On (α, k)-sets and (α, k)-hulls in the plane

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Abstract

This abstract reports first the study of upper and lower bounds for the maximum number of all the combinatorially different (α, k)-sets of an n-point set P in the plane, $0 < \alpha \leq \pi$ and $0 < 2k < n$, depending on the (fixed/variable) values of α and k, relating them with the known bounds for the maximum number of k-sets: the $O(n\sqrt[3]{k})$ upper bound from Dey [5] and the $ne^{\Omega(\sqrt{\log k})}$ lower bound from Tóth [7]; and showing also efficient algorithms for generating all of them.

Second we study the depth of a point $p \in P$ according to the (α, k)-set criterion (instead of the k-set criterion). We compute the depths of all the points of P for a given angle α, and also design a data structure for reporting the angle-interval(s) of a given depth for a point of P in $O(\log n)$ time (if it exists).

Finally, we define the (α, k)-hull of P for fixed values of α and k, and design an algorithm for computing the (α, k)-hull of P for given values of α and k. To do that, we follow the relevant ideas and techniques from Cole et al. [4]. Unfortunately, the algorithm is still no so efficient as we wish, and we believe that their complexities strong depends on the fixed values for the parameters α and k; more concretely, as α is closer to π the time complexity is close to the optimal.

1 Preliminaries

Let P be a set of n points in the plane in general position, i.e., no three points are colinear. A *wedge* is the convex region bounded by two rays with common origin with aperture angle α, $0 < \alpha \leq \pi$ and denoted by an α-wedge. An (α, k)-set of P is a k-subset K of P inside an α-wedge which contain no other point of P, and this α-wedge is denoted by (α, k)-wedge. Each (α, k)-set K has (a not-unique and directed) *associated line* defined by a point of $CH(K)$ and a point of P, and which contains a ray of the α-wedge. The associated line facilitates the counting of the number of combinatorially different (α, k)-sets of P.

To study all the different cases for the parameters α and k, we use the notation α_0 when α is fixed, and k_0 when k is fixed. When these parameters are variable, we simply use α and k. By $f_{k_0}^{\alpha_0}(n)$ we denote the maximum number of (α_0, k_0)-set of P overall n-point sets; and analogously for $f_k^{\alpha_0}(n)$, $f_{k_0}^{\alpha}(n)$, and $f_k^{\alpha}(n)$.

Related works: The first study of the (α, k)-sets was done by Claverol [1, 2] by determining upper and lower bounds on the number of (α, k)-sets for P. Here, we reproduce part of the results. Later, Erickson et al. [6] considered generalizations of the *Centerpoint Theorem* in which the half-spaces are replaced with wedges (or cones) of angle α. There are other papers in the literature focusing in this topic [5, 7].

2 Upper and lower bounds

The results presented here about the upper and lower bounds for $f_k^{\alpha}(n)$ are summarized in Figure 1 and classified into the four cases for the values (fixed or variable) of α and k. Almost all of them were obtained in [1, 2]. Due to the lack of space, we only illustrate the lower bound for α_0 and k_0.

α	k	Upper bound	Lower bound
α_0	k_0	$O(n^2)$	$f_{k_0}^{\alpha_0}(n)$
α_0	k	$O(n^3)$	$\Omega(n^3)$
α	k_0	$O(n^3\sqrt[3]{k_0})$	$\Omega(n^2 k_0)$
α	k	$O(n^4)$	$\Omega(n^4)$

Figure 1: Upper and lower bounds for $f_k^{\alpha}(n)$, with $f_{k_0}^{\alpha_0}(n)$ being a function of α_0 and k_0.

Theorem 1

$$f_{k_0}^{\alpha_0}(n) \in \Omega\left((\frac{\pi}{2\alpha_0} - 1) \left(n(k_0 - 1) - \frac{\pi}{2\alpha_0}(k_0 - 1)^2 \right) \right).$$

Proof. We construct a set P of n points in *convex position* as follows. First, put $m+1$ points $s_0, s_1, \ldots, s_m$

*Email: merce.claverol@upc.edu. Supported by MICINN PID2019-104128GB-I00/ AEI/ 10.13039/501100011033, Gen. Cat. DGR2017SGR1640.

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This work has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 734922.

in counterclockwise order on the unit circle C (red points in Figure 2), where $m+1 = \lfloor \frac{\pi}{2\alpha_0} \rfloor$. These points are equally spaced on C such that the chord $s_i s_{i+1}$ subtends the angle α_0, for $i = 0, 1, \ldots, m-1$.

We add to P the sets S_i, $i = 0, 1, \ldots, m$, each formed by s_i together with $k_0 - 2$ points of P, all of them equally spaced, and close enough to the point s_i, i.e., the distance between consecutive points, including s_i, is ε, for a small enough $\varepsilon > 0$. Notice that the length of the arc of the unit circle subtended by a central angle $2\alpha_0$ is exactly $2\alpha_0$, so we can take ε small enough such that all the points in S_i are in an arc of C of length less than α_0. In total, we are adding $(m+1)(k_0-1)$ points to P. Finally, add to P a set T of $n-(m+1)(k_0-1)$ equally spaced points, close enough between them to complete the total of n points (see Figure 2). Notice that this construction works for small angles α_0 such that $m+1 \geq 2$ and thus, $\frac{\pi}{2\alpha_0} \geq 2$, i.e, $\alpha_0 \leq \frac{\pi}{4}$.

Thus, we have α_0-wedges containing one point from T and k_0-1 points from $P \setminus T$, i.e., k_0-1-u points from S_i and u points from S_{i+1}, for some $0 \leq i \leq m-1$ and $0 \leq u \leq k_0-2$ (see Figure 2). Notice that we are not selecting the (k_0-1)-subset S_m when $u=0$. Then, for the set P, we know that the total number of combinatorially different (α_0, k_0)-sets is

$$\Omega\left((\frac{\pi}{2\alpha_0} - 1) \left(n(k_0-1) - \frac{\pi}{2\alpha_0}(k_0-1)^2 \right) \right).$$

As $\frac{\pi}{2\alpha_0}$ is constant, $f_{k_0}^{\alpha_0}(n)$ grows up with k_0. Thus, for $k_0 = \frac{n}{4}+1$, $f_{n/4+1}^{\alpha_0}(n) \in \Omega(n^2)$, which is almost the obtained upper bound. We assume that $k_0 \geq 3$ since for $k_0 = 2$ we have $\binom{n}{2}$ combinatorially different $(\alpha_0, 2)$-sets for some small value of α_0. $\square$

3 The α-wedge depth with respect to P

The α_0-wedge depth of $x \in \mathbb{R}^2$ with respect to P with the (α_0, k)-set criterion, α_0-depth for short, denoted by $Depth_{\alpha_0}^P(x)$, is defined as follows:

$$Depth_{\alpha_0}^P(x) = \min_{1 \leq k \leq n/2} \{k = |P \cap W|\},$$

for any possible closed α_0-wedge W with apex at x.

This definition is an extension of the depth with the k-set criterion, where the k-line passes through point x. In fact, if $\alpha_0 = \pi$, we obtain the depth with the k-set criterion. Thus, for any $p_i \in P$ and α_0, $0 < \alpha_0 \leq \pi$, $Depth_{\alpha_0}^P(p_i) \leq n/2$. The points $p_i \in P$ in the boundary of the convex hull of P, $CH(P)$, have $Depth_{\alpha_0}^P(p_i) = 1$. By the definition, if x is either in the exterior of $CH(P)$ or it belongs to the boundary of $CH(P)$ but $x \notin P$ and $\alpha_0 < \pi$, then $Depth_{\alpha_0}^P(x) = 0$. Moreover, for α_0 small enough, all the points $p_i \in P$ have $Depth_{\alpha_0}^P(p_i) = 1$, and there are no points x with $Depth_{\alpha_0}^P(x) \geq 2$.

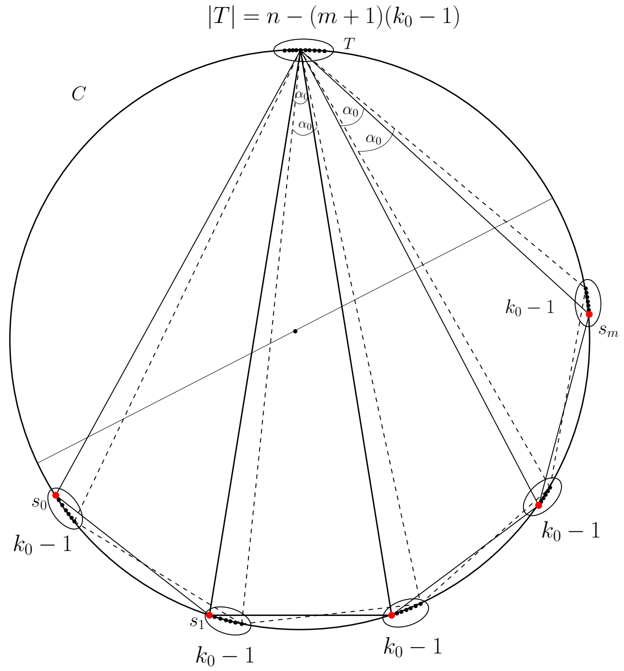

Figure 2: Equally spaced groups of points S_i on C.

Theorem 2 *Given P and α_0, the sorted list of the values $Depth_{\alpha_0}^P(p_i)$, $i = 1, \ldots, n$, can be computed in $O(n^2 \log n)$ time and $O(n)$ space. Then, $Depth_{\alpha_0}^P(p_i)$ can be computed in $O(\log n)$ time.*

3.1 Report α such that $Depth_\alpha^P(p_i) = k$

We consider the problem: Pre-compute a data structure such that given $p_i \in P$ and k, $1 \leq k \leq n/2$, then the angular interval $(\alpha_1^i, \alpha_2^i) \subseteq [0, 2\pi)$ such that for any $\alpha \in (\alpha_1^i, \alpha_2^i)$ the $Depth_\alpha^P(p_i) = k$ can be reported in $O(\log n)$ time.

Theorem 3 *In $O(n^2 \log n)$ time and $O(n^2)$ space we can compute a data structure such that for a given $p_i \in P$ and k, $1 \leq k \leq n/2$, in $O(\log n)$ time we can reported the angular interval $(\delta_i^k, \delta_i^{(k+1)}) \subseteq [0, 2\pi)$ such that for any $\alpha \in (\delta_i^k, \delta_i^{(k+1)})$ the $Depth_\alpha^P(p_i) = k$.*

4 The (α_0, k_0)-hulls

A point x inside $CH(P)$ can be characterized by the property that any line through x has at least a point of P in each of the closed half-planes determined by the line. Generalizing this definition, Cole et al. [4] defined the *k-hull of P*, for positive k, as the set of points x such that for any line through x there are at least k points of P in each closed half-plane. It is clear that the k-hull contains the $(k+1)$-hull, and if k is greater than $\lceil n/2 \rceil$ then the k-hull is empty. We extend this definition to the (α_0, k_0)-hull(P) for $0 < \alpha_0 \leq \pi$ and $1 \leq k_0 \leq n/2$, as follows.

Definition 4 *The (α_0, k_0)-hull(P) is the set of points $x \in \mathbb{R}^2$ such that any closed α_0-wedge with apex at x contains at least k_0 points of P.*

Notice that x has to be inside $CH(P)$, and then, the (α_0, k_0)-hull(P) is contained in $CH(P)$. For $\alpha_0 = \pi$, Definition 4 is equivalent to the k_0-hull of P. From the same definition, we can easily conclude that the (α_0, k_0)-hull(P) contains the $(\alpha_0, (k_0 + 1))$-hull(P).

4.1 The (α_0, k_0)-hulls for points in convex position

For points in convex position, we select n consecutive sets of k_0 consecutive points in $CH(P)$, say K, and compute the corresponding arcs, $a_{i,j}$, of adjoint circles defined by apices of the α_0-wedges containing K supported in two points, p_i and p_j, of $CH(K)$. The endpoints of the arcs occurs when a ray of the α_0-wedge bumps a point of $CH(K)$. There are $O(n)$ arcs, and the number of intersections between those arcs is at most $O(n^2)$. Doing a sweep-line we compute the (α_0, k_0)-hull(P). If α_0 is close to π, the number of intersections between the arcs is $O(n)$, and the complexities decrease accordingly. See Figure 3.

Theorem 5 *The (α_0, k_0)-hull(P) can be computed in $O(n^2 \log n)$ time and $O(n^2)$ space.*

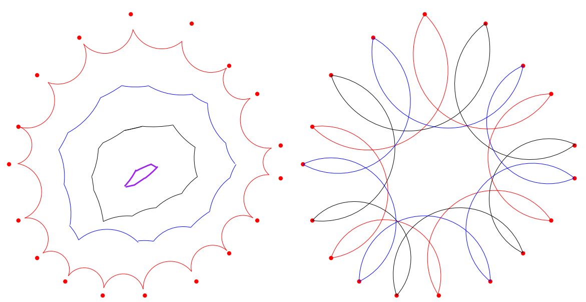

Figure 3: Left: $(90, i)$-hull(P) contains $(90, i + 1)$-hull(P), $i = 1, 2, 3, 4$, and $(90, 5)$-hull(P) is an empty; Right: the $(90, 3)$-hull(P) as the intersection of regions defined by the three cycles.

4.2 Computing the $(\alpha_0, 1)$-hull(P)

Let $e_i = p_i p_{i+1}$ be the edges of the boundary of $CH(P)$. By Definition 4, the vertices $CH(P)$ belong to the $(\alpha_0, 1)$-hull(P). The two rays of any α_0-wedge with apex at x inside $CH(P)$ which containing exactly one point of P not in $CH(P)$ has to intersect the same edge e_i of $CH(P)$. Thus, we have the next fact about the $(\alpha_0, 1)$-hull(P).

Fact 1 *The apices of the α_0-wedges containing exactly one point which rays cross the edge e_i define a polygonal-curve f_i from p_i to p_{i+1}. By definition, f_i is not self-intersecting. We call R_i the (closed) region defined by e_i and f_i, where $\overline{R_i} = CH(P) \setminus R_i$ (see Figure 4 Up). The $(\alpha_0, 1)$-hull(P) is the region $\bigcap_{i=1,\dots,m} \overline{R_i}$ (see Figure 4 Down).*

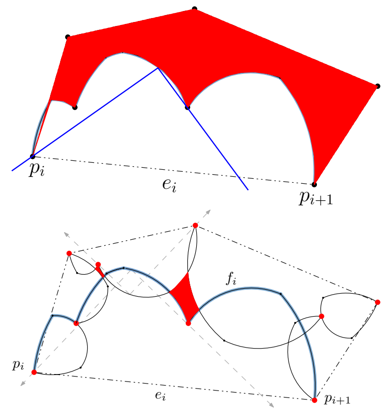

Figure 4: Up: Region $\overline{R_i}$ in red, $\alpha_0 = 90$. Down: $(90, 1)$-hull(P)= $\bigcap_{i=1,\dots,m} \overline{R_i}$ in red. In blue the polygonal-curve f_i

The $(\alpha_0, 1)$-hull(P) can have disconnected regions, as the points in $CH(P) \cap P$ in Figure 4. In fact, if $\alpha_0 < \pi$ and $p \in CH(P) \cap P$, p is an isolated point of $(\alpha_0, 1)$-hull(P). Any $p \in P$ belongs to $(\alpha_0, 1)$-hull(P) because a α_0-wedge with apex at p always contains at least p. Based on Fact 1, we describe the steps of an algorithm for computing $(\alpha_0, 1)$-hull(P) for a set P of n points in general position.

1. If $\frac{\pi}{2} \leq \alpha_0 < \pi$, in $O(n \log n)$ time and $O(n)$ space we compute the $(\alpha_0, 1)$-hull(P) as follows: compute the α_0-maximal points of P and the polygonal-curves f_i for the edges e_i (see Figure 4 Right). Alegría-Galicia et al. [3] showed the algorithm for computing the sequence of $O(n)$ arcs forming all the f_i, $i = 1 \dots, m$, where m is the number of edges of $CH(P)$. Then, in $O(n \log n)$ time and $O(n)$ space we can do a line-sweep of the arrangement of all f_i, and compute $(\alpha_0, 1)$-hull(P) formed by a (possible disconnected) region defined by the intersection of all $\overline{R_i}$, $i = 1 \dots, m$ and $CH(P)$. See Figure 4.

2. If $0 < \alpha_0 < \frac{\pi}{2}$, the computation of all the f_i, $i = 1 \dots, m$ can be done in $O(\frac{n}{\alpha_0} \log n)$ time and $O(\frac{n}{\alpha_0})$ space. But the the computation of $(\alpha_0, 1)$-hull(P) can be done in $O(n^2)$ time and space because we compute the (at most) $O(n^2)$ intersection points inside $CH(P)$ between all the f_i.

Theorem 6 *If $\pi/2 \leq \alpha_0 \leq \pi$, the $(\alpha_0, 1)$-hull(P) can be computed in $O(n \log n)$ time and $O(n)$ space. If $0 < \alpha_0 < \pi/2$, the $(\alpha_0, 1)$-hull(P) can be computed in $O(n^2)$ time and space.*

4.3 The (α_0, k_0)-hulls for points in general position

We adapt ideas in [4] to the (α_0, k_0)-hull(P) concept as follows. A (directed) line ℓ is a k_0-divider for P if ℓ has at most $k_0 - 1$ points of P strictly to its right and at most $n - k_0$ points of P strictly to its left. For any orientation $\theta \in [0, 2\pi)$ of ℓ, there is a unique k_0-divider denoted by ℓ_θ. A special k_0-divider is a k_0-divider that contains at least two points. The half-space to the left of a k_0-divider is a *special half-space*. The k_0-hull is the intersection of the special half-spaces.

The *direction of an* α_0*-wedge* with apex at $x \in \mathbb{R}^2$ is defined by the direction of its right ray (in the clockwise rotation from its apex x); and it is given by the angle θ formed by X-axis with the line containing the right ray. Let $W^\theta_{\alpha_0}$ denote a directed α_0-wedge.

Definition 7 *A directed* α_0*-wedge* $W^\theta_{\alpha_0}$ *is a* directed (α_0, k_0)-divider *for* P*, if* $W^\theta_{\alpha_0}$ *contains at most* $k_0 - 2$ *points of* P *strictly in its interior, and at most* $n - k_0$ *points of* P *strictly in its exterior. The boundary of* $W^\theta_{\alpha_0}$ *must contain at least two points of* P*. A* special (α_0, k_0)-divider *is an* (α_0, k_0)*-divider for* P *that contains at least three points on the boundary.*

Given $\theta \in [0, 2\pi)$, there are at most $O(n)$ different directed (α_0, k_0)-dividers $W^\theta_{\alpha_0}$, e.g., a set P with $O(n)$ points on the (almost vertical) right chain of $CH(P)$, $k_0 = 4$, $\alpha_0 = \pi/4$, and $\theta = 0$, see Figure 5. For $\alpha_0 = \pi$ and fixed θ, there are only two (π, k_0)-dividers.

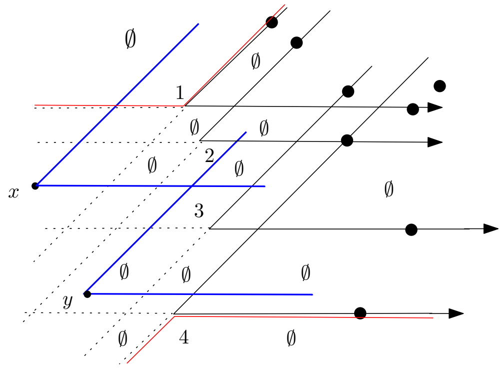

Figure 5: Different directed (α_0, k_0)-dividers $W^\theta_{\alpha_0}$.

Rotation process: An (α_0, k_0)-divider passing through p_i and p_j can be rotated anchored at p_i and p_j while its apex traces an arc $a_{i,j}$ on the *adjoint circle* defined by α_0 and segment $\overline{p_i p_j}$, until the wedge bumps a point p_k, see Figure 6. The poly-curve defined by arcs, rays, and segments from the wedges at the endpoints define an unbounded region denoted by $A_{i,j}$. The region $A_{i,j}$ has the property that for any point x in its interior there always exists an α_0-wedge with apex at x and direction in the rank between the directions of the extreme wedges which contains at most k_0 points, see Figure 6. Starting with an orientation, say $\theta = 0$, the rotation process end at the initial (α_0, k_0)-divider, and the apices of the α_0-wedges trace a cycle, which interior is the intersection of the complementary of the union of all the regions $A_{i,j}$.

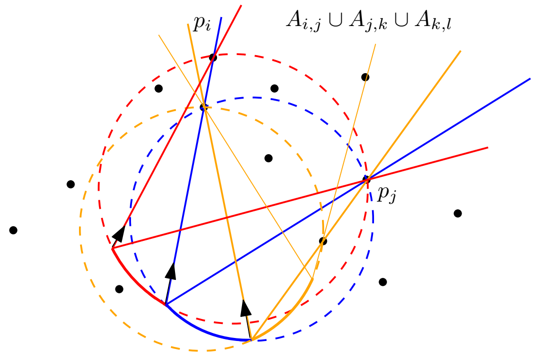

Figure 6: Three consecutive (colored) arcs.

Sketch of the algorithm: First, take the orientation $\theta = 0$, and compute the list $\mathcal{L}$ of the (α_0, k_0)-dividers with this orientation. For each (α_0, k_0)-divider, apply the rotation process above and compute the corresponding cycle, checking the used (α_0, k_0)-dividers in $\mathcal{L}$. There are at most $O(n)$ different cycles, formed by sets of arcs $a_{i,j}$. The total number of arcs $a_{i,j}$ is bounded by the number of (α_0, k_0)-sets, together with their rotations. The still no fixed question is the upper bound of the number of intersection between those arcs depending on α_0. In any case there are at most $O(n^4 {k_0}^2)$. We do a line-sweep of the arrangement of cycles and compute the region(s) of (α_0, k_0)-hull(P).

Theorem 8 *The* (α_0, k_0)*-hull(*P*) can be computed in* $O(n^4 {k_0}^2 \log n)$ *time and* $O(n^2 k_0)$ *space.*

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