceed to prove (ii). Let $Y_0 := 0$. For $s \geq 1$, consider the random variable

$$Y_s := Y_{s-1} - \mathbf{1}_{\{s\leq\tau\}}\left(X_s - X_{s-1}\right)\ .$$

By the second part of Lemma 16,

$$\mathbb{E}[Y_s|\mathcal{H}_{s-1}] \leq -\lambda\mathbf{1}_{\{s\leq\tau\}} =: y\mathbf{1}_{\{s\leq\tau\}}\ .$$

Let ξ and λ be such that $\beta \ll \xi \ll \eta, \rho \ll \lambda \ll \mu, 1/\bar{d}, p$. Similarly as before, we can apply Lemma 17 to the random variables Y_s with $-a = b = 2n^{1/4}$, $m = 1$ and $t = \xi n$ to conclude that

$$\mathbb{P}\left[(X_{\tau\wedge\xi n} - X_0) + \mathbf{1}_{\{\xi n>\tau\}}(\xi n - \tau)\lambda > \frac{\lambda}{2}\cdot\xi n\right] \geq 1 - e^{-\frac{\lambda^2}{48}\cdot\xi n^{1/2}} = 1 - o(n^{-2}). \tag{42}$$

Now, let $\mathcal{E}_2(\xi n)$ denote the event $X_{\tau\wedge\xi n} - X_0 + \mathbf{1}_{\{\xi n>\tau\}}(\xi n - \tau)\lambda > \frac{\lambda}{2}\cdot\xi n$.

Since, $\xi \ll \rho$, we then have

$$\begin{aligned}
\mathbb{P}[\tau > \rho n] \leq \mathbb{P}[\tau > \xi n] &\leq \mathbb{P}[\tau > \xi n, \mathcal{E}_2(\xi n)] + \mathbb{P}[\overline{\mathcal{E}_2(\xi n)}]\\
&\overset{(42)}{\leq} \mathbb{P}[\tau > \xi n, \mathcal{E}_2(\xi n)] + n^{-2}\ .
\end{aligned}$$

But if $\tau > \xi n$ holds simultaneously with $\mathcal{E}_2(\xi n)$, then we have $X_{\xi n} = X_{\tau\wedge\xi n} > X_0 + \frac{\lambda\xi}{2}\cdot n \geq \beta n$ which contradicts that $\tau > \xi n$. So, this event has probability 0.

Similarly, using Lemma 19 (for the first inequality) we obtain

$$\begin{aligned}
\mathbb{P}[X_\tau < \beta n] &= \mathbb{P}[X_\tau < \beta n, \tau \leq \xi n] + \mathbb{P}[X_\tau < \beta n, \tau > \xi n]\\
&\leq n^{-2} + \mathbb{P}[X_\tau < \beta n, \tau > \xi n]\\
&\leq n^{-2} + \mathbb{P}[X_\tau < \beta n, \tau > \xi n, \mathcal{E}_2(\xi n)] + \mathbb{P}[\overline{\mathcal{E}_2(\xi n)}]\\
&\overset{(42)}{=} n^{-2} + \mathbb{P}[X_\tau < \beta n, \tau > \xi n, \mathcal{E}_2(\xi n)].
\end{aligned}$$

As above, $\tau > \xi n$ and $\mathcal{E}_2(\xi n)$ are incompatible. Therefore, the second event has probability 0, whereby we deduce that $\mathbb{P}[X_\tau < \beta n] = o(1/n)$. ☐

Lemma 21. *Let $1/n \ll \nu, \rho \ll 1/\bar{d}, p \leq 1$. Suppose that $S \subseteq V$ with $|S| \leq \rho n + 1$ and $U \subseteq S$. Suppose H is a graph with vertex set S and F is a bipartite graph with vertex partition $(S, V \setminus S)$ and $|E(F)| \leq \rho n$. Let $\mathcal{D}$ be a degree sequence on V such that $\Sigma^{\mathcal{D}} \leq \bar{d}n$ and, moreover, $d(w) = d_H(w)$ for every $w \in V$ with $d(w) \geq n^{1/4}$ and $\sum_{u\in U}(d(u) - d_H(u) - d_F(u)) \geq \nu n$.*

Conditional on $G^{\mathcal{D}}[S] = H$ and on $F \subseteq G^{\mathcal{D}}$ being the set of edges between S and $V \setminus S$ that have failed to percolate in $G_p^{\mathcal{D}}$, the probability that the union of components of $G_p^{\mathcal{D}}$ that intersect U contains at most $(\nu p/(20\bar{d}))n$ vertices is $o(1)$.

Proof. We may assume that $|U| < (\nu p/(20\bar{d}))n$. Let $\hat{G}^{\mathcal{D}} := G^{\mathcal{D}} - E(F)$. Our aim is to show that $N_{\hat{G}^{\mathcal{D}}}(U) \subseteq N_{G^{\mathcal{D}}}(U)$ is typically large. Note that for every vertex $w \in N_{\hat{G}^{\mathcal{D}}}(U)$, there is at least one edge $uw \in E(\hat{G}^{\mathcal{D}})$ with $u \in U$ that has not been exposed to percolation. In the second part of the proof, we will show that many of these edges are preserved in $G_p^{\mathcal{D}}$, implying that the union of components of $G_p^{\mathcal{D}}$ that intersect U contains many vertices.

Let $K := \lfloor(\nu/5\bar{d})n\rfloor$. For every $k < K$, let $\mathcal{F}_k$ be the set of graphs G with degree sequence $\mathcal{D}$ such that $G[S] = H$, $F \subseteq G$ and $|N_{\hat{G}}(U)| = k$, where $\hat{G} := G - E(F)$. In order to estimate the probability of each $\mathcal{F}_k$ we only use edges not contained in $E(H) \cup E(F)$ for a switch.

Consider a graph in $\mathcal{F}_k$. There are at least $\nu n - k \geq 4\nu n/5$ choices for an edge $uv \in E(\hat{G})$ with $u \in U$, $v \in N_{\hat{G}}(U)$ and such that there exists $u' \neq u$ with $u' \in U$ and $u'v \in E(\hat{G})$. Since $\delta(G) \geq 1$ and $d(w) \leq n^{1/4}$ for every $w \in \{u\} \cup N_{\hat{G}}(U)$, there are at least $(n - |S \cup N_{\hat{G}}(U)|)/2 - |E(F)| - 2n^{1/2} \geq n/3$ edges $xy \in E(\hat{G})$ with $x \notin S \cup N_{\hat{G}}(U)$ and which are in distance at least 2 from uv. Thus, the total number of such switches into a graph in $\mathcal{F}_{k+1}$ is at least $\nu n^2/4$.

Given a graph in $\mathcal{F}_{k+1}$, then there are at most $(k + 1)\bar{d}n$ switches that transform it into a graph in $\mathcal{F}_k$.

Thus, for every $k < K$, we have

$$\mathbb{P}[\mathcal{F}_k] \leq \frac{(k+1)\bar{d}n}{\nu n^2/4} \cdot \mathbb{P}[\mathcal{F}_{k+1}] \leq \frac{4}{5} \cdot \mathbb{P}[\mathcal{F}_{k+1}] \,,$$

which implies

$$\mathbb{P}[\cup_{k\leq K/2}\mathcal{F}_k] \leq \mathbb{P}[\mathcal{F}_{K/2}] \sum_{i\geq 0} (4/5)^{-i} \leq (4/5)^{-K/2+1}\mathbb{P}[\mathcal{F}_K] = o(1) \,.$$

That is, with probability $1 - o(1)$, there are at least $(\nu/10\bar{d})n$ vertices that are connected to U by at least one edge in $\hat{G}^{\mathcal{D}}$. These edges have still not been exposed for percolation. Chernoff's inequality (Lemma 6) now implies that with probability $1 - o(1)$ a proportion of at least $p/2$ of them will be retained in $G_p^{\mathcal{D}}$. Therefore, with probability $o(1)$, we have $|N_{G_p^{\mathcal{D}}}(U)| \leq (\nu p/(20\bar{d}))n$. The conclusion follows. ☐

Proof of Propostion 9. We start with the first statement. Suppose there exists a set $S \subseteq V$ such that $d(v) \leq n^{1/4}$ for every $v \notin S$, and for every possible choice of G with degree sequence $\mathcal{D}$, we have $\sum_{u\in N_G[S]} d(u) \leq \alpha n$ and $\sum_{u\in V\setminus N_G[S]} d(u)(p(d(u) - 1) - 1) \leq -\mu n$.

We show that every vertex $u \in V$ is in a component of size at least γn with probability $o(1/n)$. A union bound over all vertices completes the proof.

Suppose $u \in V$. We prove the desired statement conditional on every possible neighbourhood of S. Thus let $S_0 := N[S] \cup \{u\}$ for some choice of $N[S]$. Hence $\sum_{u\in S_0} d(u) \leq 2\alpha n$ and $\sum_{u\in V\setminus S_0} d(u)(p(d(u) - 1) - 1) \leq -\mu n/2$. Moreover, for every vertex $v \in V$ with $d(v) > n^{1/4}$, all its neighbours belong to S_0. We apply the first part of Lemma 20 with $\rho = \gamma/2$. Since $\gamma \ll \mu$, there exists a $t \leq \gamma n/2$ such that $X_t = 0$ with probability $1 - o(1/n)$. Since $|S_t| \leq t + |S_0| \leq (\gamma/2 + 2\alpha)n < \gamma n$, the union of all components that intersect $\{u\} \cup N[S]$ contain less than γn vertices with probability $1 - o(1/n)$.

Now we prove the second statement. Recall that now $\Delta(\mathcal{D}) \leq n^{1/4}$. Let $S_0 := \{u_0\}$ for an arbitrary vertex $u_0 \in V$. Clearly, $\sum_{v\in S_0} d(v) \leq \alpha n$. Recall that X_t counts the number of edges between S_t and $V \setminus S_t$ in the graph $G^{\mathcal{D}}$ that have not yet been exposed for percolation. Observe that all the edges counted by X_t will belong to the same component of $G_p^{\mathcal{D}}$ if they survive percolation. Note that this component may not contain u_0.

Let β and ρ be such that $\gamma \ll \beta \ll \rho \ll \mu$. By the second part of Lemma 20, with probability $1 - o(1)$, there exists a $\tau \leq \rho n$ with $X_\tau \geq \beta n$. Recall that $\mathcal{H}_\tau$ denotes the history of the exploration process, with the corresponding choice of S_τ, H_τ and F_τ at time τ. Let U be the set of vertices from the component of $G_p^{\mathcal{D}}$ under exploration at time τ that have been already explored; that is, the ones in S_τ. Then $\sum_{u \in U}(d(u) - d_{H_\tau}(u) - d_{F_\tau}(u)) = X_\tau \geq \beta n$. Moreover, $|S_\tau| \leq \tau + 1 \leq \rho n + 1$ and $|E(F_\tau)| \leq \tau \leq \rho n$. By Lemma 21 with $\nu = \beta$, $S = S_\tau$, $H = H_\tau$ and $F = F_\tau$, with probability $1 - o(1)$, there exists a component in $G_p^{\mathcal{D}}$ with at least $(\beta p/(20\bar{d}))n \geq \gamma n$ vertices. □

9. Degree sequences with many vertices of high degree

In this section we prove Proposition 10. As in the proof of Theorem 3, we adapt our argumentation according to the structure of the degree sequence. If not stated otherwise, we always consider a degree sequence $\mathcal{D}$ on V with average degree at most $\bar{d}$, where V is a set of size n. Recall that $\bar{d}$ is assumed to be fixed. In addition, we assume $1/n \ll 1/\bar{d} \leq 1$. We start with some notation, which we use throughout this section. Let

$$\begin{aligned} T &:= \{u \in V : d(u) \leq 3\bar{d}\}, \\ S_1 &:= \{u \in V : d(u) \geq \log^2 n\}, \\ S_2 &:= \{u \in V : d(u) \geq n^{1/3}\}, \text{ and} \\ S_3 &:= \{u \in V : d(u) \geq n^{4/5}\}. \end{aligned}$$

We say $\mathcal{D}$ satisfies (D_ϵ^1) if

$$(D_\epsilon^1) \qquad \sum_{u \in S_1} d(u) \geq \epsilon n \, ,$$

and we say it satisfies (D_ϵ^3) if

$$(D_\epsilon^3) \qquad \sum_{u \in S_3} d(u) \geq \frac{\epsilon n}{10} \, .$$

9.1. Degree sequences with vertices of very high degree.

In this subsection we consider degree sequences $\mathcal{D}$ that satisfy (D_ϵ^3). We collect several results about such degree sequences, which we will use in the proof of Proposition 10.

The first lemma shows that $G^{\mathcal{D}}[S_3]$ is typically a clique.

Lemma 22. *Suppose $n \in \mathbb{N}$ and $1/n \ll 1/\bar{d} \leq 1$. Let V be a set of size n and let $\mathcal{D}$ be a degree sequence on V with $\Sigma^{\mathcal{D}} \leq \bar{d}n$. Then the probability that $G^{\mathcal{D}}[S_3]$ is a clique is at least $1 - n^{-1/11}$.*

Proof. Since $\Sigma^{\mathcal{D}} \leq \bar{d}n$, it follows that $|S_3| \leq \bar{d}n^{1/5}$. If $\mathbb{P}[uv \notin E(G^{\mathcal{D}})] \leq n^{-1/2}$ for every $u, v \in S_3$, a union bound over all pairs $u, v \in S_3$ proves the lemma.

It remains to prove that for each pair $u, v \in S_3$, we have $\mathbb{P}[uv \notin E(G^{\mathcal{D}})] \leq n^{-1/2}$. Let $\mathcal{F}^-$ be the set of graphs G on V with degree sequence $\mathcal{D}$ and $uv \notin E(G)$ and let $\mathcal{F}^+$ be the set of graphs G on V with degree sequence $\mathcal{D}$ and $uv \in E(G)$.

Suppose $G \in \mathcal{F}^-$. Since $d(u), d(v) \geq n^{4/5}$ and $\Sigma^{\mathcal{D}} \leq \bar{d}n$, there exist at least $n^{8/5}/2$ ordered pairs (x, y) with $x \in N(u)$, $y \in N(v)$ and $xy \notin E(G)$. Switching ux and vy transforms G into a graph in $\mathcal{F}^+$.

Suppose $G \in \mathcal{F}^+$, then there are at most $\bar{d}n$ switches that transform G into a graph in $\mathcal{F}^-$. Therefore, by (10), we obtain

$$\mathbb{P}[\mathcal{F}^-] \leq \frac{2\bar{d}n}{n^{8/5}} \cdot \mathbb{P}[\mathcal{F}^+] \leq n^{-1/2}.$$

□

Conditional on $G^{\mathcal{D}}[S_3]$ being a clique, $G_p^{\mathcal{D}}[S_3]$ is a binomial random graph on $|S_3|$ vertices and edge probability p. Since $p \in (0,1)$, it is an exercise to check that S_3 induces a connected graph in $G_p^{\mathcal{D}}$ with probability at least $1 - c_1^{|S_3|}$, for some $c_1 = c_1(p) < 1$. Together with Lemma 22 we obtain the following corollary.

Corollary 23. *Suppose $n \in \mathbb{N}$ and $1/n \ll 1 - c \ll p, 1/\bar{d} \leq 1$. Let V be a set of size n and let $\mathcal{D}$ be a degree sequence on V with $\Sigma^{\mathcal{D}} \leq \bar{d}n$. Then*

$$\mathbb{P}[G_p^{\mathcal{D}}[S_3] \text{ is disconnected}] \leq c^{|S_3|}.$$

The next lemma shows that, typically, the vertices in $S_2 \setminus S_3$ are connected to a vertex in S_3 in $G^{\mathcal{D}}$ if $|S_3| \geq 100$.

Lemma 24. *Suppose $n \in \mathbb{N}$ and $1/n \ll 1/\bar{d} \leq 1$. Let V be a set of size n and let $\mathcal{D}$ be a degree sequence on V with $\Sigma^{\mathcal{D}} \leq \bar{d}n$. Assume that $|S_3| \geq 100$. Then, with probability at most $1/n$, there is a vertex $u \in S_2 \setminus S_3$ which is not adjacent to a vertex in S_3.*

Proof. It suffices to show that every vertex $u \in S_2$ is adjacent to a vertex in S_3 with probability at least $1 - n^{-2}$. Let $u \in S_2$ and let $0 \leq k \leq 50$. Let $\mathcal{F}_k$ be the event that u is adjacent to exactly k vertices in S_3.

Consider a graph $G \in \mathcal{F}_{k+1}$. Clearly, there are at most $(k+1)\bar{d}n$ switches transforming G into a graph in $\mathcal{F}_k$.

Consider a graph $G \in \mathcal{F}_k$. Let x be any vertex in S_3 which is not adjacent to u (since $|S_3| \geq 100$ but $k \leq 50$ there is such a vertex). Thus there are at least $n^{1/3+4/5} = n^{17/15}$ pairs (v, y) such that $v \in N(u)$ and $y \in N(x)$. For at most n pairs $v = y$ and for at most $2\bar{d}n$ pairs, we have $vy \in E(G)$. Thus at least $n^{17/15}/2$ pairs lead to a $\{uv, xy\}$-switch transforming G into a graph in $\mathcal{F}_{k+1}$. Hence

$$\mathbb{P}[\mathcal{F}_k] \leq n^{-1/15} \cdot \mathbb{P}[\mathcal{F}_{k+1}].$$

Moreover, this implies

$$\mathbb{P}[\mathcal{F}_0] \leq n^{-50/15} \cdot \mathbb{P}[\mathcal{F}_{50}] \leq n^{-2},$$

which completes the proof. □

Recall that T is the set of vertices of degree at most $3\bar{d}$. As $\mathcal{D}$ has average degree at most $\bar{d}$, many vertices belong to T. More precisely, as every vertex in $V \setminus T$ has degree at least $3\bar{d}$ and the average degree at most $\bar{d}$, we conclude $|V \setminus T| \leq n/3$. Thus

$$|T| \geq \frac{2n}{3}\,. \tag{43}$$

The next lemma shows that many vertices in T are adjacent to a vertex in S_2 if (D_ϵ^3) holds.

Lemma 25. *Suppose $n \in \mathbb{N}$ and $1/n \ll 1000\epsilon \leq 1/\bar{d} \leq 1$. Suppose V is a set of size n and $\mathcal{D}$ is a degree sequence on V with $\Sigma^{\mathcal{D}} \leq \bar{d}n$ that satisfies (D_ϵ^3). Then,*

$$\mathbb{P}[|N(S_2) \cap T| \leq \epsilon^2 n] \leq n^{-1}\,.$$

Proof. For every $0 \leq k \leq 2\epsilon^2 n$, let $\mathcal{F}_k$ be the set of graphs with degree sequence $\mathcal{D}$ such that $|T \cap N(S_2)| = k$.

Suppose $0 \leq k \leq 2\epsilon^2 n$. Consider at graph $G \in \mathcal{F}_{k+1}$. In order to transform G into a graph in $\mathcal{F}_k$, we need to select a vertex $u \in T \cap N(S_2)$ which has at exactly one neighbour v in S_2. Then there are at most $\bar{d}n$ switches involving uv. Thus in total, there are at most $2\epsilon^2\bar{d}n^2$ switches from $\mathcal{F}_{k+1}$ to $\mathcal{F}_k$.

Suppose $G \in \mathcal{F}_k$. Recall that $k \leq 2\epsilon^2 n$. Since $\Sigma^{\mathcal{D}} \leq \bar{d}n$, we have $|S_2| \leq \bar{d}n^{2/3}$ and $|S_3| \leq \bar{d}n^{1/5}$. As (D_ϵ^3) holds and as there at most $(\bar{d})^2 n^{2/3+1/5} \leq \epsilon n/40$ edges between the vertices of S_3 and S_2, it turns out that there are at least $\epsilon n/15$ edges xy such that $x \in S_3 \subseteq S_2$ and $y \in N(S_2)$. More specifically, since $k \leq 2\epsilon^2 n$, there are least $\epsilon n/20$ edges xy with $x \in S_3$ and such that y satisfies one of the following: either $y \in N(S_2) \setminus T$ or if $y \in N(S_2) \cap T$, then it has at least two neighbours in S_2. Fix such a choice of an edge xy. Note that if we switch xy with another edge uv

such that $u \in T \setminus N(S_2)$, we can only increase the neighbourhood of S_2. Observe that there are at most $n^{2/3}$ edges uv such that $u \in T \setminus N(S_2)$ and $v \in N(y)$. Furthermore, $|T \setminus N(S_2)| \geq n/2$. Let $u \in T \setminus N(S_2)$ and $v \in N(u)$ such that $v \notin N(y)$. Then there are at least $n/2 - n^{2/3} \geq n/3$ choices for the edge uv. Observe that the $\{uv, xy\}$-switch yields a graph in $\mathcal{F}_{k+1}$ and there are at least $\epsilon n^2/60$ switches from $\mathcal{F}_k$ to $\mathcal{F}_{k+1}$. Hence

$$\mathbb{P}[\mathcal{F}_k] \leq \frac{120\epsilon^2 \bar{d} n^2}{\epsilon n^2} \cdot \mathbb{P}[\mathcal{F}_{k+1}] \leq \frac{1}{2}\mathbb{P}[\mathcal{F}_{k+1}].$$

In particular,

$$\sum_{k=0}^{\epsilon^2 n} \mathbb{P}[\mathcal{F}_k] \leq 2\mathbb{P}[\mathcal{F}_{\epsilon^2 n}] \leq 2^{-\epsilon^2 n+1},$$

which completes the proof. □

9.2. **Lighter degree sequences.** In this subsection we consider degree sequences that satisfy (D^1_ϵ) but not (D^3_ϵ). The first lemma shows that in this case, typically, the minimum degree of $G^{\mathcal{D}}[S_1]$ is large.

Lemma 26. *Suppose $n \in \mathbb{N}$ and $1/n \ll \epsilon \ll 1/\bar{d} \leq 1$. Let V be a set of size n and let $\mathcal{D}$ be a degree sequence on V with $\Sigma^{\mathcal{D}} \leq \bar{d}n$. Assume also $\mathcal{D}$ that satisfies (D^1_ϵ), but not (D^3_ϵ). Then, with probability $o(1)$, there exists a vertex $u \in S_1$ such that $d_{G^{\mathcal{D}}[S_1]}(u) \leq \min\{d(u), n^{1/6}\} \cdot \epsilon/(16\bar{d})$.*

Proof. Let $u \in S_1$ and let $K := \lfloor \min\{d(u), n^{1/6}\} \cdot \epsilon/(16\bar{d}) \rfloor$. For every $0 \leq k \leq 2K$, let $\mathcal{F}_k$ be the set of graphs G with degree sequence $\mathcal{D}$ such that $d_{G[S_1]}(u) = k$.

Suppose $G \in \mathcal{F}_k$. There are at least $d(u) - k \geq d(u)/2$ choices for an edge uv with $v \in N(u) \setminus S_1$. The degree of v is less than $\log^2 n$ and each one of its neighbours has either degree less than $n^{4/5}$ (outside S_3) or it belongs to S_3. The former have total degree less than $n^{4/5}\log^2 n$, whereas the latter have total degree at most $\epsilon n/10$ (since (D^3_ϵ) does not hold). Hence, there are at most $n^{4/5}\log^2 n + \epsilon n/10 \leq \epsilon n/5$ edges at distance 2 from v. Similarly, there are at most $n^{4/5}k + \epsilon n/10 \leq \epsilon n/5$ edges with one endpoint in $N(u) \cap S_1$. Since (D^1_ϵ) holds, there are at least $\epsilon n - 2\epsilon n/5 \geq \epsilon n/2$ edges xy with $x \in S_1 \setminus N(u)$ and $y \notin N(v)$. Performing a $\{xy, uv\}$-switch, we obtain a graph in $\mathcal{F}_{k+1}$. We conclude that there are at least $\epsilon d(u)n/4$ switches that transform G into a graph in $\mathcal{F}_{k+1}$.

If G is in $\mathcal{F}_{k+1}$, there are at most $(k+1) \cdot \bar{d}n$ switches that transform it into a graph in $\mathcal{F}_k$. Therefore, for every $0 \leq k < 2K$, we obtain

$$\mathbb{P}[\mathcal{F}_k] \leq \frac{4(k+1)\bar{d}n}{\epsilon d(u) n} \cdot \mathbb{P}[\mathcal{F}_{k+1}] \leq \frac{1}{2} \cdot \mathbb{P}[\mathcal{F}_{k+1}].$$

Since $u \in S_1$, we have $d(u) \geq \log^2 n$ and we obtain

$$\sum_{k=0}^{K-1} \mathbb{P}[\mathcal{F}_k] \leq 2^{-K}\mathbb{P}[\mathcal{F}_{2K}] \leq n^{-2}.$$

A union bound over all vertices $u \in S_1$ completes the proof. □

Lemma 27. *Suppose $n \in \mathbb{N}$ and $1/n \ll p, 1/\bar{d} \leq 1$. Let V be a set of size n and let $\mathcal{D}$ be a degree sequence on V with $\Sigma^{\mathcal{D}} \leq \bar{d}n$. For $R \subseteq V$ the following holds. Conditional on $\delta(G^{\mathcal{D}}[R]) \geq 200\log n/p$, the probability that $G^{\mathcal{D}}_p[R]$ is connected is $1 - o(1)$.*

Proof. Let $N := |R|$. Our proof strategy is to show that with high probability for every possible partition (A, B) of R, there are edges between A and B in $G^{\mathcal{D}}_p$.

Let (A, B) be a partition of R such that $\alpha := |A|/N$ and $\alpha \leq 1/2$. Let $K := \lfloor 2\alpha N \log n/p \rfloor$. For every $0 \leq k \leq 2K$, let $\mathcal{F}_k$ be the set of graphs G with degree sequence $\mathcal{D}$ such that $\delta(G[R]) \geq 200\log n/p$ and there are exactly k edges between A and B. In order to give an upper bound on $\mathbb{P}[\mathcal{F}_k]$, we will consider switches between $\mathcal{F}_k$ and $\mathcal{F}_{k+2}$.

Let $G \in \mathcal{F}_k$. We claim that there exist two subsets $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq |A|/2$ and $|B'| \geq |B|/2$ such that for every $u \in A'$ (and every $y \in B'$), there are at least $100\log n/p$

edges from u to A (and from y to B). We prove this claim for A, because the latter case is similar. Our assumption is that $0 \le k \le 2K \le 4\alpha N \log n/p$ and $\delta(G[R]) \ge 200 \log n/p$. Let $A'' \subseteq A$ be the subset that consists of all those vertices u such that $d_{G[A]}(u) < 100 \log n/p$. If $|A''| \ge |A|/2 = \alpha N/2$, then

$$e(A,B) > \frac{\alpha N}{2} \cdot \frac{100 \log n}{p} > \frac{4\alpha N \log n}{p} \ge 2K \ge k,$$

which is a contradiction. Therefore $|A''| < |A|/2$, and setting $A' = A \setminus A''$ we have $|A'| = |A \setminus A''| \ge |A|/2$. Similarly, we set $B' = B \setminus B''$.

Next we claim that the edges of $G[A]$ can be oriented in such a way so that every vertex in A' has out-degree at least $48 \log n/p$ in A. To obtain such an orientation, start consistently orienting the edges of undirected cycles in $G[A]$ until the undirected graph induces a forest. Afterwards iteratively and consistently orient maximal undirected paths in this forest. If so, the out-degree of a vertex in A is at least the in-degree minus 1. Since $d_{G[A]}(u) \ge 100 \log n/p$ for every vertex $u \in A'$, the vertex u has at least $50 \log n/p - 1 \ge 48 \log n/p$ out-neighbours. Similarly, one can also orient the edges of $G[B]$ in such a way that every vertex in B' has out-degree at least $48 \log n/p$ in B.

For each vertex in $u \in A'$, select a set $E(u)$ of exactly $48 \log n/p$ directed edges from u to a vertex in A, and analogously select $E(y)$, for each $y \in B'$. We will only count the (possible) $\{uv, xy\}$-switches with $u \in A'$, $y \in B'$, $uv \in E(u)$ and $yx \in E(y)$. For the switch to be valid, we insist on $ux, vy \notin E(G)$. Each edge ab with $a \in A, b \in B$ can only invalidate switches of the form $\{av, by\}$ with $(a,v) \in E(a)$ and of the form $\{ua, xb\}$ with $(b,x) \in E(b)$; that is, one edge ab invalidates at most

$$|E(a)| \cdot (1-\alpha)N + |E(b)| \cdot \alpha N = \frac{48 N \log n}{p}$$

switches. Hence in total the edges between A and B block at most

$$2K \cdot \frac{48 N \log n}{p} \le \frac{192 \alpha N^2 \log^2 n}{p^2}$$

possible switches. Recall that $|A'| \ge |A|/2$, $|B'| \ge |B|/2$. Thus, by using $1 - \alpha \ge 1/2$, there are at least

$$\frac{\alpha N}{2} \cdot \frac{48 \log n}{p} \cdot \frac{(1-\alpha)N}{2} \cdot \frac{48 \log n}{p} - \frac{192 \alpha N^2 \log^2 n}{p^2} \ge \frac{96 \alpha N^2 \log^2 n}{p^2} ,$$

switches that transform the graph G into a graph in $\mathcal{F}_{k+2}$. Since we only switch edges with both endpoints in R, the minimum degree in the graph induced by R stays the same.

Consider a graph in $\mathcal{F}_{k+2}$. Clearly, there are at most $(k+2)^2$ switches that transform G into a graph in $\mathcal{F}_k$. Therefore, for every $0 \le k \le 2K-2$, we conclude

$$\mathbb{P}[\mathcal{F}_k] \le \frac{p^2 (k+2)^2}{96 \alpha N^2 \log^2 n} \cdot \mathbb{P}[\mathcal{F}_{k+2}] \le \frac{1}{4} \cdot \mathbb{P}[\mathcal{F}_{k+2}] .$$

We conclude that the probability there are less than K edges in $G^{\mathcal{D}}$ between A and B is small, namely

$$\sum_{k=0}^{K-1} \mathbb{P}[\mathcal{F}_k] \le 2 \cdot 4^{-K/2} (\mathbb{P}[\mathcal{F}_{2K}] + \mathbb{P}[\mathcal{F}_{2K-1}]) \le 2^{-K+1} . \tag{44}$$

If $k \ge K$, then $\mathbb{P}[e(G_p[A,B]) = 0 \mid \mathcal{F}_k] \le (1-p)^K$. Therefore, provided $\delta(G[R]) \ge 200 \log n/p$, the probability that $e(G_p^{\mathcal{D}}[A,B]) = 0$ is at most $(1-p)^K + 2^{-K+1} \le e^{-\frac{3}{2}\alpha N \log n}$, where we used (44) and that $1 - p \le e^{-p}$.

To conclude the proof of the lemma, we use a union bound over all partitions (A,B) of R. Since for every $1 \le a \le N/2$, there are $\binom{N}{a} \le e^{a \log N}$ partitions with $|A| = a$. Conditional on

$\delta(G[R]) \geq 200 \log n/p$, the probability that $G_p^{\mathcal{D}}[R]$ is disconnected is at most

$$\sum_{a=1}^{N/2} \sum_{\substack{R=A\cup B\\ |A|=a}} \mathbb{P}[e(G_p^{\mathcal{D}}[A,B])=0] \leq \sum_{a=1}^{N/2} e^{a\log N} e^{-\frac{3}{2}a\log n} \leq \sum_{a=1}^{N/2} e^{-\frac{1}{2}a\log n} = o(1) .$$

□

9.3. **Proof of Proposition 10.** In this section we use the results from the two previous subsections to conclude the proof of Proposition 10. Let p, δ, ϵ and $\bar{d}$ be as in the statement. Let n be large enough in terms of these parameters. Let V be a set of size n. Let $\mathcal{D}$ be a degree sequence on V with $\Sigma^{\mathcal{D}} \leq \bar{d}n$. Recall that $S_1 = \{u \in V \ : \ d(u) \geq \log^2 n\}$, $S_2 = \{u \in V \ : \ d(u) \geq n^{1/3}\}$ and $S_3 = \{u \in V \ : \ d(u) \geq n^{4/5}\}$. Proposition 10 assumes that $\mathcal{D}$ satisfies (D_ϵ^1).

Case 1: Suppose $\mathcal{D}$ also satisfies (D_ϵ^3), that is, $\sum_{u\in S_3} d(u) \geq \epsilon n/10$. Let $s \geq 100$ be the smallest integer such that $\delta > 2c^s$, where c is the constant given by Corollary 23 for our choice of p and $\bar{d}$. Set $\gamma_1 := \epsilon p/(20s)$. If $|S_3| < s$, then there exists a vertex $u \in S_3$ with $d(u) \geq 2\gamma_1 n/p$, because $\mathcal{D}$ satisfies (D_ϵ^3). This implies, by a simple application of Chernoff's inequality, that $G_p^{\mathcal{D}}$ contains a star of order $\gamma_1 n$ with centre u, in particular, $G_p^{\mathcal{D}}$ contains a component of order at least $\gamma_1 n$.

Suppose now that $|S_3| \geq s$. Let $\mathcal{A}_1$ be the event that $G_p^{\mathcal{D}}[S_3]$ is connected. Then, by definition of s and by Corollary 23,

$$\mathbb{P}[\overline{\mathcal{A}_1}] \leq \frac{\delta}{2}. \tag{45}$$

Let $\mathcal{A}_2$ be the event that every vertex in $S_2 \setminus S_3$ has a neighbour in S_3 and let $\mathcal{A}_3$ be the event that $|N(S_2) \cap T| \leq \epsilon^2 n$. Then by Lemmas 24 and 25,

$$\mathbb{P}[\overline{\mathcal{A}_2} \cup \overline{\mathcal{A}_3}] \leq 2n^{-1}.$$

Let $\gamma_2 := p^2\epsilon^2 n/3$. We will show that $\mathbb{P}[L_1(G_p^{\mathcal{D}}) > \gamma_2 n \mid \mathcal{A}_2, \mathcal{A}_3] \geq 1-\delta$.

If $|N(S_3) \cap T| \geq \epsilon^2 n/2$, then a straightforward application of Chernoff's inequality combined with (45) shows that there is a component of order at least $p\epsilon^2 n/3 \geq \gamma_2 n$ in $G_p^{\mathcal{D}}$ with probability at least $1-\delta$.

If $|N(S_3) \cap T| \leq \epsilon^2 n/2$, then $|N(S_2 \setminus S_3) \cap T| \geq \epsilon^2 n/2$. Let F be a forest in $G^{\mathcal{D}}$ such that F contains $N(S_2 \setminus S_3) \cap T$, for every vertex $x_1 \in N(S_2 \setminus S_3) \cap T$, there is a path $x_1x_2x_3$ in F such that $x_2 \in S_2 \setminus S_3$ and $x_3 \in S_3$, and among all such forests, F contains as few as edges as possible. To complete the case when $\mathcal{D}$ satisfies (D_ϵ^3), we will show that $G_p^{\mathcal{D}}[S_3] \cup F_p$ contains a component of order at least $\gamma_2 n$ with probability at least $1-\delta$. Consider a realisation of $G^{\mathcal{D}}$ that satisfies $\mathcal{A}_2 \cap \mathcal{A}_3$. Observe first that whether a certain edge in F is present in F_p changes the number of vertices in $N(S_2 \setminus S_3) \cap T$ that are connected via F_p to S_3 by at most $n^{4/5}$. Thus assuming $\mathcal{A}_1$ holds, a straightforward application of McDiarmid's inequality (Lemma 7) shows that there is a component of order at least $p^2 \cdot \epsilon^2 n/3 = \gamma_2 n$ in $G_p^{\mathcal{D}}[S_3] \cup F_p$ with probability at least $1-n^{-1}$. This together with (45), completes the case when $\mathcal{D}$ satisfies (D_ϵ^3).

Case 2: Now, suppose that $\mathcal{D}$ does not satisfy Condition (D_ϵ^3). Since it satisfies (D_ϵ^1), by Lemma 26, we obtain

$$\mathbb{P}\left[\delta(G^{\mathcal{D}}[S_1]) \geq \frac{\epsilon}{16\bar{d}} \log^2 n\right] = 1 - o(1).$$

Together with Lemma 27 where S_1 plays the role of R, we conclude that

$$\mathbb{P}[G_p^{\mathcal{D}}[S_1] \text{ is connected}] = 1 - o(1). \tag{46}$$

In order to show that $G_p^{\mathcal{D}}$ contains a giant component, we will show that $|N(S_1) \cap T|$ is large. Let $K := \lfloor \epsilon n/(128\bar{d}) \rfloor$. For every $0 \leq k \leq 2K$, let $\mathcal{F}_k$ be the set of graphs with degree sequence $\mathcal{D}$ such that $|N(S_1) \cap T| = k$.

Let $G \in \mathcal{F}_k$. Using (43) and $\delta(G) \geq 1$, there are at least $|T| - k \geq 2n/3 - k \geq n/2$ choices for an edge xy with $x \in T \setminus N(S_1)$. Observe that $d(y) \leq \log^2 n$, since $x \notin N(S_1)$. Also, since $x, y \notin S_1$ and $\mathcal{D}$ does not satisfy (D^3_ϵ), we claim that there are at most

$$3\bar{d}\log^2 n + n^{4/5}\log^2 n + \epsilon n/10 \leq \frac{\epsilon n}{5}$$

edges incident to a neighbour of either x or y. Indeed, the number of edges incident to a neighbour of x is bounded by $3\bar{d}\log^2 n$, as x has no neighbours inside S_1. Now, the neighbours of y are classified either as the neighbours that belong to S_3 or those that do not. Since property (D^3_ϵ) does not hold, and there are at most $\epsilon n/10$ edges incident to any vertex in S_3, there are at most $\epsilon n/10$ edges incident to the first class of neighbours. Regarding the latter class of neighbours, there are at most $\log^2 n$ of them (as $y \notin S_1$) and each has degree at most $n^{4/5}$. Thereby, there at most $n^{4/5} \cdot \log^2 n$ such edges. Hence our claim holds.

Let uv be an edge such that $u \in S_1$, $v \notin N(y)$, and either $v \notin T$ or if $v \in T$, then there exists a $u' \in S_1$ with $u'v \in E(G)$. Since $\sum_{u \in S_1} d(u) \geq \epsilon n$, there are at least $\epsilon n - k - \epsilon n/5 \geq \epsilon n/2$ such edges. Hence, the total number of $\{xy, uv\}$-switches that transform G into a graph in $\mathcal{F}_{k+1} \cup \mathcal{F}_{k+2}$ is at least $\epsilon n^2/4$ (we transform G into a graph satisfying $\mathcal{F}_{k+2}$ if $v \in S_1$ and $y \in T \setminus N(S_1)$).

If $G \in \mathcal{F}_{k+1} \cup \mathcal{F}_{k+2}$, then there are at most $(k+2)\bar{d}n$ switches that transform G into a graph in $\mathcal{F}_k$. As before, for every $0 \leq k \leq 2K - 2$, this implies

$$\mathbb{P}[\mathcal{F}_k] \leq \frac{4(k+2)\bar{d}n}{\epsilon n^2} \cdot (\mathbb{P}[\mathcal{F}_{k+1}] + \mathbb{P}[\mathcal{F}_{k+2}]) \leq \frac{1}{4} \cdot \max\{\mathbb{P}[\mathcal{F}_{k+1}], \mathbb{P}[\mathcal{F}_{k+2}]\} .$$

Therefore,

$$\sum_{k=0}^{K-1} \mathbb{P}[\mathcal{F}_k] \leq 2^{-K}\mathbb{P}[\mathcal{F}_{2K}] \leq 2^{-K} = o(1) .$$

Hence

$$\mathbb{P}[|N(S_1) \cap T| \geq \lfloor \epsilon n/(128\bar{d}) \rfloor] = 1 - o(1) .$$

Let $\gamma_3 := \epsilon p/(130\bar{d})$. The Chernoff bound (Lemma 6) implies that

$$\mathbb{P}[|N_{G_p^{\mathcal{D}}}(S_1) \cap T| \geq p\epsilon n/(130\bar{d})] = 1 - o(1) .$$

Together with (46), this implies $\mathbb{P}[L_1(G_p^{\mathcal{D}}) \geq \gamma_3 n] \geq 1 - \delta$. Setting $\gamma := \min\{\gamma_1, \gamma_2, \gamma_3\}$, we obtain

$$\mathbb{P}[L_1(G_p^{\mathcal{D}}) \geq \gamma n] \geq 1 - \delta .$$

10. Sequences of degree sequences: proof of Theorem 1

Let $\mathfrak{D} = (\mathcal{D}_n)_{n\geq 1}$ be a sequence of degree sequences with $\mathcal{D}_n = (d_1^{(n)}, \ldots, d_n^{(n)})$. For the sake of simplicity, we write $\mathcal{D}_n = (d_1, \ldots, d_n)$ and $W(c) := W(c, \mathcal{D}_n)$. Set

$$d_{c,n} := \max\left\{\frac{\sum_{i\in V\setminus W(c)} d_i(d_i - 1)}{\sum_{i \in V\setminus W(c)} d_i}, 1\right\}.$$

We assume that $d_c := \lim_{n\to\infty} d_{c,n}$ exists for every $c \geq 1$ and that d is such that

$$d = \sup_{c\geq 1} d_c = \sup_{c\geq 1} \lim_{n\to\infty} d_{c,n} \in [1, \infty) .$$

We define the critical probability as in (5) by

$$p_{crit}(c, \mathcal{D}_n) = \min\left\{\frac{\sum_{i\in V\setminus W(c)} d_i}{\sum_{i\in V\setminus W(c)} d_i(d_i - 1)}, 1\right\} = \frac{1}{d_{c,n}} .$$

We start with the proof of part (i) and begin with a claim which states that for every large c we can replace $1/d$ by $p_{crit}(c, \mathcal{D}_n)$ provided n is large enough in terms of c.

Claim 1. *For every $\epsilon \in (0, 1/2)$, there exists c_ϵ such that for every $c \geq c_\epsilon$, there exists $n_{\epsilon,c}$ such that for every $n \geq n_{\epsilon,c}$, we have*

- *if $p < (1-\epsilon)\frac{1}{d}$, then $p < \left(1 - \frac{\epsilon}{4}\right) p_{crit}(c, \mathcal{D}_n)$.*

- *if* $p > (1+\epsilon)\frac{1}{d}$, *then* $p > \left(1+\frac{\epsilon}{4}\right) p_{crit}(c, \mathcal{D}_n)$.

Proof. Note first that $d_{c,n}$ is non-decreasing with respect to c; that is, $d_{c_2,n} \geq d_{c_1,n}$ for $c_2 \geq c_1$. This implies that $d_{c_2} = \lim_{n\to\infty} d_{c_2,n} \geq \lim_{n\to\infty} d_{c_1,n} = d_{c_1}$. Hence $(d_c)_{c\geq 1}$ is a monotone non-decreasing sequence and it converges to d. Furthermore, $d < \infty$ by Condition (a). Thus, for any $\epsilon > 0$, there exists c_ϵ such that for any $c > c_\epsilon$, we have

$$(1-\epsilon^2/2)d < d_c \leq d.$$

In turn, given c, there exists $n_{\epsilon,c}$ such that for any $n > n_{\epsilon,c}$, we have

$$(1-\epsilon^2/2)d_c < d_{c,n} < (1+\epsilon^2)d_c.$$

Therefore, for every $c \geq c_\epsilon$ and every $n \geq n_{\epsilon,c}$ we directly obtain

$$(1-\epsilon^2)d < d_{c,n} < (1+\epsilon^2)d. \tag{47}$$

Moreover, if $\epsilon < 1/2$ and $p < (1-\epsilon)\frac{1}{d}$, then

$$p < (1-\epsilon)(1+\epsilon^2)\frac{1}{d_{c,n}} < (1-\epsilon/4)\frac{1}{d_{c,n}}.$$

Similarly, if $\epsilon < 1/2$ and $p > (1+\epsilon)\frac{1}{d}$, then

$$p > (1+\epsilon)(1-\epsilon^2)\frac{1}{d_{c,n}} > (1+\epsilon/4)\frac{1}{d_{c,n}}.$$

□

In what follows we will select c_1, c_2 and η such that the hypotheses of Theorem 2 are satisfied. By Condition (b), for any $\epsilon > 0$, there exists a $c'_\epsilon \in \mathbb{N}$ such that for every $c \geq c'_\epsilon$, there exists $n'_{\epsilon,c}$ with

$$\sum_{j\in W(c)} d_j \leq \frac{\epsilon^2}{c} \cdot n\ . \tag{48}$$

We may assume that for fixed ϵ and all $c_1 \leq c_2$, we have $n_{\epsilon,c_1} \leq n_{\epsilon,c_2}$ and $n'_{\epsilon,c_2} \leq n_{\epsilon,c_2}$ as we simply can replace n_{ϵ,c_2} by $\max_{c'\leq c_2}\{n_{\epsilon,c'}, n'_{\epsilon,c'}\}$. We may also assume that $\epsilon < (64d\bar{d})^{-1}$. We choose $c_1 := \max\{c_\epsilon, c'_\epsilon\}$. Suppose $c > c_1$ and $n \geq n_{\epsilon,c}$. Next we prove that $A_2(\epsilon/4, c_1, c)$ holds for a suitable c. Note that this condition is only needed in Theorem 2 (ii). If $d_{c_1,n} \leq (1+\epsilon/5)$, then $p > (1+\epsilon)/d$ implies (by Claim 1 and $c_1 \geq c_\epsilon$) that

$$p > \left(1+\frac{\epsilon}{4}\right)\frac{1}{d_{c_1,n}} > 1,$$

and there is nothing to prove. Thus, we may assume that $d_{c_1,n} > (1+\epsilon/5)$. Hence, $d_{c,n} = \frac{\sum_{i\in V\setminus W(c)} d_i(d_i-1)}{\sum_{i\in V\setminus W(c)} d_i} \leq c$ for any $c \geq c_1$ and n sufficiently large in terms of c. It follows that

$$\begin{aligned}
\sum_{j\in W(c_1)\setminus W(c)} d_j(d_j-1) &= \sum_{j\in V\setminus W(c)} d_j(d_j-1) - \sum_{j\in V\setminus W(c_1)} d_j(d_j-1) \\
&= d_{c,n} \sum_{j\in V\setminus W(c)} d_j - d_{c_1,n} \sum_{j\in V\setminus W(c_1)} d_j \\
&= (d_{c,n} - d_{c_1,n}) \sum_{j\in V\setminus W(c_1)} d_j + d_{c,n} \sum_{j\in W(c_1)\setminus W(c)} d_j \\
&\overset{(48)}{\leq} (d_{c,n} - d_{c_1,n})\bar{d}n + \epsilon^2 dn \\
&\overset{(47)}{\leq} 3\epsilon^2 d\bar{d}n\ .
\end{aligned}$$

This in turn implies that

$$\begin{aligned}\sum_{j\in W(c_1)\setminus W(c)} d_j^2 &= \sum_{j\in W(c_1)\setminus W(c)} d_j(d_j-1) + \sum_{j\in W(c_1)\setminus W(c)} d_j \\ &\le 3\epsilon^2 d\bar{d}n + \sum_{j\in W(c_1)} d_j \\ &\stackrel{(48)}{<} 4\epsilon^2 d\bar{d}n \\ &\le \frac{\epsilon/4}{4}\cdot n\ .\end{aligned}$$

Thus $A_2(\epsilon/4, c_1, c)$ holds for all $c \ge c_1$ and $n \ge n_{\epsilon,c}$. Note that c_1 only depends on ϵ.

Let $\eta = \eta(\gamma, \epsilon/4, \bar{d})$ be as in Theorem 2. Using again Condition (b), for $n \ge n_{\eta,c_2}$ we have

$$\sum_{j\in W(c_2)} d_j \le \frac{\eta}{c_2}\cdot n\ ,$$

and thus $A_1(\eta, c_2)$ is satisfied (even if $d_{c_1,n} \le (1+\epsilon/5)$).

Also let $\rho = \rho(\epsilon/4, c_1)$ be the constant provided by Theorem 2, which in this case only depends on ϵ; that is, we can choose $\gamma \le \rho$. Let n be larger than $\max\{n_{\eta,c_2}, n_{\epsilon,c_2}\}$ and the n_0 given by Theorem 2 for the parameters $\epsilon/4, \gamma, c_1, c_2, \bar{d}$. By Claim 1, we can apply Theorem 2 with $\epsilon/4$ to $\mathcal{D}_n$ to conclude that

$$\begin{aligned}&\text{if } p < (1-\epsilon)\tfrac{1}{d}, \text{ then } \mathbb{P}[L_1(G_p^{\mathcal{D}_n}) > \gamma n] = o_n(1),\\ &\text{if } p > (1+\epsilon)\tfrac{1}{d}, \text{ then } \mathbb{P}[L_1(G_p^{\mathcal{D}_n}) > \rho n] = 1-o_n(1).\end{aligned}$$

We proceed to the proof of Theorem 1 (ii). Our aim is to apply Theorem 3. Let us check that the hypothesis are satisfied. Given δ, p and $\bar{d}$, let K be the constant provided by Theorem 3.

Suppose first that Condition (c) holds. Since $\sup_{c\ge 1} d_c = \lim_{c\to\infty} d_c = d = \infty$, there exists c_K such that for every $c \ge c_K$, we have

$$\lim_{n\to\infty} d_{c,n} = d_c > 2K\ .$$

Similarly, there exists $n_{K,c}$ such that $d_{c,n} \ge K$ for every $n \ge n_{K,c}$. For $c \ge \max\{c_K, 2\bar{d}\}$ and $n \ge n_{K,c}$, we obtain

$$\sum_{j\in V\setminus W(c)} d_j^2 \ge \sum_{j\in V\setminus W(c)} d_j(d_j-1) \ge K \sum_{j\in V\setminus W(c)} d_j \ge \frac{K}{2}\cdot n\ ,$$

and so $A_2(K, 0, c)$ does not hold. Thus Theorem 3 leads to the desired conclusion.

Suppose now that Condition (d) holds. Let c_0 be such that $f(c) \ge K$ for every $c \ge c_0$. As $f(c) \to \infty$ as $c \to \infty$ such a c_0 exists. This in turn immediately implies that $A_1(K, c)$ does not hold provided n is large enough. Again, Theorem 3 leads to the desired conclusion and this completes the proof.

We close this section with the following remark. Suppose that $\lim_{n\to\infty} n_i/n =: \lambda_i < \infty$ for all $i \ge 1$, that $\sum_{i\ge 1} \lambda_i = 1$, and that $\sum_{i\ge 1} i\lambda_i < \infty$. Then

$$d = \frac{\sum_{i\ge 1} i(i-1)\lambda_i}{\sum_{i\ge 1} i\lambda_i}.$$

This recovers the results obtained by the first author [10], Janson [13], and Bollobás and Riordan [8].

11. Application: power-law degree distributions

Power law degree distributions have attracted considerable interest as they are one of the usual characteristics of complex networks [2]. Roughly speaking, in such degree sequences the fraction of vertices that have degree equal to k (when k is large) scales like $k^{-\gamma}$, for some $\gamma > 0$.

A variety of random graph models which exhibit a power law degree distribution have been introduced in the last 15 years, mainly, in search for a sound model for complex networks. Among other properties, *robustness* is a central property that has been considered in this context; that is, how robust a random network is if several of its edges or its vertices fail.

In several random graph models with a power law degree distribution, it has been observed that if $\gamma > 3$, then there exists a critical value p_{crit} (which is bounded away from 0) for the appearance of a giant component in the bond percolation process. However, if $\gamma \leq 3$, for any fixed $p > 0$ (that is, independent of the order of the random graph), a giant component survives the random deletions with high probability. This behaviour has been observed in diverse random graph models that give rise to power-law degree distributions such as the configuration model ([3], Corollary 2.5), the preferential attachment model [7] and random graphs on the hyperbolic plane [9].

We now apply Theorem 1 in this context. This recovers a known result for power law sequences but also exemplifies how our results can be used for particular degree sequences. Consider a sequence of degree sequences $(\mathcal{D}_n)_{n\in\mathbb{N}}$, where $\mathcal{D}_n$ is a feasible degree sequence on $[n]$ and assume that it satisfies the following: for $k \geq 1$, let n_k denote the number of vertices of degree k in $\mathcal{D}_n$, then there exist positive constants $\gamma, \lambda_1, \lambda_2, k_0 > 0$ such that for every $k \geq k_0$, we have

$$\frac{\lambda_1}{k^\gamma} \leq \frac{n_k}{n} \leq \frac{\lambda_2}{k^\gamma} .$$

If so, we say that $\mathcal{D}_n$ follows a power law distribution with exponent γ.

In this section we show that power law distributions, as defined here, show the same behaviour around $\gamma = 3$. As before, we write $\mathcal{D}_n = (d_1, \dots, d_n)$ and $W(c) := W(c, \mathcal{D}_n) = \{i \ : \ d_i \geq c\}$ for every $c \geq 1$.

Let $\mathcal{D}_n$ follow a power law distribution with $\gamma > 3$. Then, there exists $\lambda_2' > 0$ such that for every $c_2 \geq k_0$, we have

$$\sum_{i\in W(c_2)} d_i = \sum_{k\geq c_2} kn_k \leq \lambda_2 n \sum_{k\geq c_2} k^{1-\gamma} \leq \frac{\lambda_2'}{c_2^{\gamma-2}} \cdot n = \lambda_2' c_2^{3-\gamma} \cdot \frac{n}{c_2} ,$$

and thus, $\mathcal{D}_n$ satisfies $A_1(\lambda_2' c_2^{3-\gamma}, c_2)$.

Moreover, there exists $\lambda_2'' > 0$ such that for all $c_2 \geq c_1 \geq k_0$, we have

$$\sum_{i\in W(c_1)\setminus W(c_2)} d_i^2 = \sum_{k=c_1}^{c_2-1} k^2 n_k \leq \lambda_2 n \sum_{k=c_1}^{c_2-1} k^{2-\gamma} \leq \lambda_2'' c_1^{3-\gamma} \cdot n ,$$

that is, $\mathcal{D}_n$ satisfies $A_2(4\lambda_2'' c_1^{3-\gamma}, c_1, c_2)$.

Provided that c_1 and c_2 are large enough and $\gamma > 3$ (so the first parameters in conditions A_1 and A_2 are arbitrarily small), we can apply Theorem 2 to determine a quantity $p_{crit} > 0$ that is bounded away from 0, such that bond percolation in $G^{\mathcal{D}_n}$ has a threshold at p_{crit}.

Now, let $\mathcal{D}_n$ follow a power law distribution with $2 < \gamma < 3$. Then, there exists $\lambda_1' > 0$ such that for every $c \geq k_0$, we have

$$\sum_{i\in W(c)} d_i = \sum_{k\geq c} kn_k \geq \lambda_1 n \sum_{k\geq c} k^{1-\gamma} \geq \lambda_1' c^{2-\gamma} \cdot n = \lambda_1' c^{3-\gamma} \cdot \frac{n}{c} ,$$

that is, $\mathcal{D}_n$ does not satisfy $A_1(\lambda_1' c^{3-\gamma}, c)$.

Provided that c_1 is large enough (so the first parameter in condition A_1 is arbitrarily large), we can apply Theorem 3 to show that bond percolation in $G^{\mathcal{D}_n}$ does not have a positive threshold.

Note that if $\gamma \leq 2$, then the average degree of $G^{\mathcal{D}_n}$ is unbounded and our results do not apply.

We finally state the "limit" version of the result for $\mathcal{D}_n$ that follows a power law distribution. Suppose that there exists $c > 0$ such that for all $k \geq 1$, we have

$$\lim_{n\to\infty} \frac{n_k}{n} = ck^{-\gamma} .$$

If $\gamma > 3$, then $d < \infty$, while if $\gamma < 3$, then $d = \infty$. So, Theorem 1 implies that in the former case we have $p_{crit} = 1/d > 0$, whereas in the latter case $p_{crit} = 0$.

It is worth to stress that our results do not provide any meaningful information at $\gamma = 3$.

12. Concluding remarks

We finish the paper with some remarks on our results.

1) Theorem 3 provides a statement that holds only with probability at least $1-\delta$. The only part of its proof that does not hold with high probability is Corollary 23. This makes it easy to construct degree sequences that show that this cannot be improved. For a given $\rho > 0$, let us consider the following degree sequence on n vertices (large enough in terms of ρ). Let $a := \lfloor 2/\rho \rfloor$ and suppose a divides $n-a$. Consider the degree sequence with a vertices of degree $(n-a)/a + a - 1$, and $n-a$ vertices of degree 1. This degree sequence is feasible and the only graph (up to isomorphism) with this degree sequence consists of a clique of size a where each of its vertices is adjacent to $n/a - 1$ vertices of degree 1. With positive probability independently of n all $\binom{a}{2}$ edges inside the clique of size a fail to percolate in $G_p^{\mathcal{D}}$. If so, $L_1(G_p^{\mathcal{D}}) \leq \rho n/2$. Thus for every $p \in [0,1)$, we have
$$\mathbb{P}[L_1(G_p^{\mathcal{D}}) < \rho n] > \delta(\rho, p) \ .$$
Observe that these degree sequences also do not satisfy $A_1(K,c)$ for all $c \geq 2K$.
2) In [15], a special role is given to vertices of degree 2. However, by considering bond percolation this special situation never appears. If most of the edges are incident to vertices of degree 2 after the bond percolation, then $p \approx 1$ and almost all vertices have degree 2 already before the percolation. In this case set $p_{crit} := 1$. Let W be the set of vertices with degree different from 2. If $\sum_{i \in W} d_i = o(n)$, then $|N[W]| = o(n)$. For every $\epsilon > 0$ and every $p < 1-\epsilon$, it follows that,
$$\sum_{i \in V \setminus N[W]} d_i(p(d_i - 1) - 1) = (n - |N[W]|)2(p-1) < -\epsilon n \ .$$
Using the first part of Proposition 9 we obtain that $G_p^{\mathcal{D}}$ has no giant component with high probability, and thus $p_{crit} = 1$.
3) The previous remark is a particular case of the case $\Sigma^{\mathcal{D}}/n \to \infty$. While it might seem natural that $p_{crit}(\mathcal{D}) \to 0$, here we provide an example for which $\Sigma^{\mathcal{D}}/n \to \infty$ and p_{crit} is bounded away from 0.

   Consider the degree sequence $\mathcal{D}$ formed by $n^{2/3}$ vertices of degree $n^{2/3}$ and $n - n^{2/3}$ vertices of degree 1. The critical condition in [15] shows that $G^{\mathcal{D}}$ has a giant component with high probability. However, it is easy to see that, with high probability, $G_p^{\mathcal{D}}$ has at least $(1-2p)n$ isolated vertices and thus we cannot expect to have a component of order larger than $2pn$. If $p \to 0$ (as $n \to \infty$), then $G_p^{\mathcal{D}}$ does not have a giant component with high probability.

References

1. W. Aiello, F. Chung, and L. Lu, *A random graph model for massive graphs*, Proceedings of the thirty-second annual ACM symposium on Theory of computing, ACM, 2000, pp. 171–180.
2. R. Albert and A.-L. Barabási, *Statistical mechanics of complex networks*, Reviews of modern physics **74** (2002), 47.
3. E. Baroni, R. van der Hofstad, and J. Komjáthy, *First passage percolation on random graphs with infinite variance degrees*, Annals of Applied Probability, to appear.
4. E.A. Bender and E.R. Canfield, *The asymptotic number of labeled graphs with given degree sequences*, J. Combinatorial Theory Ser. A **24** (1978), 296–307.
5. S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, *Complex networks: Structure and dynamics*, Physics reports **424** (2006), 175–308.
6. B. Bollobás, *A probabilistic proof of an asymptotic formula for the number of labelled regular graphs*, European J. Combin. **1** (1980), 311–316.
7. B. Bollobás and O. Riordan, *Robustness and vulnerability of scale-free random graphs*, Internet Math. **1** (2003), 1–35.
8. ———, *An old approach to the giant component problem*, J. Combin. Theory Ser. B **113** (2015), 236–260.
9. E. Candellero and N. Fountoulakis, *Bootstrap percolation and the geometry of complex networks*, Stochastic Process. Appl. **126** (2016), 234–264.
10. N. Fountoulakis, *Percolation on sparse random graphs with given degree sequence*, Internet Mathematics **4** (2007), 329–356.

11. A. Goerdt, *The giant component threshold for random regular graphs with edge faults*, Theoret. Comput. Sci. **259** (2001), 307–321.
12. H. Hatami and M. Molloy, *The scaling window for a random graph with a given degree sequence*, Random Structures Algorithms **41** (2012), 99–123.
13. S. Janson, *On percolation in random graphs with given vertex degrees*, Electron. J. Probab. **14** (2009), 87–118.
14. S. Janson and M.J. Luczak, *A new approach to the giant component problem*, Random Structures Algorithms **34** (2009), 197–216.
15. F. Joos, G. Perarnau, D. Rautenbach, and B. Reed, *How to determine if a random graph with a fixed degree sequence has a giant component*, Probab. Theory Related Fields **170** (2018), 263–310.
16. M. Kang and T. G. Seierstad, *The critical phase for random graphs with a given degree sequence*, Comb. Probab. Comput. **17** (2008), 67–86.
17. C. McDiarmid, *Concentration*, Probabilistic methods for algorithmic discrete mathematics, vol. 16, Springer, 1998, pp. 195–248.
18. M. Molloy and B. Reed, *A critical point for random graphs with a given degree sequence*, Random Structures Algorithms **6** (1995), 161–180.
19. ———, *The size of the giant component of a random graph with a given degree sequence*, Comb. Probab. Comput. **7** (1998), 295–305.
20. ———, *Graph colouring and the probabilistic method*, vol. 23, Springer Science & Business Media, 2013.
21. A. Nachmias and Y. Peres, *Critical percolation on random regular graphs*, Random Structures Algorithms **36** (2010), 111–148.
22. M. E. J. Newman, *The structure and function of complex networks*, SIAM review **45** (2003), 167–256.
23. B. Pittel, *Edge percolation on a random regular graph of low degree*, The Annals of Probability **36** (2008), 1359–1389.
24. O. Riordan, *The phase transition in the configuration model*, Comb. Probab. Comput. **21** (2012), 265–299.
25. B. Sudakov and J. Vondrak, *A randomized embedding algorithms for trees*, Combinatorica **30** (2010), 445–470.
26. N. Wormald, *Some problems in the enumeration of labelled graphs*, University of Newcastle, 1979, PhD thesis.
27. N. C. Wormald, *Models of random regular graphs*, Surveys in combinatorics, 1999 (Canterbury), London Math. Soc. Lecture Note Ser., vol. 267, Cambridge Univ. Press, Cambridge, 1999, pp. 239–298.

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