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A combinatorial expression for the group inverse of symmetric M -matrices

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Abstract: By using combinatorial techniques, we obtain an extension of the matrix-tree theorem for general symmetric M -matrices with no restrictions, this means that we do not have to assume the diagonally dominance hypothesis. We express the group inverse of a symmetric M -matrix in terms of the weight of spanning rooted forests. In fact, we give a combinatorial expression for both the determinant of the considered matrix and the determinant of any submatrix obtained by deleting a row and a column. Moreover, the singular case is obtained as a limit case when certain parameter goes to zero. In particular, we recover some known results regarding trees. As examples that illustrate our results we give the expressions for the Group inverse of any symmetric M -matrix of order two and three. We also consider the case of the cycle C_4 an example of a non-contractible situation topologically different from a tree. Finally, we obtain some relations between combinatorial numbers, such as Horadam, Fibonacci or Pell numbers and the number of spanning rooted trees on a path.

Keywords: matrix-tree theorem, spanning tree, group inverse, effective resistances, Schrödinger operator, symmetric M -matrices

MSC: 05C05, 15A09, 15A10

1 Introduction

The classical matrix–tree theorem, whose first version was proved by G. Kirchhoff in 1847, relates the principal minors of the Laplacian matrix of a network Γ , with the total weight of spanning trees of Γ . Therefore, it refers to singular, symmetric and diagonally dominant M -matrices. Since then, many generalizations and different proofs have been considered (see for instance [1] and the references therein). In particular, the matrix-tree theorem has been extended to the case of non-singular, symmetric, α -diagonally dominant M -matrices, see [12, 16, 26, 27]. By α -diagonally dominant M -matrix, we understand those diagonally dominant M -matrices that have equal excess, α , in the diagonal. Hence, the value of the determinant is expressed as a polynomial in α whose coefficients are given by the weight of spanning rooted forest. In [21], the case of non-symmetric and (non-singular) Laplacian matrices is considered for the case of complex weights. In general, it is not an easy task to compute the number of spanning trees together with their weights in the weighted case, see [25] for some explicit expression in the case that the conductances are tensorial product and the underlying graph is structured. In [28] we found expressions for enumerating spanning trees in bipartite graphs, line graphs, generalized line graphs, middle graphs, total graphs, generalized join graphs and vertex-weighted graphs.

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In this work we obtain an extension of the matrix-tree theorem for general symmetric M -matrices; which means that we remove the diagonally dominance hypothesis. The techniques we use are of combinatorial nature. Specifically, to prove our results we use an extension of the undirected version of the *all-minors theorem* by S. Chaiken [11], see also [8] for a beautiful application. We give a combinatorial expression for both the determinant of the considered matrix and the determinant of any submatrix obtained by deleting a row and a column. Thus, we express the group inverse of a symmetric M -matrix in terms of the weight of spanning rooted forests. Moreover, the singular case is obtained as a limit case when the parameter goes to zero and we recover some known result regarding trees.

Given a finite set V , the set of real valued functions on V is denoted by $\mathcal{C}(V)$. The standard inner product on $\mathcal{C}(V)$ is denoted by $\langle \cdot, \cdot \rangle$ and hence, if $u, v \in \mathcal{C}(V)$, then $\langle u, v \rangle = \sum_{x \in V} u(x)v(x)$. For any $x \in V$, $\varepsilon_x \in \mathcal{C}(V)$ stands for the Dirac function at x and 1 is the function defined by $1(x) = 1$, for any $x \in V$. On the other hand, $\omega \in \mathcal{C}(V)$ is called a *weight* if it satisfies that $\omega(x) > 0$ for any $x \in V$ and moreover $\langle \omega, \omega \rangle = n = |V|$. The set of weights on V is denoted by $\Omega(V)$.

The triple $\Gamma = (V, E, c)$ denotes a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V , whose cardinality equals n , and edge set E , in which each edge $\{x, y\}$ has been assigned a *conductance* $c(x, y) > 0$. So, the conductance can be considered as a symmetric function $c: V \times V \rightarrow [0, +\infty)$ such that $c(x, x) = 0$ for any $x \in V$ and moreover, vertex x is adjacent to vertex y iff $c(x, y) > 0$. If we label the set of vertices, then functions can be identified with vectors in \mathbb{R}^n and endomorphisms of $\mathcal{C}(V)$ with matrices of order n .

The *combinatorial Laplacian* or simply the *Laplacian* of the network Γ is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y) (u(x) - u(y)), \quad x \in V. \tag{1}$$

The matrix associated with \mathcal{L} is always a singular symmetric M -matrix. Recall that a symmetric matrix M is called a *M-matrix* if its off diagonal entries are non-positive, and its eigenvalues are non-negative. Moreover, a matrix M is called *irreducible* if it is not similar via a permutation to a block upper triangular matrix. Therefore, a symmetric matrix is irreducible if and only if its associated graph is connected.

Given $q \in \mathcal{C}(V)$, the *Schrödinger operator* on Γ with *potential* q is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$, where $qu \in \mathcal{C}(V)$ is defined as $(qu)(x) = q(x)u(x)$; see for instance [2, 6].

Given a weight $\omega \in \Omega(V)$, we called *Doob-potential determined by ω* the function given by $q_\omega = -\omega^{-1}\mathcal{L}(\omega)$. It is well-known that any Schrödinger operator is self-adjoint and moreover it is positive semi-definite iff there exist $\omega \in \Omega(V)$ and $\lambda \geq 0$ such that $q = q_\omega + \lambda$; see [2]. In addition, \mathcal{L}_q is singular iff $\lambda = 0$, in which case $\langle \mathcal{L}_{q_\omega}(v), v \rangle = 0$ iff $v = a\omega$, $a \in \mathbb{R}$. In any case, λ is the lowest eigenvalue of \mathcal{L}_q and its associated eigenfunctions are multiple of ω . In [4], some of the authors proved that any irreducible, symmetric M -matrix can be identified with a positive semi-definite Schrödinger operator on a network.

In this work we fix $\omega \in \Omega(V)$ and for each $\lambda \geq 0$, we consider the positive semi-definite Schrödinger operator \mathcal{L}_q , where $q = q_\omega + \lambda$. Then, \mathcal{L}_q is an automorphism on ω^\perp , whose inverse is called (*orthogonal*) *Green operator* and is denoted by \mathfrak{G}_q . We can extend the Green operator to $\mathcal{C}(V)$ by assigning to any $f \in \mathcal{C}(V)$, the unique $u \in \mathcal{C}(V)$ such that $\mathcal{L}_q(u) = f - \frac{1}{n}\langle \omega, f \rangle \omega$ and $\langle u, \omega \rangle = 0$. The Green operator is self-adjoint, singular and satisfies that $\mathfrak{G}_q(\omega) = 0$, see [4].

The function $G_q : V \times V \rightarrow \mathbb{R}$, defined as $G_q(x, y) = \mathfrak{G}_q(\varepsilon_y)(x)$, for any $x, y \in V$, is called *Green function*. Moreover, for $\lambda = 0$, the matrix associated with G_q is the *Moore-Penrose inverse* of \mathcal{L}_q (that in this case coincides with the *group inverse*); whereas, for $\lambda > 0$, we have that $\mathcal{L}_q^{-1} = \mathfrak{G}_q + \frac{1}{n\lambda}\omega \otimes \omega$.

Through the manuscript, we use a common technique in the context of electrical networks and Markov chains, see [17, 18]. The authors considered this technique in [4] in order to generalize the Fiedler characterization of irreducible, symmetric and diagonally dominant M -matrices as resistive inverses, see [20], to all irreducible and symmetric M -matrices or equivalently, to all positive semi-definite Schrödinger operators.

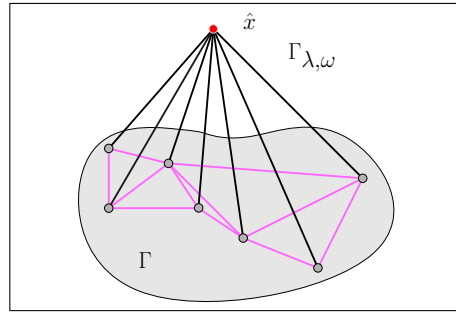


Figure 1: Host network.

Given a positive-definite Schrödinger operator on Γ , \mathcal{L}_q , the method consists in embedding the given network into a suitable host network. The new network is constructed by adding a new vertex, that represents a grounded vertex or an absorbing state in the context of Markov chains. The new vertex, \hat{x} , is joined with each vertex in the original network through a new edge whose conductance is the diagonal excess of the original M -matrix after the application of the Doob transform; see Figure 1.

Given $\lambda > 0$, $\omega \in \Omega(V)$ and $\hat{x} \notin V$, we define the *host network*, $\Gamma_{\lambda, \omega} = (V \cup \{\hat{x}\}, c_{\lambda, \omega})$ where $c_{\lambda, \omega}(x, y) = c(x, y)$ when $x, y \in V$ and $c_{\lambda, \omega}(\hat{x}, x) = c_{\lambda, \omega}(x, \hat{x}) = \lambda \omega(x)$ for any $x \in V$. We denote by $\mathcal{L}^{\lambda, \omega}$, its combinatorial Laplacian and by $\hat{\omega} \in \Omega(V \cup \{\hat{x}\})$ the weight given by $\hat{\omega}(x) = \omega(x)$ when $x \in V$ and $\hat{\omega}(\hat{x}) = 1$.

The next result, whose proof can be found in [4, Proposition 2.8], establishes the relationship between the original Schrödinger operator \mathcal{L}_q and a new semidefinite Schrödinger operator on $\Gamma_{\lambda, \omega}$.

Proposition 1.1. *If $q = q_\omega + \lambda$ and we define $\hat{q} = -\frac{1}{\hat{\omega}} \mathcal{L}^{\lambda, \omega}(\hat{\omega})$, then $\hat{q}(\hat{x}) = \lambda (n - \langle \omega, 1 \rangle)$ and $\hat{q} = q - \lambda \omega$ on V . Moreover, for any $u \in \mathbb{C}(V \cup \{\hat{x}\})$ we get that $\mathcal{L}_{\hat{q}}^{\lambda, \omega}(u)(\hat{x}) = \lambda (n u(\hat{x}) - \langle \omega, u|_V \rangle)$ and*

$$\mathcal{L}_{\hat{q}}^{\lambda, \omega}(u) = \mathcal{L}_q(u|_V) - \lambda \omega u(\hat{x}) \quad \text{on } V. \tag{2}$$

2 Matrix Tree Theorem for symmetric M -matrices

In this section we aim to present the matrix tree-theorem for positive definite Schrödinger operators and to get the relation between the number of rooted forests and the Green function of the network. Moreover, taking limits when λ goes to 0 we get the matrix tree-theorem for positive semi-definite Schrödinger operators.

If \mathcal{T} is a network such that for each pair x and y of vertices, there is exactly one path joining x and y , then \mathcal{T} is called a *weighted tree*. Therefore, \mathcal{T} is a weighted tree iff it has no cycles. Given Γ a connected network, we say that \mathcal{T} is a *spanning tree* of Γ if both networks have the same set of vertices and each edge of \mathcal{T} is also an edge of Γ with the same weight. A *forest* is a non-necessarily connected network without cycles. A *rooted forest* is a forest with one vertex marked as a *root* in each connected component. Moreover a *k-rooted forest* is a forest with k components and a vertex marked as a root in each connected component.



Figure 2: A forest (left) and a 3-rooted forest (right).

The set of vertices and the set of edges of a forest \mathcal{F} , will be denoted by $V(\mathcal{F})$ and $E(\mathcal{F})$, respectively. The *weight of a forest* \mathcal{F} is defined as the product of all weights of the edges of \mathcal{F} taking into account the weights of the vertices given by ω ; i.e.,

$$w(\mathcal{F}) = \frac{\prod_{e \in E(\mathcal{F})} w(e)}{\prod_{x \in V(\mathcal{F})} \omega^2(x)},$$

where $e = \{x, y\} \in E(\mathcal{F})$ and $w(e) = c(x, y)\omega(x)\omega(y)$ and we take the convention that the empty product equals 1.

We denote by $\mathcal{F} = \mathcal{F}(\Gamma)$ the set of rooted spanning forests of Γ and $\mathcal{F}_k = \mathcal{F}_k(\Gamma)$ the set of k -rooted spanning forest of Γ . Moreover, $\mathcal{F}_x = \mathcal{F}_x(\Gamma)$ denotes the set of rooted spanning forests of Γ such that x is a root and $\mathcal{F}_{k,x} = \mathcal{F}_{k,x}(\Gamma)$ denotes the set of k -rooted spanning forests of Γ such that x is a root. Finally, we denote by $\mathcal{F}_{x,y} = \mathcal{F}_{x,y}(\Gamma)$ the set of rooted spanning forests of Γ such that x is a root and y belongs to the same component as x and $\mathcal{F}_{k,x,y} = \mathcal{F}_{k,x,y}(\Gamma)$ the set of k -rooted spanning forest of Γ such that x is a root and y belongs to the same component as x . On the other hand, $\mathcal{F}_{k,x}^{\{y\}}$ denotes the set of rooted spanning forest of Γ such that x is a root and y does not belong to the same component as x . If $\mathcal{F} \in \mathcal{F}_k$ we denote by $r(\mathcal{F})$ the set of its roots whereas if $\mathcal{F} \in \mathcal{F}_{k,x}$ we denote by $r^*(\mathcal{F})$ the set of roots different from x .



Figure 3: A 3-rooted forest, x is a root (left) and a 3-rooted forest x is a root and y is in the same component (right).

In order to state the matrix notation we need to label the vertices and the edges of the network. So, let $\Gamma = (V, E, c)$ be a network and suppose that $V = \{1, \dots, n\}$ and $E = \{e_1, \dots, e_m\}$. Assume that each edge of Γ is assigned an orientation, which is arbitrary but fixed. Then, $e_\ell = (i, j)$, for $\ell = 1, \dots, m$ and $i, j = 1, \dots, n$, where i is the *tail* and j is the *head* of e_ℓ . We define the weight of an edge as $w(e_\ell) = c(i, j)\omega(i)\omega(j)$. Then, the weighted incidence matrix $B_\omega = (b_{\ell i}) \in \mathcal{M}_{m \times n}$ is defined as

$$b_{\ell i} = \begin{cases} \frac{1}{\omega(i)} & \text{if } i \text{ is the head of } e_\ell, \\ -\frac{1}{\omega(i)} & \text{if } i \text{ is the tail of } e_\ell, \\ 0 & \text{otherwise,} \end{cases}$$

and we denote by $W(c, \omega) \in \mathcal{M}_m$, the diagonal matrix whose diagonal elements are given by $w(e_\ell)$. Using this notation

$$L_{q_\omega} = B_\omega^\top W(c, \omega) B_\omega. \tag{3}$$

Moreover, the matrix associated with \mathcal{L}_q is given by

$$L_q = B_\omega^\top W(c, \omega) B_\omega + \lambda I.$$

If we denote by $L_q^{\lambda, \omega}$, the matrix associated with the Schrödinger operator in the host network, Equation (2) can be expressed as

$$L_q = L_q^{\lambda, \omega}(n+1|n+1), \tag{4}$$

where $L_q^{\lambda, \omega}(n+1|n+1)$ is the submatrix of $L_q^{\lambda, \omega}$ obtained by deleting the $(n+1)$ -th row and column. Moreover, the relation between weighted incidence matrices is the following

$$B_{\hat{\omega}}^{\lambda, \omega} = \begin{bmatrix} B_\omega & 0 \\ D_\omega^{-1} & -j \end{bmatrix} \quad \text{and} \quad W^{\lambda, \omega}(c, \omega) = \begin{bmatrix} W(c, \omega) & 0 \\ 0 & \lambda D_\omega^2 \end{bmatrix},$$

where $B_{\hat{\omega}}^{\lambda, \omega} \in \mathcal{M}_{(m+n) \times (n+1)}(\mathbb{R})$, $W^{\lambda, \omega}(c, \omega) \in \mathcal{M}_{m+n}(\mathbb{R})$ and \mathbf{j} denotes the all ones vector. Therefore, if $B_{\hat{\omega}}^{\lambda, \omega}(n+1)$ denotes the submatrix of $B_{\hat{\omega}}^{\lambda, \omega}$ obtained by deleting the $(n+1)$ -th column, we get

$$\begin{aligned} L_q &= B_{\hat{\omega}}^{\lambda, \omega}(n+1)^T W^{\lambda, \omega}(c, \omega) B_{\hat{\omega}}^{\lambda, \omega}(n+1) \\ &= \begin{bmatrix} B_{\omega}^T & D_{\omega}^{-1} \end{bmatrix} \begin{bmatrix} W(c, \omega) & 0 \\ 0 & \lambda D_{\omega}^2 \end{bmatrix} \begin{bmatrix} B_{\omega} \\ D_{\omega}^{-1} \end{bmatrix} = B_{\omega}^T W(c, \omega) B_{\omega} + \lambda I. \end{aligned}$$

Note, that the above matrices are similar to the ones introduced in [16] in relation with the normalized Laplacian. Notice that with our interpretation these matrices acquire a real meaning.

Observe that if $\hat{\mathcal{F}}$ is a spanning tree on $\Gamma_{\lambda, \omega}$ with vertex $n+1$ as a root and k edges of type $\{n+1, i\}$, the forest generated in Γ , \mathcal{F} , has k connected components and each i belongs to a different component and hence it can be considered as a root of the forest, see Figure 4.

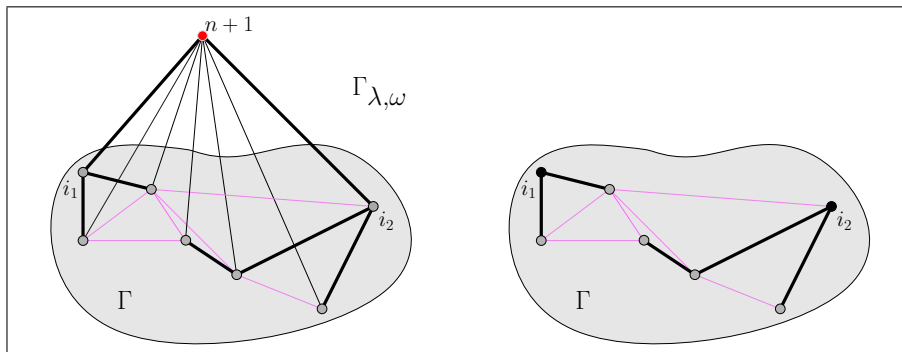


Figure 4: A spanning tree $\hat{\mathcal{F}}$ (left) and the induced 2-rooted forest \mathcal{F}_2 (right).

With this interpretation of $\hat{\mathcal{F}}$, we can state some generalization of the matrix-tree Theorem. This extension considers not only weights in the edges, as in the previous works, but also weights on the vertices. Moreover, the result applies to general irreducible symmetric M -matrices. Although the proof is based on the *all minor tree Theorem* due to Chaiken, [11], we follow here the clearer terminology introduced in [8].

Call a subset $J = \{(i_1, j_1), \dots, (i_\ell, j_\ell)\} \subset V \times V$ component-disjoint if $i_p \neq i_q$ and $j_p \neq j_q$ for every $p \neq q$ and $i_1 < i_2 < \dots < i_\ell$; denote $\sum J = \sum_{p=1}^{\ell} (i_p + j_p)$. Denote by τ_J the permutation of $\{1, 2, \dots, \ell\}$ defined by the condition $j_{\tau_J(1)} < j_{\tau_J(2)} < \dots < j_{\tau_J(\ell)}$.

A spanning forest, \mathcal{F} , is called J -admissible if it has $\ell = |J|$ connected components, and every component contains exactly one vertex from the set $\{i_1, \dots, i_\ell\}$, and exactly one, from $\{j_1, \dots, j_\ell\}$ (the two may coincide if the sets intersect). The vertices i_1, \dots, i_ℓ will be considered as roots. Denote by $\sigma_J^{\mathcal{F}}$ a permutation of V such that i_p and $j_{\sigma_J^{\mathcal{F}}(p)}$ lie in the same component of \mathcal{F} , for every $p = 1, 2, \dots, \ell$.

For a matrix M of order n and a component-disjoint set J , denote by $M(J)$ the submatrix of M obtained by deletion of the rows i_1, \dots, i_ℓ and the columns j_1, \dots, j_ℓ . For any permutation τ , $\epsilon(\tau)$ will denote its sign.

Theorem 2.1. [8, 11] For any component-disjoint subset $J \subset V \times V$ it is verified

$$(-1)^{\sum J} \det(L_{q_\omega}(J)) = \sum_{\mathcal{F}} \epsilon(\tau_J \circ \sigma_J^{\mathcal{F}}) w(\mathcal{F}) \prod_{p=1}^{\ell} \omega(i_p) \omega(j_p),$$

where the sum is taken over the set of all J -admissible forest \mathcal{F} of Γ and \circ means the composition of two permutations.

The cases in which $\ell = 1, 2$ have an special meaning for our purposes.

Proposition 2.2. *Let $\lambda > 0$ and $\omega \in \Omega(V)$. Then, the determinant of the matrix associated with a positive definite Schrödinger operator, \mathcal{L}_q , is given by*

$$\det(\mathcal{L}_q) = \sum_{k=1}^n a_k \lambda^k,$$

with

$$a_k = \sum_{\mathcal{F} \in \mathcal{F}_k} w(\mathcal{F}) \prod_{i_p \in r(\mathcal{F})} \omega(i_p)^2.$$

Proof. From Equation (4) and Theorem 2.1 taking $J = \{(n + 1, n + 1)\}$ and keeping in mind that $\hat{\omega}(n + 1) = 1$, we get that

$$\det(\mathcal{L}_q) = \det\left(\mathcal{L}_{\hat{q}}^{\lambda, \omega}(J)\right) = \sum_{\hat{\mathcal{F}}} \hat{w}(\hat{\mathcal{F}}),$$

where $\hat{\mathcal{F}}$ is a J -admissible forest in $\Gamma_{\lambda, \omega}$ rooted at $n + 1$. Notice that in this case the permutations are the identity. Moreover, if $k, k = 1, \dots, n$, is the number of edges of the form $e_p = \{n + 1, i_p\}, p = 1, \dots, k$, in $\hat{\mathcal{F}}$ and \mathcal{F} is the induced k -rooted spanning forest in $\Gamma, k = 1, \dots, n$ with $r(\mathcal{F}) = \{i_1, \dots, i_k\}$ as set of roots, we have that

$$\hat{w}(\hat{\mathcal{F}}) = \frac{\prod_{e_p} \lambda \omega(i_p)^2 \prod_{e \in E(\mathcal{F})} w(e)}{\prod_{j \in V} \omega(j)^2} = \lambda^k w(\mathcal{F}) \prod_{p=1}^k \omega(i_p)^2. \quad \square$$

Observe that

$$\det(\mathcal{L}_q) = \det(\mathcal{L}_{q_\omega} + \lambda I),$$

and hence a_n must equal 1 and $a_{n-1} = \text{tr}(\mathcal{L}_{q_\omega}) = \sum_{x, y \in V} c(x, y) \frac{\omega(x)}{\omega(y)}$. This values can also be deduced from the fact that there is only one forest with n connected components and there are $2m$ rooted forest with $n - 1$ components. Moreover, notice that the above result is also valid for $\lambda = 0$, since in this case $\det(\mathcal{L}_{q_\omega}) = 0$.

Proposition 2.3. *Let $\lambda > 0$ and $\omega \in \Omega(V)$. Then, for any $t, s \in V$,*

$$(-1)^{t+s} \det(\mathcal{L}_q(t|s)) = \omega(s)\omega(t) \sum_{k=1}^n b_k(t, s) \lambda^{k-1},$$

where

$$b_k(t, s) = b_k(s, t) = \sum_{\mathcal{F} \in \mathcal{F}_{k, t, s}} w(\mathcal{F}) \prod_{i_p \in r^*(\mathcal{F})} \omega(i_p)^2.$$

In particular, when $t = s$,

$$\det(\mathcal{L}_q(t|t)) = \sum_{k=1}^n \lambda^{k-1} \sum_{\mathcal{F} \in \mathcal{F}_{k, t}} w(\mathcal{F}) \prod_{i_p \in r(\mathcal{F})} \omega(i_p)^2.$$

Proof. From Equation (4) and Theorem 2.1 taking $J = \{(t, n + 1), (n + 1, s)\}$ and keeping in mind that $\hat{\omega}(n + 1) = 1$, we get that

$$(-1)^{t+s} \det(\mathcal{L}_q(t|s)) = (-1)^{t+s} \det\left(\mathcal{L}_{\hat{q}}^{\lambda, \omega}(J)\right) = \sum_{\hat{\mathcal{F}}} w(\hat{\mathcal{F}}) \omega(t) \omega(s),$$

where $\hat{\mathcal{F}}$ is a 2-rooted spanning forest in $\hat{\Gamma}$ rooted at t and $n + 1$ and such that t and s belong to the same connected component. Moreover,

$$\hat{w}(\hat{\mathcal{F}}) = \frac{\prod_{e_p} \lambda \omega(i_p)^2 \prod_{e \in \mathcal{F}} w(e)}{\prod_{j \in V} \omega(j)^2} = \lambda^k \prod_{i_p} \omega(i_p)^2 w(\mathcal{F}),$$

where $k \in \{0, \dots, n - 1\}$ is the number of edges of the form $e_p = \{n + 1, i_p\}$, $p = 1, \dots, k$, and \mathcal{F} is the induced $(k + 1)$ -rooted spanning forest in Γ , $k = 0, \dots, n - 1$ with $r(\mathcal{F}) = \{t, i_1, \dots, i_k\}$ as set of roots and t and s belong to the same connected component; and hence

$$(-1)^{t+s} \det(L_q(t|s)) = \omega(s)\omega(t) \sum_{k=0}^{n-1} \lambda^k \sum_{\mathcal{F} \in \mathcal{F}_{k+1,t,s}} w(\mathcal{F}) \prod_{i_p \in r^*(\mathcal{F})} \omega(i_p)^2,$$

where \mathcal{F} is a $(k + 1)$ -rooted spanning forest in Γ with $t \in r(\mathcal{F})$ as set of roots and t, s belong to the same connected component. When $t = s$, the formula follows straightforward. \square

Observe that the above result for the specific case of the normalized Laplacian was obtained by Chung and Zhao in [16, Theorem 3], using an intricate proof based on Cauchy–Binet Theorem. Chebotarev *et al.*, in a serie of papers, also obtained a similar result for the case of weighted (di)graphs for the matrix $L + \lambda I$, [12, 13].

The following result represents a full weighted version of the classical matrix tree Theorem, [7, 22]. It can be obtained either directly from Chaiken’s Theorem or taking limit when λ goes to 0 in the previous result.

Corollary 2.4. *Let $\omega \in \Omega(V)$. Then, for any $t, s \in V$,*

$$(-1)^{t+s} \det(L_{q_\omega}(t|s)) = \omega(s)\omega(t) \sum_{\mathcal{F}} w(\mathcal{F}),$$

where \mathcal{F} runs on the set of spanning trees of Γ .

3 The Group inverse of symmetric M -matrices

Using the matrix–tree theorems in the preceding section we can write down a combinatorial expression for the group inverse of any irreducible symmetric M -matrix. To do this we consider the associated Schrödinger operator, \mathcal{L}_q , and then we get its Green function in terms of the weights of rooted spanning forests in Γ .

Proposition 3.1. *Let $\lambda \geq 0$ and $\omega \in \Omega(V)$. Then, for any $t, s \in V$,*

$$G_q(t, s) = \omega(s)\omega(t) \frac{\sum_{k=2}^n \lambda^{k-2} (nb_k(t, s) - a_k)}{n \sum_{k=1}^n \lambda^{k-1} a_k},$$

where

$$b_k(t, s) = b_k(s, t) = \sum_{\mathcal{F} \in \mathcal{F}_{k,t,s}} w(\mathcal{F}) \prod_{i_p \in r^*(\mathcal{F})} \omega(i_p)^2 \text{ and } a_k = \sum_{\mathcal{F} \in \mathcal{F}_k} w(\mathcal{F}) \prod_{i_p \in r(\mathcal{F})} \omega(i_p)^2.$$

In particular, $nb_1(t, s) = a_1$, and hence b_1 is independent of the variables t, s . Moreover, for $\lambda = 0$

$$G_{q_\omega}(t, s) = \omega(s)\omega(t) \frac{(nb_2(t, s) - a_2)}{na_1}.$$

Proof. Applying Theorems 2.3 and 2.2 to the adjoint formula for the inverse of a non-singular matrix, we get that

$$L_q^{-1}(t, s) = \omega(s)\omega(t) \frac{\sum_{k=1}^n b_k(t, s)\lambda^{k-1}}{\lambda \sum_{k=1}^n a_k \lambda^{k-1}}.$$

Observe that, $nb_1(t, s) = a_1$, since

$$b_1(t, s) = \sum_{\mathcal{F} \in \mathcal{F}_{1,t,s}} w(\mathcal{F}) \prod_{i_p \in r(\mathcal{F})^*} \omega(i_p)^2 = \sum_{\mathcal{F} \text{ spanning tree}} w(\mathcal{F})$$

and

$$a_1 = \sum_{\mathcal{F} \in \mathcal{F}_1} w(\mathcal{F}) \prod_{i_p \in r(\mathcal{F})} \omega(i_p)^2 = \sum_{\substack{\mathcal{F} \text{ spanning} \\ \text{tree}}} w(\mathcal{F}) \sum_{j \in V} \omega^2(j) = n \sum_{\substack{\mathcal{F} \text{ spanning} \\ \text{tree}}} w(\mathcal{F}).$$

Hence, we get that

$$\frac{G_q(t, s)}{\omega(s)\omega(t)} = \frac{L_q^{-1}(t, s)}{\omega(s)\omega(t)} - \frac{1}{n\lambda} = \frac{\sum_{k=1}^n \lambda^{k-1} (nb_k(t, s) - a_k)}{n\lambda \sum_{k=1}^n \lambda^{k-1} a_k} = \frac{\sum_{k=2}^n \lambda^{k-2} (nb_k(t, s) - a_k)}{n \sum_{k=1}^n \lambda^{k-1} a_k}.$$

The result for $\lambda = 0$ follows by taking limit in the above expression. □

The following result contains the particular case in which ω is the constant weight. So, the operator is $\mathcal{L}_q = \mathcal{L} + \lambda I$ where \mathcal{L} is the combinatorial Laplacian of a network. This result coincides with the ones obtained in [13, 26, 27].

Proposition 3.2. *Let $\lambda \geq 0$. Then, for any $t, s \in V$,*

$$G_q(t, s) = \frac{\sum_{k=2}^n \lambda^{k-2} \left(n \sum_{\mathcal{F} \in \mathcal{F}_{k,t,s}} w(\mathcal{F}) - \sum_{\mathcal{F} \in \mathcal{F}_k} w(\mathcal{F}) \right)}{n \sum_{k=1}^n \lambda^{k-1} \sum_{\mathcal{F} \in \mathcal{F}_k} w(\mathcal{F})}.$$

In particular, for $\lambda = 0$,

$$G_{q_\omega}(t, s) = \frac{n \sum_{\mathcal{F} \in \mathcal{F}_{2,t,s}} w(\mathcal{F}) - \sum_{\mathcal{F} \in \mathcal{F}_2} w(\mathcal{F})}{n \sum_{\mathcal{F} \in \mathcal{F}_1} w(\mathcal{F})}.$$

For graphs we obtain the expression for the Group inverse of the combinatorial Laplacian L that has been widely considered in the literature, [13, 16].

Proposition 3.3. *Let $\lambda \geq 0$ and \mathcal{L} the combinatorial Laplacian of a graph. Then, for any $t, s \in V$,*

$$G_\lambda(t, s) = \frac{\sum_{k=2}^n \lambda^{k-2} \left(n|\mathcal{F}_{k,t,s}| - |\mathcal{F}_k| \right)}{n \sum_{k=1}^n \lambda^{k-1} |\mathcal{F}_k|}.$$

It means that for $\lambda = 0$,

$$G(t, s) = \frac{\left(n|\mathcal{F}_{2,t,s}| - |\mathcal{F}_2| \right)}{n|\mathcal{F}_1|}.$$

In the following example we show how to compute the combinatorial coefficients a_k and $b_k(x_t, x_s)$ in the case of a star on n vertices with center at x_1 and conductances $c_i, i = 1, \dots, n - 1$, see Figure 5. In [9, Corollary 5.3] the expression for $G_\lambda(x_t, x_s)$ was obtained by the authors using different techniques.

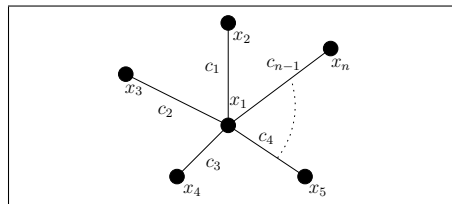


Figure 5: A star on n vertices.

Proposition 3.4. For a star network with n vertices centered at x_1 and conductances $c_i, i = 1, \dots, n - 1$, it is satisfied that

$$a_k = (n - k + 1) \sum_{1 \leq i_1 < \dots < i_{n-k} \leq n-1} c_{i_1} \dots c_{i_{n-k}}, \quad k = 1, \dots, n.$$

Moreover, for any $k = 2, \dots, n$, we have

$$b_k(x_1, x_1) = \sum_{1 \leq i_1 < \dots < i_{n-k} \leq n-1} c_{i_1} \dots c_{i_{n-k}},$$

for $1 < j \leq n$,

$$b_k(x_1, x_j) = \sum_{1 \leq i_1 < \dots < i_{n-k} \leq n-1} c_{i_1} \dots c_{i_{n-k}} - \sum_{\substack{1 \leq i_1 < \dots < i_{n-k} \leq n-1 \\ i_\ell \neq j}} c_{i_1} \dots c_{i_{n-k}},$$

for $1 < j \leq n$,

$$b_k(x_j, x_j) = \sum_{1 \leq i_1 < \dots < i_{n-k} \leq n-1} c_{i_1} \dots c_{i_{n-k}} + (n - k) \sum_{\substack{1 \leq i_1 < \dots < i_{n-k} \leq n-1 \\ i_\ell \neq j}} c_{i_1} \dots c_{i_{n-k}},$$

for $1 < s, j \leq n, s \neq j$

$$b_k(x_j, x_s) = \sum_{1 \leq i_1 < \dots < i_{n-k} \leq n-1} c_{i_1} \dots c_{i_{n-k}} + \sum_{\substack{1 \leq i_1 < \dots < i_{n-k} \leq n-1 \\ i_\ell \neq j, s}} c_{i_1} \dots c_{i_{n-k}} - \sum_{\substack{1 \leq i_1 < \dots < i_{n-k} \leq n-1 \\ i_\ell \neq j}} c_{i_1} \dots c_{i_{n-k}} - \sum_{\substack{1 \leq i_1 < \dots < i_{n-k} \leq n-1 \\ i_\ell \neq s}} c_{i_1} \dots c_{i_{n-k}}.$$

To end this section we consider the computation of some small cases that will show the spirit of the results.

The Group inverse for K_2 . Observe that this case encompasses the case of any 2×2 symmetric M -matrix. Consider $\Gamma = (V, E, c)$, where $V = \{x_1, x_2\}$, $E = \{e = (x_1, x_2)\}$ and $c = c(x_1, x_2) > 0$. Moreover, we consider the weight $\omega_1 = \omega(x_1)$, $\omega_2 = \omega(x_2)$, where $\omega_1^2 + \omega_2^2 = 2$, the parameter $\lambda \geq 0$ and $q = q_\omega + \lambda$. In this case,

$$L_q = \begin{bmatrix} c \frac{\omega_2}{\omega_1} + \lambda & -c \\ -c & c \frac{\omega_1}{\omega_2} + \lambda \end{bmatrix}.$$

For $\lambda > 0$, we can compute directly the inverse of L_q , obtaining

$$(L_q)^{-1} = \frac{1}{\lambda \left(\frac{2c}{\omega_1 \omega_2} + \lambda \right)} \begin{bmatrix} c \frac{\omega_1}{\omega_2} + \lambda & c \\ c & c \frac{\omega_2}{\omega_1} + \lambda \end{bmatrix} = \frac{\omega_1 \omega_2}{\lambda (2c + \lambda \omega_1 \omega_2)} \begin{bmatrix} c \frac{\omega_1}{\omega_2} + \lambda & c \\ c & c \frac{\omega_2}{\omega_1} + \lambda \end{bmatrix}.$$

On the other hand, we get that

$$2b_1(x_i, x_j) = a_1 = 2 \frac{c}{\omega_1 \omega_2}, \quad a_2 = 1, \\ b_2(x_1, x_1) = \frac{1}{\omega_1^2}, \quad b_2(x_1, x_2) = 0, \quad b_2(x_2, x_2) = \frac{1}{\omega_2^2}.$$

In Figure 6 we can see the existing trees. Therefore, keeping in mind that $a_1 + \lambda a_2 = \frac{2c}{\omega_1 \omega_2} + \lambda$, we get that

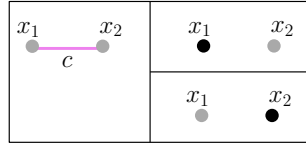


Figure 6: The network K_2 (left) and the collection of 2-rooted forests for K_2 (right).

$$(L_q)^{-1}(x_1, x_1) = \frac{\omega_1^2 (b_1(x_1, x_1) + \lambda b_2(x_1, x_1))}{\lambda(a_1 + \lambda a_2)} = \frac{\frac{c\omega_1}{\omega_2} + \lambda}{\lambda \left(\frac{2c}{\omega_1\omega_2} + \lambda \right)} = \frac{\omega_1\omega_2}{\lambda(2c + \lambda\omega_1\omega_2)} \left(\frac{c\omega_1}{\omega_2} + \lambda \right),$$

$$(L_q)^{-1}(x_1, x_2) = \frac{\omega_1\omega_2 (b_1(x_1, x_2) + \lambda b_2(x_1, x_2))}{\lambda(a_1 + \lambda a_2)} = \frac{c}{\lambda \left(\frac{2c}{\omega_1\omega_2} + \lambda \right)} = \frac{\omega_1\omega_2 c}{\lambda(2c + \lambda\omega_1\omega_2)},$$

$$(L_q)^{-1}(x_2, x_2) = \frac{\omega_2^2 (b_1(x_2, x_2) + \lambda b_2(x_2, x_2))}{\lambda(a_1 + \lambda a_2)} = \frac{\frac{c\omega_2}{\omega_1} + \lambda}{\lambda \left(\frac{2c}{\omega_1\omega_2} + \lambda \right)} = \frac{\omega_1\omega_2}{\lambda(2c + \lambda\omega_1\omega_2)} \left(\frac{c\omega_2}{\omega_1} + \lambda \right).$$

Moreover,

$$G_q(x_1, x_1) = \frac{\omega_1^2 (2b_2(x_1, x_1) - a_2)}{2(a_1 + \lambda a_2)} = \frac{\omega_2^2}{2 \left(\frac{2c}{\omega_1\omega_2} + \lambda \right)} = \frac{\omega_1\omega_2^3}{2(2c + \lambda\omega_1\omega_2)},$$

$$G_q(x_1, x_2) = \frac{\omega_1\omega_2 (2b_2(x_1, x_2) - a_2)}{2(a_1 + \lambda a_2)} = \frac{-\omega_1\omega_2}{2 \left(\frac{2c}{\omega_1\omega_2} + \lambda \right)} = \frac{-\omega_1^2\omega_2^2}{2(2c + \lambda\omega_1\omega_2)},$$

$$G_q(x_2, x_2) = \frac{\omega_2^2 (2b_2(x_2, x_2) - a_2)}{2(a_1 + \lambda a_2)} = \frac{\omega_1^2}{2 \left(\frac{2c}{\omega_1\omega_2} + \lambda \right)} = \frac{\omega_1^3\omega_2}{2(2c + \lambda\omega_1\omega_2)}.$$

Finally, for $\lambda = 0$

$$G_{q_\omega}(x_1, x_1) = \frac{\omega_1\omega_2^3}{4c}, \quad G_{q_\omega}(x_1, x_2) = \frac{-\omega_1^2\omega_2^2}{4c}, \quad G_{q_\omega}(x_2, x_2) = \frac{\omega_2\omega_1^3}{4c}.$$

The Group inverse for K_3 . Now, we consider any symmetric M -matrix of order 3. For that, Let $\Gamma = (V, E, c)$, where $V = \{x_1, x_2, x_3\}$, $E = \{e_1 = (x_1, x_2), e_2 = (x_2, x_3), e_3 = (x_3, x_1)\}$ and $c_1 = c(x_1, x_2)$, $c_2 = c(x_2, x_3)$, $c_3 = c(x_3, x_1) > 0$. Moreover, we consider the weight $\omega_1 = \omega(x_1)$, $\omega_2 = \omega(x_2)$, $\omega_3 = \omega(x_3)$, where $\sum_{k=1}^3 \omega_k^2 = 3$, $w_i = c_i\omega_i\omega_{i+1}$, the parameter $\lambda \geq 0$ and $q = q_\omega + \lambda$. In this case,

$$L_q = \begin{bmatrix} c_1 \frac{\omega_2}{\omega_1} + c_3 \frac{\omega_3}{\omega_1} + \lambda & -c_1 & -c_3 \\ -c_1 & c_1 \frac{\omega_1}{\omega_2} + c_2 \frac{\omega_3}{\omega_2} + \lambda & -c_2 \\ -c_3 & -c_2 & c_3 \frac{\omega_1}{\omega_3} + c_2 \frac{\omega_2}{\omega_3} + \lambda \end{bmatrix}.$$

Next, we compute the coefficients b and a involved in the formula of the group inverse. We also represent the trees or forests of any order in Figures 7-9.

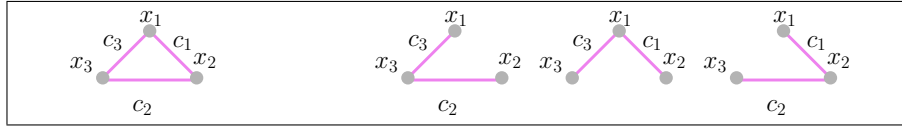


Figure 7: The network K_3 (left) and the collection of spanning trees for K_3 (right).

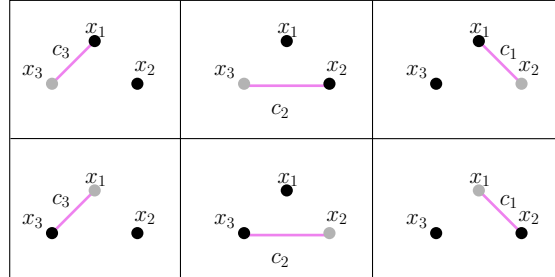


Figure 8: The collection of 2-rooted forests for K_3 .

$$a_1 = 3 \prod_{i=1}^3 c_i \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} \right),$$

$$a_2 = \prod_{i=1}^3 c_i \left(\frac{\omega_2^2(\omega_1^2 + \omega_3^2)}{w_1 w_2} + \frac{\omega_1^2(\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{\omega_3^2(\omega_1^2 + \omega_2^2)}{w_2 w_3} \right),$$

$$a_3 = 1 = \prod_{i=1}^3 c_i \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3}.$$

To compute coefficients $b_k(x_i, x_j)$ it is enough to count the k -rooted forests displayed at Figures 8 and 9 with x_i as a root and x_j at the same component as x_i .

$$b_1(x_i, x_j) = \frac{a_1}{3},$$

$$b_2(x_1, x_1) = \prod_{i=1}^3 c_i \left(\frac{\omega_2^2}{w_1 w_2} + \frac{\omega_3^2}{w_2 w_3} + \frac{(\omega_2^2 + \omega_3^2)}{w_1 w_3} \right),$$

$$b_2(x_2, x_2) = \prod_{i=1}^3 c_i \left(\frac{\omega_1^2}{w_1 w_3} + \frac{\omega_3^2}{w_2 w_3} + \frac{(\omega_1^2 + \omega_3^2)}{w_1 w_2} \right),$$

$$b_2(x_3, x_3) = \prod_{i=1}^3 c_i \left(\frac{\omega_1^2}{w_1 w_3} + \frac{\omega_2^2}{w_1 w_2} + \frac{(\omega_1^2 + \omega_2^2)}{w_2 w_3} \right),$$

$$b_2(x_1, x_2) = \prod_{i=1}^3 c_i \frac{\omega_3^2}{w_2 w_3}, \quad b_2(x_1, x_3) = \prod_{i=1}^3 c_i \frac{\omega_2^2}{w_1 w_2}, \quad b_2(x_2, x_3) = \prod_{i=1}^3 c_i \frac{\omega_1^2}{w_1 w_3},$$

$$b_3(x_i, x_i) = \frac{1}{\omega_i^2} = \prod_{i=1}^3 c_i \frac{\prod_{j \neq i} \omega_j^2}{w_1 w_2 w_3}, \quad b_3(x_i, x_j) = 0, \text{ for all } i \neq j.$$

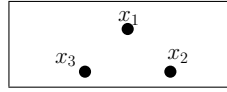


Figure 9: The collection of 3-rooted forests for K_3 .

Applying Theorems 2.3 and 2.2, we get that the inverse of L_q for K_3 is

$$L_q^{-1}(x_1, x_1) = \omega_1^2 \frac{\left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) + \lambda \left(\frac{\omega_2^2}{w_1 w_2} + \frac{\omega_3^2}{w_2 w_3} + \frac{(\omega_2^2 + \omega_3^2)}{w_1 w_3}\right) + \lambda^2 \frac{\omega_2^2 \omega_3^2}{w_1 w_2 w_3}}{3\lambda \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) + \lambda^2 \left(\frac{\omega_2^2(\omega_1^2 + \omega_3^2)}{w_1 w_2} + \frac{\omega_1^2(\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{\omega_3^2(\omega_1^2 + \omega_2^2)}{w_2 w_3}\right) + \lambda^3 \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3}},$$

$$L_q^{-1}(x_2, x_2) = \omega_2^2 \frac{\left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) + \lambda \left(\frac{\omega_1^2}{w_1 w_3} + \frac{\omega_3^2}{w_2 w_3} + \frac{(\omega_1^2 + \omega_3^2)}{w_1 w_2}\right) + \lambda^2 \frac{\omega_1^2 \omega_3^2}{w_1 w_2 w_3}}{3\lambda \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) + \lambda^2 \left(\frac{\omega_2^2(\omega_1^2 + \omega_3^2)}{w_1 w_2} + \frac{\omega_1^2(\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{\omega_3^2(\omega_1^2 + \omega_2^2)}{w_2 w_3}\right) + \lambda^3 \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3}},$$

$$L_q^{-1}(x_3, x_3) = \omega_3^2 \frac{\left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) + \lambda \left(\frac{\omega_1^2}{w_1 w_3} + \frac{\omega_2^2}{w_1 w_2} + \frac{(\omega_1^2 + \omega_2^2)}{w_2 w_3}\right) + \lambda^2 \frac{\omega_1^2 \omega_2^2}{w_1 w_2 w_3}}{3\lambda \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) + \lambda^2 \left(\frac{\omega_2^2(\omega_1^2 + \omega_3^2)}{w_1 w_2} + \frac{\omega_1^2(\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{\omega_3^2(\omega_1^2 + \omega_2^2)}{w_2 w_3}\right) + \lambda^3 \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3}},$$

$$L_q^{-1}(x_1, x_2) = \omega_1 \omega_2 \frac{\left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) + \lambda \frac{\omega_3^2}{w_2 w_3}}{3\lambda \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) + \lambda^2 \left(\frac{\omega_2^2(\omega_1^2 + \omega_3^2)}{w_1 w_2} + \frac{\omega_1^2(\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{\omega_3^2(\omega_1^2 + \omega_2^2)}{w_2 w_3}\right) + \lambda^3 \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3}},$$

$$L_q^{-1}(x_1, x_3) = \omega_1 \omega_3 \frac{\left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) + \lambda \frac{\omega_2^2}{w_1 w_2}}{3\lambda \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) + \lambda^2 \left(\frac{\omega_2^2(\omega_1^2 + \omega_3^2)}{w_1 w_2} + \frac{\omega_1^2(\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{\omega_3^2(\omega_1^2 + \omega_2^2)}{w_2 w_3}\right) + \lambda^3 \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3}},$$

$$L_q^{-1}(x_2, x_3) = \omega_2 \omega_3 \frac{\left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) + \lambda \frac{\omega_1^2}{w_1 w_3}}{3\lambda \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3}\right) + \lambda^2 \left(\frac{\omega_2^2(\omega_1^2 + \omega_3^2)}{w_1 w_2} + \frac{\omega_1^2(\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{\omega_3^2(\omega_1^2 + \omega_2^2)}{w_2 w_3}\right) + \lambda^3 \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3}}.$$

Moreover, from Proposition 4.1 we get that the expression for the Green function for K_3 and $\lambda \geq 0$ is

$$3G_q(x_1, x_1) = \frac{\omega_1^2 \left(\frac{\omega_2^4}{w_1 w_2} + \frac{\omega_3^4}{w_2 w_3} + \frac{(\omega_2^2 + \omega_3^2)^2}{w_1 w_3} + \lambda \frac{\omega_2^2 \omega_3^2 (\omega_2^2 + \omega_3^2)}{w_1 w_2 w_3} \right)}{3 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} \right) + \lambda \left(\frac{\omega_2^2 (\omega_1^2 + \omega_3^2)}{w_1 w_2} + \frac{\omega_1^2 (\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{\omega_3^2 (\omega_1^2 + \omega_2^2)}{w_2 w_3} \right) + \lambda^2 \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3}},$$

$$3G_q(x_2, x_2) = \frac{\omega_2^2 \left(\frac{\omega_1^4}{w_1 w_3} + \frac{\omega_3^4}{w_2 w_3} + \frac{(\omega_1^2 + \omega_3^2)^2}{w_1 w_2} + \lambda \frac{\omega_1^2 \omega_3^2 (\omega_1^2 + \omega_3^2)}{w_1 w_2 w_3} \right)}{3 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} \right) + \lambda \left(\frac{\omega_2^2 (\omega_1^2 + \omega_3^2)}{w_1 w_2} + \frac{\omega_1^2 (\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{\omega_3^2 (\omega_1^2 + \omega_2^2)}{w_2 w_3} \right) + \lambda^2 \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3}},$$

$$3G_q(x_3, x_3) = \frac{\omega_3^2 \left(\frac{\omega_1^4}{w_1 w_3} + \frac{\omega_2^4}{w_1 w_2} + \frac{(\omega_1^2 + \omega_2^2)^2}{w_2 w_3} + \lambda \frac{\omega_1^2 \omega_2^2 (\omega_1^2 + \omega_2^2)}{w_1 w_2 w_3} \right)}{3 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} \right) + \lambda \left(\frac{\omega_2^2 (\omega_1^2 + \omega_3^2)}{w_1 w_2} + \frac{\omega_1^2 (\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{\omega_3^2 (\omega_1^2 + \omega_2^2)}{w_2 w_3} \right) + \lambda^2 \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3}},$$

$$3G_q(x_1, x_2) = \frac{\omega_1 \omega_2 \left(\frac{\omega_3^4}{w_2 w_3} - \frac{\omega_2^2 (\omega_1^2 + \omega_3^2)}{w_1 w_2} - \frac{\omega_1^2 (\omega_2^2 + \omega_3^2)}{w_1 w_3} - \lambda \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3} \right)}{3 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} \right) + \lambda \left(\frac{\omega_2^2 (\omega_1^2 + \omega_3^2)}{w_1 w_2} + \frac{\omega_1^2 (\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{\omega_3^2 (\omega_1^2 + \omega_2^2)}{w_2 w_3} \right) + \lambda^2 \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3}},$$

$$3G_q(x_1, x_3) = \frac{\omega_1 \omega_3 \left(\frac{\omega_2^4}{w_1 w_2} - \frac{\omega_1^2 (\omega_2^2 + \omega_3^2)}{w_1 w_3} - \frac{\omega_3^2 (\omega_1^2 + \omega_2^2)}{w_2 w_3} - \lambda \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3} \right)}{3 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} \right) + \lambda \left(\frac{\omega_2^2 (\omega_1^2 + \omega_3^2)}{w_1 w_2} + \frac{\omega_1^2 (\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{\omega_3^2 (\omega_1^2 + \omega_2^2)}{w_2 w_3} \right) + \lambda^2 \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3}},$$

$$3G_q(x_2, x_3) = \frac{\omega_1 \omega_2 \left(\frac{\omega_1^4}{w_1 w_3} - \frac{\omega_2^2 (\omega_1^2 + \omega_3^2)}{w_1 w_2} - \frac{\omega_3^2 (\omega_1^2 + \omega_2^2)}{w_2 w_3} - \lambda \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3} \right)}{3 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} \right) + \lambda \left(\frac{\omega_2^2 (\omega_1^2 + \omega_3^2)}{w_1 w_2} + \frac{\omega_1^2 (\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{\omega_3^2 (\omega_1^2 + \omega_2^2)}{w_2 w_3} \right) + \lambda^2 \frac{\omega_1^2 \omega_2^2 \omega_3^2}{w_1 w_2 w_3}}.$$

In particular, for $\lambda = 0$, we get that

$$\frac{G_q(x_1, x_1)}{\omega_1^2} = \frac{\frac{\omega_2^4}{w_1 w_2} + \frac{\omega_3^4}{w_2 w_3} + \frac{(\omega_2^2 + \omega_3^2)^2}{w_1 w_3}}{9 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} \right)}, \quad \frac{G_q(x_1, x_2)}{\omega_1 \omega_2} = \frac{\frac{\omega_3^4}{w_2 w_3} - \frac{\omega_2^2 (\omega_1^2 + \omega_3^2)}{w_1 w_2} - \frac{\omega_1^2 (\omega_2^2 + \omega_3^2)}{w_1 w_3}}{9 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} \right)},$$

$$\frac{G_q(x_2, x_2)}{\omega_2^2} = \frac{\frac{\omega_1^4}{w_1 w_3} + \frac{\omega_3^4}{w_2 w_3} + \frac{(\omega_1^2 + \omega_3^2)^2}{w_1 w_2}}{9 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} \right)}, \quad \frac{G_q(x_1, x_3)}{\omega_1 \omega_3} = \frac{\frac{\omega_2^4}{w_1 w_2} - \frac{\omega_1^2 (\omega_2^2 + \omega_3^2)}{w_1 w_3} - \frac{\omega_3^2 (\omega_1^2 + \omega_2^2)}{w_2 w_3}}{9 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} \right)},$$

$$\frac{G_q(x_3, x_3)}{\omega_3^2} = \frac{\frac{\omega_1^4}{w_1 w_3} + \frac{\omega_2^4}{w_1 w_2} + \frac{(\omega_1^2 + \omega_2^2)^2}{w_2 w_3}}{9 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} \right)}, \quad \frac{G_q(x_2, x_3)}{\omega_2 \omega_3} = \frac{\frac{\omega_1^4}{w_1 w_3} - \frac{\omega_2^2 (\omega_1^2 + \omega_3^2)}{w_1 w_2} - \frac{\omega_3^2 (\omega_1^2 + \omega_2^2)}{w_2 w_3}}{9 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} \right)}.$$

4 The group inverse for trees

Consider now a weighted tree with constant weights on the vertex set, \mathcal{T} . For any $k \geq 1$ and any set of edges $\mathcal{E}_k \subset E(\mathcal{T})$ satisfying $|\mathcal{E}_k| = k - 1$, we let $\mathcal{T} \setminus \mathcal{E}_k$ denote the graph obtained from \mathcal{T} by deleting the edges of \mathcal{E}_k . Observe that, $\mathcal{T} \setminus \mathcal{E}_k$ is a forest that has k connected components, that will be denoted by $C_i(\mathcal{E}_k)$, $i = 1, \dots, k$. For each set $\mathcal{E}_k \in E(\mathcal{T})$ and a vertex $t \in \mathcal{T}$, we denote $C_i^t(\mathcal{E}_k)$, $i = 1, \dots, k - 1$, the set of vertices in the i -th connected component of $\mathcal{T} \setminus \mathcal{E}_k$ that does not contain vertex t . Moreover, for any pair of vertices $t, s \in V(\mathcal{T})$, we denote by $\mathcal{E}_k^{t \rightarrow s} \subset E(\mathcal{T})$ a set of $k - 1$ edges that are not in the unique path from t to s in \mathcal{T} , $\mathcal{P}_{t,s}$. Finally, denote by $w(\mathcal{E}_k) = \prod_{e \in \mathcal{E}_k} c(e)$.

Corollary 4.1. *Let $\lambda \geq 0$. Then, for any $t, s \in \mathcal{T}$*

$$G_\lambda(t, s) = \frac{\sum_{k=2}^n \lambda^{k-2} \left(n \sum_{\mathcal{E}_k^{t \rightarrow s}} \frac{\prod_{i=1}^{k-1} |C_i^t(\mathcal{E}_k^{t \rightarrow s})|}{w(\mathcal{E}_k^{t \rightarrow s})} - \sum_{\mathcal{E}_k} \frac{\prod_{i=1}^k |C_i(\mathcal{E}_k)|}{w(\mathcal{E}_k)} \right)}{n \sum_{k=1}^n \lambda^{k-1} \sum_{\mathcal{E}_k} \frac{\prod_{i=1}^k |C_i(\mathcal{E}_k)|}{w(\mathcal{E}_k)}}.$$

In particular, for $\lambda = 0$ we have that

$$G(t, s) = \frac{1}{n^2} \sum_{e \in E(\mathcal{T})} \frac{|C_1^t(e)|^2}{c(e)} - \frac{1}{n} \sum_{e \in \mathcal{P}_{t,s}} \frac{|C_1^t(e)|}{c(e)}.$$

Proof. It is enough to observe that for any $t, s \in V(\mathcal{T})$, and any choice of edges $\mathcal{E}_k^{t \rightarrow s} = \{e_1, \dots, e_{k-1}\}$, there exist $\prod_{i=1}^{k-1} |C_i^t(\mathcal{E}_k^{t \rightarrow s})|$ different k -rooted forest with t as a root and $\prod_{i=1}^k |C_i(\mathcal{E}_k)|$ different k -rooted forest whose weights are given by

$$w(\mathcal{T} \setminus \mathcal{E}_k^{t \rightarrow s}) = \frac{\prod_{e \in E(\mathcal{T})} c(e)}{w(\mathcal{E}_k^{t \rightarrow s})} \text{ and } w(\mathcal{T} \setminus \mathcal{E}_k) = \frac{\prod_{e \in E(\mathcal{T})} c(e)}{w(\mathcal{E}_k)}$$

and hence the results follow by applying Proposition 4.1. □

Observe that the result in the above Corollary for $\lambda = 0$, coincides with the one obtained in [23, Corollary 7.2.6], see also [24, Lemma 2.14].

In particular, if we consider a path on n vertices and keeping in mind that $|C_1^t(e_k)| = k$ for any e_k whose initial vertex is $k \leq t - 1$ and $|C_1^t(e_k)| = n - k$ for any e_k whose initial vertex is $k \geq t$, we recover the expression for the Green function obtained by the authors in [5]

$$G(t, s) = \frac{1}{n^2} \left(\sum_{k=1}^{\min\{t,s\}-1} \frac{k^2}{c(e_k)} + \sum_{k=\min\{t,s\}}^{n-1} \frac{(n-k)^2}{c(e_k)} \right) - \frac{1}{n} \sum_{k=\min\{t,s\}}^{\max\{t,s\}} \frac{(n-k)}{c(e_k)}.$$

Proposition 4.2. *Let $\lambda > 0$ and \mathcal{L} the combinatorial Laplacian of a weighted tree, \mathcal{T} . Then, for any $t, s \in V(\mathcal{T})$,*

$$L_\lambda^{-1}(t, s) = \frac{1 + \sum_{k=2}^n \lambda^{k-1} \sum_{\mathcal{E}_k^{t \rightarrow s}} w^{-1}(\mathcal{E}_k^{t \rightarrow s}) \prod_{i=1}^{k-1} |C_i^t(\mathcal{E}_k^{t \rightarrow s})|}{\lambda \left(n + \sum_{k=2}^n \lambda^{k-1} \sum_{\mathcal{E}_k} w^{-1}(\mathcal{E}_k) \prod_{i=1}^k |C_i(\mathcal{E}_k)| \right)}.$$

In particular,

$$L_\lambda^{-1}(t, t) = \frac{1 + \sum_{k=2}^n \lambda^{k-1} \sum_{\mathcal{E}_k} w^{-1}(\mathcal{E}_k) \prod_{i=1}^{k-1} |C_i^t(\mathcal{E}_k)|}{\lambda \left(n + \sum_{k=2}^n \lambda^{k-1} \sum_{\mathcal{E}_k} w^{-1}(\mathcal{E}_k) \prod_{i=1}^k |C_i(\mathcal{E}_k)| \right)}.$$

For the case of a path with constant conductances the authors obtained in [3] the inverse of L_λ by using difference equation techniques and comparing both expressions we can state the following in some sense surprising identities.

Corollary 4.3. *Let $\lambda > 0$ with the above notations, we get that*

$$n + \sum_{k=2}^n \lambda^{k-1} \sum_{\mathcal{E}_k} \prod_{i=1}^k |C_i(\mathcal{E}_k)| = U_{n-1} \left(1 + \frac{\lambda}{2} \right)$$

and for $t \leq s$

$$1 + \sum_{k=2}^n \lambda^{k-1} \sum_{\mathcal{E}_k^{t \rightarrow s}} \prod_{i=1}^{k-1} |C_i^t(\mathcal{E}_k^{t \rightarrow s})| = V_{t-1} \left(1 + \frac{\lambda}{2} \right) V_{n-s} \left(1 + \frac{\lambda}{2} \right),$$

where $U_n(x)$ and $V_n(x)$ are the second and third kind Chebyshev polynomials.

It is notable that different values of the parameter λ enable us to give more or less importance to the length of the different walks between vertices, [10, 15]. Moreover, we can obtain some relations between combinatorial numbers and the number of spanning rooted trees. First, let us define the *Horadam numbers* $H_n(r, s)$ as the solution of the following three term recurrence sequence

$$H_{n+2}(r, s) = rH_{n+1}(r, s) + sH_n(r, s), \quad H_0(r, s) = 0, \quad H_1(r, s) = 1, \quad r, s \in \mathbb{Z}.$$

See [19] for properties of Horadam numbers. In particular, it is well known that $H_n(1, 1) = F_n$, the n -th Fibonacci number; $H_n(2, 1) = P_n$, the n -th Pell number; $H_n(1, 2) = J_n$, the n -th Jacobsthal number and $H_n(2, -1) = n$. Moreover, if we allow $r, s \in \mathbb{R}$, we can obtain the relation between Horadam solutions and the second kind Chebyshev polynomials: If $s < 0$, $H_n(r, s) = (\sqrt{-s})^{n-1} U_{n-1}(\frac{r}{2\sqrt{-s}})$ and if $s > 0$, $H_{2n}(r, s) = rs^{n-1} U_{n-1}(1 + \frac{r^2}{2s})$.

Corollary 4.4. *For a path on n vertices whose conductances equal 1 and using the above notations, it is satisfied that*

$$n + \sum_{k=2}^n \lambda^{k-1} \sum_{\mathcal{E}_k} \prod_{i=1}^k |C_i(\mathcal{E}_k)| = \frac{1}{\lambda^n} H_{2n}(\lambda, \lambda) = \frac{1}{\sqrt{\lambda}} H_{2n}(\sqrt{\lambda}, 1) = \lambda^{n-1} H_{2n}(1, \lambda^{-1}).$$

In particular for $\lambda = 1, 4$ and $\frac{1}{2}$, we get

$$\begin{aligned} n + \sum_{k=2}^n \sum_{\mathcal{E}_k} \prod_{i=1}^k |C_i(\mathcal{E}_k)| &= F_{2n}, \\ n + \sum_{k=2}^n 4^{k-1} \sum_{\mathcal{E}_k} \prod_{i=1}^k |C_i(\mathcal{E}_k)| &= \frac{1}{2} P_{2n}, \\ n + \sum_{k=2}^n \frac{1}{2^{k-1}} \sum_{\mathcal{E}_k} \prod_{i=1}^k |C_i(\mathcal{E}_k)| &= \frac{1}{2^{n-1}} J_{2n}. \end{aligned}$$

Observe that the case $\lambda = 1$ gives that F_{2n} is the total number of spanning rooted forests of the path, see [14] for a different proof of this result. On the other hand, P_{2n} represents the number of rooted spanning forest taking into account a weight associated with the number of components of the forest, so the more components the greater the weight. In contrast, for the numbers J_{2n} the less components the greater the weight.

5 The Group inverse for C_4

In this section we consider a cycle of order 4, and hence we look for the inverse or the Green function of some especial class of symmetric M -matrix of order 4. Let $\Gamma = (V, E, c)$, where $V = \{x_1, x_2, x_3, x_4\}$, $E = \{e_1 =$

$(x_1, x_2), e_2 = (x_2, x_3), e_3 = (x_3, x_4), e_4 = (x_4, x_1)$ and $c_1 = c(x_1, x_2), c_2 = c(x_2, x_3), c_3 = c(x_3, x_4), c_4 = c(x_4, x_1) > 0$. Moreover, we consider the weight $\omega_1 = \omega(x_1), \omega_2 = \omega(x_2), \omega_3 = \omega(x_3), \omega_4 = \omega(x_4)$, where $\sum_{k=1}^4 \omega_k^2 = 4, w_i = c_i \omega_i \omega_{i+1}$, the parameter $\lambda \geq 0$ and $q = q_\omega + \lambda$. In this case,

$$L_q = \begin{bmatrix} c_1 \frac{\omega_2}{\omega_1} + c_4 \frac{\omega_4}{\omega_1} + \lambda & -c_1 & 0 & -c_4 \\ -c_1 & c_1 \frac{\omega_1}{\omega_2} + c_2 \frac{\omega_3}{\omega_2} + \lambda & -c_2 & 0 \\ 0 & -c_2 & c_2 \frac{\omega_2}{\omega_3} + c_3 \frac{\omega_4}{\omega_3} + \lambda & -c_3 \\ -c_4 & 0 & -c_3 & c_4 \frac{\omega_1}{\omega_4} + c_3 \frac{\omega_3}{\omega_4} + \lambda \end{bmatrix}.$$

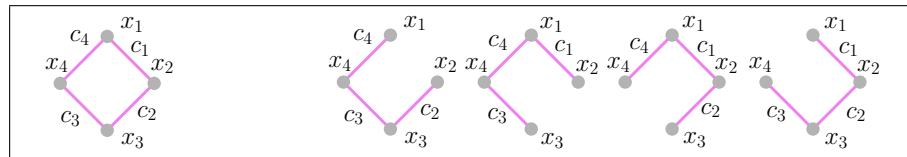


Figure 10: The network C_4 (left) and the collection of spanning trees for C_4 (right).

Next, we compute the coefficients b and a involved in the formula of the group inverse. The corresponding spanning trees or rooted forests of any order are represented in Figures 10-13.

$$a_1 = 4 \prod_{i=1}^4 c_i \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4} \right),$$

$$a_2 = \prod_{i=1}^4 c_i \left(\frac{\omega_2^2(\omega_1^2 + \omega_3^2 + \omega_4^2)}{w_1 w_2} + \frac{\omega_1^2(\omega_2^2 + \omega_3^2 + \omega_4^2)}{w_1 w_4} + \frac{\omega_3^2(\omega_1^2 + \omega_2^2 + \omega_4^2)}{w_2 w_3} + \frac{\omega_4^2(\omega_1^2 + \omega_2^2 + \omega_3^2)}{w_3 w_4} \right. \\ \left. + \frac{(\omega_2^2 + \omega_3^2)(\omega_1^2 + \omega_4^2)}{w_1 w_3} + \frac{(\omega_3^2 + \omega_4^2)(\omega_1^2 + \omega_2^2)}{w_2 w_4} \right),$$

$$a_3 = \prod_{i=1}^4 c_i \left(\frac{\omega_2^2 \omega_3^2 (\omega_1^2 + \omega_4^2)}{w_1 w_2 w_3} + \frac{\omega_1^2 \omega_2^2 (\omega_3^2 + \omega_4^2)}{w_1 w_2 w_4} + \frac{\omega_1^2 \omega_4^2 (\omega_2^2 + \omega_3^2)}{w_1 w_3 w_4} + \frac{\omega_3^2 \omega_4^2 (\omega_1^2 + \omega_2^2)}{w_2 w_3 w_4} \right),$$

$$a_4 = 1.$$

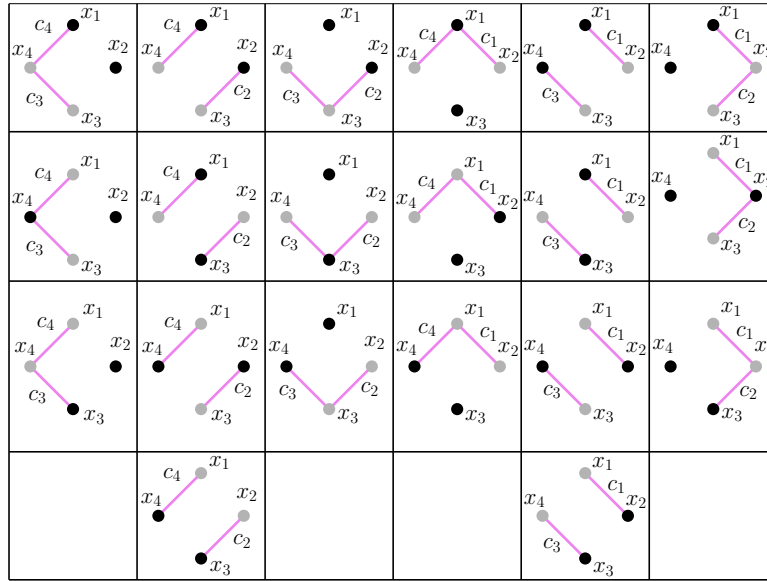


Figure 11: The collection of 2-rooted forests for C_4 .

To compute the coefficient $b_k(x_i, x_j)$ it is enough to count the k -rooted forests displayed at Figures 11-13 with x_i as a root and x_j at the same component as x_i .

$$b_1(x_i, x_j) = \frac{a_1}{4},$$

$$b_2(x_1, x_1) = \prod_{i=1}^4 c_i \left(\frac{\omega_2^2}{w_1 w_2} + \frac{\omega_3^2}{w_2 w_3} + \frac{\omega_4^2}{w_3 w_4} + \frac{(\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{(\omega_3^2 + \omega_4^2)}{w_2 w_4} + \frac{(\omega_2^2 + \omega_3^2 + \omega_4^2)}{w_1 w_4} \right),$$

$$b_2(x_2, x_2) = \prod_{i=1}^4 c_i \left(\frac{\omega_1^2}{w_1 w_4} + \frac{\omega_3^2}{w_2 w_3} + \frac{\omega_4^2}{w_3 w_4} + \frac{(\omega_1^2 + \omega_4^2)}{w_1 w_3} + \frac{(\omega_3^2 + \omega_4^2)}{w_2 w_4} + \frac{(\omega_1^2 + \omega_3^2 + \omega_4^2)}{w_1 w_2} \right),$$

$$b_2(x_3, x_3) = \prod_{i=1}^4 c_i \left(\frac{\omega_1^2}{w_1 w_4} + \frac{\omega_2^2}{w_1 w_2} + \frac{\omega_4^2}{w_3 w_4} + \frac{(\omega_1^2 + \omega_4^2)}{w_1 w_3} + \frac{(\omega_1^2 + \omega_2^2)}{w_2 w_4} + \frac{(\omega_1^2 + \omega_2^2 + \omega_4^2)}{w_2 w_3} \right),$$

$$b_2(x_4, x_4) = \prod_{i=1}^4 c_i \left(\frac{\omega_1^2}{w_1 w_4} + \frac{\omega_2^2}{w_1 w_2} + \frac{\omega_3^2}{w_2 w_3} + \frac{(\omega_1^2 + \omega_2^2)}{w_2 w_4} + \frac{(\omega_2^2 + \omega_3^2)}{w_1 w_3} + \frac{(\omega_1^2 + \omega_2^2 + \omega_3^2)}{w_3 w_4} \right),$$

$$b_2(x_1, x_2) = \prod_{i=1}^4 c_i \left(\frac{\omega_3^2}{w_2 w_3} + \frac{\omega_4^2}{w_3 w_4} + \frac{(\omega_3^2 + \omega_4^2)}{w_2 w_4} \right),$$

$$b_2(x_1, x_3) = \prod_{i=1}^4 c_i \left(\frac{\omega_2^2}{w_1 w_2} + \frac{\omega_4^2}{w_3 w_4} \right),$$

$$b_2(x_1, x_4) = \prod_{i=1}^4 c_i \left(\frac{\omega_2^2}{w_1 w_2} + \frac{\omega_3^2}{w_2 w_3} + \frac{(\omega_2^2 + \omega_3^2)}{w_1 w_3} \right),$$

$$b_2(x_2, x_3) = \prod_{i=1}^4 c_i \left(\frac{\omega_1^2}{w_1 w_4} + \frac{\omega_4^2}{w_3 w_4} + \frac{(\omega_1^2 + \omega_4^2)}{w_1 w_3} \right),$$

$$b_2(x_2, x_4) = \prod_{i=1}^4 c_i \left(\frac{\omega_1^2}{w_1 w_4} + \frac{\omega_3^2}{w_2 w_3} \right),$$

$$b_2(x_3, x_4) = \prod_{i=1}^4 c_i \left(\frac{\omega_1^2}{w_1 w_4} + \frac{\omega_2^2}{w_1 w_2} + \frac{(\omega_1^2 + \omega_2^2)}{w_2 w_4} \right),$$

$$b_3(x_1, x_1) = \prod_{i=1}^4 c_i \left(\frac{\omega_2^2 \omega_3^2}{w_1 w_2 w_3} + \frac{\omega_2^2 (\omega_3^2 + \omega_4^2)}{w_1 w_2 w_4} + \frac{\omega_2^2 \omega_3^2}{w_1 w_2 w_3} + \frac{\omega_4^2 (\omega_2^2 + \omega_3^2)}{w_2 w_3 w_4} \right),$$

$$b_3(x_2, x_2) = \prod_{i=1}^4 c_i \left(\frac{\omega_1^2 \omega_4^2}{w_1 w_2 w_3} + \frac{\omega_1^2 (\omega_3^2 + \omega_4^2)}{w_1 w_2 w_4} + \frac{\omega_3^2 \omega_4^2}{w_2 w_3 w_4} + \frac{\omega_3^2 (\omega_1^2 + \omega_2^2)}{w_1 w_2 w_3} \right),$$

$$b_3(x_3, x_3) = \prod_{i=1}^4 c_i \left(\frac{\omega_1^2 \omega_4^2}{w_1 w_3 w_4} + \frac{\omega_4^2 (\omega_1^2 + \omega_2^2)}{w_2 w_3 w_4} + \frac{\omega_1^2 \omega_2^2}{w_1 w_2 w_4} + \frac{\omega_2^2 (\omega_1^2 + \omega_4^2)}{w_1 w_2 w_3} \right),$$

$$b_3(x_4, x_4) = \prod_{i=1}^4 c_i \left(\frac{\omega_1^2 \omega_2^2}{w_1 w_2 w_4} + \frac{\omega_3^2 (\omega_1^2 + \omega_2^2)}{w_2 w_3 w_4} + \frac{\omega_2^2 \omega_3^2}{w_1 w_2 w_3} + \frac{\omega_1^2 (\omega_2^2 + \omega_3^2)}{w_1 w_3 w_4} \right),$$

$$b_3(x_1, x_3) = b_3(x_2, x_4) = 0,$$

$$b_3(x_1, x_4) = \prod_{i=1}^4 c_i \frac{\omega_2^2 \omega_3^2}{w_1 w_2 w_3}, \quad b_3(x_2, x_3) = \prod_{i=1}^4 c_i \frac{\omega_1^2 \omega_4^2}{w_1 w_3 w_4}, \quad b_3(x_3, x_4) = \prod_{i=1}^4 c_i \frac{\omega_1^2 \omega_2^2}{w_1 w_2 w_4},$$

$$b_4(x_i, x_i) = \frac{1}{\omega_i^2}, \quad b_4(x_i, x_j) = 0, \quad \text{for all } j \neq i.$$

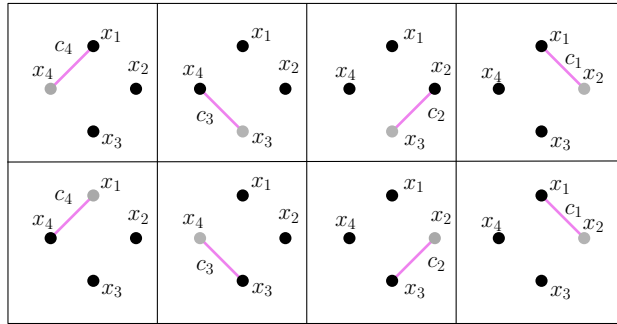


Figure 12: The collection of 3-rooted forests for C_4 .

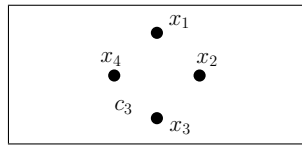


Figure 13: The 4-rooted forest for C_4 .

Hence, we could compute the term involved in the expression for the Green function and the inverse of L_q for C_4 . However, due to the length of these expressions we only give explicitly the Green kernel for the case of $\lambda = 0$.

$$4b_2(x_1, x_1) - a_2 = \prod_{i=1}^4 c_i \left(\frac{\omega_2^4}{w_1 w_2} + \frac{\omega_3^4}{w_2 w_3} + \frac{\omega_4^4}{w_3 w_4} + \frac{(\omega_2^2 + \omega_3^2)^2}{w_1 w_3} + \frac{(\omega_3^2 + \omega_4^2)^2}{w_2 w_4} + \frac{(\omega_2^2 + \omega_3^2 + \omega_4^2)^2}{w_1 w_4} \right),$$

$$4b_2(x_2, x_2) - a_2 = \prod_{i=1}^4 c_i \left(\frac{\omega_1^4}{w_1 w_4} + \frac{\omega_3^4}{w_2 w_3} + \frac{\omega_4^4}{w_3 w_4} + \frac{(\omega_1^2 + \omega_4^2)^2}{w_1 w_3} + \frac{(\omega_3^2 + \omega_4^2)^2}{w_2 w_4} + \frac{(\omega_1^2 + \omega_3^2 + \omega_4^2)^2}{w_1 w_2} \right),$$

$$4b_2(x_3, x_3) - a_2 = \prod_{i=1}^4 c_i \left(\frac{\omega_1^4}{w_1 w_4} + \frac{\omega_2^4}{w_1 w_2} + \frac{\omega_4^4}{w_3 w_4} + \frac{(\omega_1^2 + \omega_4^2)^2}{w_1 w_3} + \frac{(\omega_1^2 + \omega_2^2)^2}{w_2 w_4} + \frac{(\omega_1^2 + \omega_2^2 + \omega_4^2)^2}{w_2 w_3} \right),$$

$$4b_2(x_4, x_4) - a_2 = \prod_{i=1}^4 c_i \left(\frac{\omega_1^4}{w_1 w_4} + \frac{\omega_2^4}{w_1 w_2} + \frac{\omega_3^4}{w_2 w_3} + \frac{(\omega_2^2 + \omega_3^2)^2}{w_1 w_3} + \frac{(\omega_1^2 + \omega_2^2)^2}{w_2 w_4} + \frac{(\omega_1^2 + \omega_2^2 + \omega_3^2)^2}{w_3 w_4} \right),$$

$$\begin{aligned}
4b_2(x_1, x_2) - a_2 &= \prod_{i=1}^4 c_i \left(\frac{\omega_3^4}{w_2 w_3} + \frac{\omega_4^4}{w_3 w_4} + \frac{(\omega_3^2 + \omega_4^2)^2}{w_2 w_4} - \frac{(\omega_2^2 + \omega_3^2)(\omega_1^2 + \omega_4^2)}{w_1 w_3} \right. \\
&\quad \left. - \frac{\omega_2^2(\omega_1^2 + \omega_3^2 + \omega_4^2)}{w_1 w_2} - \frac{\omega_1^2(\omega_2^2 + \omega_3^2 + \omega_4^2)}{w_1 w_4} \right), \\
4b_2(x_1, x_3) - a_2 &= \prod_{i=1}^4 c_i \left(\frac{\omega_2^4}{w_1 w_2} + \frac{\omega_4^4}{w_3 w_4} - \frac{\omega_1^2(\omega_2^2 + \omega_3^2 + \omega_4^2)}{w_1 w_4} - \frac{\omega_3^2(\omega_1^2 + \omega_2^2 + \omega_4^2)}{w_2 w_3} \right. \\
&\quad \left. - \frac{(\omega_2^2 + \omega_3^2)(\omega_1^2 + \omega_4^2)}{w_1 w_3} - \frac{(\omega_3^2 + \omega_4^2)(\omega_1^2 + \omega_2^2)}{w_2 w_4} \right), \\
4b_2(x_1, x_4) - a_2 &= \prod_{i=1}^4 c_i \left(\frac{\omega_2^4}{w_1 w_2} + \frac{\omega_3^4}{w_2 w_3} + \frac{(\omega_2^2 + \omega_3^2)^2}{w_1 w_3} - \frac{\omega_1^2(\omega_2^2 + \omega_3^2 + \omega_4^2)}{w_1 w_4} \right. \\
&\quad \left. - \frac{\omega_4^2(\omega_1^2 + \omega_2^2 + \omega_3^2)}{w_3 w_4} - \frac{(\omega_3^2 + \omega_4^2)(\omega_1^2 + \omega_2^2)}{w_2 w_4} \right), \\
4b_2(x_2, x_3) - a_2 &= \prod_{i=1}^4 c_i \left(\frac{\omega_1^4}{w_1 w_4} + \frac{\omega_4^4}{w_3 w_4} + \frac{(\omega_1^2 + \omega_4^2)^2}{w_1 w_3} - \frac{\omega_2^2(\omega_1^2 + \omega_3^2 + \omega_4^2)}{w_1 w_2} \right. \\
&\quad \left. - \frac{\omega_3^2(\omega_1^2 + \omega_2^2 + \omega_4^2)}{w_2 w_3} - \frac{(\omega_3^2 + \omega_4^2)(\omega_1^2 + \omega_2^2)}{w_2 w_4} \right), \\
4b_2(x_2, x_4) - a_2 &= \prod_{i=1}^4 c_i \left(\frac{\omega_1^4}{w_1 w_4} + \frac{\omega_3^4}{w_2 w_3} - \frac{\omega_2^2(\omega_1^2 + \omega_3^2 + \omega_4^2)}{w_1 w_2} - \frac{\omega_4^2(\omega_1^2 + \omega_2^2 + \omega_3^2)}{w_3 w_4} \right. \\
&\quad \left. - \frac{(\omega_2^2 + \omega_3^2)(\omega_1^2 + \omega_4^2)}{w_1 w_3} - \frac{(\omega_3^2 + \omega_4^2)(\omega_1^2 + \omega_2^2)}{w_2 w_4} \right), \\
4b_2(x_3, x_4) - a_2 &= \prod_{i=1}^4 c_i \left(\frac{\omega_1^4}{w_1 w_4} + \frac{\omega_2^4}{w_1 w_2} - \frac{(\omega_1^2 + \omega_2^2)^2}{w_2 w_4} - \frac{\omega_3^2(\omega_1^2 + \omega_2^2 + \omega_4^2)}{w_2 w_3} \right. \\
&\quad \left. - \frac{\omega_4^2(\omega_1^2 + \omega_2^2 + \omega_3^2)}{w_3 w_4} - \frac{(\omega_2^2 + \omega_3^2)(\omega_1^2 + \omega_4^2)}{w_1 w_3} \right).
\end{aligned}$$

Therefore, using again the expression obtained in Proposition 4.1, we get the Green function for C_4 and $\lambda = 0$.

$$\frac{G(x_1, x_1)}{\omega_1^2} = \frac{\frac{\omega_2^4}{w_1 w_2} + \frac{\omega_3^4}{w_2 w_3} + \frac{\omega_4^4}{w_3 w_4} + \frac{(\omega_2^2 + \omega_3^2)^2}{w_1 w_3} + \frac{(\omega_3^2 + \omega_4^2)^2}{w_2 w_4} + \frac{(\omega_2^2 + \omega_3^2 + \omega_4^2)^2}{w_1 w_4}}{16 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4} \right)},$$

$$\frac{G(x_2, x_2)}{\omega_2^2} = \frac{\frac{\omega_1^4}{w_1 w_4} + \frac{\omega_3^4}{w_2 w_3} + \frac{\omega_4^4}{w_3 w_4} + \frac{(\omega_1^2 + \omega_4^2)^2}{w_1 w_3} + \frac{(\omega_3^2 + \omega_4^2)^2}{w_2 w_4} + \frac{(\omega_1^2 + \omega_3^2 + \omega_4^2)^2}{w_1 w_2}}{16 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4} \right)},$$

$$\frac{G(x_3, x_3)}{\omega_3^2} = \frac{\frac{\omega_1^4}{w_1 w_4} + \frac{\omega_2^4}{w_1 w_2} + \frac{\omega_4^4}{w_3 w_4} + \frac{(\omega_1^2 + \omega_4^2)^2}{w_1 w_3} + \frac{(\omega_1^2 + \omega_2^2)^2}{w_2 w_4} + \frac{(\omega_1^2 + \omega_2^2 + \omega_4^2)^2}{w_2 w_3}}{16 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4} \right)},$$

$$\frac{G(x_4, x_4)}{\omega_4^2} = \frac{\frac{\omega_1^4}{w_1 w_4} + \frac{\omega_2^4}{w_1 w_2} + \frac{\omega_3^4}{w_2 w_3} + \frac{(\omega_2^2 + \omega_3^2)^2}{w_1 w_3} + \frac{(\omega_1^2 + \omega_2^2)^2}{w_2 w_4} + \frac{(\omega_1^2 + \omega_2^2 + \omega_3^2)^2}{w_3 w_4}}{16 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4} \right)},$$

$$\frac{G(x_1, x_2)}{\omega_1 \omega_2} = \frac{\frac{\omega_3^4}{w_2 w_3} + \frac{\omega_4^4}{w_3 w_4} + \frac{(\omega_3^2 + \omega_4^2)^2}{w_2 w_4} - \frac{(\omega_2^2 + \omega_3^2)(\omega_1^2 + \omega_4^2)}{w_1 w_3} - \frac{\omega_2^2(\omega_1^2 + \omega_3^2 + \omega_4^2)}{w_1 w_2} - \frac{\omega_1^2(\omega_2^2 + \omega_3^2 + \omega_4^2)}{w_1 w_4}}{16 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4} \right)},$$

$$\frac{G(x_1, x_3)}{\omega_1 \omega_3} = \frac{\frac{\omega_2^4}{w_1 w_2} + \frac{\omega_4^4}{w_3 w_4} - \frac{\omega_1^2(\omega_2^2 + \omega_3^2 + \omega_4^2)}{w_1 w_4} - \frac{\omega_3^2(\omega_1^2 + \omega_2^2 + \omega_4^2)}{w_2 w_3} - \frac{(\omega_2^2 + \omega_3^2)(\omega_1^2 + \omega_4^2)}{w_1 w_3} - \frac{(\omega_3^2 + \omega_4^2)(\omega_1^2 + \omega_2^2)}{w_2 w_4}}{16 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4} \right)},$$

$$\frac{G(x_1, x_4)}{\omega_1 \omega_4} = \frac{\frac{\omega_2^4}{w_1 w_2} + \frac{\omega_3^4}{w_2 w_3} + \frac{(\omega_2^2 + \omega_3^2)^2}{w_1 w_3} - \frac{\omega_1^2(\omega_2^2 + \omega_3^2 + \omega_4^2)}{w_1 w_4} - \frac{\omega_4^2(\omega_1^2 + \omega_2^2 + \omega_3^2)}{w_3 w_4} - \frac{(\omega_3^2 + \omega_4^2)(\omega_1^2 + \omega_2^2)}{w_2 w_4}}{16 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4} \right)},$$

$$\frac{G(x_2, x_3)}{\omega_2 \omega_3} = \frac{\frac{\omega_1^4}{w_1 w_4} + \frac{\omega_4^4}{w_3 w_4} + \frac{(\omega_1^2 + \omega_4^2)^2}{w_1 w_3} - \frac{\omega_2^2(\omega_1^2 + \omega_3^2 + \omega_4^2)}{w_1 w_2} - \frac{\omega_3^2(\omega_1^2 + \omega_2^2 + \omega_4^2)}{w_2 w_3} - \frac{(\omega_3^2 + \omega_4^2)(\omega_1^2 + \omega_2^2)}{w_2 w_4}}{16 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4} \right)},$$

$$\frac{G(x_2, x_4)}{\omega_2 \omega_4} = \frac{\frac{\omega_1^4}{w_1 w_4} + \frac{\omega_3^4}{w_2 w_3} - \frac{\omega_2^2(\omega_1^2 + \omega_3^2 + \omega_4^2)}{w_1 w_2} - \frac{\omega_4^2(\omega_1^2 + \omega_2^2 + \omega_3^2)}{w_3 w_4} - \frac{(\omega_2^2 + \omega_3^2)(\omega_1^2 + \omega_4^2)}{w_1 w_3} - \frac{(\omega_3^2 + \omega_4^2)(\omega_1^2 + \omega_2^2)}{w_2 w_4}}{16 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4} \right)},$$

$$\frac{G(x_3, x_4)}{\omega_3 \omega_4} = \frac{\frac{\omega_1^4}{w_1 w_4} + \frac{\omega_2^4}{w_1 w_2} + \frac{(\omega_1^2 + \omega_2^2)^2}{w_2 w_4} - \frac{\omega_3^2(\omega_1^2 + \omega_2^2 + \omega_4^2)}{w_2 w_3} - \frac{\omega_4^2(\omega_1^2 + \omega_2^2 + \omega_3^2)}{w_3 w_4} - \frac{(\omega_2^2 + \omega_3^2)(\omega_1^2 + \omega_4^2)}{w_1 w_3}}{16 \left(\frac{1}{w_1} + \frac{1}{w_2} + \frac{1}{w_3} + \frac{1}{w_4} \right)}.$$

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