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Apunts

Introduction to the Numerical Solution of the Navier-Stokes equations.

Module 1: Introduction

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Introduction to the Numerical Solution of the Navier-Stokes equations

Module 1. Introduction

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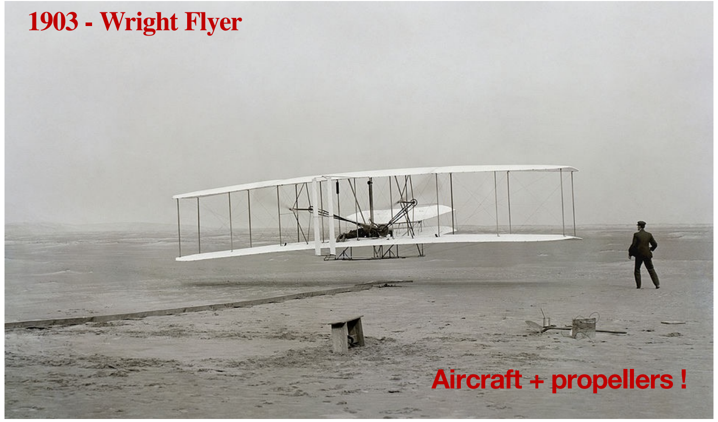
MÀSTER UNIVERSITARI EN ENGINYERIA AERONÀUTICA ESEIAAT

Module 1. Introduction

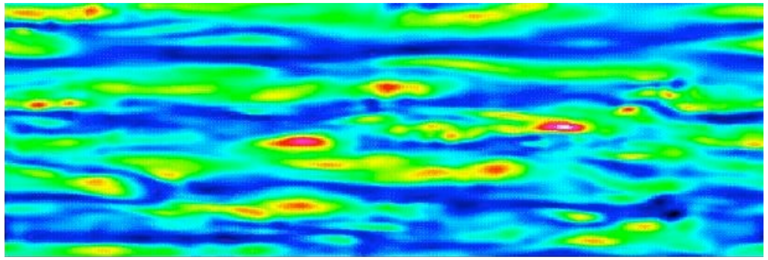
The gap between NS and practical fluid mechanics

1500 - Leonardo Da Vinci
1687 - Newton *Principia*
1707 - Euler equations
1752 - d'Alembert paradox: zero drag for potential flow
1822 - Navier Stokes equations
1895 - Reynolds average (RANS)
1903 - Wright Flyer (Aircraft+propellers)
1904 - Prandtl boundary layer (solves d'Alembert paradox)
1910 - Richardson, hand made CFD attempt; hierarchy of eddies
1969 - Saturn V rocket
1981 - F117 stealth first flight
1987 - Kim, Moin & Moser, DNS channel flow



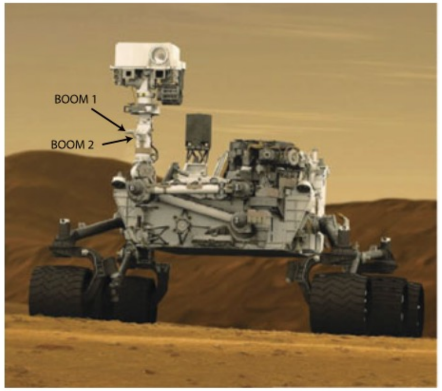
- The challenge of understanding complex and turbulent flows is at least 500 years old.
- Despite advances in computational fluid mechanics and the huge computing power available today, we can not say that we have fully solved the problem of turbulence yet.
- Actually, the first true solutions of turbulent flows are from 1980s
- However, many engineering problems involving very difficult fluid mechanics issues have been solved with experimental means or simplified methods
- Consider for instance 1903 Wright Brothers aircraft. It would have been designed exactly the same if Navier Stokes equations haven't existed
- Nowadays, the gap between practical and theoretical approaches is closing and Aerospace engineers must master computational aspects and software development as well as theoretical fluid mechanics

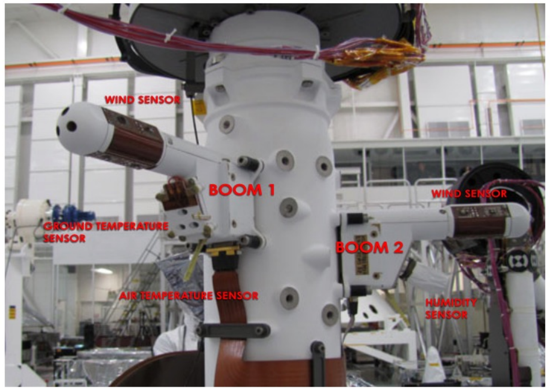
1981 - F117 stealth

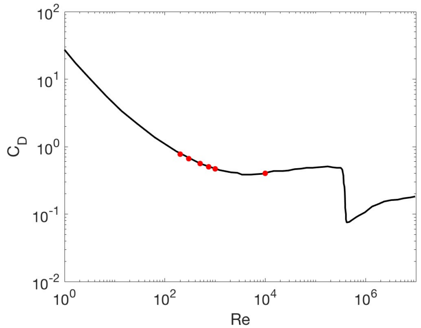


1987 - Kim, Moin & Moser DNS channel flow

1.1 An example of the application of NS equations to a challenging design







Fluid dynamics and heat transfer in the wake of a sphere

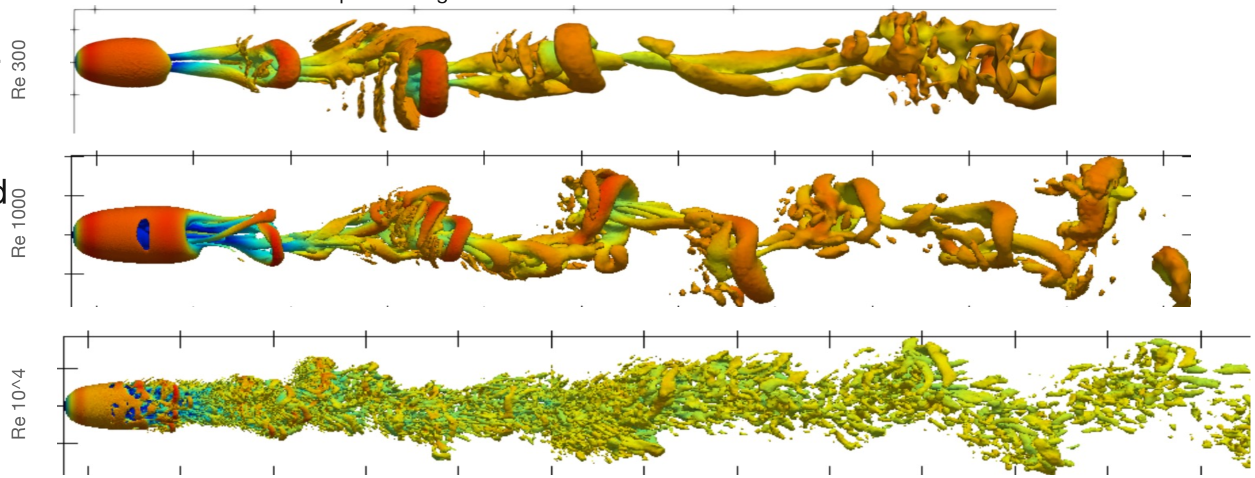
Rodriguez, I.; Lehmkuhl, O.; Soria, M.; Gómez, S.; Dominguez, M.; Kowalski, L.

https://doi.org/10.1016/j.ijheatfluidflow.2019.02.004

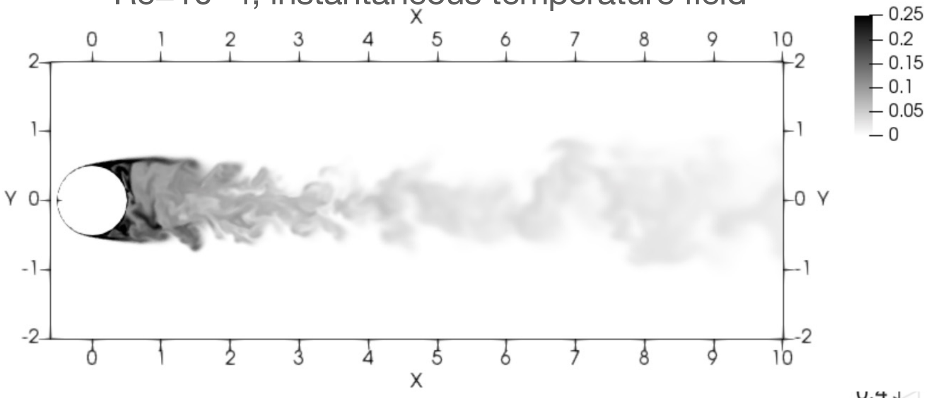
Analyzing the performance of a miniature 3D wind sensor for Mars,
Dominguez, M.; Kowalski, L.; Jimenez, V.; Rodriguez, I.; Soria, M.; Bermejo, S.; Pons,

https://doi.org/10.3390/s20205912

- Using CFD, high fidelity numerical simulations can be carried out. Consider for instance the next generation anemometer for Mars can be carried out
- To do so, it is important to have a very good understanding of concepts such as the pressure-velocity coupling or the boundary conditions.
- In this course you will learn how to implement these concepts in your own code, as well as to verify the solutions obtained

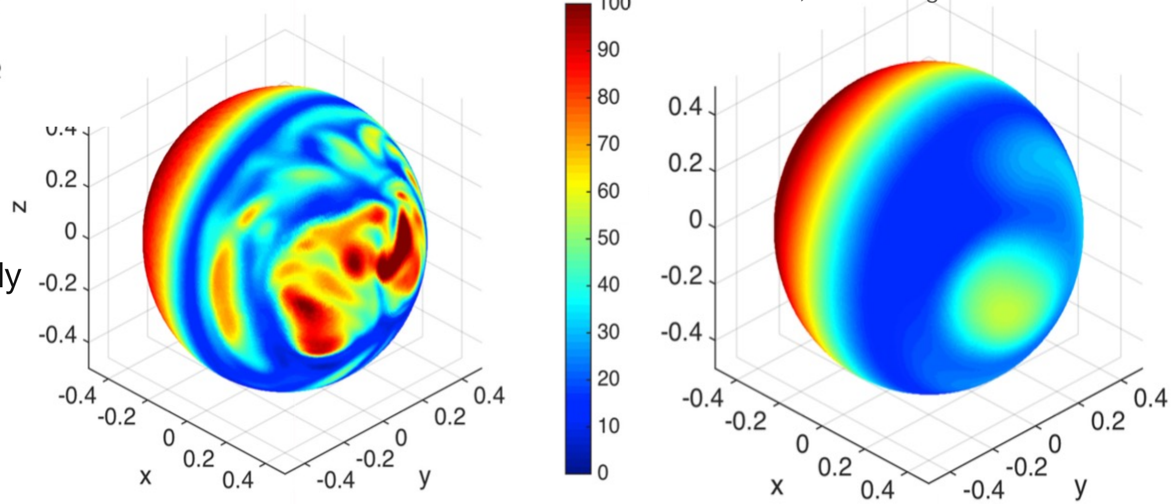


Re=10^4, instantaneous temperature field



instantaneous Nusselt

Re=10^4, time averaged Nusselt



With NS we can compute heat transfer and accurately predict the behavior at the detached zones
In Mars, despite the low density, continuous formulation is still valid for objects of about 10mm

Incompressible Navier-Stokes equations:

- Mass conservation equation (continuity equation)
- Momentum equations
- Energy equation (not needed for incompressible flow)

They are obtained from:

- Mass conservation principle
- Newton's law
- Substantial derivative
- Surface forces in a fluid element subject to a stress state
- Stress / strain relation for a Newtonian fluid

You have studied this in previous Degree and Master subjects but here we will do a quick review

A few words about notation:

$\boldsymbol{u}$	- Velocity vector
u_1, u_2, u_3	- Velocity vector components
u, v, w	- Velocity vector components
p	- Pressure field
x, y, z	- Position
t	- Time

(both notations will be used)

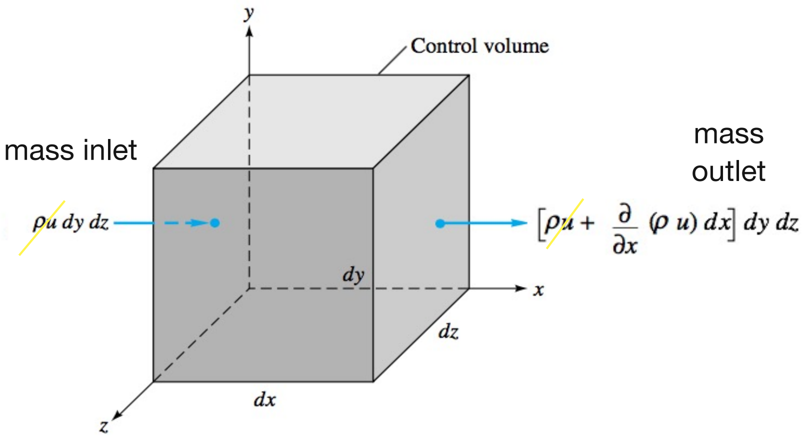
Recall: $$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

Einstein summation convention that implies summation over a set of indexed terms in a formula

Also called *continuity equation* because it requires no assumptions except continuum media (and mass conservation principle)

Mass increment = mass inlet - mass outlet

Mass increment + Mas outlet - Mass inlet = 0



$$\frac{\partial \rho}{\partial t} dx\, dy\, dz + \frac{\partial}{\partial x}(\rho u_1)\, dx\, dy\, dz + \frac{\partial}{\partial y}(\rho u_2)\, dx\, dy\, dz + \frac{\partial}{\partial z}(\rho u_3)\, dx\, dy\, dz$$

incompressible

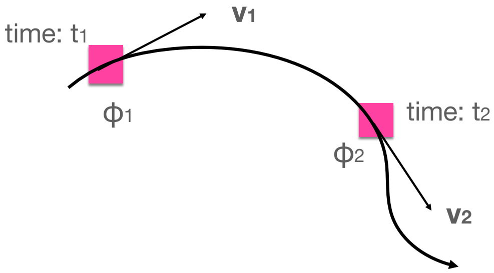
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

incompressible: $\nabla \cdot \boldsymbol{u} = 0$ or $\frac{\partial u_j}{\partial x_j} = 0$

divergence of velocity vector is 0

Recall: $\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$

Consider a generic property ϕ in a fluid field: ϕ =f(t,x,y,z)



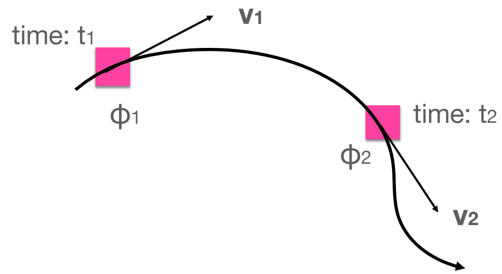
$$\phi_2 = \phi_1 + \frac{\partial \phi}{\partial t}(t_2 - t_1) + \frac{\partial \phi}{\partial x}(x_2 - x_1) + \frac{\partial \phi}{\partial y}(y_2 - y_1) + \frac{\partial \phi}{\partial z}(z_2 - z_1) + \ldots$$

Time rate of change of ϕ, following a moving fluid element:

$$\frac{\phi_2 - \phi_1}{t_2 - t_1} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\underbrace{\left(\frac{x_2 - x_1}{t_2 - t_1}\right)}_{u_1} + \frac{\partial \phi}{\partial y}\underbrace{\left(\frac{y_2 - y_1}{t_2 - t_1}\right)}_{u_2} + \frac{\partial \phi}{\partial z}\underbrace{\left(\frac{z_2 - z_1}{t_2 - t_1}\right)}_{u_3} + \ldots$$

Time rate of change of ϕ, following a moving fluid element:

$$\frac{\phi_2 - \phi_1}{t_2 - t_1} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\left(\frac{x_2 - x_1}{t_2 - t_1}\right) + \frac{\partial \phi}{\partial y}\left(\frac{y_2 - y_1}{t_2 - t_1}\right) + \frac{\partial \phi}{\partial z}\left(\frac{z_2 - z_1}{t_2 - t_1}\right) + \ldots$$



If time increment tends to zero, we have:

$$Lim_{t_2 \to t_1} \frac{x_2 - x_1}{t_2 - t_1} = u$$ velocity of the fluid element

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + u_1 \frac{\partial \phi}{\partial x} + u_2 \frac{\partial \phi}{\partial y} + u_3 \frac{\partial \phi}{\partial z}$$

Substantial *or total or material derivative* — Time derivative — Convection or advection

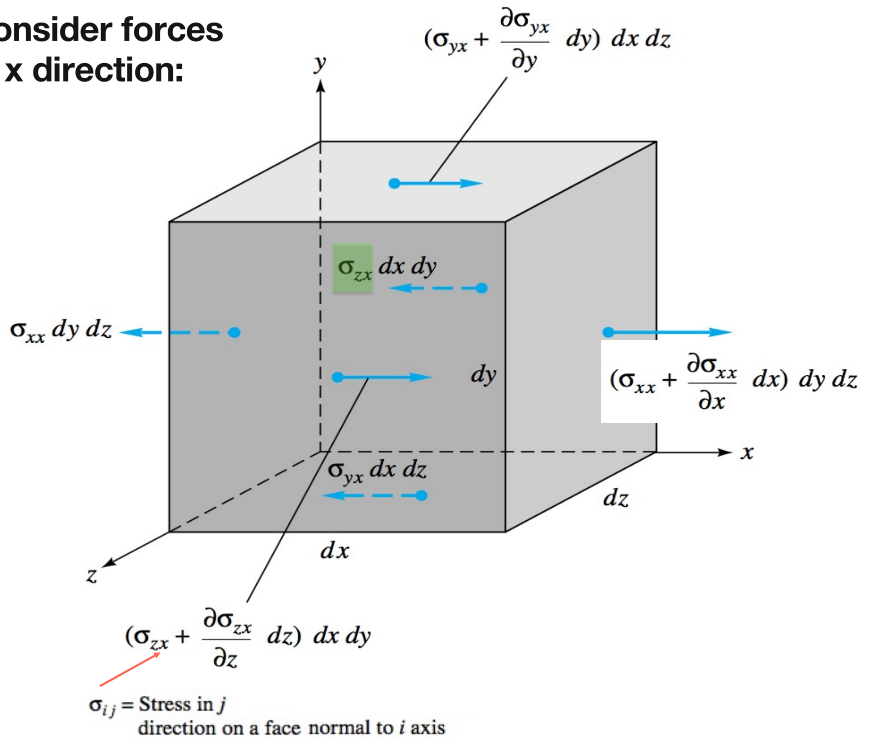
in general: →

$$\frac{D}{Dt} := \frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} + u_3 \frac{\partial}{\partial z}$$
$$= \frac{\partial}{\partial t} + (\boldsymbol{u} \cdot \nabla)$$
$$= \frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}$$

When the property **ϕ is one of the velocity components**, the operator becomes **non-linear** 😱

E.g., σ_{zx} : face normal to z, force in x direction

Consider forces in x direction:



As an example, we collect all the forces in x direction:

$$F_x = F_1 =$$

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx - \sigma_{xx}\right) dy\, dz$$

$$+ \left(\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dx - \sigma_{yx}\right) dx\, dz$$

$$+ \left(\sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z} dx - \sigma_{zx}\right) dx\, dy =$$

$$= \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right) dx\, dy\, dz$$

Thus, it is not the stress but its gradient what causes the net force on the fluid element

Newton's law on x axis, from previous slide:

$$m a_x = F_x$$

Where:

$$F_x = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \right) dx\, dy\, dz$$

$$m = \rho\, dx\, dy\, dz$$

Mass for an infinitesimal control volume

$$a_x = \frac{D u_x}{D t}$$

$$= \frac{\partial u_x}{\partial t} + \boxed{u_j \frac{\partial u_x}{\partial x_j}}$$

Acceleration for an infinitesimal control volume

Substantial derivatives of velocity are non-linear !
Velocity field is transporting itself
This will have very important consequences

With the previous expressions for the force and acceleration $ma_x = F_x$ becomes:

$$\rho \left(\frac{\partial u_x}{\partial t} + u_j \frac{\partial u_x}{\partial x_j} \right) dx\, dy\, dz = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \right) dx\, dy\, dz$$

↓

$$\rho \left(\frac{\partial u_x}{\partial t} + u_j \frac{\partial u_x}{\partial x_j} \right) = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \right)$$

Now, we need to express the stress tensor σ as a function of **u** and **p**

Stress tensor is due to hydrostatic pressure p plus viscous stresses τ_{ij} due to motion with velocity gradients:

$$\sigma_{ij} = \begin{vmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{vmatrix}$$

For Newtonian fluids, the viscous stresses can be expressed as a function of the velocity field as:

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad \tau_{xx} = \lambda\,(\nabla \cdot \boldsymbol{u}) + 2\mu \frac{\partial u}{\partial x}$$

$$\tau_{zy} = \tau_{yz} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad \tau_{yy} = \lambda\,(\nabla \cdot \boldsymbol{u}) + 2\mu \frac{\partial v}{\partial y}$$

$$\tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \tau_{zz} = \lambda\,(\nabla \cdot \boldsymbol{u}) + 2\mu \frac{\partial w}{\partial z}$$

Bulk viscosity coefficient or second viscosity coefficient, **not needed** in **incompressible flow**

$$\rho\left(\frac{\partial u_x}{\partial t} + u_j\frac{\partial u_x}{\partial x_j}\right) = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} =$$

As:

$$\sigma_{ij} = \begin{vmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{vmatrix}$$

$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{zy} = \tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$= -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial}{\partial x}\frac{\partial u_1}{\partial x} + \frac{\partial}{\partial x}\frac{\partial u_1}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u_2}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u_1}{\partial y} + \frac{\partial}{\partial z}\frac{\partial u_1}{\partial z} + \frac{\partial}{\partial z}\frac{\partial u_3}{\partial x}\right)$$

$$= -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial}{\partial x}\frac{\partial u_1}{\partial x} + \frac{\partial}{\partial x}\frac{\partial u_1}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u_2}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u_1}{\partial y} + \frac{\partial}{\partial z}\frac{\partial u_1}{\partial z} + \frac{\partial}{\partial z}\frac{\partial u_3}{\partial x} \right)$$

Changing the order of the derivatives, the three terms together become: $\frac{\partial}{\partial x} \nabla \cdot \boldsymbol{u} = 0$

So the final incompressible momentum NS equation *for each of the i=1,2,3 velocity components* is:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

Recall:

μ dynamic viscosity

$\nu = \frac{\mu}{\rho}$ kinematic viscosity

$$\frac{\partial u_j}{\partial x_j} = 0$$ Mass conservation "continuity"

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$ Momentum

We expand again the momentum equation and take a closer look:

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} + u_3 \frac{\partial u_1}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu \left(\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} \right)$$

$$\frac{\partial u_2}{\partial t} + u_1 \frac{\partial u_2}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} + u_3 \frac{\partial u_2}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_2} + \nu \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2} \right)$$

$$\frac{\partial u_3}{\partial t} + u_1 \frac{\partial u_3}{\partial x_1} + u_2 \frac{\partial u_3}{\partial x_2} + u_3 \frac{\partial u_3}{\partial x_3} = -\frac{1}{\rho}\frac{\partial p}{\partial x_3} + \nu \left(\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2} \right)$$

Time derivative | Convection (or advection) non-linear 😱 | Pressure gradient | Diffusion / viscous

This apparently rather simple equation has attracted a huge interest in the last almost 200 years.

Millennium Prize Problems

P versus NP problem
Hodge conjecture
Poincaré conjecture (solved)
Riemann hypothesis
Yang–Mills existence and mass gap
Navier–Stokes existence and smoothness
Birch and Swinnerton-Dyer conjecture

1M$ prize

EXISTENCE & SMOOTHNESS OF THE NAVIER–STOKES EQUATION [1]

CHARLES L. FEFFERMAN

The Euler and Navier–Stokes equations describe the motion of a fluid in $\mathbb{R}^n (n = 2$ or $3)$. These equations are to be solved for an unknown velocity vector $u(x,t) = (u_i(x,t))_{1 \leqslant i \leqslant n} \in \mathbb{R}^n$ and pressure $p(x,t) \in \mathbb{R}$, defined for position $x \in \mathbb{R}^n$ and time $t \geqslant 0$. We restrict attention here to incompressible fluids filling all of $\mathbb{R}^n$. The *Navier–Stokes* equations are then given by

$$\frac{\partial}{\partial t} u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t) \qquad (x \in \mathbb{R}^n,\ t \geqslant 0) \tag{1}$$

$$div\ u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \qquad (x \in \mathbb{R}^n,\ t \geqslant 0) \tag{2}$$

with initial conditions

$$u(x,0) = u^\circ(x) \qquad (x \in \mathbb{R}^n)\,. \tag{3}$$

. . .

(A) Existence and smoothness of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence–free vector field satisfying (4). Take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t)$, $u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (6), (7).

(B) Existence and smoothness of Navier–Stokes solutions in $\mathbb{R}^3/\mathbb{Z}^3$. Take $\nu > 0$ and $n = 3$. Let $u^\circ(x)$ be any smooth, divergence–free vector field satisfying (8); we take $f(x,t)$ to be identically zero. Then there exist smooth functions $p(x,t)$, $u_i(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$ that satisfy (1), (2), (3), (10), (11).

(C) Breakdown of Navier–Stokes solutions on $\mathbb{R}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence–free vector field $u^\circ(x)$ on $\mathbb{R}^3$, and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (4), (5), for which there exist no solutions (p,u) of (1), (2), (3), (6), (7) on $\mathbb{R}^3 \times [0,\infty)$.

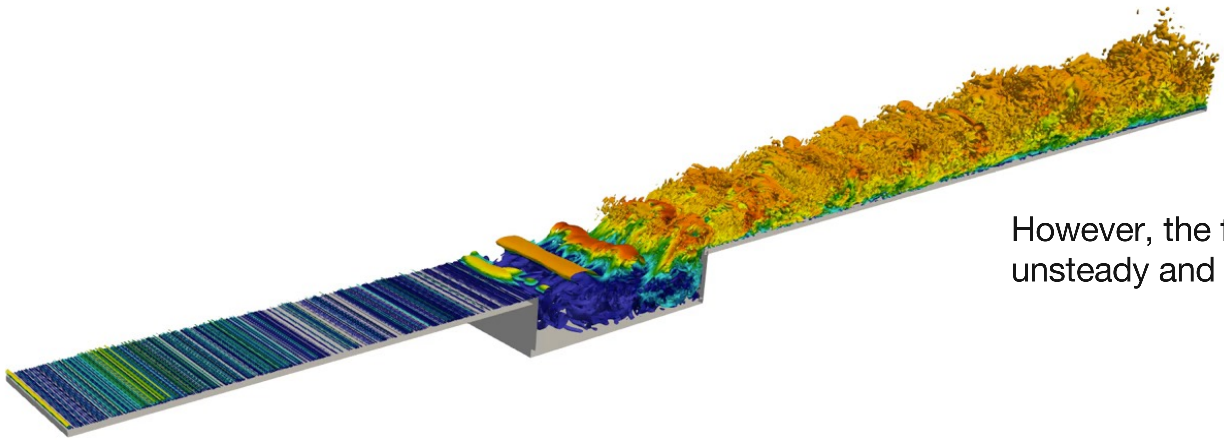
(D) Breakdown of Navier–Stokes Solutions on $\mathbb{R}^3/\mathbb{Z}^3$. Take $\nu > 0$ and $n = 3$. Then there exist a smooth, divergence–free vector field $u^\circ(x)$ on $\mathbb{R}^3$, and a smooth $f(x,t)$ on $\mathbb{R}^3 \times [0,\infty)$, satisfying (8), (9), for which there exist no solutions (p,u) of (1), (2), (3), (10), (11) on $\mathbb{R}^3 \times [0,\infty)$.

[1] - http://www.esi2.us.es/~mbilbao/pdffiles/navier_stokes.pdf

Why are they so difficult even to solve numerically ?

- Non-linear. The origin of the non-linearity is the advective term (Velocity field is transporting itself)
- Three-dimensional and unsteady (even with 2D steady boundary conditions)
- Strongly coupled. One variable affects all the others. Note that the viscous term doesn't couple the velocity components, only advection (and indirectly, the pressure gradient as we will see)
- Chaotic

2D/3D: Even in a 2D situation (such as this gap infinitely wide) the equations have to be treated as 3D



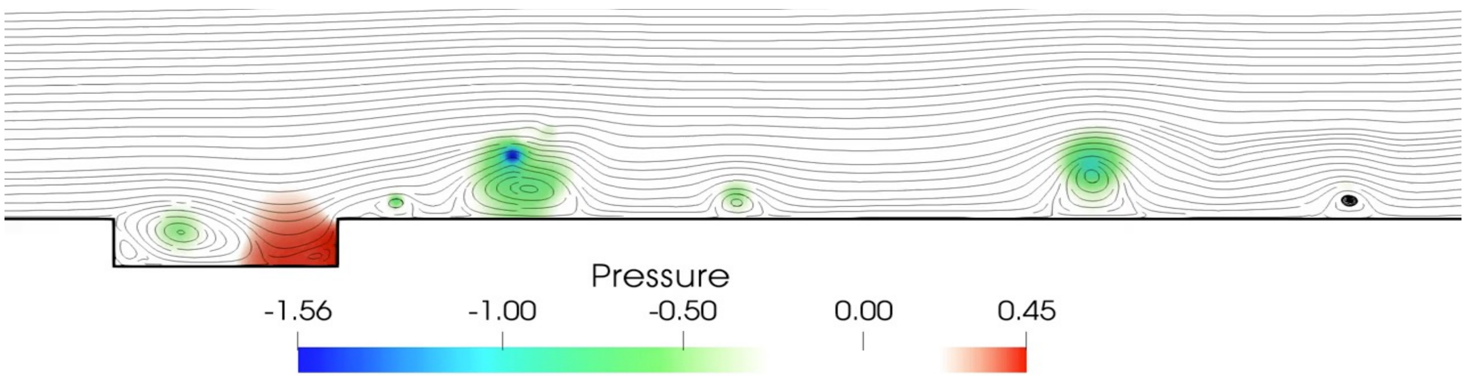
However, the flow becomes unsteady and three-dimensional

A developed boundary layer profile is imposed as inlet conditions (homogeneous in the spanwise direction)

Martín, R.; Soria, M.; Lehmkuhl, O.; Duben, A.; Gorobets, A.

https://doi.org/10.1177/1475472X19871534

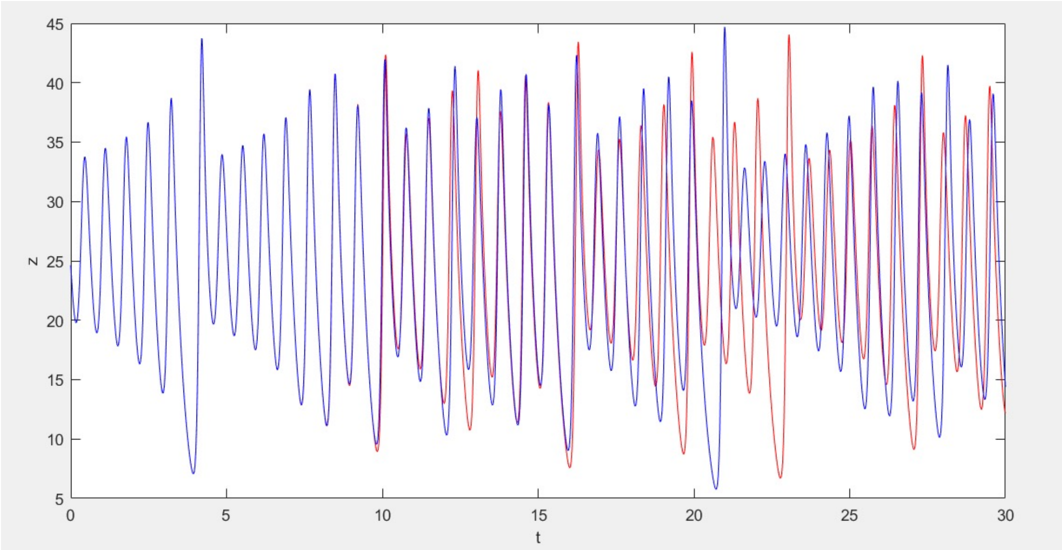
On the other hand, if the equations are rewritten to assume the flow to be 2D, the result is totally different.
An important mechanism known as vortex stretching is only captured by the full 3D equations



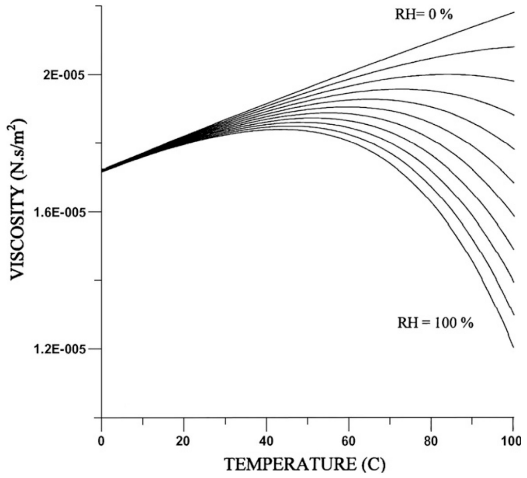
Turbulent flows are Chaos.
Lorenz 1961
Butterfly effect

On some dynamic systems (NS, in turbulent conditions among them), even very small perturbations grow.
This means **the behavior of such systems can not be predicted even if their governing laws are known**.
However, the statistic properties of the flow (e.g., mean drag and lift can be predicted)

```
clear
close all

% Lorentz system
a=10;
c=8/3;
b=28;

dydt=@(t,y) [ a*(y(2)-y(1)) ; y(1)*(b-y(3))-y(2); …
            y(1)*y(2)-c*y(3) ];

y01=[-4.9,-3.81,24.63];
y02=y01 + [0.002,0,0]; % small perturbation

[T1,Y1]=ode45(dydt,[0 30],y01);

plot(T1,Y1(:,3),'r');
hold on
[T2,Y2]=ode45(dydt,[0 30],y02);
plot(T2,Y2(:,3),'b');

xlabel('t'); ylabel('z');
```



Compressibility effects are obviously not described by this incompressible model.
There are many other issues, such as properties that depend on the composition and temperature (e.g., viscosity)



Also humidity in some cases !

- In aerodynamics, turbulence of the incoming fluid can have a significant effect,

but all these can be included in our model.

- **Otherwise, the incompressible NS equations describe very well the fluid flow (laminar and turbulent)**

Recall the momentum equation:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

We introduce *generic* reference values for length, time, velocity and pressure

$$x = \tilde{x}L \quad t = \tilde{t}T_0 \quad u = \tilde{u}V \quad p = \tilde{p}P$$

$\tilde{\varphi}$ – dimensionless variable

By simple substitution we express momentum eq. as:

$$V\frac{\partial \tilde{u}_i}{\partial t} + V^2\, \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{P}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \nu\, V\, \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j}$$

Now we need to express the derivatives in dimensionless terms. For each expression ϕ we will use:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} = \frac{1}{L}\frac{\partial \phi}{\partial \tilde{x}}$$

The generic dimensionless momentum equation is now:

$$\frac{V}{T_0}\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \frac{V^2}{L}\tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{P}{\rho L}\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \nu \frac{V}{L^2}\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j \partial \tilde{x}_j}$$

Suitable reference values (V,L,P and T_0) must be chosen for each situation.

For a flow around a solid body, the usual selection for the references is:
- L is a characteristic body length (such as the chord)
- V is the velocity of the body (or the unperturbed flow in a wind tunnel)
- There are no external candidates for the pressure and time, so we form reference pressure and time as:

$$P = \rho V^2 \qquad T_0 = \frac{L}{V}$$

$$\frac{V}{T_0}\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \frac{V^2}{L}\tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{P}{\rho L}\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \nu\frac{V}{L^2}\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j \partial \tilde{x}_j}$$

$$\frac{V^2}{L}\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \frac{V^2}{L}\tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{V^2}{L}\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \nu\frac{V}{L^2}\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j \partial \tilde{x}_j}$$

We introduce the Reynolds number: $Re = \frac{VL}{\nu}$

$$\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re}\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_j \partial \tilde{x}_j}$$

$$\frac{\partial \tilde{u}_j}{\partial \tilde{x}_j} = 0$$

Re is the only dimensionless parameter in external flows, no additional numbers arise from the continuity equation

- In other subjects of this course you will see that a different set of equations is used to model turbulent flows.

- Actually, time-accurate three-dimensional incompressible Navier-Stokes equations describe both laminar and incompressible flows.

- However, turbulent flows are very complex, and the number of mesh nodes needed to describe them is very large. Hence, the computational cost involved is usually too large.

- In order to reduce this computational cost, turbulence models are used. You can learn more about this in the subject “Advanced Aerodynamics”.

As with any other set of PDEs, different numerical methods can be used to obtain their solution. Some of them are:

Finite differences - The main idea is to replace the derivatives appearing in the differential equation by finite differences that approximate them.

Finite volumes - The equations are integrated over finite volumes and then the equation terms that contain a divergence are transformed to surface integrals. Historically, this is the method preferred for fluid mechanic problems and our choice for this course. It can be used in unstructured meshes and used for complex geometries. Open Foam, for instance, is based on finite volumes.

Finite elements - As in finite volumes, the domain is divided into small (but finite-sized) elements but then applies variational formulations to derive the discrete equations that are eventually solved. Recently, many CFD codes have been based on finite elements and perhaps it will be the dominant approach in the future. You have more information about finite element methods in other Master subjects.