

CONSTRUCTING TURING COMPLETE EULER FLOWS IN DIMENSION 3

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ABSTRACT. Can every physical system simulate any Turing machine? This is a classical problem which is intimately connected with the *undecidability* of certain physical phenomena. Concerning fluid flows, Moore asked in [19] if hydrodynamics is capable of performing computations. More recently, Tao launched a programme based on the Turing completeness of the Euler equations to address the blow up problem in the Navier-Stokes equations. In this direction, the undecidability of some physical systems has been studied in recent years, from the quantum gap problem [9] to quantum field theories [13]. To the best of our knowledge, the existence of undecidable particle paths of 3D fluid flows has remained an elusive open problem since Moore's works in the early 1990's. In this article we construct a Turing complete stationary Euler flow on a Riemannian S^3 and speculate on its implications concerning Tao's approach to the blow up problem in the Navier-Stokes equations.

1. INTRODUCTION

In the book *The Emperor's new mind* [21] Roger Penrose returns to the artificial intelligence debate to convince us that creativity cannot be presented as the output of a "mind" representable as a Turing machine. This idea, which is platonic in nature and highly philosophical, evolves into more tangible questions such as: *What kind of physics might be non-computational?*

The ideas of the book are a source of inspiration and can be taken to several landscapes and levels of complexity: Is hydrodynamics capable of performing computations? (Moore [19]). Given the Hamiltonian of a quantum many-body system, is there an algorithm to check if it has a spectral gap? (this is known as *the spectral gap problem*, recently proved to be undecidable [9]). And last but not least, can a mechanical system (including a fluid flow) simulate a universal Turing machine (*universality*)? (Tao [28, 29, 30]).

This last question has been analyzed related to the conjecture of the regularity of the Navier-Stokes equations [27], which is one of the unsolved problems in the Clay's millennium list. In [31] Tao suggests a connection between a potential blow-up of the Navier-Stokes equations and Turing completeness and fluid computation. It is interesting to mention that another of the one million dollars problem on the same list whose resolution is still pending is the *P versus NP problem*, which concerns the complexity of systems. Grosso modo, the question is if any problem whose solution can be *verified* by an algorithm polynomial in time ("of type *NP*") can also be *solved* by another algorithm polynomial in time ("of type *P*"). The delicate distinction between verification and solution has opened up an intricate scenery combining research in theoretical computer science, physics and mathematics. Although there is no apparent relation between these two celebrated problems, understanding a fluid flow as a Turing machine may shed some light on their connection.

On the other hand, undecidability of systems is everywhere and also on the invisible fine line between geometry and physics: As proven by Freedman [13] non-abelian topological quantum field theories exhibit the mathematical features (combinatorics) necessary to support an NP-hard model. This relates topological quantum field theory and the Jones polynomial (as described by

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Witten [32]) to the $P \neq NP$ problem. Other undecidable problems on the crossroads of geometry and physics are the stability of an n -body system [18], the problem of finding an Einstein metric for a fixed 4-fold as observed by Wolfram [33], ray tracing problems in 3D optical systems [25], or neural networks [26]. Fundamental questions at the heart of low dimensional geometry and topology such as verifying the equivalence of two finitely specified 4-manifolds [33] or the problem of computing the genus of a knot [1] have also been proven to be undecidable and NP -hard problems, respectively.

In this article we address the appearance of undecidable phenomena in fluid dynamics proving the existence of Turing complete fluid flows on a Riemannian 3-dimensional sphere. Our novel strategy fusions the computational power of symbolic dynamics with fine techniques in contact topology and its connection with hydrodynamics unveiled by Sullivan, Etnyre and Ghrist more than two decades ago. The type of flows that we consider are stationary solutions to the Euler equations, which describe the dynamics of an inviscid incompressible flow in equilibrium. Its close companion, the Navier-Stokes equations, describe the dynamics of the viscid case. We end up this article discussing an application to these equations.

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2. EULER EQUATIONS AND BELTRAMI FIELDS

Euler equations model the dynamics of an incompressible fluid flow without viscosity. Even if they are classically considered on \mathbb{R}^3 , they can be formulated on any 3-dimensional Riemannian manifold (M, g) . The div and curl operators follow the same mnemonics as the classical ones in general coordinates from vector calculus (for an introduction to the topic see [2, 22]). The equations read as:

$$\begin{cases} \frac{\partial}{\partial t} X + \nabla_X X &= -\nabla p, \\ \text{div } X &= 0, \end{cases}$$

where p stands for the inner pressure and X is the velocity field of the fluid (a non-autonomous vector field on M). Here $\nabla_X X$ denotes the covariant derivative of X along X . A solution to the Euler equations is called stationary whenever X does not depend on time, i.e., $\frac{\partial}{\partial t} X = 0$. As shown by the celebrated Arnold's structure theorem [2], among stationary solutions, Beltrami fields play a central role. We say that a divergence-free vector field X on (M, g) is Beltrami if

$$\text{curl } X = fX,$$

with $f \in C^\infty(M)$. When f is non-vanishing, we call those vector fields *rotational*. Hopf fields on \mathbb{S}^3 and ABC flows on \mathbb{T}^3 are examples of rotational (actually with constant factor) Beltrami fields.

The Euler equations can be defined in higher dimensions [2], an extension that is very useful to show that the steady Euler flows exhibit remarkable universality features. In [7] we established a sort of universality of the stationary Euler flows by proving that any non-autonomous dynamics can be embedded into a steady Euler flow of high dimension. To achieve this, we used stationary solutions of Beltrami type (which can be defined on any odd dimensional manifold) and a remarkable connection with contact geometry [12]: a correspondence principle between Beltrami fields and Reeb flows of contact structures in arbitrary dimension, which we describe in the next section. In our construction we also benefited from the h -principle in contact topology via novel Reeb embedding theorems which are key to establish the universality properties, see [7] for details.

3. VIEWING FLUID FLOWS THROUGH A CONTACT MIRROR

A contact structure on an odd dimensional manifold M^{2n+1} is determined by a hyperplane distribution ξ given (at least locally) by the kernel of a one form α such that $\alpha \wedge (d\alpha)^n \neq 0$ (condition known as *maximal non-integrability*). We will assume that the distribution is co-oriented (i.e., its normal bundle is oriented). This condition is equivalent to having a global one form defining the contact structure (called a defining contact form). For a fixed contact form we define its associated

Reeb field R by the equations $\alpha(R) = 1$, $\iota_R d\alpha = 0$. Contact geometry is often seen as the odd dimensional analogue of symplectic geometry. Indeed, symplectic and contact manifolds are related by several constructions. In particular, the contactization of an exact symplectic manifold $(M, d\lambda)$ is defined as the manifold $\mathbb{R} \times M$ equipped with the contact structure $\xi_\lambda = \ker(dt + \lambda)$. A key result in contact geometry is the existence of a Darboux theorem: the only local invariant of a contact structure is the dimension. The most simple proof is given by the following path method result in the contact realm:

Theorem 3.1 (Gray stability theorem [15]). *Let ξ_t , $t \in [0, 1]$, be a smooth homotopy of contact structures on a closed contact manifold M . Then there is an isotopy ψ_t of M such that $\psi_t^* \xi_0 = \xi_t$ for each $t \in [0, 1]$. Moreover, if the family is constant in the complement of a compact set K , then the diffeomorphisms ψ_t are the identity away from K .*

There is a surprising connection between contact geometry and hydrodynamics: in short, any non-vanishing Beltrami field can be reparametrized as the Reeb vector field of a contact structure. This geometrical discovery was suggested by Sullivan and proved by Etnyre and Ghrist [12].

In order to understand this remarkable correspondence it is convenient to rewrite the Euler equations in a *dual language*. Duality is given by contraction with the Riemannian metric g . With the one form α defined as $\alpha := g(X, \cdot)$ and the Bernoulli function as $B := p + \frac{1}{2}g(X, X)$, the steady Euler equations become,

$$\begin{cases} \iota_X d\alpha = -dB, \\ d\iota_X \mu = 0, \end{cases}$$

where μ is the Riemannian volume form.

Observe that:

- The equation $\text{curl } X = fX$, with $f \in C^\infty(M)$, satisfied by a Beltrami vector field on a 3-manifold, can be equivalently written as $d\alpha = f\iota_X \mu$. Assume that X is rotational, e.g., $f > 0$, then if X does not vanish on M we infer that

$$\alpha \wedge d\alpha = f\alpha \wedge \iota_X \mu > 0,$$

thus proving that α defines a contact structure on M .

- Obviously, X satisfies $\iota_X(d\alpha) = \iota_X \iota_X \mu = 0$, so $X \in \ker d\alpha$ and therefore it is a reparametrization of the Reeb vector field by the function $\alpha(X) = g(X, X)$, i.e., $R = \frac{X}{\alpha(X)}$.

These observations prove one of the implications of the following theorem, which is due to Etnyre and Ghrist [12]. This result will be a key instrumental tool to construct a Turing complete Euler flow on a Riemannian \mathbb{S}^3 in Section 7.

Theorem 3.2. *Any non-vanishing rotational Beltrami field is a reparametrization of a Reeb vector field for some contact form. Conversely, any reparametrization of a Reeb vector field of a contact structure is a non-vanishing rotational Beltrami field for some Riemannian metric and volume form.*

Beyond this characterization of (non-vanishing, rotational) Beltrami fields in terms of contact geometry, a natural question is whether a general volume preserving vector field is a solution to the Euler equations for some Riemannian metric. More concretely, to address this problem we introduced in [23] the following definition:

Definition 3.3. Let M be a (not necessarily closed) manifold endowed with a volume form μ . A volume preserving vector field X is *Eulerisable* if there is a metric g on M for which X satisfies the stationary Euler equations for some Bernoulli function $B : M \rightarrow \mathbb{R}$.

Since the Eulerisable fields are stationary solutions of the Euler equations on some (M, g) , they exist for all time. With this definition one gets rid of the metric (which is no longer fixed as in the standard setting of the Euler equations), which allows one to exploit the enormous geometric wealth of the Euler equations. The contact mirror stated in Theorem 3.2, which also holds on manifolds with boundary [6], is just the first instance of a series of striking results that establish connections with geodesible flows [24, 8] or Sullivan's theory of foliated cycles [23]. Moreover, as mentioned in Section 2, the Eulerisable flows have proved to be flexible enough to encode any

non-autonomous dynamics by increasing the dimension of the ambient manifold (“universality”), see [7].

4. AN EXCURSION TO CONTACT TOPOLOGY

The goal of this section is to prove the following result in contact topology, which is a key ingredient for the proof of the main result in this article. All along this section D_ρ is a 2-dimensional disk of radius ρ . If $\rho = 1$ we just omit it to write D .

Theorem 4.1. *Let (M, ξ) be a contact 3-manifold and $\varphi : D \rightarrow D$ an area-preserving diffeomorphism of the disk which is the identity (in a neighborhood of) the boundary. Then there exists a defining contact form α whose associated Reeb vector field R exhibits a Poincaré section with first return map conjugated to φ .*

Combining this result with the contact/Beltrami correspondence (Theorem 3.2) we obtain a metric g on M for which R is a Beltrami field. This yields the following corollary:

Corollary 4.2. *Let M be a 3-manifold. Then, given any area-preserving diffeomorphism $\varphi : D \rightarrow D$ of the disk which is the identity (in a neighborhood of) the boundary, there exists a metric g on M such that φ can be realized as the first return map of some Beltrami field X on (M, g) , up to conjugation.*

Remark 4.3. To prove Corollary 4.2 we need to use the well known fact (since the works of Martinet) that any 3-manifold admits a contact structure. In higher dimensions only almost contact manifolds admit contact structures and thus the existence of contact structures on a given manifold is topologically obstructed [4].

Proof of Theorem 4.1.

We divide the proof in two steps. The first one realizes the diffeomorphism φ as the first-return map of a Reeb vector field on a solid torus. This result is not new, but we provide an alternative (and simpler) argument to the proof presented by Bramham in [5, Chapter 4]. In the second step we globalize the previous construction to obtain a Reeb field on any 3-manifold.

Step 1: Constructing a Reeb mapping torus.

Let us denote by λ the one form $r^2 d\phi$, where (r, ϕ) are polar coordinates on the disk D . In particular, the form $dt + \lambda$ on $D \times [0, 1]$ (t is the coordinate on $[0, 1]$) defines a contact form. Since the diffeomorphism φ is area-preserving and the identity in a neighborhood of ∂D , it is isotopic to the identity and the time-1 flow of a Hamiltonian (non-autonomous) vector field. More precisely, there is a family of diffeomorphisms φ_t , which are the identity in a neighborhood of ∂D for all $t \in [0, 1]$, such that $\varphi_1 = \varphi$, $\varphi_0 = \text{id}$, and this family is generated by a family of compactly supported vector fields X_t so that

$$\iota_{X_t} d\lambda = dH_t, \quad (4.1)$$

where H_t is a family of functions (Hamiltonians) of the disk. For each $t \in [0, 1]$, H_t is obviously constant on a neighborhood of ∂D . Additionally, we can safely assume that φ_t is the identity for $t < \delta$ and $t > 1 - \delta$, which implies that H_t is constant on D (t -dependent) for $t < \delta$ and $t > 1 - \delta$. Accordingly, redefining H_t if necessary, we can assume that $H_t = 0$ in a neighborhood of ∂D for all $t \in [0, 1]$, and $H_t = 0$ on the whole D for $t < \delta$ and $t > 1 - \delta$. Let us now define the function

$$\begin{aligned} \tilde{H} : D \times [0, 1] &\longrightarrow \mathbb{R} \\ (a, z) &\longmapsto \tilde{H}(a, z) := H_z(a), \end{aligned}$$

and the one form

$$\tilde{\alpha} := (\tilde{H} + C)dz + \lambda$$

on the cylindrical set $D \times [0, 1]$, where C is a positive constant. We claim that for a large enough constant C , $\tilde{\alpha}$ is a contact form. Indeed, a straightforward computation shows that

$$\tilde{\alpha} \wedge d\tilde{\alpha} = Cd\lambda \wedge dz + \tilde{H}d\lambda \wedge dz - d\tilde{H} \wedge \lambda \wedge dz, \quad (4.2)$$

which is obviously positive if $C > C_0$, a constant that only depends on the C^1 -norm of \tilde{H} on D . Additionally, the Reeb field of this contact form is a multiple of $\frac{\partial}{\partial z} + X$, where X is defined as $X(a, z) := X_z(a)$. This is equivalent to checking the condition $\iota_{\frac{\partial}{\partial z} + X} d\tilde{\alpha} = 0$. Indeed, we have

$$\begin{aligned} \iota_{\frac{\partial}{\partial z} + X} d\tilde{\alpha} &= \iota_{\frac{\partial}{\partial z} + X} (d\lambda + dH_z \wedge dz) \\ &= \iota_X d\lambda - dH_z + (\iota_X dH_z) dz = 0, \end{aligned}$$

where we have used Equation (4.1) to cancel out the first two terms in the second equality, and the third summand in the equality vanishes by contracting the same equation with X . In particular, the flow of the Reeb vector field of $\tilde{\alpha}$ is a reparametrization of the flow of $\frac{\partial}{\partial z} + X$, whose time-one map is given by φ .

By the construction of the Hamiltonian family H_t , we conclude that the contact form $\tilde{\alpha}$ is equal to $Cdz + \lambda$ on a neighborhood of the boundary of the set $D \times [0, 1]$, and it descends to the quotient (the solid torus $D \times \mathbb{S}^1$, where the coordinate z goes to a coordinate θ in \mathbb{S}^1). Still denoting the contact form in the quotient as $\tilde{\alpha}$, it is obvious that near the boundary of the solid torus, $\tilde{\alpha}$ is $Cd\theta + r^2 d\phi$.

Step 2: Global extension.

Let (M, ξ) be a contact 3-manifold and take a circle γ transverse to the contact structure (which always exists and can be chosen C^0 close to any given closed curve). It is standard that there are coordinates (r', ϕ, θ) in a neighborhood $U = D_\rho \times S^1$ of the circle $\gamma = \{0\} \times S^1$ such that ξ is defined by the kernel of the contact form

$$\beta_0 = C(d\theta + r'^2 d\phi).$$

Here D_ρ is a 2-dimensional disk of small enough radius ρ , the coordinates are the standard angle θ on \mathbb{S}^1 and polar coordinates (r', ϕ) on D_ρ , and C is the large constant introduced in Step 1. In particular, multiplying by a suitable positive factor if necessary, we can take a global contact form β defining ξ such that $\beta|_U = \beta_0$. Now we observe that the contact form $\tilde{\alpha}$ obtained in Step 1 can be constructed on a disk D_ρ of arbitrary radius using a ρ -rescaling of D (a conjugation): $\Phi_\rho : D \times \mathbb{S}^1 \rightarrow D_\rho \times \mathbb{S}^1$, with $(r', \phi, \theta) := \Phi(r, \phi, \theta) = (\rho r, \phi, \theta)$. Since $C > 0$ is any large enough constant, it is clear that we can take the radius ρ to be

$$\rho = C^{-1/2}.$$

The (large) constant C is fixed in what follows.

Using the conjugation Φ_ρ we define the contact form $\tilde{\alpha}' := \Phi_{\rho*} \tilde{\alpha}$ on U . Specifically,

$$\tilde{\alpha}' = (\tilde{H}' + C)d\theta + Cr'^2 d\phi,$$

where $\tilde{H}' \equiv \tilde{H}'(r', \phi, \theta) = \tilde{H}(C^{1/2}r', \phi, \theta)$. Obviously the associated Reeb field is a multiple of $\Phi_{\rho*} \left(\frac{\partial}{\partial z} + X \right)$; in particular, its first return map on the section $D_\rho \times \{0\}$ is given by $\Phi_\rho \circ \varphi \circ \Phi_\rho^{-1}$. Additionally, by construction,

$$\tilde{\alpha}' = C(d\theta + r'^2 d\phi)$$

in a neighborhood of the boundary of the solid torus U . $\tilde{\alpha}'$ can then be extended to the complement $M \setminus U$ as β because $\tilde{\alpha}' = \beta_0$ on ∂U . In summary, denoting by α this globally defined contact form on M , we conclude that it coincides with $\tilde{\alpha}'$ in U , and its associated Reeb field in this set has a first return map that is conjugated to φ .

It remains to show that the contact structure defined by $\ker \alpha$ is homotopic through contact structures to ξ . Indeed, let us define the family of one forms

$$a_t := ((1-t)\tilde{H}' + C)d\theta + Cr'^2 d\phi$$

in the toroidal set U . A straightforward computation shows that a_t is a contact form for all $t \in [0, 1]$. Moreover, $a_0 = \alpha$ and $a_1 = C(d\theta + r'^2 d\phi) = \beta_0$. Noticing that $\alpha = \beta$ in $M \setminus U$, this yields a global homotopy of contact forms that interpolates α with β , which immediately implies an homotopy of contact structures ξ_t such that $\xi_0 = \ker \alpha$ and $\xi_1 = \xi$. Applying Theorem 3.1, we deduce that $\ker \alpha$ is contactomorphic to ξ . The theorem then follows. \square

Remark 4.4. An easy modification of the proof of Step 2 allows us to choose the defining contact form β in the complement of the toroidal set U . More precisely, given any contact form β defining the contact structure ξ , there is another defining contact form α such that $\alpha = \tilde{\alpha}'$ on U and $\alpha = \beta$ in the complement of a neighborhood of U .

A similar proof provides an equivalent statement for higher dimensional contact manifolds. It reads as follows:

Theorem 4.5. *Let (M, ξ) be a contact $(2n + 1)$ -manifold and $\varphi : D \rightarrow D$ a symplectomorphism of the $2n$ -ball which is the identity (in a neighborhood of) the boundary. Then there exists a defining contact form α whose associated Reeb vector field R exhibits a Poincaré section with first return map conjugated to φ .*

5. TURING MACHINES AND SYMBOLIC DYNAMICS

The key tool to construct a dynamical system that simulates a Turing machine is symbolic dynamics. Our goal in this section is to recall some basic properties of Turing machines and to introduce Moore's theory [19] on the connection between diffeomorphisms of manifolds and computation. In particular, we shall show that suitable generalizations of the shift map are enough to simulate universal Turing machines. This paves the way to construct a Turing complete area-preserving diffeomorphism of the disk, as we shall see in Section 6.

Remark 5.1. As discussed in several parts of the literature [11, 10], there are misleading intuitions that lead to conclude that the shift map can simulate a universal Turing machine. This happens when we accept to take initial points that are not constructible, i.e., they contain as initial information all the computations of the Turing machine instead of just its initial tape. See Section 5.2 for a detailed explanation.

5.1. Turing machines. A Turing machine is defined via the following data:

- A finite set Q of "states" including an initial state q_0 and a halting state q_{halt} .
- A finite set Σ which is the "alphabet" with cardinality at least two.
- A transition function $\delta : (Q \times \Sigma) \rightarrow (Q \times \Sigma \times \{-1, 0, 1\})$.

Let us denote by $q \in Q$ the current state, and by $t = (t_n)_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ the current tape. For a given Turing machine $(Q, q_0, q_{halt}, \Sigma, \delta)$ and an input tape $s = (s_n)_{n \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ the machine runs applying the following algorithm:

- (1) Set the current state q as the initial state and the current tape t as the input tape.
- (2) If the current state is q_{halt} then halt the algorithm and return t as output. Otherwise compute $\delta(q, t_0) = (q', t'_0, \varepsilon)$, with $\varepsilon \in \{-1, 0, 1\}$.
- (3) Replace q with q' and t_0 with t'_0 .
- (4) Replace t by the ε -shifted tape, then return to step (2). Following Moore [19], our convention is that $\varepsilon = 1$ (resp. $\varepsilon = -1$) corresponds to the left shift (resp. the right shift).

In particular, the space of all possible internal states of a Turing machine is given by $\mathcal{P} := Q \times \Sigma^{\mathbb{Z}}$. The transition function δ induces a global transition function $\Delta : Q \setminus \{q_{halt}\} \times \Sigma^{\mathbb{Z}} \rightarrow \mathcal{P}$, which sends an internal state in \mathcal{P} to the internal state obtained after applying a step of the algorithm.

It is convenient to decompose the transition function δ in three components that we can denote by F_1, F_2, F_3 :

$$\begin{aligned} F_1 : Q \times \Sigma &\longrightarrow Q \\ F_2 : Q \times \Sigma &\longrightarrow \Sigma \\ F_3 : Q \times \Sigma &\longrightarrow \{-1, 0, 1\}. \end{aligned}$$

The first component F_1 tells you the new state q' in terms of the current state q and the tape value t_0 . The second component F_2 computes the new value of the tape cell t'_0 in terms of the state q and the tape value t_0 . Finally, the last component F_3 tells you if the tape should stay, shift to the left or shift to the right in terms of the current state and the tape value t_0 .

As we shall see in Section 5.2, the reversible condition plays a crucial role to construct well behaved dynamical systems that simulate universal computation. Reversibility of a Turing machine

can be defined in several equivalent ways. The definition in terms of the global transition function Δ is suitable for our discussion.

Definition 5.2. A Turing machine $T = (Q, q_0, q_{halt}, \Sigma, \delta)$ is *reversible* if the global transition function Δ is injective.

5.2. Generalized shifts and Turing machine simulation. In this section we introduce Moore's theory of generalized shifts [19], which will be instrumental to construct a Turing complete area-preserving diffeomorphism of the disk in Section 6.

First, an important remark on the necessity of this theory is in order: let us elaborate on the reason why the shift map is not suitable to perform universal computation. Indeed, consider a Turing machine with some given initial input. We can associate to it a sequence $(q_i)_{i \in \mathbb{N}}$ of states, where q_i is the state of the machine at the step i . If the machine reaches the halting state at a step j , we define $q_i = q_{halt}$ for all $i \geq j$. Using this sequence, we can construct another sequence $(p_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ by setting $p_i = 0$ if $i < 0$, $p_i = 0$ if $i \geq 0$ and $q_i \neq q_{halt}$ and $p_i = 1$ if $q_i = q_{halt}$. Iterating this sequence by the standard (left) shift map, it is clear that the Turing machine halts if and only if the shift map finds a digit 1 in position 0 at some iteration. The problem is, however, that the initial sequence is not constructible; it is a priori an undecidable problem to construct the whole sequence $(q_i)_{i \in \mathbb{N}}$.

To introduce the notion of generalized shift, let A be an alphabet and $S \in A^{\mathbb{Z}}$ an infinite sequence. A generalized shift is specified by two maps F and G which depend on *finitely many* positions of S . Denote by $D_F = \{i, \dots, i + r - 1\}$ and $D_G = \{j, \dots, j + l - 1\}$ the sets of positions on which F and G depend, respectively. They have cardinality $r \geq 1$ and $l \geq 1$, respectively. Obviously, these functions take a finite number of different values since they depend on a finite number of positions. The function G modifies the sequence only at the positions indicated by D_G :

$$G : A^l \longrightarrow A^l \\ (s_j \dots s_{j+l-1}) \longmapsto (s'_j \dots s'_{j+l-1})$$

Here $s_j \dots s_{j+l-1}$ are the symbols at the positions $j, \dots, j + l - 1$ of an infinite sequence $S \in A^{\mathbb{Z}}$.

On the other hand, the function F assigns to the finite subsequence (s_i, \dots, s_{i+r-1}) of the infinite sequence $S \in A^{\mathbb{Z}}$ an integer:

$$F : A^r \longrightarrow \mathbb{Z}$$

A generalized shift $\phi : A^{\mathbb{Z}} \rightarrow A^{\mathbb{Z}}$ is then defined as follows:

- Compute $F(S)$ and $G(S)$.
- Modify S changing the positions in D_G by the function $G(S)$, obtaining a new sequence S' .
- Shift S' by $F(S)$ positions. That is, we obtain a new sequence $s''_n = s'_{n+F(S)}$ for all $n \in \mathbb{Z}$.

The sequence S'' is then $\phi(S)$. For example, the standard shift is obtained by taking G to be the identity and $F \equiv 1$. For later convenience, when taking a sequence in $A^{\mathbb{Z}}$, we will write a point to denote that the symbol at the right of that point is the symbol at position 0. In particular, the sequence (s_n) can be denoted by $(\dots s_{-1} \cdot s_0 s_1 \dots)$.

The remarkable property of generalized shifts is that they can simulate any Turing machine in the following sense:

Definition 5.3. We say that a generalized shift ϕ with alphabet A is conjugated to a Turing machine $T = (Q, q_0, q_{halt}, \Sigma, \delta)$ if there is an injective map

$$\varphi : \mathcal{P} \rightarrow A^{\mathbb{Z}}$$

such that the global transition function of the Turing machine is given by

$$\Delta = \varphi^{-1} \phi \varphi. \tag{5.1}$$

We recall that $\mathcal{P} = Q \times \Sigma^{\mathbb{Z}}$ is the space of all possible internal states of T .

The following result, which was first proved in [19], establishes that any Turing machine is conjugated to a generalized shift. Although this result is relatively standard, we include a proof because it helps to elucidate the connection between Turing machines and Generalized Shifts.

Lemma 5.4. *Given a Turing machine $T = (Q, q_0, q_{halt}, \Sigma, \delta)$, there is a generalized shift conjugated to it.*

Proof. Recall that F_1, F_2, F_3 denote the three components of the transition function δ of the Turing machine T . Let us construct a generalized shift whose alphabet is given by $A := \Sigma \cup Q$, i.e., both the alphabet and the set of states of the Turing machine T . First, notice that to every internal state $(q, (t_i)_{i \in \mathbb{Z}}) \in \mathcal{P}$ of T , we can assign the sequence $(\dots t_{-1}.qt_0t_1 \dots)$ in $A^{\mathbb{Z}}$.

Now let us define the maps F and G , which will depend only on the three positions $-1, 0, 1$, i.e., $D_F = D_G = \{-1, 0, 1\}$. For a sequence $(s_n) \in A^{\mathbb{Z}}$, denote by $a := (s_{-1}s_0s_1)$ the subsequence of symbols in positions $-1, 0, 1$. If this subsequence a is not of the form $(t_{-1}qt_0)$ with $t_{-1}, t_0 \in \Sigma$ and $q \in Q$, then we define $F(a) := 0$ and $G(a) := a$. Otherwise, we have $a = t_{-1}qt_0$ for some symbols t_{-1}, t_0 in Σ and $q \in Q$. Setting $q' := F_1(q, t_0)$ and $t'_0 := F_2(q, t_0)$, we can define F and G as:

$$F(a) := F_3(q, t_0),$$

and

$$G(a) := \begin{cases} t_{-1}.t'_0q' & \text{if } F(a) = 1, \\ q'.t_{-1}t'_0 & \text{if } F(a) = -1, \\ t_{-1}.q't'_0 & \text{if } F(a) = 0. \end{cases} \quad (5.2)$$

Although the function F only depends on the positions a_0 and a_1 of a , we have chosen $D_F = D_G = \{-1, 0, 1\}$ so that both domains are the same. These maps F and G define a generalized shift ϕ as explained above.

Finally, given an internal state $(q, (t_i)_{i \in \mathbb{Z}}) \in \mathcal{P}$, it is straightforward to check that a step of the Turing machine T corresponds to $\varphi^{-1}\phi\varphi$, where

$$\begin{aligned} \varphi : \mathcal{P} &\longrightarrow A^{\mathbb{Z}} \\ (q, (t_i)_{i \in \mathbb{Z}}) &\longmapsto s = \dots t_{-1}.qt_0t_1 \dots \end{aligned}$$

is an injective map. By definition, this proves that T is conjugated to ϕ , and the lemma follows. \square

The main property that we introduce in this section is that a reversible Turing machine is conjugated to a bijective generalized shift. This fact was stated in [19] without a proof. This is not immediately clear, since the global transition function of a Turing machine is not defined for the halting state q_{halt} , which prevents to extend the generalized shift when the symbol in position 0 is q_{halt} . In order to fix this issue, and to obtain a bijective generalized shift via Lemma 5.4, we can extend the transition function by setting

$$\delta(q_{halt}, t_0) := (q_0, t_0, 0), \quad (5.3)$$

which ensures that δ is defined in all the domain of states Q . This also guarantees that Δ extends to all \mathcal{P} in an injective way. When necessary, we shall assume that the global transition function has been extended this way without further mention. Moreover, Equation (5.1) is satisfied on the whole domain \mathcal{P} .

Lemma 5.5. *A reversible Turing machine T , whose transition function has been extended as above, is conjugated to a bijective generalized shift.*

Proof. By the previous discussion, the global transition function Δ of T is injective and defined on the whole domain of states, so the conjugation specified in Equation (5.1) also holds on the whole domain. Accordingly, the generalized shift map ϕ is injective when restricted to the subset $\varphi(\mathcal{P}) \subset A^{\mathbb{Z}}$ because it is conjugated to the injective map Δ ; and it is also injective on its complement set, where it is the identity map. This shows that the generalized shift ϕ is injective, and in fact bijective by [19, Lemma 2], which completes the proof of the lemma. \square

As shown by Moore, the relevance of generalized shifts comes from the fact that they are conjugated to maps of the square Cantor set, which allows one to use the machinery of symbolic dynamics (compare with the particular case of the standard shift map). We recall the following:

Definition 5.6. The square Cantor set is the product set $C^2 := C \times C \subset I^2$, where C is the (standard) Cantor ternary set in the unit interval $I = [0, 1]$. Additionally, we say that a Cantor block is a block of the form $B = [\frac{a}{3^i}, \frac{a+1}{3^i}] \times [\frac{b}{3^j}, \frac{b+1}{3^j}]$, where i, j are nonnegative integers and $a < 3^i, b < 3^j$ are nonnegative integers such that there are points of C^2 in the interior of B .

It is clear that for given i, j we can find a finite amount of disjoint Cantor blocks whose union contains all the points of the square Cantor set. In what follows, we shall consider generalized shifts with alphabet $A = \{0, 1\}$. Actually, as proved in [19, Lemma 1], this can always be assumed. Given an infinite sequence $s = (\dots s_{-1} \cdot s_0 s_1 \dots) \in A^{\mathbb{Z}}$, we can associate to it an explicitly constructible point in the square Cantor set. The usual way to do this is to express the coordinates of the assigned point in base 3: the coordinate y corresponds to the expansion (y_1, y_2, \dots) where $y_i = 0$ if $s_i = 0$ and $y_i = 2$ if $s_i = 1$. Analogously, the coordinate x corresponds to the expansion (x_1, x_2, \dots) in base 3 where $x_i = 0$ if $s_{-i} = 0$ and $x_i = 2$ if $s_{-i} = 1$.

The aforementioned assignment between infinite sequences and points in the square Cantor set is key to prove a fundamental lemma that we borrow from Moore's work [19]. Combined with Lemma 5.5 it will be key to construct a Turing complete area-preserving diffeomorphism of the disk in Section 6.

Lemma 5.7 (Moore). *Any generalized shift is conjugated to the restriction to the square Cantor set of a piecewise linear map of I^2 . This map consists of k finitely many area-preserving linear components defined on Cantor blocks, with k bounded as:*

$$k \leq n^{|D_F \cup D_G| + \max|F|}.$$

Here $n := |A|$. If the generalized shift is bijective, then the image blocks are pairwise disjoint.

Remark 5.8. In the literature, there have been other attempts to simulate a reversible Turing machine by means of selecting the space of states of the machine as a constructible choice of coordinates in the square I^2 and extending the global transition function from that set of points to a bijective map of I^2 . However, these other models do not provide a continuous [25] or a compactly supported extension [14] or they increase the dimension [28]. In Section 6 we will show that Moore's approach has the advantage that it can be used to promote the map constructed in Lemma 5.7 to a smooth (area-preserving) diffeomorphism of the disk.

Let us briefly explain the main ideas of the construction of the map in Lemma 5.7. If we fix our attention on a single Cantor block, the piecewise linear map is constructed as the composition of two linear maps. The first one is a translation (depending on the function G of the generalized shift), which sends a block onto another one. Next, using the function F , we get an integer which tells us how many shifts have to be applied to the block. The action of the shift map on a block can be obtained by restriction of a positive or negative power of the horseshoe map. This second well known map, is a composition of a translation, a rotation and a rescaling in each coordinate. We finish with the following example, which illustrates Lemma 5.7.

Example 5.9. A simple example of a generalized shift and its associated piecewise linear map can be constructed as follows. Consider a generalized shift with alphabet $\{0, 1\}$, and such that $D_F = D_G = \{-1, 0\}$. We define the functions F and G as: $G(0.1) = 0.1, G(1.1) = 0.0, G(0.0) = 0.1, G(1.0) = 1.1$ and $F(0.1) = F(0.0) = -1, F(1.1) = F(1.0) = 0$. By assigning letters to the Cantor blocks corresponding to each possible finite string of two elements, the associated map can be represented by blocks. Denote by A, B, C and D the Cantor blocks whose corresponding sequences have in positions $-1, 0$ respectively the pairs $(0.1), (1.1), (0.0)$ and (1.0) ; the position of these blocks in the square is computed following the assignment that we introduced before, thus obtaining Figure 1 (in the same figure we also represent the images of the blocks). The piecewise linear map can be explicitly written as:

$$(x, y) \longmapsto \begin{cases} (3x, y/3) & \text{if } (x, y) \in A \\ (3(x - 2/3), 1/3(y - 2/3)) & \text{if } (x, y) \in B \\ (x, y + 2/3) & \text{if } (x, y) \in C \cup D \end{cases}$$

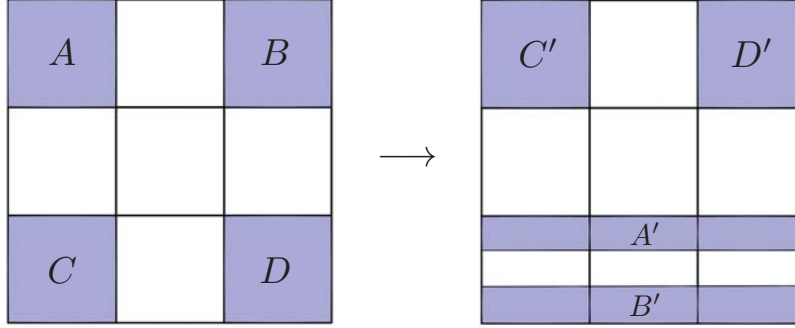


FIGURE 1. Blocks map in the unit square

6. AN AREA-PRESERVING DIFFEOMORPHISM OF THE DISK THAT IS TURING COMPLETE

The goal of this section is to construct an area-preserving diffeomorphism of the disk that simulates a universal Turing machine. The main tool is the generalized shifts introduced in Section 5.2 and their connection with piecewise linear maps of Cantor blocks. In this direction, a first simple observation is that if we choose a set of disjoint blocks containing all the Cantor set, they lie in the unit square with some gaps in between that do not contain points of the square Cantor set. We will use these gaps to extend the piecewise linear map constructed in Lemma 5.7 to an area-preserving diffeomorphism of the disk, provided that the generalized shift is bijective.

6.1. Smoothing the map. For any generalized shift, Lemma 5.7 establishes the existence of a piecewise linear map defined on finitely many Cantor blocks whose action on the square Cantor set is conjugated to the generalized shift. In [19, Theorem 12], Moore sketches an argument to extend this map to a diffeomorphism of the disk. In the following proposition, using standard arguments, we show that this map can be done area-preserving as long as the generalized shift is bijective.

Proposition 6.1. *For each bijective generalized shift and its associated map of the square Cantor set ϕ , there exists an area-preserving diffeomorphism of the disk $\varphi : D \rightarrow D$ which is the identity in a neighborhood of ∂D and whose restriction to the square Cantor set is conjugated to ϕ .*

Proof. For simplicity we use the same notation ϕ for the generalized shift and its associated map of the square Cantor set obtained via Lemma 5.7. This map is defined on a finite disjoint union of Cantor blocks (that contain the whole square Cantor set), and the images of these blocks are pairwise disjoint because the generalized shift is bijective. Taking an open neighborhood D (diffeomorphic to a disk) of the square I^2 , our goal is to extend ϕ to the whole D .

To this end, we start by choosing a set contained in D of disjoint (small enough) open neighborhoods B_i of each Cantor block. Since the map ϕ , which is piecewise linear, is obviously defined on each neighborhood, it maps each B_i onto a neighborhood V_i of the images of the Cantor blocks. Obviously, V_i are pairwise disjoint and have the same area as B_i .

This immediately yields a diffeomorphism $F : \bigcup B_i \rightarrow \bigcup V_i$ that preserves the standard area form $\omega_{std} = dx \wedge dy$. We claim that this map extends to a diffeomorphism of the disk that is isotopic to the identity. To prove this, we construct a family of maps $F_t : \bigcup B_i \rightarrow D$ such that $F_1 = F$, $F_0 = \text{id}$, and F_t is a diffeomorphism into its image for each $t \in [0, 1]$. To construct this family we first define $F_t^{(1)}$ for $t \in [0, 1/3]$ to be an homothety in each open ball B_i , which contracts each ball to a ball of small enough area δ . Specifically, taking a point $p_i \in B_i$ such that B_i is star shaped with respect to p_i (this is possible because B_i is a neighborhood of a Cantor block), for each $x \in B_i$

$$F_t^{(1)}(x) := p_i + \lambda_t(x - p_i),$$

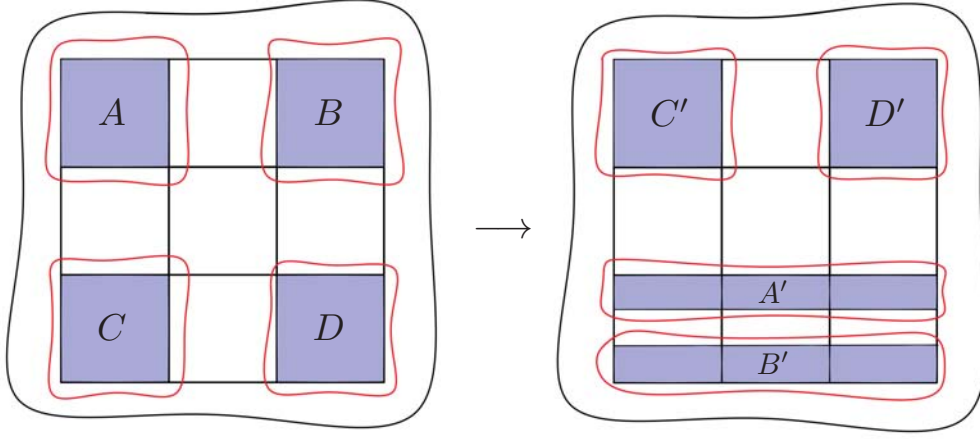


FIGURE 2. Blocks map by open balls

where λ_t is a smooth function on B_i such that $\lambda_0 = 1$ and $\lambda_{1/3} < \delta$. Next we choose different points q_i inside the image balls V_i and construct $F_t^{(2)} : \bigcup F_{1/3}^{(1)}(B_i) \rightarrow D$ for $t \in [1/3, 2/3]$ to be a map such that $F_{1/3}^{(2)} = \text{id}$ and $F_{2/3}^{(2)}$ sends each ball $F_{1/3}^{(1)}(B_i)$ inside a δ -neighborhood of q_i (contained in V_i). Finally, we define an expansion $F_t^{(3)} : \bigcup F_{2/3}^{(2)}(F_{1/3}^{(1)}(B_i)) \rightarrow D$ for $t \in [2/3, 1]$, analogous to $F_t^{(1)}$, so that $F_{2/3}^{(3)} = \text{id}$ and $F_1^{(3)}(F_{2/3}^{(2)}(F_{1/3}^{(1)}(B_i))) = V_i$. The map F_t is then obtained as:

$$F_t := \begin{cases} F_t^{(1)} & \text{for } t \in [0, 1/3], \\ F_t^{(2)} \circ F_{1/3}^{(1)} & \text{for } t \in [1/3, 2/3], \\ F_t^{(3)} \circ F_{2/3}^{(2)} \circ F_{1/3}^{(1)} & \text{for } t \in [2/3, 1]. \end{cases}$$

Now, if we set F_t to be the identity in a neighborhood of the boundary of D , the homotopy extension property allows us to extend F_t to a family of diffeomorphisms φ_t of the disk such that $\varphi_1|_{\bigcup B_i} = F$, $\varphi_0 = \text{id}$ and φ_t is the identity near the boundary of D for all t .

The standard area form ω_{std} is sent to another area form that we denote by $\omega_1 := \varphi_1^* \omega_{std}$. Notice that $\omega_1 = \omega_{std}$ on $\bigcup B_i$ because F is area-preserving. Additionally, we can interpolate linearly between these two area forms:

$$\omega_t := t\omega_1 + (1-t)\omega_{std}.$$

Of course, ω_t is nondegenerate for all $t \in [0, 1]$. Noticing that both forms have the same area since $\int_D \omega_{std} = \int_{\varphi^* D} \omega_{std} = \int_D \varphi^* \omega_{std} = \int_D \omega_1$, it follows that $\omega_1 - \omega_{std}$ is an exact 2-form. Applying Moser's path method we then obtain a family of diffeomorphisms $G_t : D \rightarrow D$, $G_0 = \text{id}$, such that $G_t^* \omega_t = \omega_{std}$ for all $t \in [0, 1]$. Moreover, we can assume that $G_t|_{\bigcup B_i} = \text{id}$ because $\omega_t = \omega_{std}$ for all $t \in [0, 1]$. Finally, the diffeomorphism $\varphi := \varphi_1 \circ G_1$ satisfies the required conditions, i.e., $\varphi|_{\bigcup B_i} = F$ and $\varphi^* \omega_{std} = \omega_{std}$. The proposition follows noticing that D can be identified with the unit disk in \mathbb{R}^2 , after applying a suitable diffeomorphism. \square

6.2. A Turing complete area-preserving diffeomorphism of the disk.

We are now ready to establish the existence of a Turing complete area-preserving diffeomorphism of the disk that is the identity on the boundary. We remark that our notion of Turing completeness is slightly different from the one used in [28, 7], see Remark 6.3 below, but it has the same computational power. Key to the proof are Proposition 6.1 and the constructibility of the unique point in the square Cantor associated to an infinite sequence in $\{0, 1\}^{\mathbb{Z}}$, cf. Section 5.2. In

the proof we also make use of an instrumental result (Lemma 6.4) allowing us to show that our area-preserving diffeomorphism can check a finite substring of the output of a Turing machine.

Theorem 6.2. *There exists a Turing complete area-preserving diffeomorphism φ of the disk that is the identity in a neighborhood of the boundary. Specifically, for any integer $k \geq 0$, given a Turing machine $T = (Q, q_0, q_{halt}, \Sigma, \delta)$, an input tape $(t_n) \in \Sigma^{\mathbb{Z}}$ and a finite string $(t_{-k}^*, \dots, t_k^*) \in \Sigma^{2k+1}$, there is an explicitly constructible point $p \in D$ and an explicitly constructible open set $U \subset D$ such that the orbit of φ through p intersects U if and only if T halts with an output tape whose positions $-k, \dots, k$ correspond to t_{-k}^*, \dots, t_k^* .*

Proof. The first observation is that there are several constructions of reversible universal Turing machines. For instance, in [20] there is an explicit construction with 17 states and an alphabet of 5 symbols. In fact, it is known [3] that for any Turing machine (and in particular for a universal one) there is a reversible Turing machine doing the same computations. Hence, let us denote by T_{un} some reversible universal Turing machine. By Lemma 5.4, we can associate to T_{un} a conjugated generalized shift ϕ , which is, in fact, bijective in view of Lemma 5.5. Applying Proposition 6.1, we can construct an area-preserving diffeomorphism φ of the disk D which is the identity in a neighborhood of ∂D and whose restriction to the square Cantor set is conjugated to ϕ .

We claim that the map φ is Turing complete. Indeed, given a Turing machine $T = (Q, q_0, q_{halt}, \Sigma, \delta)$ and a finite part of the output tape $(t_{-k}^*, \dots, t_k^*) \in \Sigma^{2k+1}$, Lemma 6.4 below allows us to construct another Turing machine T' which reads the output of T . Since T_{un} is universal, it can simulate the evolution of T' . In particular, given an input (q, t) of T' there is an explicit input (\hat{q}, \hat{t}) of T_{un} , with $\hat{q} \in Q_{un}$ and $\hat{t} \in \Sigma_{un}^{\mathbb{Z}}$ (here, Q_{un} and Σ_{un} are the space of states and the alphabet of T_{un} , respectively), such that T' halts with the aforementioned input if and only if T_{un} halts with input (\hat{q}, \hat{t}) .

As explained in the proof of Lemma 5.4 we obtain an explicit sequence $s = (\dots s_{-1}.s_0s_1\dots) \in A^{\mathbb{Z}}$, where $A = \Sigma_{un} \cup Q_{un}$, from the input (\hat{q}, \hat{t}) of T_{un} , which defines a unique point $p \in D$ via the correspondence introduced in Section 5.2. For example, taking $\{0, 1\}$ to be the alphabet A of the generalized shift ϕ (which can always be done as mentioned before), then the coordinates of p are given by

$$p = (x, y) = \left(\sum_{i=1}^{\infty} \frac{2s_{-i}}{3^i}, \sum_{i=1}^{\infty} \frac{2s_{i-1}}{3^i} \right).$$

(In general, it is an expansion in base $2r + 1$ where r is the number of symbols of the alphabet.) Finally, we can take the open set U to be a neighborhood of the Cantor blocks (and hence a disjoint set) that contain all the points associated to sequences in $A^{\mathbb{Z}}$ whose symbols in positions $-1, 0, 1$ are of the form $(a_{-1}q_{halt}^{un}a_1)$ for some $a_{\pm 1} \in A$. Here q_{halt}^{un} is the halting state of T_{un} . Of course, this set U exists because the Cantor blocks are pairwise disjoint, and it is explicitly constructible. This completes the proof of the theorem. \square

Remark 6.3. There is a key technical difference between the diffeomorphism φ we construct in Theorem 6.2 and the Turing complete diffeomorphism of \mathbb{T}^4 constructed in [28]. In Tao's construction, the point p depends only on the Turing machine T and the input (q, t) . Then, for any given finite string $t^* := (t_{-k}^*, \dots, t_k^*)$ there is some open set U_{t^*} such that the orbit through p intersects U_{t^*} if and only if T halts with input (q, t) and output whose positions $-k, \dots, k$ correspond to t^* . In contrast, in the diffeomorphism φ we construct in Theorem 6.2, the point p depends on all the information: the Turing machine T , the input (q, t) and the finite string $t^* = (t_{-k}^*, \dots, t_k^*)$. In particular, if we pick another finite string t_2^* , the point p will be different. Additionally, U is always the same, i.e., a neighborhood of those blocks associated to the halting state of T_{un} .

Finally, we prove the lemma that is used in the proof of Theorem 6.2: given a Turing machine T and a finite string (t_{-k}^*, \dots, t_k^*) , one can construct a Turing machine T' which halts with a given input if and only if T halts with the same input and with the output tape having in positions $(-k, \dots, k)$ the fixed symbols (t_{-k}^*, \dots, t_k^*) . This is intuitively clear, one simply needs to construct a machine T' that works exactly as T , but when T reaches the halting state, T' reads the positions $-k, \dots, k$ to

compare with (t_{-k}^*, \dots, t_k^*) . This is formalized in the following lemma (which is probably standard in the theory of Turing machines).

Lemma 6.4. *Let $T = (Q, q_0, q_{halt}, \Sigma, \delta)$ be a Turing machine. For any $k \geq 0$ and finite string $(t_{-k}^*, \dots, t_k^*) \in \Sigma^{2k+1}$, there is a Turing machine T' which halts with input (q_0, t) if and only if the machine T with input (q_0, t) halts with coefficients t_{-k}^*, \dots, t_k^* in positions $-k, \dots, k$ in the output tape.*

Proof. Fix a Turing machine $T = (Q, q_0, q_{halt}, \Sigma, \delta)$ and a finite string (t_{-k}^*, \dots, t_k^*) . As before, F_1, F_2, F_3 denote the three components of the transition function $\delta : Q \times \Sigma \rightarrow Q \times \Sigma \times \{-1, 0, 1\}$. To define the Turing machine T' , take as alphabet $\Sigma' := \Sigma$ and as set of states $Q' := Q \sqcup \{r_0, \dots, r_{3k}, q_{nohalt}\}$, where r_j and q_{nohalt} simply denote new states we include in the space. The initial and halting states of T' are the same as for T . The idea is to use the states r_0, \dots, r_{3k} as “reading states” that will check if the final output is the desired one.

Let us denote the current state of the Turing machine by (q, t) and by t_0 the symbol in the central position. The transition function δ' can be defined as follows. For $q \in Q \setminus \{q_{halt}\}$, if $F_1(q, t_0) \in Q \setminus \{q_{halt}\}$ then we set $\delta'(q, t_0) := \delta(q, t_0)$ and if $F_1(q, t_0) = q_{halt}$ we define $\delta'(q, t) := (r_0, F_2(q, t_0), F_3(q, t_0))$. This way, when T reaches a halting state, T' will reach the state r_0 .

Now we define the transition function δ' for $q \in \{r_0, \dots, r_{3k}, q_{nohalt}\}$ as:

$$\delta'(r_i, t_0) := \begin{cases} (r_{i+1}, t_0, -1) & \text{if } t_0 = t_{-i}^* \\ (q_{nohalt}, t_0, 0) & \text{otherwise} \end{cases}, \text{ for } i = 0, \dots, k-1. \quad (6.1)$$

$$\delta'(r_i, t_0) := \begin{cases} (r_{i+1}, t_0, +1) & \text{if } t_0 = t_{i-2k}^* \\ (q_{nohalt}, t_0, 0) & \text{otherwise} \end{cases}, \text{ for } i = k, \dots, 3k-1, \quad (6.2)$$

and $\delta'(r_{3k}, t_0) := (q_{halt}, t_0, 0)$ if $t_0 = t_k^*$ and $(q_{nohalt}, t_0, 0)$ otherwise. Finally, we define δ' for q_{nohalt} so that the machine gets trapped in a loop, e.g. we can set $\delta'(q_{nohalt}, t_0) := (q_{nohalt}, t_0, 0)$ for any symbol t_0 .

Let us check that T' satisfies the required property. Suppose that T halts with a given input (q_0, t) . Denote by $t^h := (\dots t_{-1}^h \cdot t_0^h t_1^h \dots)$ the output tape of T , i.e., the tape when T reaches the halting state. By the construction, the machine T' with input (q_0, t) will reach the state r_0 with tape t^h instead of halting. By Equation (6.1), if $t_0^h = t_0^*$ the machine will shift the tape to the right and change the current state to r_1 . If the symbol t_0^h does not correspond to t_0^* , then T enters a loop through the state q_{nohalt} and will never halt.

After shifting to the right, the current tape is now $(\dots t_{-2}^h \cdot t_{-1}^h t_0^h \dots)$ and the current state is r_1 . Again by Equation 6.1, the machine enters a loop unless $t_{-1}^h = t_{-1}^*$, in which case we shift to the right and change to state r_2 . Iterating this process, the machine reaches the state r_k if and only if $t_{-i}^h = t_{-i}^*$ for each $i = 0, 1, \dots, k-1$. The current tape is then $(\dots t_{-(k+1)}^h \cdot t_{-k}^h \dots)$. Similarly, by Equation (6.2) for states r_j with $j = k, \dots, 3k-1$, at each step the machine is at the state r_j with current tape $(\dots t_{j-2k-1}^h \cdot t_{j-2k}^h \dots)$, and it checks if $t_{j-2k}^h = t_{j-2k}^*$, in which case it shifts to the left with new state r_{j+1} . Finally, the machine reaches the state r_{3k} if and only if $(t_{-k}^h \dots t_{k-1}^h) = (t_{-k}^* \dots t_{k-1}^*)$, and the current tape becomes $(\dots t_{k-1}^h \cdot t_k^h \dots)$. By the definition of δ' at r_{3k} , the machine halts if and only if $t_k^h = t_k^*$ or else enters a loop. It is then obvious that T' halts with input (q_0, t) if and only if T halts and its output satisfies that $(t_{-k}^h \dots t_k^h) = (t_{-k}^* \dots t_k^*)$, which completes the proof of the lemma. \square

7. TURING COMPLETENESS OF FLUID FLOWS

In this last section we use the Turing complete area-preserving diffeomorphism constructed in Theorem 6.2 to establish the existence of an Eulerisable field in \mathbb{S}^3 which is Turing complete. In the proof we use Etnyre-Ghrist’s contact mirror, cf. Theorem 3.2, and the realization Theorem 4.1 which allows one to embed a diffeomorphism of the disk as the return map of a Reeb flow.

7.1. Embedding diffeomorphisms as cross sections of Beltrami flows. In [7] we constructed a Turing complete Eulerisable flow on \mathbb{S}^{17} using a new h -principle for Reeb embeddings; the dimension 17 is essentially sharp with this approach. In contrast, the ideas we introduced in Sections 4, 5 and 6 allow us to reduce the dimension to 3, as shown in the following theorem, which is the main result of this work.

Theorem 7.1. *There exists an Eulerisable flow X in \mathbb{S}^3 that is Turing complete in the following sense. For any integer $k \geq 0$, given a Turing machine T , an input tape t , and a finite string (t_{-k}^*, \dots, t_k^*) of symbols of the alphabet, there exist an explicitly constructible point $p \in \mathbb{S}^3$ and an open set $U \subset \mathbb{S}^3$ such that the orbit of X through p intersects U if and only if T halts with an output tape whose positions $-k, \dots, k$ correspond to the symbols t_{-k}^*, \dots, t_k^* . The metric g that makes X a stationary solution of the Euler equations can be assumed to be the round metric in the complement of an embedded solid torus.*

Proof. By Theorem 6.2, there exists a Turing complete area-preserving diffeomorphism φ of the disk which is the identity in a neighborhood of the boundary. Take the standard contact sphere $(\mathbb{S}^3, \xi_{std})$ and apply Theorem 4.1 to obtain a defining contact form α whose Reeb field X exhibits an invariant solid torus \mathcal{T} where the first return map on a disk section is conjugated to φ via a diffeomorphism $\Phi : D \rightarrow D$. By Remark 4.4 we can assume that in the complement of a neighborhood V of $\overline{\mathcal{T}}$, the one form α coincides with the standard contact form α_{std} of \mathbb{S}^3 . In particular, X coincides with a Hopf field in $\mathbb{S}^3 \setminus V$. When applying the contact/Beltrami correspondence in Lemma 3.2, we obtain a Riemannian metric g which coincides with the round one (as done also in [7]) on $\mathbb{S}^3 \setminus V$. By construction of the metric, X satisfies the equation $\text{curl}_g X = X$, so it is a stationary solution of the Euler equations on (\mathbb{S}^3, g) .

Finally, let us check that X satisfies the stated Turing completeness property. Take a Turing machine T with an input t and a finite string (t_{-k}^*, \dots, t_k^*) of symbols. Denoting by $D_0 = \{0\} \times D \subset \mathbb{S}^3$ the transverse section in \mathcal{T} where the first return map of X is conjugated to φ , we can find the point $p \in D_0$ and the set $U_0 \subset D_0$ (open as a subset of D) defined as $p := \Phi(p_*)$ and $U_0 := \Phi(U_*)$, where p_* and U_* are, respectively, the point and open set given by Theorem 6.2. We then take the open set

$$U := \bigcup_{t \in (-\varepsilon_0, \varepsilon_0)} \phi_t(U_0),$$

where $\varepsilon_0 > 0$ is a small enough constant, and ϕ_t is the flow defined by X . It is then clear that the point $p \in \mathbb{S}^3$ and the set $U \subset \mathbb{S}^3$ satisfy that T will halt with the given output positions if and only if the orbit of X through p intersects the open set U , thus completing the proof of the theorem. \square

7.2. Final remark: the Navier-Stokes equations. The Navier-Stokes equations describe the dynamics of an incompressible fluid flow with viscosity. On a Riemannian 3-manifold (M, g) they read as

$$\begin{cases} \frac{\partial u}{\partial t} + \nabla_u u - \nu \Delta u = -\nabla p, \\ \text{div } u = 0, \\ u(t=0) = u_0, \end{cases} \quad (7.1)$$

where $\nu > 0$ is the viscosity. Here all the differential operators are computed with respect to the metric g , and Δ is the Hodge Laplacian (whose action on a vector field is defined as $\Delta u := (\Delta u^b)^\sharp$).

In this final section we analyze what happens with the vector field X constructed in Theorem 7.1 when taken as initial condition for the Navier-Stokes equations with the metric g that makes X a steady Euler flow. Specifically, using that $\text{curl}_g(X) = X$, the solution to Equation (7.1) with $u_0 = MX$, $M > 0$ a real constant, is easily seen to be

$$\begin{cases} u(\cdot, t) = MX(\cdot)e^{-\nu t}, \\ p(\cdot, t) = c_0 - \frac{1}{2}M^2e^{-2\nu t}\|X\|_g^2, \end{cases} \quad (7.2)$$

for any constant c_0 . The integral curves (fluid particle paths) of the non-autonomous field u solve the ODE

$$\frac{dx(t)}{dt} = Me^{-\nu t}X(x(t)).$$

Accordingly, reparametrizing the time as

$$\tau(t) := \frac{M}{\nu}(1 - e^{-\nu t}),$$

we show that the solution $x(t)$ can be written in terms of the solution $y(\tau)$ of the ODE

$$\frac{dy(\tau)}{d\tau} = X(y(\tau)),$$

as

$$x(t) = y(\tau(t)).$$

When $t \rightarrow \infty$ the new “time” τ tends to $\frac{M}{\nu}$, and hence the integral curve $x(t)$ of the Navier-Stokes equations travels the orbit of X just for the time interval $\tau \in [0, \frac{M}{\nu})$. In particular, the flow of the solution u only simulates a finite number of steps of a given Turing machine, so we cannot deduce the Turing completeness of the Navier-Stokes equations using the vector field MX as initial condition. More number of steps of a Turing machine can be simulated if $\nu \rightarrow 0$ (the vanishing viscosity limit) or $M \rightarrow \infty$ (the L^2 norm of the initial datum blows up). For example, to obtain a universal Turing simulation we can take a family $\{M_k X\}_{k \in \mathbb{N}}$ of initial data for the Navier-Stokes equations, where $M_k \rightarrow \infty$ is a sequence of positive numbers. The energy (L^2 norm) of this family is not uniformly bounded, thus raising the challenging question of whether there exists an initial datum of finite energy that gives rise to a Turing complete solution of the Navier-Stokes equations.

REFERENCES

- [1] I. Agol, J. Hass, W. Thurston. *The computational complexity of knot genus and spanning area*. Trans. Amer. Math. Soc. 358 (2006) 3821–3850.
- [2] V. I. Arnold, B. Khesin. *Topological Methods in Hydrodynamics*. Springer, New York, 1999.
- [3] C.H. Bennett. *Logical Reversibility of Computation*. IBM J. Res. Develop. 17 (1973) 525–532.
- [4] M. Borman, Y. Eliashberg, E. Murphy. *Existence and classification of overtwisted contact structures in all dimensions*. Acta Math. 215 (2015) 281–361.
- [5] B. Bramham. *Pseudoholomorphic Foliations for Area Preserving Disc Maps*. Ph.D.-thesis, New York University 2008.
- [6] R. Cardona, E. Miranda, D. Peralta-Salas. *Euler flows and singular geometric structures*. Philos. Trans. Roy. Soc. A 377 (2019), no. 2158, 20190034, 15 pp.
- [7] R. Cardona, E. Miranda, D. Peralta-Salas, F. Presas. *Universality of Euler flows and flexibility of Reeb embeddings*. Preprint (2019) arXiv:1911.01963.
- [8] K. Cieliebak, E. Volkov. *A note on the stationary Euler equations of hydrodynamics*. Ergodic Theor. & Dynam. Sys. 37 (2017) 454–480.
- [9] T. Cubitt, D. Perez-Garcia, M. Wolf. *Undecidability of the spectral gap*. Nature 528 (2015) 207–211.
- [10] J.C. Delvenne. *What is a universal computing machine?*. Appl. Math. Comput. 215 (2009) 1368–1374.
- [11] J.C. Delvenne, P. Kurka, V.D. Blondel. *Computational Universality in Symbolic Dynamical Systems*. In M. Margenstern, editor, MCU 2004, number 3354 in Lecture Notes in Computer Science, pages 104–115. Springer-Verlag, 2005.
- [12] J. Etnyre, R. Ghrist. *Contact topology and hydrodynamics I. Beltrami fields and the Seifert conjecture*. Nonlinearity 13 (2000) 441–458.
- [13] M. Freedman. *P/NP, and the quantum field computer*. Proc. Natl. Acad. Sci. USA 95 (1998) 98–101.
- [14] D.S. Graca, M.L. Campagnolo, J. Buescu. *Robust simulations of Turing machines with analytic maps and flows*. Lecture Notes in Computer Science 3526 (2005) 169–179.
- [15] J. W. Gray. *Some global properties of contact structures*. Ann. of Math. 69 (1959) 421–450.
- [16] R. Lutz. *Structures de contact sur les fibrés principaux en cercles de dimension trois*. Ann. Inst. Fourier (Grenoble) 27 (1977) 559–564.
- [17] J. Martinet. *Formes de contact sur les variétés de dimension 3*. Proc. Liverpool Singularities Sympos. II, Lecture Notes in Math. 209, Springer-Verlag, Berlin (1971), 142–163.
- [18] C. Moore. *Undecidability and Unpredictability in Dynamical Systems*. Phys. Rev. Lett. 64 (1990) 2354–2357.
- [19] C. Moore. *Generalized shifts: unpredictability and undecidability in dynamical systems*. Nonlinearity 4 (1991) 199–230.
- [20] K. Morita, Y. Yamaguchi. *A universal reversible Turing machine*. In: DurandLose, J., Margenstern, M. (eds.) MCU 2007. LNCS, vol. 4664, pp. 90–98. Springer, Heidelberg (2007).
- [21] R. Penrose. *The emperor’s new mind. Concerning computers, minds, and the laws of physics*. With a foreword by Martin Gardner. The Clarendon Press, Oxford University Press, New York, 1989. xiv+466 pp. ISBN: 0-19-851973-7
- [22] D. Peralta-Salas. *Selected topics on the topology of ideal fluid flows*. Int. J. Geom. Methods Mod. Phys. 13 (2016), suppl., 1630012, 23 pp.
- [23] D. Peralta-Salas, A. Rechtman, F. Torres de Lizaur. *A characterization of 3D steady Euler flows using commuting zero-flux homologies*. To appear in Ergodic Theor. & Dynam. Sys. (2021).
- [24] A. Rechtman. *Existence of periodic orbits for geodesible vector fields on closed 3-manifolds*. Ergodic Theor. & Dynam. Sys. 30 (2010) 1817–1841.

- [25] J.H. Reif, J.D. Tygar, A. Yoshida. *Computability and complexity of ray tracing*. Discrete Comput. Geom. 11 (1994) 265–288.
- [26] H.T. Siegelmann, E.D. Sontag. *On the computational power of neural nets*. J. Comput. Syst. Sci. 50 (1995) 132–150.
- [27] T. Tao. *Finite time blowup for an averaged three-dimensional Navier-Stokes equation*. J. Amer. Math. Soc. 29 (2016), no. 3, 601–674.
- [28] T. Tao. *On the universality of potential well dynamics*. Dyn. PDE 14 (2017) 219–238.
- [29] T. Tao. *On the universality of the incompressible Euler equation on compact manifolds*. Discrete and Continuous Dynamical Systems - A (2018) 38 (3) : 1553–1565.
- [30] T. Tao. *On the universality of the incompressible Euler equation on compact manifolds, II. Nonrigidity of Euler flows*. Preprint arXiv:1902.0631 (2019).
- [31] T. Tao. *Searching for singularities in the Navier-Stokes equations*. Nature Reviews Physics 1 (2019) 418–419.
- [32] E. Witten. *Quantum field theory and the Jones polynomial*. Comm. Math. Phys. 121 (1989) 351–399.
- [33] S. Wolfram. *Undecidability and intractability in theoretical physics*. Phys. Rev. Lett. 54 (1985) 735–738.

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