

The edge labeling of higher order Voronoi diagrams

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Abstract

We present an edge labeling of order- k Voronoi diagrams, $V_k(S)$, of point sets S in the plane, and study properties of the regions defined by them. Among them, we show that $V_k(S)$ has a small orientable cycle and path double cover, and we identify configurations that cannot appear in $V_3(S)$.

1 Introduction

Let S be a set of n points in general position in the plane (no three collinear, and no four cocircular), and let $1 \leq k \leq n - 1$ be an integer. The order- k Voronoi diagram of S , $V_k(S)$, is a subdivision of the plane into cells, also called faces, such that the points in the same cell have the same k nearest points of S , also called k nearest neighbors. Voronoi diagrams have applications in a broad range of disciplines, see e.g. [1]. The most studied Voronoi diagrams of point sets S are $V_1(S)$, the classic Voronoi diagram, and $V_{n-1}(S)$, the furthest point Voronoi diagram, which only has unbounded faces. Many properties of $V_k(S)$ were obtained by Lee [8], we also mention [4, 9, 10, 13] among the sources on the structure of $V_k(S)$.

An edge that delimits a cell of $V_k(S)$ is a (possibly unbounded) segment of the perpendicular bisector of two points of S . This well-known observation induces a natural labeling of the edges of $V_k(S)$ with the following rule:

- **Edge rule:** An edge of $V_k(S)$ in the perpendicular bisector of points $i, j \in S$ has labels i and j . We put the label i on the side (half-plane) of the edge that contains point i and label j on the other side.

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See Figure 1. Based on the edge rule, we prove a vertex rule and a face rule. $V_k(S)$ contains two types of vertices, denoted as type I and type II (also called new and old vertices [8]), that are defined below.

- **Vertex rule:** Let v be a vertex of $V_k(S)$ and let $\{i, j, \ell\} \in S$ be the set of labels of the edges incident to v . The cyclic order of the labels of the edges around v is i, i, j, j, ℓ, ℓ if v is of type I, and it is i, j, ℓ, i, j, ℓ if v is of type II.

- **Face rule:** In each face of $V_k(S)$, the edges that have the same label i are consecutive, and these labels i are either all in the interior of the face, or are all in the exterior of the face.

Note that when walking along the boundary of a face, in its interior (exterior), a change in the labels of its edges appears whenever we reach a vertex of type II (type I), or possibly at consecutive unbounded edges of an unbounded face, see Figure 1.

Edges with same label i enclose a region $R_k(i)$ consisting of all the points of the plane that have point $i \in S$ as one of their k nearest neighbors from S . The union of all these regions $R_k(i)$ is a k -fold covering of the plane. $R_k(i)$ is related to the k -th nearest point Voronoi diagram ($k - NP VD$) of S , that assigns to each point of the plane its k -nearest neighbor from S [10]. This diagram is also called k -th degree Voronoi diagram in [4]. The region of a point i in $k - NP VD$ is $R_k(i) \setminus R_{k-1}(i)$. In [5] it was proved that $R_k(i)$ is star-shaped. We further observe that $R_1(i)$ is contained in its kernel, and we identify the reflex (convex) vertices on its boundary $B_k(i)$ as vertices of type II (type I).

We also show that every $V_k(S)$ admits an *orientable double cover* [7] of its edges using, precisely, the cycles and paths $\bigcup_{i \in S} B_k(i)$. A cycle and path double cover of a graph G is a collection of cycles and paths \mathcal{C} such that every edge of G belongs to precisely two elements of \mathcal{C} . Paths are needed in the double cover \mathcal{C} of $V_k(S)$ due to the unbounded edges. A double cover \mathcal{C} is orientable if an orientation can be assigned to each element of \mathcal{C} such that for every edge e of G , the two cycles, resp. paths, that cover e are oriented in opposite directions through e [7].

The *small cycle double cover conjecture* states that every simple bridgeless graph on n vertices has a cycle double cover with at most $n - 1$ cycles [2]. Seyf-

farth [12] proved this conjecture for 4-connected planar graphs, and also proved that any simple bridgeless planar graph of size n has a cycle double cover with at most $3\lfloor(n-1)/2\rfloor$ cycles [11]. We show in Property 7 that a higher order Voronoi diagram admits a much smaller cycle and path double cover compared to its number of vertices.

We present several more new properties of $V_k(S)$. All of them rely on the edge labeling and on elementary geometric arguments in the plane. This technique also allows us to obtain new proofs of known results about $V_k(S)$. Other techniques used previously also apply projections of points to \mathbb{R}^d and hyperplane arrangements, among others.

We finally focus on the edge labeling of $V_3(S)$ and show that certain configurations cannot appear in $V_3(S)$. We omit all proofs in this abstract.

We define some notation. The points of S are $\{1, \dots, n\}$. A set of k neighbors defining a cell of $V_k(S)$ is denoted by $P_k \subset S$ and the corresponding face is denoted by $f^b(P_k)$. If the face is bounded, we denote it by $f^b(P_k)$. Note that not every possible P_k determines a face in $V_k(S)$. The (perpendicular) bisector of two points $i, j \in S$ is denoted b_{ij} . And an edge of $V_k(S)$ on b_{ij} is denoted \bar{b}_{ij} . A vertex of $V_k(S)$ is the intersection of three bisectors b_{ab} , b_{bc} and b_{ca} . Equivalently, it is the center of the circle through $a, b, c \in S$. There are two types of vertices in $f^b(P_k)$, and hence in $V_k(S)$: If $a \in P_k$ and $b, c \in S \setminus P_k$ we say that it is of type I; and if $a, b \in P_k$ and $c \in S \setminus P_k$, we say that it is of type II. It is well known that a vertex of type I in $V_k(S)$ is also a vertex of $V_{k+1}(S)$, and a vertex of type II in $V_k(S)$ is also a vertex of $V_{k-1}(S)$ [8]. Further, the three edges of $V_k(S)$ around the vertex alternate in cyclic order with the three edges of $V_{k+1}(S)$ (resp. $V_{k-1}(S)$).

The edges and vertices of $V_k(S)$ form (a drawing of) a graph, which also contains unbounded edges. When considering the union of $V_k(S)$ and $V_{k+1}(S)$ (resp. $V_{k-1}(S)$), the graph induced by $V_{k+1}(S)$ ($V_{k-1}(S)$) in a cell $f^b(P_k)$ of $V_k(S)$ is the subgraph of $V_{k+1}(S)$ ($V_{k-1}(S)$) whose vertices and edges are contained in $f^b(P_k)$.

2 Properties of the labeling of $V_k(S)$

Property 1 *For $k > 1$, every bounded face $f^b(P_k)$ of $V_k(S)$ contains at least two and at most $n - k$ vertices of type I, and at least two and at most k vertices of type II.*

The lower bounds in Property 1 were already given in [8]. In particular, they imply that for $k > 1$, $V_k(S)$ does not contain triangles, as also proved in [9] with a different method. As for the upper bounds, there exist point sets S such that some face of $V_k(S)$ has exactly $n - k$ vertices of type I and k vertices of type II.

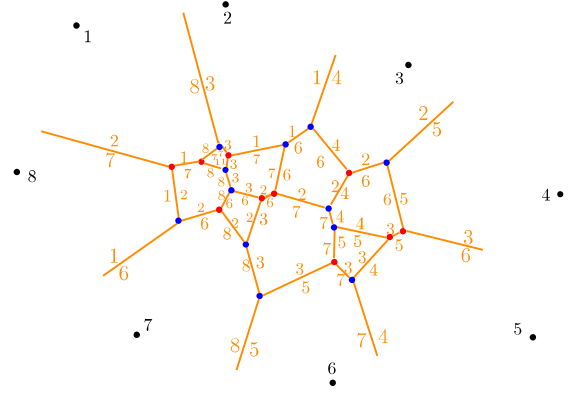


Figure 1: The edge labeling of $V_3(S)$ for a set S of eight points in convex position. Vertices of type I are drawn in blue, and vertices of type II in red.

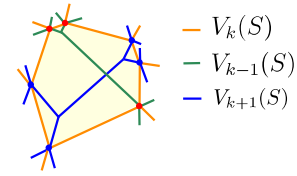


Figure 2: The graphs induced by $V_{k-1}(S)$ and $V_{k+1}(S)$ in a face $f^b(P_k)$ are trees. Each of them creates a subdivision of $f^b(P_k)$ into regions.

The next property concerns the graphs induced by $V_{k+1}(S)$ and $V_{k-1}(S)$ inside $f^b(P_k)$. It is known that these graphs are trees [4, 8, 13], see Figure 2.

Property 2 *Let $f^b(P_k)$ be a bounded cell of $V_k(S)$, for $k > 1$. Then, on its boundary, not all vertices of the same type are consecutive. Equivalently, inside $f^b(P_k)$, there is an edge of $V_{k+1}(S)$ that crosses an edge of $V_{k-1}(S)$.*

The graphs induced by $V_{k+1}(S)$ and $V_{k-1}(S)$ inside a bounded face $f^b(P_k)$ of $V_k(S)$ determine a subdivision of $f^b(P_k)$ into regions, see Figure 2. All the regions induced by $V_{k+1}(S)$ in $f^b(P_k)$ have the same k nearest neighbors from S , and a different $(k+1)$ -nearest neighbor. Similarly, all the regions induced by $V_{k-1}(S)$ in $f^b(P_k)$ have the same k nearest neighbors and differ in the k -nearest neighbor. Property 3 describes the labeling of the edges in the boundary of such regions, from which the vertex rule in the introduction can be deduced. There are two types of boundary edges in the regions: edges of $f^b(P_k)$ and edges of $V_{k+1}(S)$ (resp. $V_{k-1}(S)$).

Property 3 *Let $f^b(P_k)$ be a bounded cell of $V_k(S)$, $k > 1$, and let R be a region in the subdivision of $f^b(P_k)$ induced by $V_{k+1}(S)$ (resp. $V_{k-1}(S)$). The edges of $V_{k+1}(S)$ ($V_{k-1}(S)$) in the boundary of R have*

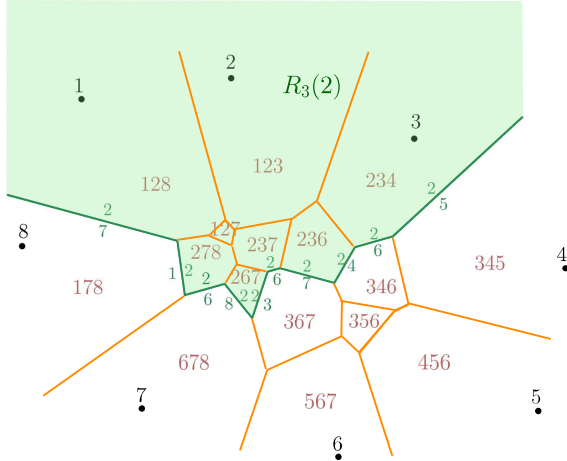


Figure 3: $V_3(S)$ for S the point set in Figure 1; in each face, its three nearest neighbors are indicated. In green, the region $R_3(2)$ formed by all the cells of $V_3(S)$ that have point 2 as one of their three nearest neighbors. All the cells in $R_3(2)$ contain the label 2. The boundary $B_3(2)$ of $R_3(2)$ is formed by all the edges that have the label 2 and this label is always inside $R_3(2)$. Vertices of $B_3(2)$ with an incident edge lying in the interior of $R_3(2)$ are of type II, and the other vertices of $B_3(2)$ are of type I.

the unique $(k+1)$ -nearest neighbor (resp. k -nearest neighbor) of the points of R as label inside (outside) R . The edges of $f^b(P_k)$ in the boundary of R have this label outside (inside) R .

Figure 4 illustrates Property 3 and the vertex rule.

We now describe properties of the region $R_k(i)$ formed by all the faces of $V_k(S)$ having point $i \in S$ as one of their defining points; and of its boundary $B_k(i)$. First, we observe that for every $k > 1$, and for every point $i \in S$, $R_{k-1}(i) \subset R_k(i)$.

Property 4 $R_k(i)$ is a connected region. Furthermore, for any bounded face $f^b(P_k)$ contained in $R_k(i)$, $R_k(i) \setminus f^b(P_k)$ is a connected region.

Property 5 The boundary $B_k(i)$ of a region $R_k(i)$ is formed by all the edges of $V_k(S)$ that have the label i , and this label is always inside $R_k(i)$. $B_k(i)$ is either a cycle, or one or more paths whose first and last edge are unbounded edges of $V_k(S)$.

Property 5 is illustrated in Figure 3.

Property 6 The vertices of $B_k(i)$ that are incident to an edge of $V_k(S)$ lying in the interior of $R_k(i)$ are of type II, and the remaining vertices of $B_k(i)$ are of type I. Moreover, for $k > 1$, if $B_k(i)$ is a cycle, then it encloses at least three faces of $V_k(S)$. If $B_k(i)$ has r reflex vertices, then it encloses at least r faces of $V_k(S)$.

Let \mathcal{C} be the collection of paths and cycles in $\bigcup_{i \in S} B_k(i)$ of $V_k(S)$. We prove that \mathcal{C} forms a double cover of $V_k(S)$. Let f_k^∞ be the number of unbounded faces (or unbounded edges) in $V_k(S)$, also equal to the number of $(k-1)$ -edges of S [4, 13]. It is known that f_k^∞ is at least $2k+1$ [6] and at most $O(n\sqrt[3]{k})$ [3]. We use precisely f_k^∞ paths for \mathcal{C} .

Property 7 $V_k(S)$, which has f_k^∞ unbounded faces, has an orientable cycle and path double cover consisting of f_k^∞ paths and of at most $\max\{n-2k-1, 2k-n-1\}$ cycles.

This double cover of a higher order Voronoi diagram is small compared to its number of vertices; from [4, 8], we deduce that $V_k(S)$ has at least $(2k-1)n-2k^2$ vertices, for $k < (n-2)/2$. When S is in convex position, then $V_k(S)$ has this number of vertices.

The bound on the number of cycles in Property 7 is attained for the point sets S from [6] which have $2k+1$ $(k-1)$ -edges. Property 7 also shows that for $k = \lfloor n/2 \rfloor$ and $k = \lceil n/2 \rceil$, $V_k(S)$ has an orientable path double cover with f_k^∞ paths. This also holds for point sets in convex position and any value of k :

Property 8 Let S be a set of n points in convex position. Then, $V_k(S)$ has an orientable path double cover consisting of n paths.

Property 9 For every $i \in S$, the region $R_k(i)$ of $V_k(S)$ is star-shaped. Furthermore, the face $R_1(i)$ of $V_1(S)$ is contained in its kernel.

Property 10 Let v be a vertex of $B_k(i)$. If v is of type I, then it is a convex vertex of $B_k(i)$; and if v is of type II, it is a reflex vertex of $B_k(i)$.

3 Not too many alternating hexagons in $V_3(S)$

A hexagon in $V_k(S)$ is called alternating, if its vertices alternate between type I and type II, see Figure 5. By Property 1, $V_2(S)$ contains no alternating hexagons. We consider then $V_3(S)$. Figure 5 shows that it is not possible to label the edges of the hexagons of $V_3(S)$ while complying with the properties of the labeling.

Property 11 Let v be a vertex of type I in $V_3(S)$. Then, at most two of the three incident faces to v are alternating hexagons.

Property 12 Any alternating hexagon in $V_3(S)$ is adjacent to at most three other alternating hexagons.

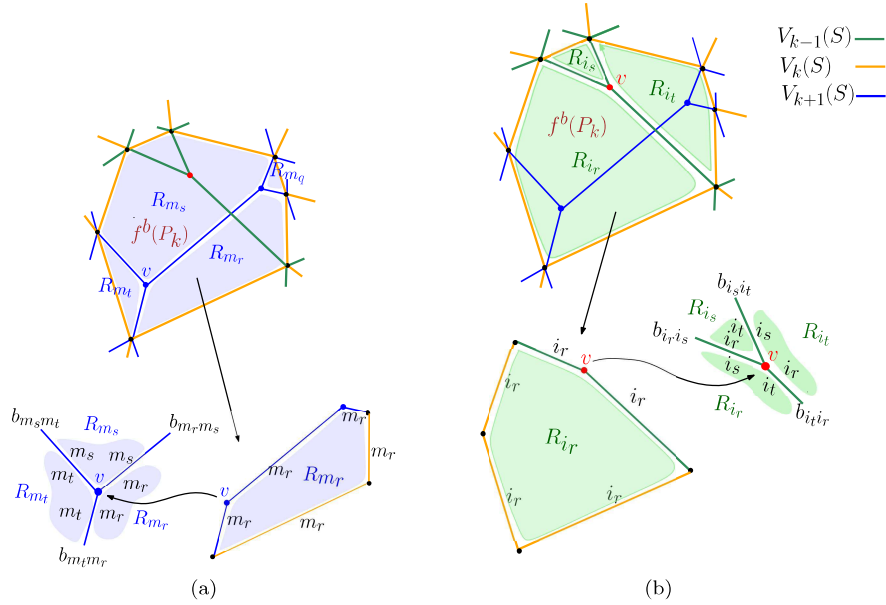


Figure 4: (a) The graph induced by $V_{k+1}(S)$ in $f^b(P_k)$ divides $f^b(P_k)$ into regions R_{m_i} delimited by bisectors between the $(k+1)$ -nearest neighbor $m_i \in S$ of the points of the region, and another point of S . The cyclic order of the labels of the edges around a vertex v incident with R_{m_s} , R_{m_r} and R_{m_t} , is $m_r, m_s, m_s, m_t, m_t, m_r$. (b) The graph induced by $V_{k-1}(S)$ in $f^b(P_k)$ divides $f^b(P_k)$ into regions R_{i_j} delimited by bisectors between the k -nearest neighbor $i_j \in S$ of the points of R_{i_j} , and another point of S . The cyclic order of the labels of the edges around v is $i_r, i_s, i_t, i_r, i_s, i_t$.

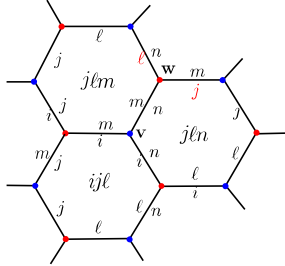


Figure 5: Three alternating hexagons incident to vertex v of type I. The labels around w shown in red do not comply with the properties of the labeling.

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