Asymptotic Analysis 0 (0) 1
IOS Press

Time decay for several porous thermoviscoelastic systems of Moore-Gibson-Thompson type

Noelia Bazarra [a], José R. Fernández [a *], Antonio Magaña [b] and Ramón Quintanilla [b]
[a] *Departamento de Matemática Aplicada I, Universidade de Vigo, 36310 Vigo, Spain*
E-mails: nbazarra@uvigo.es, jose.fernandez@uvigo.es
[b] *Departamento de Matemáticas, Universitat Politécnica de Catalunya, E.S.E.I.A.A.T., Colom 11, 08222 Terrassa, Barcelona, Spain*
E-mails: antonio.magana@upc.edu, ramon.quintanilla@upc.edu

Abstract. In this paper, we consider several problems arising in the theory of thermoelastic bodies with voids. Four particular cases are considered depending on the choice of the constitutive tensors, assuming different dissipation mechanisms determined by Moore-Gibson-Thompson-type viscosity. For all of them, the existence and uniqueness of solutions are shown by using semigroup arguments. The energy decay of the solutions is also analyzed for each case.

Keywords: Thermoviscoelasticity, Existence and uniqueness, Energy decay, Dissipation mechanisms

1. Introduction

It is accepted that the thermoelasticity with voids is the easiest extension of the well-known theory of thermoelasticity. In this case, the usual variables to take into account are the macroscopic deformations, the temperature and the new variable which is given by the volume fraction. This last component clarifies the porous structure and it can be seen as the component of the microstructure of the material. It is worth recalling that Cowin [1] and Cowin and Nunziato [2] proposed this theory for the isothermal case and that Ieşan [3] extended it by including the thermal effects also in the linear case. This theory is currently widely accepted by the scientist community as a model to describe the behavior of thermoelastic solids with small distributed pores. We can say that the quantity of contributions in this theory is huge. In fact, in recent days there has been a big interest to understand the case where we can consider a double porous structure (see [4–7] among others). In view of the kind of materials described by this theory, its physical applications could include rocks, soils, ceramics and woods, as well as biological materials as dentures or bones. The study of the decay of perturbations of this kind of materials has been developed in recent years and we can cite [8–16] among others.

Fourier's law is usually considered in engineering to describe the heat conduction phenomena; however, the juxtaposition of this constitutive law with the usual heat equation brings us to the instantaneous propagation of the thermal waves. That is, the perturbations are instantaneously felt everywhere in the solid. Of course, this effect violates the causality principle and many scientists have proposed alternative constitutive laws to overcome the paradox. The most known law is the one proposed by Cattaneo and Maxwell, who introduced a relaxation (or delay) parameter into the Fourier law, bringing to a damped hyperbolic heat conduction equation. Green and Nagdhi [17–19] proposed three alternative constitutive laws based on the axioms of the thermomechanics. It is worth saying that the main difference among them consists in the choice of the independent variables. The most general is the one called "type III theory", which contains the other two as limit cases. However, the linear equation for the type III theory also brings to the instantaneous propagation of the heat. Therefore, it also violates the causality principle. That was the reason why in [20] the author proposed the introduction of a relaxation parameter, bringing to a heat conduction equation of the type of the one called Moore-Gibson-Thompson. Another alternative theory was proposed by Gurtin [21], where the equations should satisfy the invariance of the entropy with respect to the time reversal. Although this theory has deserved few attention in the literature, it has been extended to several thermomechanical situations (see [22–24] among others).

In the present contribution, we consider the equations of the porous thermoelasticity invariant under time reversal, and we specify several examples of constitutive systems. In particular, we recover the Moore-Gibson-Thompson thermoelasticity [20, 25–30], but we also obtain some other systems which are new in the study of porous thermoelasticity. In this paper, we analyze the time decay of four one-dimensional systems obtained in this way. To be precise, we first consider the case corresponding to the Moore-Gibson-Thompson heat conduction theory and we obtain that generically we can expect the exponential decay. The second case corresponds to assume a similar dissipation mechanism on the volume fraction, but this mechanism (see system (2)) is weak and we see that the solutions do not decay in an exponential way; however, in the case that we impose a stronger mechanism of dissipation on the volume fraction (see system (3)) we obtain (generically) again the exponential decay[1]. The last case we analyze corresponds to assume the dissipation mechanism on the displacement, where we also obtain the exponential decay of solutions[2]. We believe that the interest of our contribution is given by the introduction of a mechanism of dissipation which is different compared with the usual mechanisms proposed in porous-thermoelasticity, but, at the same time, it can be obtained from the usual formulations of thermomechanics. We clarify the implications of the different mechanism when we apply it in each component of the system.

The structure of this paper is the following. In the next section we obtain the basic equations and systems we will study. In the other sections, we analyze the decay for each case. In all of them we obtain the existence and uniqueness of solutions by means of the semigroup arguments, which is also used to prove the exponential stability of solutions. In order to show the slow decay of solutions, we apply the Hurwitz rule in a similar way as in [31].

2. Basic Equations

In this section we establish the systems of partial differential equations that we want to study. To this end, we begin with the evolution equations:

$$\begin{aligned} \rho \ddot{u}_i &= t_{ij,j}, \\ J\ddot{\phi} &= h_{i,i} + g, \\ T_0 \dot{\eta} &= q_{i,i}, \end{aligned}$$

[1] In fact, the introduction of a relaxation parameter could also be motivated here to overcome the unbounded speed of propagation of the volume fraction in the viscous case.

[2] A similar comment to the previous footnote could be also done here.

where ρ is the mass density assumed to be positive, u_i is the displacement vector, t_{ij} is the stress tensor, J is the equilibrated inertia, ϕ is the volume fraction, h_i is the equilibrated stress, g is the equilibrated body force, T_0 is the reference temperature (that we assume equal to one to simplify the calculations), η is the entropy and q is the heat flux vector.

We consider now the thermoelastic theory proposed by Gurtin, where we assume that the entropy is invariant under time reversal, and we restrict our attention to materials with a center of symmetry. Therefore, the constitutive equations are (see [21–24]):

$$
\begin{aligned}
t_{ij} &= \int_{-\infty}^{t} \left[G_{ijmn}(t-s)\dot{e}_{mn}(s) + B_{ij}(t-s)\dot{\phi}(s) - l_{ij}(t-s)\dot{\theta}(s)\right] ds,\\
h_i &= \int_{-\infty}^{t} \left[A_{ij}(t-s)\dot{\phi}_{,j}(s) + M_{ij}(t-s)\theta_{,j}(s)\right] ds,\\
g &= \int_{-\infty}^{t} \left[-B_{ij}(t-s)\dot{e}_{ij}(s) - b(t-s)\dot{\phi}(s) + m(t-s)\dot{\theta}(s)\right] ds,\\
\eta &= \int_{-\infty}^{t} \left[l_{ij}(t-s)\dot{e}_{ij}(s) + a(t-s)\dot{\theta}(s) + m(t-s)\dot{\phi}(s)\right] ds,\\
q_i &= \int_{-\infty}^{t} \left[k_{ij}(t-s)\theta_{,j}(s) + M_{ji}(t-s)\dot{\phi}_{,j}(s)\right] ds,
\end{aligned}
$$

where $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ is the linearized strain tensor.

We can define different problems by considering particular choices of the constitutive tensors G_{ijmn}, B_{ij}, l_{ij}, A_{ij}, M_{ij}, b, m, a and k_{ij}. For all choices, we assume that tensors G_{ijmn}, l_{ij}, A_{ij} and k_{ij} are symmetric, that is,

$$
G_{ijmn} = G_{mnij} = G_{jimn}, \quad l_{ij} = l_{ji}, \quad A_{ij} = A_{ji}, \quad k_{ij} = k_{ji}.
$$

First, we assume that

$$
\begin{aligned}
&G_{ijmn}(\boldsymbol{x},s) = G_{ijmn}(\boldsymbol{x}), \quad B_{ij}(\boldsymbol{x},s) = B_{ij}(\boldsymbol{x}), \quad l_{ij}(\boldsymbol{x},s) = l_{ij}(\boldsymbol{x}),\\
&A_{ij}(\boldsymbol{x},s) = A_{ij}(\boldsymbol{x}), \quad M_{ij}(\boldsymbol{x},s) = M_{ij}(\boldsymbol{x}), \quad m(\boldsymbol{x},s) = m(\boldsymbol{x}), \quad a(\boldsymbol{x},s) = a(\boldsymbol{x}),\\
&b(\boldsymbol{x},s) = b(\boldsymbol{x}), \quad k_{ij}(\boldsymbol{x},s) = k^*_{ij}(\boldsymbol{x}) + (\tau^{-1}k_{ij}(\boldsymbol{x}) - k^*_{ij}(\boldsymbol{x}))e^{-\tau^{-1}s},
\end{aligned}
$$

where τ is a positive constant. From now on, we also assume that the solutions vanish at $t = -\infty$ (see [33]).

If we denote $\hat{u}_i = u_i + \tau\dot{u}_i$ and $\hat{\phi} = \phi + \tau\dot{\phi}$, then we obtain the system:

$$
\begin{aligned}
\rho\ddot{\hat{u}}_i &= \left(G_{ijmn}\hat{u}_{m,n} + B_{ij}\hat{\phi} - l_{ij}(\theta + \tau\dot{\theta})\right)_{,j},\\
J\ddot{\hat{\phi}} &= \left(A_{ij}\hat{\phi}_{,j} + M_{ij}(\alpha_{,j} + \tau\theta_{,j})\right)_{,i} - B_{ij}\hat{u}_{i,j} - b\hat{\phi} + m(\theta + \tau\dot{\theta}),\\
\tau a\ddot{\theta} + a\dot{\theta} &= -l_{ij}\dot{\hat{u}}_{i,j} - m\dot{\hat{\phi}} + \left(M_{ji}\hat{\phi}_{,j} + k^*_{ij}\alpha_{,j} + k_{ij}\theta_{,j}\right)_{,i},
\end{aligned}
$$

where $\alpha(\boldsymbol{x},t) = \alpha^0(\boldsymbol{x}) + \int_0^t \theta(\boldsymbol{x},s)\,ds$ denotes the thermal displacement.

In the rest of paper, in order to simplify the notation, we omit the hat over variables u_i and ϕ.

This is a system of hyperbolic equations with only a dissipative mechanism which is at the temperature. We could obtain existence, uniqueness and stability of solutions under suitable assumptions and initial and boundary conditions. To save a very large contribution we focus our attention to one-dimensional isotropic and homogeneous materials. Thus, our system becomes

$$
\left.\begin{array}{l}
\rho\ddot{u} = \mu u_{xx} + B\phi_x - l(\theta + \tau\dot{\theta})_x, \\
J\ddot{\phi} = A\phi_{xx} + M(\alpha_{xx} + \tau\theta_{xx}) - Bu_x - b\phi + m(\theta + \tau\dot{\theta}), \\
\tau a\ddot{\theta} + a\dot{\theta} = -l\dot{u}_x - m\dot{\phi} + M\phi_{xx} + k^*\alpha_{xx} + k\theta_{xx}.
\end{array}\right\} \tag{1}
$$

This system will be studied in the next section.

The second system we consider can be obtained if we assume that

$$
\begin{array}{l}
G_{ijmn}(\boldsymbol{x}, s) = G_{ijmn}(\boldsymbol{x}), \quad B_{ij}(\boldsymbol{x}, s) = B_{ij}(\boldsymbol{x}), \quad l_{ij}(\boldsymbol{x}, s) = l_{ij}(\boldsymbol{x}), \\
A_{ij}(\boldsymbol{x}, s) = A_{ij}(\boldsymbol{x}), \quad M_{ij}(\boldsymbol{x}, s) = M_{ij}(\boldsymbol{x}), \quad m(\boldsymbol{x}, s) = m(\boldsymbol{x}), \quad a(\boldsymbol{x}, s) = a(\boldsymbol{x}), \\
b(\boldsymbol{x}, s) = b(\boldsymbol{x}), \quad k_{ij}(\boldsymbol{x}, s) = k_{ij}(\boldsymbol{x}), \quad b(\boldsymbol{x}, s) = b^*(\boldsymbol{x}) + (\tau^{-1}b(\boldsymbol{x}) - b^*(\boldsymbol{x}))e^{-\tau^{-1}s}.
\end{array}
$$

So, we obtain the system:

$$
\begin{array}{l}
\rho\ddot{u}_i = \left(G_{ijmn}u_{m,n} + B_{ij}(\phi + \tau\dot{\phi}) - l_{ij}\theta\right)_{,j}, \\
\tau J\dddot{\phi} + J\ddot{\phi} = -B_{ij}u_{i,j} - b^*\phi - b\dot{\phi} + m\theta + \left(M_{ij}\alpha_{,j} + A_{ij}\phi_{,j} + \tau A_{ij}\dot{\phi}_{,j}\right)_{,i}, \\
a\dot{\theta} = -l_{ij}\dot{u}_{i,j} - m(\dot{\phi} + \tau\ddot{\phi}) + \left(M_{ji}\phi_{,j} + \tau M_{ij}\dot{\phi}_{,j} + k_{ij}\alpha_{,j}\right)_{,i}.
\end{array}
$$

The one-dimensional version of this system reads:

$$
\left.\begin{array}{l}
\rho\ddot{u} = \mu u_{xx} + B(\phi_x + \tau\dot{\phi}_x) - l\theta_x, \\
\tau J\dddot{\phi} + J\ddot{\phi} = -Bu_x - b\dot{\phi} + m\theta - b^*\phi + A\phi_{xx} + M\alpha_{xx} + \tau A\dot{\phi}_{xx}, \\
a\dot{\theta} = -l\dot{u}_x - m(\dot{\phi} + \tau\ddot{\phi}) + M\phi_{xx} + M\tau\dot{\phi}_{xx} + k\alpha_{xx}.
\end{array}\right\} \tag{2}
$$

It is worth noting that this is a system of three hyperbolic equations with only one dissipative mechanism which is located in the Laplacian of the volume fraction.

The third system we consider corresponds to assume that

$$
\begin{array}{l}
G_{ijmn}(\boldsymbol{x}, s) = G_{ijmn}(\boldsymbol{x}), \quad B_{ij}(\boldsymbol{x}, s) = B_{ij}(\boldsymbol{x}), \quad l_{ij}(\boldsymbol{x}, s) = l_{ij}(\boldsymbol{x}), \quad b(\boldsymbol{x}, s) = b(\boldsymbol{x}), \\
M_{ij}(\boldsymbol{x}, s) = M_{ij}(\boldsymbol{x}), \quad m(\boldsymbol{x}, s) = m(\boldsymbol{x}), \quad a(\boldsymbol{x}, s) = a(\boldsymbol{x}), \quad b(\boldsymbol{x}, s) = b(\boldsymbol{x}), \\
k_{ij}(\boldsymbol{x}, s) = k_{ij}(\boldsymbol{x}), \quad A_{ij}(\boldsymbol{x}, s) = A^*_{ij}(\boldsymbol{x}) + (\tau^{-1}A_{ij}(\boldsymbol{x}) - A^*_{ij}(\boldsymbol{x}))e^{-\tau^{-1}s}.
\end{array}
$$

Therefore, our system takes the form:

$$
\begin{array}{l}
\rho\ddot{u}_i = \left(G_{ijmn}u_{m,n} + B_{ij}(\phi + \tau\dot{\phi}) - l_{ij}\theta\right)_{,j}, \\
\tau J\dddot{\phi} + J\ddot{\phi} = -B_{ij}u_{i,j} - b\phi - b\tau\dot{\phi} + m\theta + \left(M_{ij}\alpha_{,j} + A^*_{ij}\phi_{,j} + A_{ij}\dot{\phi}_{,j}\right)_{,i}, \\
a\dot{\theta} = -l_{ij}\dot{u}_{i,j} - m(\dot{\phi} + \tau\ddot{\phi}) + \left(M_{ji}\phi_{,j} + M_{ij}\tau\dot{\phi}_{,j} + k_{ij}\alpha_{,j}\right)_{,i},
\end{array}
$$

and its one-dimensional homogeneous version is

$$
\left.\begin{array}{l}
\rho\ddot{u} = \mu u_{xx} + B(\phi_x + \tau\dot{\phi}_x) - l\theta_x, \\
\tau J\dddot{\phi} + J\ddot{\phi} = -Bu_x - b\dot{\phi} + m\theta - b^*\phi + A^*\phi_{xx} + A\dot{\phi}_{xx} + M\alpha_{xx}, \\
a\dot{\theta} = -l\dot{u}_x - m(\dot{\phi} + \tau\ddot{\phi}) + M\phi_{xx} + M\tau\dot{\phi}_{xx} + k\alpha_{xx}.
\end{array}\right\} \tag{3}
$$

In the case that we assume

$$\begin{array}{l} k_{ij}(\boldsymbol{x},s)=k_{ij}(\boldsymbol{x}),\quad b(\boldsymbol{x},s)=b(\boldsymbol{x}),\quad B_{ij}(\boldsymbol{x},s)=B_{ij}(\boldsymbol{x}),\quad l_{ij}(\boldsymbol{x},s)=l_{ij}(\boldsymbol{x}),\\ A_{ij}(\boldsymbol{x},s)A_{ij}(\boldsymbol{x}),\quad M_{ij}(\boldsymbol{x},s)=M_{ij}(\boldsymbol{x}),\quad m(\boldsymbol{x},s)=m(\boldsymbol{x}),\quad a(\boldsymbol{x},s)=a(\boldsymbol{x}),\\ b(\boldsymbol{x},s)=b(\boldsymbol{x}),\quad G_{ijmn}(\boldsymbol{x},s)=G^*_{ijmn}(\boldsymbol{x})+\left(\tau^{-1}G_{ijmn}(\boldsymbol{x})-G^*_{ijmn}(\boldsymbol{x})\right)e^{-\tau^{-1}s}, \end{array}$$

we obtain the following system:

$$\begin{array}{l} \tau\rho\dddot{u}_i+\rho\ddot{u}_i=\left(G^*_{ijmn}u_{m,n}+G_{ijmn}\dot{u}_{m,n}+B_{ij}\phi-l_{ij}\theta\right)_{,j},\\ J\ddot{\phi}=-B_{ij}(u_{i,j}+\tau\dot{u}_{ij})-b\phi+m\theta+\left(M_{ij}\alpha_{,j}+A_{ij}\phi_{,j}\right)_{,i},\\ a\dot{\theta}=-l_{ij}(\dot{u}_{i,j}+\tau\ddot{u}_{i,j})-m\dot{\phi})+\left(M_{ji}\phi_{,j}+k_{ij}\alpha_{,j}\right)_{,i}, \end{array}$$

and its one-dimensional version is

$$\left.\begin{array}{l} \tau\rho\dddot{u}+\rho\ddot{u}=\mu^*u_{xx}+\mu\dot{u}_{xx}+B\phi_x-l\theta_x,\\ J\ddot{\phi}=-B(u_x+\tau\dot{u}_x)-b\phi+m\theta+A\phi_{xx}+M\alpha_{xx},\\ a\dot{\theta}=-l(\dot{u}_x+\ddot{u}_x)-m\dot{\phi}+M\phi_{xx}+k\alpha_{xx}. \end{array}\right\}\tag{4}$$

Again, we obtain a system of three hyperbolic equations in a way that the only dissipative mechanism is given by the viscosity.

3. First system

The aim of this section is to analyze the problem obtained by the system (1) we have proposed with the boundary conditions:

$$u(0,t)=u(\pi,t)=\phi_x(0,t)=\phi_x(\pi,t)=\alpha_x(0,t)=\alpha_x(\pi,t)=0,\tag{5}$$

and the initial conditions:

$$\begin{array}{l} u(x,0)=u^0(x),\quad \dot{u}(x,0)=v^0(x),\quad \phi(x,0)=\phi^0(x),\quad \dot{\phi}(x,0)=\psi^0(x),\\ \alpha(x,0)=\alpha^0(x),\quad \dot{\alpha}(x,0)=\theta^0(x),\quad \ddot{\alpha}(x,0)=\xi^0(x). \end{array}$$

We study this problem assuming also that

$$\int_0^\pi \phi^0(x)\,dx=\int_0^\pi \psi^0(x)\,dx=\int_0^\pi \alpha^0(x)\,dx=\int_0^\pi \theta^0(x)\,dx=\int_0^\pi \xi^0(x)\,dx=0.$$

We note that, in this case, the integrals of ϕ, $\dot{\phi}$, α, $\theta,\dot{\theta}$ are null for every time.

In this section, we assume that

$$\begin{array}{l} \mu>0,\quad \mu b>B^2,\quad k^*>0,\quad k>k^*\tau,\quad \rho>0,\quad J>0,\quad a>0,\quad A>0,\\ Ak^*>M^2. \end{array}$$

We do not impose any restriction on the sign of l nor M, but we need to assume that they are different from zero.

We study the above problem in the Hilbert space

$$\mathcal{H} = H_0^1 \times L^2 \times H_*^1 \times L_*^2 \times H_*^1 \times H_*^1 \times L_*^2,$$

where

$$L_*^2 = \{ f \in L^2 \,;\, \int_0^\pi f(x)\, dx = 0 \}, \quad H_*^1 = H^1 \cap L_*^2.$$

We define the inner product

$$\begin{aligned} < (u,v,\phi,\psi,\alpha,\theta,\xi),(u^*,v^*,\phi^*,\psi^*,\alpha^*,\theta^*,\xi^*) >= \frac{1}{2}\int_0^\pi \Big(\rho v \overline{v^*} + \mu u_x \overline{u_x^*} + J\psi\overline{\psi^*} \\ + A\phi_x\overline{\phi_x^*} + B(u_x\overline{\phi^*} + \overline{u_x^*}\phi) + b\phi\overline{\phi^*} + a(\tau\xi+\theta)\overline{(\tau\xi^*+\theta^*)} + M((\alpha_x+\tau\theta_x)\overline{\phi_x^*} \\ + \overline{(\alpha_x^*+\tau\theta_x^*)}\phi_x) + k^*(\alpha_x+\tau\theta_x)\overline{(\alpha_x^*+\tau\theta_x^*)} + \tau\overline{k}\theta_x\overline{\theta_x^*} \Big)\, dx, \end{aligned}$$

where $\overline{k} = k - \tau k^*$ and, in general, a bar over the elements of the Hilbert space represents their conjugated. This product is equivalent to the usual one in our space.

If we define the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu D^2}{\rho} & 0 & \frac{BD}{\rho} & 0 & 0 & -\frac{lD}{\rho} & \frac{-l\tau D}{\rho} \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ \frac{-BD}{J} & 0 & \frac{AD^2-b}{J} & 0 & \frac{MD^2}{J} & \frac{M\tau D^2+m}{J} & \frac{m\tau}{J} \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & \frac{-lD}{\tau a} & \frac{MD^2}{\tau a} & \frac{-m}{\tau a} & \frac{k^*D^2}{\tau a} & \frac{kD^2}{\tau a} & -\tau^{-1} \end{pmatrix},$$

where D means the spatial derivative. Thus, we can write our problem as

$$\frac{dU}{dt} = \mathcal{A}U, \quad U(0) = U^0,$$

where $U = (u,v,\phi,\psi,\alpha,\theta,\xi)$ and $U^0 = (u^0,v^0,\phi^0,\psi^0,\alpha^0,\theta^0,\xi^0)$.

Theorem 3.1. *The operator $\mathcal{A}$ generates a contractive semigroup.*

Proof. We first note that the domain of the operator is

$$\begin{aligned} & v \in H_0^1, \quad \psi \in H_*^1, \quad \theta \in H_*^1, \quad \xi \in H_*^1, \\ & A\phi + M\alpha + M\tau\theta \in H^2, \quad u \in H^2, \\ & M\phi + k^*\alpha + k\theta \in H^2. \end{aligned}$$

It is clear that the domain is a dense subspace.

After a direct application of the boundary conditions, keeping in mind that $k > k^*\tau$ (and so, $\overline{k} > 0$) we have, for every $U \in D(\mathcal{A})$,

$$Re < \mathcal{A}U, U >= \frac{-\overline{k}}{2} \int_0^{\pi} |\theta_x|^2 \, dx \leqslant 0.$$

To finish the proof we only need to show that zero belongs to the resolvent of the operator. Let us consider $(f_1, f_2, \dots, f_7)$ in the Hilbert space. We need to solve

$$\begin{aligned}
&v = f_1, \quad \psi = f_3, \quad \theta = f_5, \quad \xi = f_6,\\
&\mu u_{xx} + B\phi_x - l(\theta_x + \tau\xi_x) = \rho f_2,\\
&-Bu_x - b\phi + m(\theta + \tau\theta) + M(\alpha_{xx} + \tau\theta_{xx}) + A\phi_{xx} = J f_4,\\
&-lv_x - m\psi + M\phi_{xx} + k^*\alpha_{xx} + k\theta_{xx} - a\xi = \tau a f_7.
\end{aligned}$$

Since we have the expressions for v, ϕ, θ and ξ we substitute them into the others to obtain the system:

$$\begin{aligned}
&\mu u_{xx} + B\phi_x = \rho f_2 - l(f_{5,x} + \tau f_{6,x}),\\
&-Bu_x - b\phi + M\alpha_{xx} + A\phi_{xx} = J f_4 - m(f_5 + \tau f_6) - M\tau f_{5,xx},\\
&M\phi_{xx} + k^*\alpha_{xx} = \tau a f_7 + l f_{1,x} + m f_3 - k f_{5,xx} + a f_6.
\end{aligned}$$

It is not difficult to solve this system with the help of the Fourier series. We know that

$$\begin{aligned}
&f_1 = \sum_{n\geqslant 1} f_1^n \sin nx, \quad f_2 = \sum_{n\geqslant 1} f_2^n \sin nx, \quad f_3 = \sum_{n\geqslant 1} f_3^n \cos nx, \quad f_4 = \sum_{n\geqslant 1} f_4^n \cos nx,\\
&f_5 = \sum_{n\geqslant 1} f_5^n \cos nx, \quad f_6 = \sum_{n\geqslant 1} f_6^n \cos nx, \quad f_7 = \sum_{n\geqslant 1} f_7^n \cos nx,
\end{aligned}$$

where

$$\begin{aligned}
&\sum_{n\geqslant 1} n^2 (f_1^n)^2 < \infty, \quad \sum_{n\geqslant 1} (f_2^n)^2 < \infty, \quad \sum_{n\geqslant 1} n^2 (f_3^n)^2 < \infty, \quad \sum_{n\geqslant 1} (f_4^n)^2 < \infty, ,\\
&\sum_{n\geqslant 1} n^2 (f_5^n)^2 < \infty, \quad \sum_{n\geqslant 1} n^2 (f_6^n)^2 < \infty, \quad \sum_{n\geqslant 1} (f_7^n)^2 < \infty.
\end{aligned}$$

We are looking for solutions of the form:

$$\begin{aligned}
&u = \sum_{n\geqslant 1} u^n \sin nx, \quad v = \sum_{n\geqslant 1} v^n \sin nx, \quad \phi = \sum_{n\geqslant 1} \phi^n \cos nx, \quad \psi = \sum_{n\geqslant 1} \psi^n \cos nx,\\
&\alpha = \sum_{n\geqslant 1} \alpha^n \cos nx, \quad \theta = \sum_{n\geqslant 1} \theta^n \cos nx, \quad \xi = \sum_{n\geqslant 1} \xi^n \cos nx,
\end{aligned}$$

where

$$\begin{aligned}
&\sum_{n\geqslant 1} n^2 (u^n)^2 < \infty, \quad \sum_{n\geqslant 1} n^2 (\phi^n)^2 < \infty, \quad \sum_{n\geqslant 1} n^2 (\alpha^n)^2 < \infty, \quad \sum_{n\geqslant 1} (\theta^n)^2 < \infty, ,\\
&\sum_{n\geqslant 1} (v^n)^2 < \infty, \quad \sum_{n\geqslant 1} (\psi^n)^2 < \infty, \quad \sum_{n\geqslant 1} (\xi^n)^2 < \infty.
\end{aligned}$$

Therefore, we have to solve the system:

$$\begin{aligned}
&-n^2\mu u^n - nB\phi^n = \rho f_2^n + nl(f_5^n + \tau f_6^n),\\
&-Bnu^n - b\phi^n - Mn^2\alpha^n - An^2\phi^n = Jf_4^n - m(f_5^n + \tau f_6^n) + n^2 M\tau f_5^n,\\
&-Mn^2\phi^n - k^* n^2\alpha^n = \tau a f_7^n + lnf_1^n + mf_3^n + kn^2 f_5^n + af_6^n.
\end{aligned}$$

The solution to this system has the following form:

$$\begin{aligned}
&u^n = (n^3(M^2 f_5 l + M^2 f_6 l\tau + BMf_5 k - Af_5 k^* l - BMf_5 k^*\tau - Af_6 k^* l\tau)\\
&\quad + n^2(-M^2 f_2\rho + BMf_1 l + Af_2 k^*\rho) + n(BMaf_6 - BJf_4 k^* + BMf_3 m\\
&\quad + Bf_5 k^* m - bf_5 k^* l + BMaf_7\tau + Bf_6 k^* m\tau - bf_6 k^* l\tau) + bf_2 k^*\rho)/(n^2\Delta),\\
&\phi^n = -(n^3(Mf_5 k\mu - Mf_5 k^*\mu\tau) + Mf_1 l\mu n^2\\
&\quad + n(-Bf_5 k^* l + Maf_6\mu - Jf_4 k^*\mu + Mf_3 m\mu + f_5 k^* m\mu\\
&\quad - Bf_6 k^* l\tau + Maf_7\mu\tau + f_6 k^* m\mu\tau) + Bf_2 k^*\rho)/(n\Delta),\\
&\alpha^n = (n^4(Af_5 k\mu - M^2 f_5\mu\tau) + Af_1 l\mu n^3 + n^2(-B^2 f_5 k - BMf_5 l - JMf_4\mu + Aaf_6\mu n\\
&\quad + Af_3 m\mu + Mf_5 m\mu + bf_5 k\mu - BMf_6 l\tau + Aaf_7\mu\tau + Mf_6 m\mu\tau)\\
&\quad + n(-B^2 f_1 ln + BMf_2 n\rho + bf_1 l\mu n) + bf_3 m\mu - B^2 f_3 m - B^2 af_6 - B^2 af_7\tau + abf_6\mu\\
&\quad + abf_7\mu\tau)/(n^2\Delta),
\end{aligned}$$

where we have dropped the super-indices n to make easier the presentation, and we denoted $\Delta = ((\mu M^2 - Ak^*\mu)n^2 + k^* B^2 - bk^*\mu)$.

In view of the convergences of $f_1, f_2, \ldots, f_7$ we see that u^n, ϕ^n and α^n satisfy the required conditions. Hence, we have that zero belongs to the resolvent of the operator

Therefore, in view of the Lumer-Phillips corollary to the Hille-Yosida theorem, we conclude the proof. $\square$

Thus, we conclude the following existence and uniqueness result.

Theorem 3.2. *Assume that $U^0 \in \mathcal{D}(\mathcal{A})$. Then, there exists a unique solution $U \in C^1([0,\infty);\mathcal{H}) \cap C^0([0,\infty);\mathcal{D}(\mathcal{A}))$ to system (1) such that the required initial and boundary conditions are satisfied.*

We also have the following theorem which states the exponential decay of the solution.

Theorem 3.3. *If we assume that l and M are different from zero, then there exist two positive constants N and ω such that*

$$||U(t)\| \leqslant N||U(0)||e^{-\omega t}.$$

Proof. To prove this theorem we only need to show that the imaginary axis is contained in the resolvent of the operator $\mathcal{A}$ and that the asymptotic condition:

$$\lim_{|\lambda|\to\infty} \|(i\lambda I - \mathcal{A})^{-1}\| < \infty$$

is satisfied. Our proof follows from the arguments proposed by Liu and Zheng in their book [32]. If we assume that the imaginary axis is not contained in the resolvent of the operator, then there exist a sequence of real numbers $\lambda_n \to \lambda$ with $|\lambda| > |\lambda_n|$ and a sequence of unitary vectors $U_n =$

$(u_n, v_n, \phi_n, \psi_n, \alpha_n, \theta_n, \xi)$ in the domain of the operator such that

$$i\lambda_n u_n - v_n \to 0 \quad \text{in} \quad H^1, \tag{6}$$

$$i\lambda_n \rho v_n - \mu D^2 u_n - BD\phi_n + lD\theta_n + l\tau D\xi_n \to 0 \quad \text{in} \quad L^2, \tag{7}$$

$$i\lambda_n \phi_n - \psi_n \to 0 \quad \text{in} \quad H^1, \tag{8}$$

$$\begin{aligned} &i\lambda_n J\psi_n + BDu_n - AD^2\phi_n + b\phi_n - MD^2\alpha_n - (M\tau D^2 + m)\theta_n \\ &\qquad -m\tau\xi_n \to 0 \quad \text{in} \quad L^2, \end{aligned} \tag{9}$$

$$i\lambda_n \alpha_n - \theta_n \to 0 \quad \text{in} \quad H^1, \tag{10}$$

$$i\lambda_n \theta_n - \xi_n \to 0 \quad \text{in} \quad H^1, \tag{11}$$

$$\begin{aligned} &i\tau a\lambda_n \xi_n + lDv_n - MD^2\phi_n + m\psi_n - k^* D^2\alpha_n - kD^2\theta_n \\ &\qquad +a\xi_n \to 0 \quad \text{in} \quad L^2. \end{aligned} \tag{12}$$

If we take into account the dissipation we obtain that $D\theta_n \to 0$ in L^2. Therefore, we also have that $\lambda_n D\alpha_n \to 0$ in L^2 and $\lambda_n^{-1} D\xi_n \to 0$ in L^2. Now, we want to prove that $D\phi_n$ tends to zero in L^2. To this end, we need to do some steps. The first one is obtained after the multiplication of convergence (9) by $\lambda_n^{-1}\xi_n$, and so we find that $iJ < \psi_n, \xi_n > \to 0$. Then, we see that $< \lambda_n \psi_n, \theta_n >$ also tends to zero. The second step follows after multiplication of convergence (12) by ϕ_n to obtain:

$$i\lambda_n l < Du_n, \phi_n > +M||D\phi_n||^2 + im\lambda_n \|\phi_n\|^2 \to 0.$$

In order to prove that $D\phi_n$ tends to zero, it will be sufficient to show that $< Du_n, \phi_n >$ tends to a real number. To show it, we multiply convergence (9) by ϕ_n to find that

$$-J\|\psi_n\|^2 + B < Du_n, \phi_n > +A\|D\phi_n\|^2 + b\|\phi_n\|^2 \to 0. \tag{13}$$

We see that $< Du_n, \phi_n >$ tends to a number with null imaginary part. It then follows that $D\phi_n \to 0$ in L^2. We have that convergence (13) implies that $\psi_n \to 0$ in L^2. Now, we multiply convergence (12) by $\lambda_n^{-1}\xi_n$ and we find that

$$i\tau a\|\xi_n\|^2 + il < Du_n, \xi_n > \to 0.$$

Since

$$< Du_n, \xi_n > = - < u_n, D\xi_n > = -\lim < \frac{v_n}{i\lambda_n}, \lambda_n iD\theta_n > = -\lim < \frac{v_n}{i}, iD\theta_n > = 0,$$

therefore we also obtain that $\|\xi_n\| \to 0$.

Now, we want to prove that Du_n tends to zero. To this end we multiply convergence (12) by $\lambda_n^{-1} Du_n$ and we find that

$$il\|Du_n\|^2 + < MD\psi_n + k^* D\alpha_n + kD\theta_n, \lambda_n^{-1} D^2 u_n > \to 0.$$

From convergence (7) we see that $\lambda_n^{-1}D^2u_n$ is bounded. It then follows that $\|Du_n\| \to 0$. We only need to multiply convergence (7) by u_n to prove that $v_n \to 0$ in L^2.

Therefore, we have arrived to a contradiction because we had assumed that the elements of the sequence have a unit norm. Hence, we conclude that the imaginary axis is contained in the resolvent of the operator.

Now, we need to prove the asymptotic condition. If we assume again that this is not true, then there exist a sequence of real numbers such that its absolute value tends to infinity and a sequence of unit vectors in the domain such that convergences (6)-(12) hold. So, we can use the arguments used above because we have only required in the proof that λ_n does not tend to zero. The proof is complete. □

4. Second system

In this section we analyze the system (2) with the same boundary conditions used in the previous section but with the following initial conditions:

$$\begin{array}{l} u(x,0) = u^0(x), \quad \dot{u}(x,0) = v^0(x), \quad \phi(x,0) = \phi^0(x), \quad \dot{\phi}(x,0) = \psi^0(x), \\ \ddot{\phi}(x,0) = \xi^0(x), \quad \alpha(x,0) = \alpha^0(x), \quad \dot{\alpha}(x,0) = \theta^0(x). \end{array} \tag{14}$$

Again, we assume that

$$\int_0^\pi \phi^0(x)\,dx = \int_0^\pi \psi^0(x)\,dx = \int_0^\pi \xi^0(x)\,dx = \int_0^\pi \alpha^0(x)\,dx = \int_0^\pi \theta^0(x)\,dx = 0.$$

Therefore, the integrals of ϕ, $\dot{\phi}$, $\ddot{\phi}$, α and θ vanish for every time and we can apply Poincare's inequality.

In this section, we assume that

$$\begin{array}{l} \mu > 0, \quad \mu b^* > B^2, \quad b^* > 0, \quad b > b^*\tau, \quad \rho > 0, \quad J > 0, \quad a > 0, \quad A > 0, \\ Ak > M^2. \end{array}$$

In this case, we can study the existence of solutions in the space:

$$\mathcal{H} = H_0^1 \times L^2 \times H_*^1 \times H_*^1 \times L_*^2 \times H_*^1 \times L_*^2,$$

and the inner product is given by

$$\begin{aligned} < (u,v,\phi,\psi,\xi,\alpha,\theta),(u^*,v^*,\phi^*,\psi^*,\xi^*,\alpha^*,\theta^*) >= \frac{1}{2}\int_0^\pi \Big(\rho v\overline{v^*} + \mu u_x\overline{u_x^*} \\ +J(\psi+\tau\xi)\overline{(\psi^*+\tau\xi^*)} + B(u_x\overline{(\phi^*+\tau\psi^*)} + \overline{u_x^*}(\phi+\tau\psi)) + \tau\overline{b}\psi\overline{\psi^*} + a\theta^2 \\ +k^*\alpha_x\overline{\alpha_x^*} + b^*(\phi+\tau\psi)\overline{(\phi^*+\tau\psi^*)} + A(\phi_x+\tau\psi_x)\overline{(\phi_x^*+\tau\psi_x^*)} \\ +M(\alpha_x\overline{(\phi_x^*+\tau\psi^*)} + \overline{\alpha_x^*}(\phi_x+\tau\psi_x))\Big)\,dx, \end{aligned}$$

where $\overline{b} = b - \tau b^*$ is positive due to the assumption $b > \tau b^*$. Again, the product is equivalent to the usual one in $\mathcal{H}$. We can write our problem in the form

$$\frac{dU}{dt} = \mathcal{A}U, \quad U(0) = U^0,$$

where $U = (u, v, \phi, \psi, \xi, \alpha, \theta)$ and $U^0 = (u^0, v^0, \phi^0, \psi^0, \xi^0, \alpha^0, \theta^0)$, and the matrix operator $\mathcal{A}$ is now given by

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu D^2}{\rho} & 0 & \frac{BD}{\rho} & \frac{\tau BD}{\rho} & 0 & 0 & \frac{-lD}{\rho} \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ \frac{-BD}{\tau J} & 0 & \frac{AD^2-b^*}{\tau J} & \frac{\tau AD^2-b}{\tau J} & -\tau^{-1} & \frac{MD^2}{\tau J} & \frac{m}{\tau J} \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & \frac{-lD}{a} & \frac{MD^2}{a} & \frac{\tau MD^2-m}{a} & \frac{-m\tau}{a} & \frac{kD^2}{a} & 0 \end{pmatrix}.$$

The domain of this operator is determined by the elements of the Hilbert space such that

$$\begin{aligned} & v \in H_0^1, \quad \psi \in H_*^1, \quad \theta \in H_*^1, \quad u \in H^2, \\ & M\alpha + A\phi + \tau A\psi \in H^2, \\ & M\phi + M\tau\psi + k^*\alpha \in H^2. \end{aligned}$$

We also note that

$$Re < \mathcal{A}U, U >= \frac{-\overline{b}}{2} \int_0^\pi |\psi|^2 \, dx.$$

Following the arguments of the previous section it is not difficult to prove that zero belongs to the resolvent of the operator $\mathcal{A}$. Therefore, we conclude that $\mathcal{A}$ generates a contractive semigroup and so, the following existence and uniqueness result.

Theorem 4.1. *Assume that $U^0 \in \mathcal{D}(\mathcal{A})$. Then, there exists a unique solution $U \in C^1([0, \infty); \mathcal{H}) \cap C^0([0, \infty); \mathcal{D}(\mathcal{A}))$ to system (2) such the required initial and boundary conditions are satisfied.*

The remaining objective of this section is to show that the solution to this problem does not decay uniformly in an exponential way.

Let us suppose that there exists a solution to system (2) with the above boundary and initial conditions of the form

$$u = A^* e^{\omega t} \sin(nx), \quad \phi = B^* e^{\omega t} \cos(nx), \quad \alpha = C^* e^{\omega t} \cos(nx), \tag{15}$$

such that $\mathrm{Re}(\omega) > -\epsilon$ for all positive ϵ small enough. This fact implies that a solution ω as near as desired to the imaginary axis can be found, and, hence, it is impossible to have uniform exponential decay on the solutions to problem (2), (5) and (14).

Imposing that u, ϕ and α in system (2) are as above, we obtain the following homogeneous system on the unknowns A^*, B^* and C^*:

$$\begin{pmatrix} \mu n^2 + \rho\omega^2 & Bn(\tau\omega + 1) & -ln\omega \\ Bn & J\tau\omega^3 + J\omega^2 + b\omega + An^2(\tau\omega + 1) + b^* & Mn^2 - m\omega \\ ln\omega & M(\tau\omega + 1)n^2 + m\tau\omega^2 + m\omega & kn^2 + a\omega^2 \end{pmatrix} \begin{pmatrix} A^* \\ B^* \\ C^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{16}$$

This linear system has nontrivial solutions if and only if the determinant of the coefficients matrix is zero. In that case, ω would be a root of the following seventh degree polynomial:

$$p(z) = a_0 z^7 + a_1 z^6 + a_2 z^5 + a_3 z^4 + a_4 z^3 + a_5 z^2 + a_6 z + a_7, \tag{17}$$

where

$$\begin{aligned}
a_0 &= aJ\rho\tau,\\
a_1 &= aJ\rho,\\
a_2 &= \tau\left(n^2\left(aA\rho + J\left(a\mu + k\rho + l^2\right)\right) + m^2\rho\right) + ab\rho,\\
a_3 &= n^2\left(aA\rho + J\left(a\mu + k\rho + l^2\right)\right) + ab^*\rho + m^2\rho,\\
a_4 &= n^2\left(\tau\left(a\left(A\mu n^2 - B^2\right) + n^2\left(Ak\rho + Al^2 + Jk\mu - M^2\rho\right) - 2Blm + \mu m^2\right) + b\left(a\mu + k\rho + l^2\right)\right),\\
a_5 &= n^2\left(a\left(A\mu n^2 - B^2\right) + b^*\left(a\mu + k\rho + l^2\right) + n^2\left(A\left(k\rho + l^2\right) + Jk\mu - M^2\rho\right) - 2Blm + \mu m^2\right),\\
a_6 &= n^4\left(bk\mu - \tau\left(\mu n^2\left(M^2 - Ak\right) + B^2 k\right)\right),\\
a_7 &= n^4\left(\mu n^2\left(Ak - M^2\right) + k(b^*\mu - B^2)\right).
\end{aligned}$$

We want to prove that there are roots of $p(z)$ as near to the complex axis as desired. Equivalently, we would prove that, for any $\epsilon > 0$, there are roots located on the right side of the vertical line $\mathrm{Re}(z) = -\epsilon$. So, it will be sufficient to show that there exists a root with positive real part for polynomial $p(z-\epsilon)$. We use the Routh-Hurwitz theorem to show it (see, for example, Dieudonné [34]). It claims that, if $a_0 > 0$, then all the roots of polynomial $p(z)$ have negative real part if and only if a_7 and all the leading diagonal minors of matrix

$$\tilde{\mathcal{A}} = \begin{pmatrix}
a_1 & a_0 & 0 & 0 & 0 & 0 & 0\\
a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0\\
a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0\\
a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1\\
0 & 0 & a_7 & a_6 & a_5 & a_4 & a_3\\
0 & 0 & 0 & 0 & a_7 & a_6 & a_5\\
0 & 0 & 0 & 0 & 0 & 0 & a_7
\end{pmatrix}$$

are positive. From the assumptions over the coefficients, it is clear that a_7 is positive. Let Λ_i for $i = 1, 2, \ldots, 7$ be the leading diagonal minors of matrix $\tilde{\mathcal{A}}$ corresponding to polynomial $p(z-\epsilon)$.

Direct calculations prove that there exists n large enough that makes $\Lambda_2 < 0$. In fact, this minor is a second degree polynomial with respect to n. To be precise,

$$\begin{aligned}
\Lambda_2 = -2aJ\rho\tau^2\epsilon\left(aA\rho + J\left(a\mu + k\rho + l^2\right)\right)n^2 + a^2J\rho^2\left(b - b^*\tau\right) - 112a^2J^2\rho^2\tau^2\epsilon^3\\
+ 48a^2J^2\rho^2\tau\epsilon^2 - 2aJ\rho^2\epsilon\left(ab\tau + 3aJ + m^2\tau^2\right).
\end{aligned}$$

Notice that the main coefficient of Λ_2 is negative.

This argument proves that a uniform rate of decay of exponential type cannot be obtained for all the solutions and, hence, the decay of the solutions is slow.

It is worth noting that in reference [10] the authors proved the exponential stability of the solutions in the case of type II thermo-porous-elasticity with weak porous dissipation. Therefore, we have seen a difference in the behavior when we change the usual porous dissipation by the Moore-Gibson-Thompson-porous dissipation.

5. Third system

We now consider the third system (3) with the same initial and boundary conditions employed in the previous section. We also assume the conditions on the initial data for ϕ^0, ψ^0, ξ^0, α^0 and θ^0.

In this section we assume the following conditions on the constitutive coefficients:

$$\begin{aligned}&\mu > 0, \quad \mu b > B^2, \quad A^* > 0, \quad A > A^*\tau, \quad \rho > 0, \quad J > 0, \quad a > 0,\\ &A^* k > M^2,\end{aligned}$$

and we also use the previous definition for the Hilbert space $\mathcal{H}$; however, the inner product is slightly different:

$$\begin{aligned}< (u,v,\phi,\psi,\xi,\alpha,\theta),(u^*,v^*,\phi^*,\psi^*,\xi^*,\alpha^*,\theta^*) >= \frac{1}{2}\int_0^\pi \Big(\rho v\overline{v^*} + \mu u_x \overline{u_x^*}\\ +J(\psi+\tau\xi)\overline{(\psi^*+\tau\xi^*)} + B(u_x\overline{(\phi^*+\tau\psi^*)} + \overline{u_x^*}(\phi+\tau\psi))\\ +A^*(\phi_x+\tau\psi_x)\overline{(\phi_x^*+\tau\psi^*)_x} + \tau\overline{A}\psi_x\overline{\psi_x^*} + b(\phi+\tau\psi)\overline{(\phi+\tau\psi)}\\ +M(\alpha_x\overline{(\phi_x^*+\tau\psi^*)} + \overline{\alpha_x^*}(\phi_x+\tau\psi_x)) + a\theta^2 + k\alpha_x\overline{\alpha_x^*}\Big)\, dx,\end{aligned}$$

where $\overline{A} = A - \tau A^*$ is positive due to the condition $A > A^*\tau$. The problem is really similar to the one studied in the previous section but, in this case, the matrix operator $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu D^2}{\rho} & 0 & \frac{BD}{\rho} & \frac{\tau BD}{\rho} & 0 & 0 & \frac{-lD}{\rho} \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ \frac{-BD}{\tau J} & 0 & \frac{A^*D^2-b}{\tau J} & \frac{\tau AD^2-b\tau}{J} & -\tau^{-1} & \frac{MD^2}{\tau J} & \frac{m}{\tau J} \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & \frac{-lD}{a} & \frac{MD^2}{a} & \frac{\tau MD^2-m}{a} & \frac{-m\tau}{a} & \frac{kD^2}{a} & 0 \end{pmatrix}.$$

The domain of this operator is determined by the elements of the Hilbert space such that

$$\begin{aligned}&v \in H_0^1, \quad \psi \in H_*^1, \quad \theta \in H_*^1, \quad u \in H^2,\\ &M\alpha + A^*\phi + A\psi \in H^2,\\ &M\phi + M\tau\psi + k\alpha \in H^2,\end{aligned}$$

which is a dense subspace in $\mathcal{H}$. We also have

$$Re < \mathcal{A}U, U >= -\frac{\overline{A}}{2}\int_0^\pi |\psi_x|^2\, dx.$$

Again, we can prove that zero belongs to the resolvent of the operator, and we obtain an existence and uniqueness result for the solutions to this problem that we state in the following.

Theorem 5.1. *Assume that $U^0 \in \mathcal{D}(\mathcal{A})$. Then, there exists a unique solution $U \in C^1([0,\infty);\mathcal{H}) \cap C^0([0,\infty);\mathcal{D}(\mathcal{A}))$ to system (3) such that the required initial and boundary conditions are satisfied.*

Now, we prove the main result of this section; that is, the exponential stability of the solutions.

Theorem 5.2. *If we assume that l and M are different from zero, then the solutions to the problem determined by system (3) decay in an exponential way; that is, there exist two positive constants N and ω such that*

$$||U(t)| \leqslant N||U(0)||e^{-\omega t}.$$

Proof. The proof follows a structure which is rather similar to the one shown in Theorem 3.3. In this case, we have

$$i\lambda_n u_n - v_n \to 0 \quad \text{in} \quad H^1, \tag{18}$$
$$i\lambda_n \rho v_n - \mu D^2 u_n - \tau BD\phi_n - BD\phi_n + lD\theta_n \to 0 \quad \text{in} \quad L^2, \tag{19}$$
$$i\lambda_n \phi_n - \psi_n \to 0 \quad \text{in} \quad H^1, \tag{20}$$
$$i\lambda_n \psi_n - \xi_n \to 0 \quad \text{in} \quad H^1, \tag{21}$$
$$\begin{aligned} &iJ\lambda_n\xi_n + BDu_n - A^*D^2\phi_n + b\phi_n - AD^2\psi_n + b\tau\psi_n + J\xi_n \\ &\qquad -MD^2\alpha_n - m\theta_n \to 0 \quad \text{in} \quad L^2, \end{aligned} \tag{22}$$
$$i\lambda_n \alpha_n - \theta_n \to 0 \quad \text{in} \quad H^1, \tag{23}$$
$$\begin{aligned} &ia\lambda_n\theta_n + lDv_n - MD^2\phi_n - \tau MD^2\xi_n + m\psi_n + m\tau\xi_n \\ &\qquad -kD^2\alpha_n \to 0 \quad \text{in} \quad L^2. \end{aligned} \tag{24}$$

We note that the dissipation implies that $\psi_n \to 0$ in H^1, and then $\lambda_n\phi_n \to 0$ in H^1 and $\lambda_n^{-1}\xi_n$ also tends to zero in H^1. If we multiply convergence (24) by $\lambda_n^{-1}\xi_n$, we obtain that $< \theta_n, \xi_n > \to 0$ and then $< \alpha_n, \lambda_n\xi_n > \to 0$. Now, we multiply convergence (22) by α_n to see that

$$B < Du_n, \alpha_n > + M||D\alpha_n||^2 - i\lambda_n m||\alpha_n||^2 \to 0.$$

We will conclude that $D\alpha_n \to 0$ if we show that the real part of $< Du_n, \alpha_n >$ tends to zero. To this end, we multiply convergence (24) by α_n to see that

$$-a||\theta_n||^2 + i\lambda_n l < Du_n, \alpha_n > + k||D\alpha_n||^2 \to 0.$$

Then, we have that $Re < Du_n, \alpha_n > \to 0$ and therefore $D\alpha_n, \theta_n \to 0$ in L^2.

If we multiply convergence (22) by $\lambda_n^{-1}\xi_n$ we get

$$i\tau J||\xi_n||^2 + B < Du_n, \lambda_n^{-1}\xi_n > \to 0.$$

Since $< Du_n, \lambda_n^{-1}\xi_n > \to 0$, it follows that $\xi_n \to 0$ in L^2.

In order to prove that Du_n tends to zero, we multiply convergence (24) by $\lambda_n^{-1}Du_n$ to see

$$il||Du_n||^2 + < MD\phi_n + \tau MD\psi_n + kD\alpha_n, \lambda_n^{-1}D^2u_n > \to 0.$$

Moreover, we can see from convergence (19) that $\lambda_n^{-1}D^2u_n$ is bounded to obtain that $Du_n \to 0$ in L^2. We also obtain from here that $v_n \to 0$ in L^2_*.

Again, we obtain a contradiction and we see that the imaginary axis is contained in the resolvent of the operator. We can show the asymptotic condition by using the same arguments employed in the proof of Theorem 3.3 because the key feature is that λ_n does not tend to zero. Therefore, the proof is finished. □

6. Fourth system

To conclude the analysis of this paper we consider the fourth system (4); however, in order to simplify the calculations, we modify slightly the boundary conditions in the following form:

$$u_x(0,t) = u_x(\pi,t) = \phi(0,t) = \phi(\pi,t) = \alpha(0,t) = \alpha(\pi,t) = 0,$$

with the initial conditions:

$$\begin{aligned}&u(x,0) = u^0(x), \quad \dot{u}(x,0) = v^0(x), \quad \ddot{u}(x,0) = d^0(x), \quad \phi(x,0) = \phi^0(x),\\ &\dot{\phi}(x,0) = \psi^0(x), \quad \alpha(x,0) = \alpha^0(x), \quad \dot{\alpha}(x,0) = \theta^0(x).\end{aligned}$$

In this case, we assume that

$$\int_0^\pi u^0(x)\,dx = \int_0^\pi v^0(x)\,dx = \int_0^\pi d^0(x)\,dx = 0.$$

Again, the integrals of u, $\dot{u}$ and $\ddot{u}$ vanish for every time.

Regarding the constitutive coefficients, we assume that

$$\mu^* > 0, \quad \mu > \tau\mu^*, \quad \mu^* b > B^2, \quad \rho > 0, \quad J > 0, \quad a > 0, \quad A > 0, \quad Ak > M^2,$$

and we use the Hilbert space

$$\mathcal{H} = H^1_* \times H^1_* \times L^2_* \times H^1_0 \times L^2 \times H^1_0 \times L^2,$$

with the inner product given by

$$\begin{aligned}< (u,v,d,\phi,\psi,\alpha,\theta),(u^*,v^*,d^*,\phi^*,\psi^*,\alpha^*,\theta^*) >= \frac{1}{2}\int_0^\pi \Big(\rho(\tau d+v)\overline{(\tau d^*+v^*)}\\ +\overline{\mu} v_x\overline{v^*_x} + J\psi\overline{\psi^*} + \mu^*(u_x+\tau v_x)\overline{(u^*_x+\tau v^*_x)} + A\phi_x\overline{\phi^*_x} + b\phi\overline{\phi^*} + a\theta\overline{\theta^*}\\ +B((u_x+\tau v_x)\overline{\phi^*} + \overline{(u^*_x+\tau v^*_x)}\phi) + k\alpha_x\overline{\alpha^*_x} + M(\alpha_x\overline{\phi^*_x} + \overline{\alpha^*_x}\phi_x)\Big)\,dx,\end{aligned}$$

where $\overline{\mu} = \mu - \tau\mu^*$ is positive due to the assumption $\mu > \tau\mu^*$. Again, this product is equivalent to the usual one in $\mathcal{H}$. As in the previous section, we can write our problem in the form

$$\frac{dU}{dt} = \mathcal{A}U, \quad U(0) = U^0,$$

where $U = (u, v, d, \phi, \psi, \alpha, \theta)$ and $U^0 = (u^0, v^0, d^0, \phi^0, \psi^0, \alpha^0, \theta^0)$, and $\mathcal{A}$ being the matrix operator given by

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ \frac{\mu^* D^2}{\tau\rho} & \frac{\mu D^2}{\tau\rho} & -\tau^{-1} & \frac{BD}{\tau\rho} & 0 & 0 & \frac{-lD}{\tau\rho} \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ \frac{-BD}{J} & \frac{-\tau BD}{J} & 0 & \frac{AD^2-b}{J} & 0 & \frac{MD^2}{J} & \frac{m}{J} \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & \frac{-lD}{a} & -\frac{\tau lD}{a} & \frac{MD^2}{a} & -\frac{m}{a} & \frac{kD^2}{a} & 0 \end{pmatrix}.$$

We note that the domain of the operator $\mathcal{A}$ is given

$$\begin{aligned} & d \in H^1_*, \quad \psi \in H^1_0, \quad \theta \in H^1_0, \quad \alpha, \phi \in H^2, \\ & \mu^* u + \mu v \in H^2. \end{aligned}$$

We also note that

$$Re < \mathcal{A}U, U >= -\frac{1}{2}\int_0^\pi \overline{\mu}|v_x|^2\, dx,$$

and we can prove that zero belongs to the resolvent of the operator $\mathcal{A}$ by using similar arguments to the ones proposed in Section 3. Therefore, we can prove that $\mathcal{A}$ generates a contractive semigroup, which gives us the existence and uniqueness of solutions to our problem. It is stated in the following.

Theorem 6.1. *Assume that $U^0 \in \mathcal{D}(\mathcal{A})$. Then, there exists a unique solution $U \in C^1([0,\infty);\mathcal{H}) \cap C^0([0,\infty);\mathcal{D}(\mathcal{A}))$ to system (4) such that the required initial and boundary conditions are satisfied.*

Our next aim is to prove the exponential decay of solutions to this problem. To this end, we need the following lemmata.

Lemma 6.2. *The imaginary axis is contained in the resolvent of the operator $\mathcal{A}$.*

Proof. As in the previous sections, we assume that the thesis of the lemma is not true. Therefore, there exist a sequence of real number $\lambda_n \to \lambda \neq 0$ and a sequence of elements in the domain of the operator such that

$$i\lambda_n u_n - v_n \to 0 \quad \text{in} \quad H^1, \tag{25}$$

$$i\lambda_n v_n - d_n \to 0 \quad \text{in} \quad H^1, \tag{26}$$

$$i\tau\rho\lambda_n d_n - \mu^* D^2 u_n - \mu D^2 v_n + \rho d_n - BD\phi_n - lD\theta_n \to 0 \quad \text{in} \quad L^2, \tag{27}$$

$$i\lambda_n \phi_n - \psi_n \to 0 \quad \text{in} \quad H^1, \tag{28}$$

$$\begin{aligned} & iJ\lambda_n\psi_n + BDu_n + \tau BDv_n - AD^2\phi_n + b\phi_n - MD^2\alpha_n \\ & \qquad -m\theta_n \to 0 \quad \text{in} \quad L^2, \end{aligned} \tag{29}$$

$$i\lambda_n\alpha_n - \theta_n \to 0 \quad \text{in} \quad H^1, \tag{30}$$

$$ia\lambda_n\theta_n + lDv_n + \tau lDd_n - MD^2\phi_n + m\psi_n - kD^2\alpha_n \to 0 \quad \text{in} \quad L^2. \tag{31}$$

In view of the dissipation we see that $Dv_n \to 0$ in L^2 and therefore, $\lambda_n Du_n \to 0$ in L^2 and $\lambda_n^{-1}Dd_n$ also tends to zero in L^2.

We first note that

$$B\phi_n - l\theta_n \to 0 \quad \text{in} \quad L^2. \tag{32}$$

After the multiplication of convergence (27) by $\lambda_n^{-1}D\alpha_n$ we get

$$-<\mu^* D^2u_n + \mu D^2v_n, \lambda_n^{-1}D\alpha_n> +B<D\phi_n, \lambda_n^{-1}D\alpha_n> -l<D\theta_n, \lambda_n^{-1}D\alpha_n> \to 0.$$

Moreover, since

$$<\mu^* D^2u_n + \mu D^2v_n, \lambda_n^{-1}D\alpha_n> = -<\mu^* Du_n + \mu Dv_n, \lambda_n^{-1}D^2\alpha_n>$$

and $\lambda_n^{-1}D^2\alpha_n$ is bounded, we obtain that

$$\lim \frac{il}{B}\|D\alpha_n\|^2 = \lim \lambda_n^{-1} <D\phi_n, D\alpha_n> .$$

The case $B=0$ implies that $D\alpha_n \to 0$ in L^2. In the generic case, we multiply convergence (29) by ϕ_n to see that

$$i\lambda_n J<\psi_n, \phi_n> +A\|D\phi_n\|^2 + b\|\phi_n\|^2 + M<D\alpha_n, D\phi_n> -m<\theta_n, \phi_n> \to 0.$$

Then, in view of (32) we see that the imaginary part of $<D\alpha_n, D\phi_n>$ tends to zero. Therefore, we also obtain that $D\alpha_n \to 0$ in L^2. If we multiply convergence (31) by α_n we also see that $\theta_n \to 0$ in L^2.

Now, we have to prove that ϕ_n also tends to zero. If we multiply convergence (31) by ϕ_n we see

$$ia\lambda_n<\theta_n, \phi_n> +M\|D\phi_n\|^2 + i\lambda_n m\|\phi_n\|^2 \to 0.$$

So we also see that $D\phi_n \to 0$ in L^2 and, after multiplication of convergence (29) by ϕ_n, we find that $\psi_n \to 0$ in L^2. Since we have arrived at a contradiction, the lemma is proved.
□

Lemma 6.3. *The operator $\mathcal{A}$ satisfies the following asymptotic condition:*

$$\lim_{|\lambda|\to\infty} \|(i\lambda I - \mathcal{A})^{-1}\| < \infty. \tag{33}$$

Proof. Again, we use a contradiction argument. Assuming that the thesis does not hold, there exist a sequence of real numbers λ_n such that its absolute value becomes unbounded, and a sequence of elements in the domain of the operator $\mathcal{A}$ with unit norm satisfying the asymptotic condition (33). From the dissipation we find that

$$\lambda_n Du_n, Dv_n, \lambda_n^{-1} Dd_n \to 0 \quad \text{in} \quad L^2.$$

If we multiply convergence (27) by $\lambda_n^{-1} d_n$ we see that $d_n \to 0$ in L^2. Now, we consider the product of convergence (27) by $\lambda_n^{-1} D\alpha_n$ (keeping in mind that $\lambda_n^{-1} D^2 \alpha_n$ is bounded), and we get

$$i\tau\rho < d_n, D\alpha_n > -\lambda_n^{-1} B < D\phi_n, D\alpha_n > -i\rho \|D\alpha_n\|^2 \to 0.$$

The second term clearly tends to zero and

$$\lim < d_n, D\alpha_n >= -\lim < Dd_n, \alpha_n >= -\lim < \lambda_n^{-1} Dd_n, i\theta_n > .$$

Thus, we also see that $< d_n, D\alpha_n > \to 0$ and therefore, $\|D\alpha_n\| \to 0$. Then, we can proceed as in the proof of the previous lemma to obtain that θ_n, $D\phi_n$ and ψ_n tend to zero in L^2. Again, it leads to a contradiction and the lemma is proved. □

As a consequence of the previous lemmata, we derive the following stability result.

Theorem 6.4. *If we assume that l and M are different from zero, then the solutions to the problem determined by system (4) decay in an exponential way; that is, there exist two positive constants N and ω such that*

$$||U(t)\| \leqslant N||U(0)||e^{-\omega t}.$$

Acknowledgments

The authors thank to the anonymous referee her/his criticism that allowed us to improve this contribution.

The work of J.R. Fernández has been partially supported by Ministerio de Ciencia, Innovación y Universidades under the research project PGC2018-096696-B-I00 (FEDER, UE).

The work of A. Magaña and R. Quintanilla is supported by the Ministerio de Ciencia, Innovación y Universidades under the research project “Análisis matemático aplicado a la termomecánica” (Ref. PID2019-105118GB-I00).

References

[1] S. C. Cowin, The viscoelastic behavior of linear elastic materials with voids, *J. Elasticity*, **15** (1985), 185–191.
[2] S. C. Cowin and J. W. Nunziato, Linear elastic materials with voids, *J. Elasticity*, **13** (1983), 125–147.
[3] D. Ieşan, A theory of thermoelastic materials with voids, *Acta Mechanica*, **60** (1986), 67–89.
[4] D. Ieşan and R. Quintanilla, On a theory of thermoelastic materials with double porosity structure, *J. Thermal Stresses*, **37** (2014), 1017–1036.
[5] R. Kumar and R. Vohra, Effect of hall current in thermoelastic materials with double porosity structure, *Int. J. Appl. Mech. Engrg.*, **22** (2017), 303–319.

[6] N. Bazarra, J. R. Fernández, M. C. Leseduarte, A. Magaña and R. Quintanilla, On the thermoelasticity with two porosities: asymptotic behavior, *Math. Mech. Solids*, **24** (2019), 2713–2725.
[7] A. Magaña and R. Quintanilla, Exponential decay in one-dimensional type II/III thermoelasticity with porosities, *Math. Meth. Appl. Sci.*, **43** (2020), 6921–6937.
[8] B. Feng and T. A. Apalara, Optimal decay for a porous elasticity system with memory, *J. Math. Anal. Appl.*, **470** (2019), 1108–1128.
[9] B. Feng and M. Yin, Decay of solutions for a one-dimensional porous elasticity system with memory: the case of nonequal wave speeds, *Math. Mech. Solids*, **24** (2019), 2361-2373.
[10] M. C. Leseduarte, A. Magaña and R. Quintanilla, On the time decay of solutions in porous-thermo-elasticity of type II, *Discrete Cont. Dyn. Systems - B*, **13** (2010), 375–391.
[11] A. Magaña and R. Quintanilla, On the exponential decay of solutions in one-dimensional generalized porous-thermo-elasticity, *Asymptotic Anal.*, **49** (2006), 173–187.
[12] A. Magaña and R. Quintanilla, On the time decay of solutions in porous-elasticity with quasi-static microvoids, *J. Math. Anal. Appl.*, **331** (2007), 617–630.
[13] A. Magaña and R. Quintanilla, Exponential stability in three-dimensional type III thermo- porous-elasticity with microtemperatures, *J. Elasticity*, **139** (2020), 153–161.
[14] A. Miranville and R. Quintanilla, Exponential decay in one-dimensional type II thermoviscoelasticity with voids, *Appl. Math. Letters*, **94** (2019), 30–37.
[15] A. Miranville and R. Quintanilla, Exponential decay in one-dimensional type II thermoviscoelasticity with voids, *J. Comput. Appl. Math.*, **368** (2020), 112573.
[16] S. Nicaise and J. Valein, Stabilization of non-homogeneous elastic materials with voids, *J. Math. Anal. Appl.*, **387** (2012), 1061–1087.
[17] A. E. Green and P. M. Naghdi, On undamped heat waves in an elastic solid, *J. Thermal Stresses*, **15** (1992), 253–264.
[18] A. E. Green and P. M. Naghdi, Thermoelasticity without energy dissipation, *J. Elasticity*, **31** (1993), 189–208.
[19] A. E. Green and P. M. Naghdi, A verified procedure for construction of theories of deformable media. I. Classical continuum physics, II. Generalized continua, III. Mixtures of interacting continua, *Proc. Royal Society London A*, **448** (1995), 335–356, 357–377, 378–388.
[20] R. Quintanilla, Moore-Gibson-Thompson thermoelasticity, *Math. Mech. Solids*, **24** (2019), 4020–4031.
[21] M. E. Gurtin, Time-reversal and symmetry in the thermodynamics of materials with memory, *Arch. Rat. Mech. Anal.*, **44** (1972), 387–399.
[22] R. Borghesani and A. Morro, Time-reversal and thermodynamics of electromagnetic fields in materials with memory, *Annali Mat. Pura Appl.*, **99** (1974), 65–80.
[23] D. Ieşan and R. Quintanilla, On the theory of interactions with memory, *J. Thermal Stresses*, **25** (2002), 1161–1177.
[24] M. Ciarletta and A. Scalia, On some theorems in the linear theory of viscoelastic materials with voids, *J. Elasticity*, **25** (1991), 149–158.
[25] M. Conti, V. Pata and R. Quintanilla, Thermoelasticity of Moore-Gibson-Thompson type with history dependence in the temperature, *Asymptotic Anal.*, **120** (2020), 1–21.
[26] M. Pellicer and R. Quintanilla, On uniqueness and instability for some thermomechanical problems involving the Moore–Gibson–Thompson equation, *Zeit. Ang. Math. Phys.*, **71** (2020), 84.
[27] M. Conti, V. Pata, M. Pellicer and R. Quintanilla, On the analiticity of the MGT-viscoelastic plate with heat conduction, *J. Differential Equations*, **269** (2020), 7862–7880.
[28] N. Bazarra, J. R. Fernández, A. Magaña and R. Quintanilla, A poro-thermoelastic problem with dissipative heat conduction, *J. Thermal Stresses*, **43** (2020), 1415–1436.
[29] H. Kumar and S Mukhopadyay, Thermoelastic damping analysis in microbeam resonators based on Moore-Gibson-Thompson generalized thermoelasticity theory, *Acta Mechanica*, **231** (2020), 3003–3015.
[30] J. R. Fernández and R. Quintanilla, Moore-Gibson-Thompson theory for thermoelastic dielectrics, *Appl. Math. Mech.*, **42** (2021), 309–316.
[31] R. Quintanilla, A condition on the delay parameters in the one-dimensional dual-phase-lag thermoelastic theory, *J. Thermal Stresses*, **26** (2003), 713–721.
[32] Z. Liu and S. Zheng, *Semigroups associated with dissipative systems*, Chapman & Hall/CRC Research Notes in Mathematics, vol. 398, Chapman & Hall/CRC, Boca Raton, FL, 1999.
[33] M. Conti, V. Pata, M. Pellicer, R. Quintanilla, A new approach to MGT-thermoviscoelasticity, *Discrete Cont. Dyn. Sys.*, **41** (2021), 4645-4666.
[34] M. J. Dieudonné, *La Theorie Analytique des Polynomes d'une Variable (A Coefficients Quelconques)*, Gauthier-Villars, 1938.