An a priori error analysis of a porous strain gradient model

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Received XXXX, revised XXXX, accepted XXXX
Published online XXXX

Key words Strain-gradient, porosity, finite elements, discrete stability, a priori error estimates, numerical behavior.
MSC (2010) 65M60, 65M15, 65M12, 74K10, 74F05, 74M15

In this work, we consider, from the numerical point of view, a boundary-initial value problem for non-simple porous elastic materials. The mechanical problem is written as a coupled hyperbolic linear system in terms of the displacement and porosity fields. The resulting variational formulation is used to approximate the solution by the finite element method and the implicit Euler scheme. A discrete stability property and a priori error estimates are proved, from which the linear convergence of the numerical scheme is deduced under adequate regularity conditions. Finally, some numerical simulations are presented to show the accuracy of the finite element scheme studied previously, the evolution of the discrete energy and the behavior of the solution.

1 Introduction

The porous structure of the materials has a great influence in the behavior of elastic materials. For this reason, it has been developed in a relevant way the study of porous-elastic materials. Nunziato and Cowin [14] proposed a non-linear theory with a matrix material which is elastic and where the interstices are void of material. Cowin and Nunziato [4, 14] derived the linear theory. A systematic development for this theory can be found in the book of Ieşan [7]. We can see a porous-elastic material as the combination of the macroscopic structure (the elastic deformation) and the microscopic structure (the porosity). They are coupled and it is of interest to know how the introduction of a dissipative mechanism in one of these structures is carried out to the other component. A big quantity of contributions in this line has been obtained in the recent years (see, among others, [8–10, 12, 13, 15–17]).

The introduction of higher order gradients in the basic postulates of elasticity has also been considered in the second part of the last century. It seems that it was motivated by the desire to obtain more detailed models for the response of the materials to stimuli. The works of Green-Rivlin [5], Mindlin [11] or Toupin [18] should be considered in this line as several pioneering contributions. The theories including second gradient of the displacement are usually called strain-gradient theories.

We here want to center our attention to the linear strain gradient porous-elasticity, in the case that we consider second gradient of the displacement and second gradient of the volume fraction field among the independent variables. This theory has been recently proposed by Ieşan [6], and the purpose of this paper is to analyze numerically this problem when several dissipation mechanisms are introduced in the system. It is of interest to determine how the solutions decay in time in this case.

The paper is structured as follows. In Section 2, we describe the basic equations we are going to work with and we derive the variational formulation of this mechanical problem. An existence and uniqueness result proved in [1] is recalled. Then, in Section 3 we introduce a fully discrete approximation by using the finite element method and the implicit Euler scheme. A discrete stability property and a priori error estimates are shown. The linear convergence of the approximation is derived under suitable additional regularity conditions. Finally, in Section 4 some numerical simulations are presented to demonstrate the accuracy of the algorithm, the discrete energy decay and the behavior of the solution depending on the viscosity coefficient.

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2 The porous-elastic model

First of all, we recall the evolution and constitutive equations which govern the theory we are going to deal with. We follow the guidelines proposed by Ieşan [6].

Our analysis is focused in the one-dimensional problem, whose evolution equations are

$$\begin{aligned}\rho\ddot{u} &= \tau_x - \mu_{xx},\\ J\ddot{\varphi} &= \chi_x - \sigma_{xx} + g.\end{aligned}$$

Here, u is the displacement, φ is the fraction of volume, τ is the stress, μ is the hyperstress, χ is the equilibrated stress vector, σ is the equilibrated hyperstress tensor and g the equilibrated body force. As usual, ρ stands for the mass density and J for the product of the mass density by the equilibrated inertia, and both are assumed to be positive.

The primary constitutive equations are given by

$$\begin{aligned}\tau &= au_x + b\varphi + \beta\varphi_{xx} + a^*\dot{u}_x,\\ \mu &= \kappa_1 u_{xx} + \gamma\varphi_x + \kappa_1^*\dot{u}_{xx},\\ \chi &= \gamma u_{xx} + \alpha\varphi_x,\\ \sigma &= \beta u_x + d\varphi + \kappa_2\varphi_{xx},\\ g &= -bu_x - \xi\varphi - d\varphi_{xx} - \xi^*\dot{\varphi}.\end{aligned}$$

The conditions for the constitutive coefficients a, b, β, κ_1, γ, α, d, κ_2, κ_1^*, a^*, ξ and ξ^* will be stated below.

Without loss of generality, we suppose that the spatial variable x lies in the interval $[0,\ell]$ and that the time t goes from 0 to T, where $T>0$ denotes the final time.

Therefore, introducing the constitutive equations into the evolution equations we find that the resulting linear system of equations is the following:

$$\left.\begin{aligned}\rho\ddot{u} &= au_{xx} + b\varphi_x - \kappa_1 u_{xxxx} - \eta\varphi_{xxx} - \kappa_1^*\dot{u}_{xxxx} + a^*\dot{u}_{xx},\\ J\ddot{\varphi} &= \eta u_{xxx} - bu_x + \delta\varphi_{xx} - \xi\varphi - \kappa_2\varphi_{xxxx} - \xi^*\dot{\varphi},\end{aligned}\right\} \tag{1}$$

where we have simplified the notation defining $\eta = \gamma - \beta$ and $\delta = \alpha - 2d$.

We note that from the linear system (1) we can obtain three different problems with porosity: porous-hyperviscoelastic problems (assuming that $\kappa_1^* \neq 0$ and $a^* = \xi^* = 0$), porous-viscoelastic problems (assuming that $a^* \neq 0$ and $\kappa_1^* = \xi^* = 0$) and weak viscoporous-elastic problems (assuming that $\xi^* \neq 0$ and $a^* = \kappa_1^* = 0$). Proceeding in an analogous way, we could also analyze the hyperviscoporosity and viscoporosity cases, which are rather similar to the hyperviscoelastic and viscoelastic ones with porosity.

The following set of boundary and initial conditions are imposed, for a.e. $x \in (0,\ell)$ and $t \in (0,T)$:

$$\begin{aligned}&u(0,t) = u(\ell,t) = u_x(0,t) = u_x(\ell,t) = 0,\\ &\varphi(0,t) = \varphi_x(0,t) = \varphi(\ell,t) = \varphi_x(\ell,t) = 0,\end{aligned} \tag{2}$$

and

$$u(x,0) = u_0(x), \quad \dot{u}(x,0) = v_0(x), \quad \varphi(x,0) = \varphi_0(x), \quad \dot{\varphi}(x,0) = \psi_0(x). \tag{3}$$

The constitutive coefficients satisfy the following conditions:

$$\rho > 0, \quad J > 0, \quad a > 0, \quad \kappa_1 > 0, \quad \kappa_2 > 0, \quad a\xi > b^2, \quad \delta\kappa_1 > \eta^2 > 0. \tag{4}$$

In what follows, we obtain the variational formulation of the above mechanical problem. Let $Y = L^2(0,\ell)$, and denote by $(\cdot,\cdot)$ the scalar product in this space, with corresponding norm $\|\cdot\|_Y$. Moreover, let us denote $E = H^1(0,\ell)$ and $V = H_0^2(0,\ell)$, with norms $\|\cdot\|_E$ and $\|\cdot\|_V$, respectively.

Let us denote by $v = \dot{u}$ and $\psi = \dot{\varphi}$ the velocity and porosity speed. Therefore, integrating by parts it leads to the following variational formulation for problem (1), (2) and (3).

Find the velocity field $v : [0,T] \to V$ and the porosity speed $\psi : [0,T] \to V$ such that $v(0) = v_0$, $\psi(0) = \psi_0$, and, for a.e. $t \in (0,T)$ and $w,\ r \in V$,

$$\begin{aligned}&\rho(\dot{v}(t),w) + a(u_x(t),w_x) + \kappa_1(u_{xx}(t),w_{xx}) + a^*(v_x(t),w_x) + \kappa_1^*(v_{xx}(t),w_{xx})\\ &\qquad = b(\varphi_x(t),w) + \eta(\varphi_{xx}(t),w_x),\end{aligned} \tag{5}$$

$$\begin{aligned}&J(\dot{\psi}(t),r) + \delta(\varphi_x(t),r_x) + \kappa_2(\varphi_{xx}(t),r_{xx}) + \xi^*(\psi(t),r) + \xi(\varphi(t),r)\\ &\qquad = -\eta(u_{xx}(t),r_x) - b(u_x(t),r),\end{aligned} \tag{6}$$

where the displacement and the porosity field u and φ are recovered from the equations:

$$u(t) = \int_0^t v(s)\,ds + u_0, \quad \varphi(t) = \int_0^t \psi(s)\,ds + \varphi_0. \tag{7}$$

Proceeding as in [1], we could prove the following result which states the existence of a unique solution to the above problem.

Theorem 2.1 *Assume that the constitutive coefficients satisfy conditions (4). If we also assume that one of the coefficients a^*, κ_1^* or ξ^* is positive, then there exists a unique solution*

$$u,\, \varphi \in \mathcal{C}^2([0,T]; L^2(0,\ell)) \cap \mathcal{C}^1([0,T]; H^2(0,\ell)) \cap \mathcal{C}^0([0,T]; H^4(0,\ell))$$

to problem (1)-(3). Moreover, if we also assume that $a^* \neq 0$, then the energy of the problem decays exponentially.

It was also proved in [1] that, in the hyperviscosity case (i.e. $a^* = \xi^* = 0$), the energy decays in a slow way. Moreover, for the weak viscoporosity case ($a^* = \kappa_1^* = 0$), if $J\kappa_1 = \rho\kappa_2$ then the energy decays exponentially but, on the contrary, the energy decays in a slow way.

3 Fully discrete approximations: an a priori error analysis

In this section, we introduce the fully discrete approximation of variational problem (5)-(7). First, in order to obtain the spatial approximation, we assume that the interval $[0,\ell]$ is divided into M subintervals $a_0 = 0 < a_1 < \ldots < a_M = \ell$ of length $h = a_{i+1} - a_i = \ell/M$. Therefore, to approximate the variational space V, we define the finite dimensional space $V^h \subset V$ given by

$$V^h = \{w^h \in \mathcal{C}^1([0,\ell])\ ;\ w^h_{|_{[a_i,a_{i+1}]}} \in P_3([a_i,a_{i+1}]) \quad i = 0,\ldots,M-1,$$
$$w^h(0) = w^h(\ell) = w^h_x(0) = w^h_x(\ell) = 0\},$$

where $P_3([a_i,a_{i+1}])$ represents the space of polynomials of degree less or equal to three in the subinterval $[a_i,a_{i+1}]$; i.e. the finite element space V^h is made of C^1 and piecewise cubic functions. Here, $h > 0$ denotes the spatial discretization parameter. Furthermore, let the discrete initial data u_0^h, v_0^h, φ_0^h and ψ_0^h be defined as

$$u_0^h = \mathcal{P}^h u_0, \quad v_0^h = \mathcal{P}^h v_0, \quad \varphi_0^h = \mathcal{P}^h \varphi_0, \quad \psi_0^h = \mathcal{P}^h \psi_0,$$

where $\mathcal{P}^h$ is the classical finite element interpolation operator over V^h (see [3]).

Secondly, we consider a uniform partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, with step size $k = T/N$ and nodes $t_n = n\,k$ for $n = 0,1,\ldots,N$.

Therefore, using the well-known implicit Euler scheme, the fully discrete approximations of the above variational problem are the following.

Find the discrete velocity $v^{hk} = \{v_n^{hk}\}_{n=0}^N \subset V^h$ and the discrete porosity speed $\psi^{hk} = \{\psi_n^{hk}\}_{n=0}^N \subset V^h$ such that $v_0^{hk} = v_0^h$, $\psi_0^{hk} = \psi_0^h$ and, for all w^h, $r^h \in V^h$ and $n = 1,\ldots,N$,

$$\rho((v_n^{hk} - v_{n-1}^{hk})/k, w^h) + a((u_n^{hk})_x, w_x^h) + \kappa_1((u_n^{hk})_{xx}, w_{xx}^h) + a^*((v_n^{hk})_x, w_x^h)$$
$$+\kappa_1^*((v_n^{hk})_{xx}, w_{xx}^h) = b((\varphi_n^{hk})_x, w^h) + \eta((\varphi_n^{hk})_{xx}, w_x^h), \tag{8}$$
$$J((\psi_n^{hk} - \psi_{n-1}^{hk})/k, r^h) + \delta((\varphi_n^{hk})_x, r_x^h) + \kappa_2((\varphi_n^{hk})_{xx}, r_{xx}^h) + \xi^*(\psi_n^{hk}, r^h)$$
$$+\xi(\varphi_n^{hk}, r^h) = -\eta((u_n^{hk})_{xx}, r_x^h) - b((u_n^{hk})_x, r^h), \tag{9}$$

where the discrete displacement and the discrete porosity u_n^{hk} and φ_n^{hk} are now recovered from the equations:

$$u_n^{hk} = u_{n-1}^{hk} + k v_n^{hk}, \quad \varphi_n^{hk} = \varphi_{n-1}^{hk} + k\psi_n^{hk}. \tag{10}$$

It is straightforward to obtain that this fully discrete problem has a unique solution applying the well-known Lax Milgram lemma and the required assumptions on the constitutive parameters.

We will prove now a discrete stability property.

Lemma 3.1 *Let the assumptions of Theorem 2.1 hold. Then, the sequences $\{u^{hk}, v^{hk}, \varphi^{hk}, \psi^{hk}\}$, generated by discrete problem (8)-(10), satisfy the stability estimate:*

$$\|v_n^{hk}\|_Y^2 + \|u_n^{hk}\|_V^2 + \|\psi_n^{hk}\|_Y^2 + \|\varphi_n^{hk}\|_V^2 \leq C,$$

where C is a positive constant which is independent of the discretization parameters h and k.

P r o o f. Taking v_n^{hk} as a test function in variational equation (8) we find that

$$\rho((v_n^{hk}-v_{n-1}^{hk})/k, v_n^{hk}) + a((u_n^{hk})_x, (v_n^{hk})_x) + \kappa_1((u_n^{hk})_{xx}, (v_n^{hk})_{xx}) + a^*((v_n^{hk})_x, (v_n^{hk})_x)$$
$$+\kappa_1^*((v_n^{hk})_{xx}, (v_n^{hk})_{xx}) - b((\varphi_n^{hk})_x, (v_n^{hk})) - \eta((\varphi_n^{hk})_{xx}, (v_n^{hk})_x) = 0.$$

Therefore, taking into account that

$$\rho((v_n^{hk}-v_{n-1}^{hk})/k, v_n^{hk}) \geq \frac{\rho}{2k}\Big\{\|v_n^{hk}\|_Y^2 - \|v_{n-1}^{hk}\|_Y^2\Big\},$$
$$a((u_n^{hk})_x, (v_n^{hk})_x) = \frac{a}{2k}\Big\{\|(u_n^{hk})_x\|_Y^2 - \|(u_{n-1}^{hk})_x\|_Y^2 + \|(u_n^{hk}-u_{n-1}^{hk})_x\|_Y^2\Big\},$$
$$\kappa_1((u_n^{hk})_{xx}, (v_n^{hk})_{xx}) = \frac{\kappa_1}{2k}\Big\{\|(u_n^{hk})_{xx}\|_Y^2 - \|(u_{n-1}^{hk})_{xx}\|_Y^2 + \|(u_n^{hk}-u_{n-1}^{hk})_{xx}\|_Y^2\Big\},$$
$$\eta((\varphi_n^{hk})_{xx}, (v_n^{hk})_x) = -\eta((\varphi_n^{hk})_x, (v_n^{hk})_{xx}),$$

and using several times Cauchy's inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2, \quad a, b, \epsilon \in \mathbb{R},\ \epsilon > 0, \tag{11}$$

after some easy algebraic manipulations we find that

$$\frac{\rho}{2k}\Big\{\|v_n^{hk}\|_Y^2 - \|v_{n-1}^{hk}\|_Y^2\Big\} + \frac{a}{2k}\Big\{\|(u_n^{hk})_x\|_Y^2 - \|(v_{n-1}^{hk})_x\|_Y^2 + \|(u_n^{hk}-u_{n-1}^{hk})_x\|_Y^2\Big\}$$
$$+\frac{\kappa_1}{2k}\Big\{\|(u_n^{hk})_{xx}\|_Y^2 - \|(v_{n-1}^{hk})_{xx}\|_Y^2 + \|(u_n^{hk}-u_{n-1}^{hk})_{xx}\|_Y^2\Big\}$$
$$\leq C\Big(\|(\varphi_n^{hk})_x\|_Y^2 + \|v_n^{hk}\|_Y^2\Big) - \eta((\varphi_n^{hk})_x, (v_n^{hk})_{xx}). \tag{12}$$

Proceeding in a similar form we obtain the following estimates for the discrete porosity speed:

$$\frac{J}{2k}\Big\{\|\psi_n^{hk}\|_Y^2 - \|\psi_{n-1}^{hk}\|_Y^2\Big\} + \frac{\delta}{2k}\Big\{\|(\varphi_n^{hk})_x\|_Y^2 - \|(\varphi_{n-1}^{hk})_x\|_Y^2 + \|(\varphi_n^{hk}-\varphi_{n-1}^{hk})_x\|_Y^2\Big\}$$
$$+\frac{\kappa_2}{2k}\Big\{\|(\varphi_n^{hk})_{xx}\|_Y^2 - \|(\varphi_{n-1}^{hk})_{xx}\|_Y^2 + \|(\varphi_n^{hk}-\varphi_{n-1}^{hk})_{xx}\|_Y^2\Big\}$$
$$\leq C\Big(\|(u_n^{hk})_x\|_Y^2 + \|\psi_n^{hk}\|_Y^2\Big) - \eta((u_n^{hk})_{xx}, (\psi_n^{hk})_x). \tag{13}$$

Combining the estimates (12) and (13) we have

$$\frac{\rho}{2k}\Big\{\|v_n^{hk}\|_Y^2 - \|v_{n-1}^{hk}\|_Y^2\Big\} + \frac{J}{2k}\Big\{\|\psi_n^{hk}\|_Y^2 - \|\psi_{n-1}^{hk}\|_Y^2\Big\}$$
$$+\frac{a}{2k}\Big\{\|(u_n^{hk})_x\|_Y^2 - \|(u_{n-1}^{hk})_x\|_Y^2 + \|(u_n^{hk}-u_{n-1}^{hk})_x\|_Y^2\Big\}$$
$$+\frac{\kappa_1}{2k}\Big\{\|(u_n^{hk})_{xx}\|_Y^2 - \|(u_{n-1}^{hk})_{xx}\|_Y^2 + \|(u_n^{hk}-u_{n-1}^{hk})_{xx}\|_Y^2\Big\}$$
$$+\frac{\delta}{2k}\Big\{\|(\varphi_n^{hk})_x\|_Y^2 - \|(\varphi_{n-1}^{hk})_x\|_Y^2 + \|(\varphi_n^{hk}-\varphi_{n-1}^{hk})_x\|_Y^2\Big\}$$
$$+\frac{\kappa_2}{2k}\Big\{\|(\varphi_n^{hk})_{xx}\|_Y^2 - \|(\varphi_{n-1}^{hk})_{xx}\|_Y^2 + \|(\varphi_n^{hk}-\varphi_{n-1}^{hk})_{xx}\|_Y^2\Big\}$$
$$\leq C\Big(\|(\varphi_n^{hk})_x\|_Y^2 + \|v_n^{hk}\|_Y^2 + \|(u_n^{hk})_x\|_Y^2 + \|\psi_n^{hk}\|_Y^2\Big)$$
$$-\eta((\varphi_n^{hk})_x, (v_n^{hk})_{xx}) - \eta((u_n^{hk})_{xx}, (\psi_n^{hk})_x).$$

Now, observing that

$$\eta((\varphi_n^{hk})_x, (v_n^{hk})_{xx}) + \eta((u_n^{hk})_{xx}, (\psi_n^{hk})_x) = \frac{\eta}{k}\Big(((\varphi_n^{hk})_x, (u_n^{hk})_{xx}) - ((\varphi_{n-1}^{hk})_x, (u_{n-1}^{hk})_{xx})$$
$$+((\varphi_n^{hk}-\varphi_{n-1}^{hk})_x, (u_n^{hk}-u_{n-1}^{hk})_{xx})\Big),$$

taking into account that $\delta\kappa_1 > \eta^2$ we have

$$\frac{\delta}{2k}\|(\varphi_n^{hk}-\varphi_{n-1}^{hk})_x\|_Y^2 + \frac{\kappa_1}{2k}\|(u_n^{hk}-u_{n-1}^{hk})_{xx}\|_Y^2 + \frac{\eta}{k}((\varphi_n^{hk}-\varphi_{n-1}^{hk})_x, (u_n^{hk}-u_{n-1}^{hk})_{xx}) \geq 0.$$

Thus, multiplying the previous estimates by k and summing up to n we find that

$$\begin{aligned}&\rho\|v_n^{hk}\|_Y^2+J\|\psi_n^{hk}\|_Y^2+a\|(u_n^{hk})_x\|_Y^2+\kappa_1\|(u_n^{hk})_{xx}\|_Y^2+\delta\|(\varphi_n^{hk})_x\|_Y^2\\&\quad+\kappa_2\|(\varphi_n^{hk})_{xx}\|_Y^2+2\eta((\varphi_n^{hk})_x,(u_n^{hk})_{xx})\\&\leq Ck\sum_{j=1}^n\Big(\|(\varphi_j^{hk})_x\|_Y^2+\|v_j^{hk}\|_Y^2+\|(u_j^{hk})_x\|_Y^2+\|\psi_j^{hk}\|_Y^2\Big)\\&\quad+C\Big(\|v_0^h\|_Y^2+\|\psi_0^h\|_Y^2+\|u_0^h\|_V^2+\|\varphi_0^h\|_V^2\Big).\end{aligned}$$

Keeping in mind again that $\delta\kappa_1>\eta^2$ we obtain

$$\kappa_1\|(u_n^{hk})_{xx}\|_Y^2+\delta\|(\varphi_n^{hk})_x\|_Y^2+2\eta((\varphi_n^{hk})_x,(u_n^{hk})_{xx})\geq C_1\|(u_n^{hk})_{xx}\|_Y^2+C_2\|(\varphi_n^{hk})_x\|_Y^2,$$

for given positive constants C_1 and C_2. Finally, if we use a discrete version of Gronwall's inequality (see [2]) we conclude the discrete stability property. □

In the rest of the section, we will prove some a priori error estimates on the numerical errors $v_n-v_n^{hk}$, $u_n-u_n^{hk}$, $\psi_n-\psi_n^{hk}$ and $\varphi_n-\varphi_n^{hk}$. Moreover, for a continuous function f we use the notation $f_n=f(t_n)$.

Theorem 3.2 *Let the assumptions of Theorem 2.1 still hold. If we denote by (u,v,φ,ψ) the solution to problem (5)-(7) and by $(u^{hk},v^{hk},\varphi^{hk},\psi^{hk})$ the solution to problem (8)-(10) , then we have the following a priori error estimates, for all $w^h=\{w_j^h\}_{j=0}^N$, $r^h=\{r_j^h\}_{j=0}^N\subset V^h$,*

$$\begin{aligned}&\max_{0\leq n\leq N}\Big\{\|v_n-v_n^{hk}\|_Y^2+\|(u_n-u_n^{hk})_x\|_Y^2+\|(u_n-u_n^{hk})_{xx}\|_Y^2+\|\psi_n-\psi_n^{hk}\|_Y^2\\&\quad+\|(\varphi_n-\varphi_n^{hk})_x\|_Y^2+\|(\varphi_n-\varphi_n^{hk})_{xx}\|_Y^2\Big\}\\&\leq Ck\sum_{j=1}^N\Big(\|\dot v_j-(v_j-v_{j-1})/k\|_Y^2+\|\dot u_j-(u_j-u_{j-1})/k\|_V^2+\|v_j-w_j^h\|_V^2\\&\quad+\|\dot\psi_j-(\psi_j-\psi_{j-1})/k\|_Y^2+\|\dot\varphi_j-(\varphi_j-\varphi_{j-1})/k\|_V^2+\|\psi_j-r_j^h\|_V^2\Big)\\&\quad+C\max_{0\leq n\leq N}\|v_n-w_n^h\|_Y^2+C\max_{0\leq n\leq N}\|\psi_n-r_n^h\|_Y^2\\&\quad+\frac{C}{k}\sum_{j=1}^{N-1}\Big(\|v_j-w_j^h-(v_{j+1}-w_{j+1}^h)\|_Y^2+\|\psi_j-r_j^h-(\psi_{j+1}-r_{j+1}^h)\|_Y^2\Big)\\&\quad+C\Big(\|v_0-v_0^h\|_Y^2+\|u_0-u_0^h\|_V^2+\|\psi_0-\psi_0^h\|_Y^2+\|\varphi_0-\varphi_0^h\|_V^2\Big),\end{aligned}$$

where C is a positive constant which does not depend on parameters h and k.

P r o o f. First, we obtain the error estimates for the velocity field. Therefore, we subtract variational equation (5) at time $t=t_n$ for a test function $w=w^h\in V^h\subset V$ and discrete variational equation (8) to obtain

$$\begin{aligned}&\rho(\dot v_n-(v_n^{hk}-v_{n-1}^{hk})/k,w^h)+a((u_n-u_n^{hk})_x,w_x^h)+\kappa_1((u_n-u_n^{hk})_{xx},w_{xx}^h)\\&\quad+a^*((v_n-v_n^{hk})_x,w_x^h)+\kappa_1^*((v_n-v_n^{hk})_{xx},w_{xx}^h)-b((\varphi_n-\varphi_n^{hk})_x,w^h)\\&\quad+\eta((\varphi_n-\varphi_n^{hk})_{xx},w_x^h)=0\quad\forall w^h\in V^h,\end{aligned}$$

and so, we have

$$\begin{aligned}&\rho(\dot v_n-(v_n^{hk}-v_{n-1}^{hk})/k,v_n-v_n^{hk})+a((u_n-u_n^{hk})_x,(v_n-v_n^{hk})_x)\\&\quad+\kappa_1((u_n-u_n^{hk})_{xx},(v_n-v_n^{hk})_{xx})+a^*((v_n-v_n^{hk})_x,(v_n-v_n^{hk})_x)\\&\quad+\kappa_1^*((v_n-v_n^{hk})_{xx},(v_n-v_n^{hk})_{xx})-b((\varphi_n-\varphi_n^{hk})_x,v_n-v_n^{hk})\\&\quad-\eta((\varphi_n-\varphi_n^{hk})_{xx},(v_n-v_n^{hk})_x)\\&=\rho(\dot v_n-(v_n^{hk}-v_{n-1}^{hk})/k,v_n-w^h)+a((u_n-u_n^{hk})_x,(v_n-w^h)_x)\\&\quad+\kappa_1((u_n-u_n^{hk})_{xx},(v_n-w^h)_{xx})+a^*((v_n-v_n^{hk})_x,(v_n-w^h)_x)\\&\quad+\kappa_1^*((v_n-v_n^{hk})_{xx},(v_n-w^h)_{xx})-b((\varphi_n-\varphi_n^{hk})_x,v_n-w^h)\\&\quad-\eta((\varphi_n-\varphi_n^{hk})_{xx},(v_n-w^h)_x)\quad\forall w^h\in V^h.\end{aligned}$$

Taking into account that

$$
\begin{aligned}
&(\dot v_n-(v_n^{hk}-v_{n-1}^{hk})/k, v_n-v_n^{hk})=(\dot v_n-(v_n-v_{n-1})/k, v_n-v_n^{hk})\\
&\qquad+((v_n-v_{n-1})/k-(v_n^{hk}-v_{n-1}^{hk})/k, v_n-v_n^{hk}),\\
&\big((v_n-v_{n-1})/k-(v_n^{hk}-v_{n-1}^{hk})/k, v_n-v_n^{hk}\big)\geq \frac{1}{2k}\Big\{\|v_n-v_n^{hk}\|_Y^2-\|v_{n-1}-v_{n-1}^{hk}\|_Y^2\Big\},\\
&((u_n-u_n^{hk})_x,(v_n-v_n^{hk})_x)\geq((u_n-u_n^{hk})_x,(\dot u_n-(u_n-u_{n-1})/k)_x)\\
&\qquad+\frac{1}{2k}\Big\{\|(u_n-u_n^{hk})_x\|_Y^2-\|(u_{n-1}-u_{n-1}^{hk})_x\|_Y^2\Big\},\\
&((u_n-u_n^{hk})_{xx},(v_n-v_n^{hk})_{xx})\geq((u_n-u_n^{hk})_{xx},(\dot u_n-(u_n-u_{n-1})/k)_{xx})\\
&\qquad+\frac{1}{2k}\Big\{\|(u_n-u_n^{hk})_{xx}\|_Y^2-\|(u_{n-1}-u_{n-1}^{hk})_{xx}\|_Y^2\\
&\qquad+\|(u_n-u_n^{hk}-(u_{n-1}-u_{n-1}^{hk}))_{xx}\|_Y^2\Big\},\\
&-\eta((\varphi_n-\varphi_n^{hk})_{xx},(v_n-v_n^{hk})_x)=\eta((\varphi_n-\varphi_n^{hk})_x,(v_n-v_n^{hk})_{xx}),
\end{aligned}
$$

and using several times Cauchy's inequality (11) it follows

$$
\begin{aligned}
&\frac{\rho}{2k}\Big\{\|v_n-v_n^{hk}\|_Y^2-\|v_{n-1}-v_{n-1}^{hk}\|_Y^2\Big\}+\eta((\varphi_n-\varphi_n^{hk})_x,(v_n-v_n^{hk})_{xx})\\
&\quad+\frac{a}{2k}\Big\{\|(u_n-u_n^{hk})_x\|_Y^2-\|(u_{n-1}-u_{n-1}^{hk})_x\|_Y^2\Big\}\\
&\quad+\frac{\kappa_1}{k}\Big\{\|(u_n-u_n^{hk})_{xx}\|_Y^2-\|(u_{n-1}-u_{n-1}^{hk})_{xx}\|_Y^2+\|(u_n-u_n^{hk}-(u_{n-1}-u_{n-1}^{hk}))_{xx}\|_Y^2\Big\}\\
&\leq C\Big(\|\dot v_n-(v_n-v_{n-1})/k\|_Y^2+\|\dot u_n-(u_n-u_{n-1})/k\|_V^2+\|v_n-w^h\|_V^2\\
&\quad+\|(u_n-u_n^{hk})_x\|_Y^2+\|v_n-v_n^{hk}\|_Y^2+\|(u_n-u_n^{hk})_{xx}\|_Y^2+\|(\varphi_n-\varphi_n^{hk})_x\|_Y^2\\
&\quad+\|(\varphi_n-\varphi_n^{hk})_{xx}\|_Y^2+((v_n-v_{n-1}-(v_n^{hk}-v_{n-1}^{hk}))/k, v_n-w^h)\Big)\quad\forall w^h\in V^h.
\end{aligned}
\tag{14}
$$

Now, proceeding in an analogous way we obtain the following estimates for the porosity speed:

$$
\begin{aligned}
&\frac{J}{2k}\Big\{\|\psi_n-\psi_n^{hk}\|_Y^2-\|\psi_{n-1}-\psi_{n-1}^{hk}\|_Y^2\Big\}+\eta((\psi_n-\psi_n^{hk})_x,(u_n-u_n^{hk})_{xx})\\
&\quad+\frac{\delta}{2k}\Big\{\|(\varphi_n-\varphi_n^{hk})_x\|_Y^2-\|(\varphi_{n-1}-\varphi_{n-1}^{hk})_x\|_Y^2+\|(\varphi_n-\varphi_n^{hk}-(\varphi_{n-1}-\varphi_{n-1}^{hk}))_x\|_Y^2\Big\}\\
&\quad+\frac{\kappa_2}{k}\Big\{\|(\varphi_n-\varphi_n^{hk})_{xx}\|_Y^2-\|(\varphi_{n-1}-\varphi_{n-1}^{hk})_{xx}\|_Y^2\Big\}\\
&\leq C\Big(\|\dot\psi_n-(\psi_n-\psi_{n-1})/k\|_Y^2+\|\dot\varphi_n-(\varphi_n-\varphi_{n-1})/k\|_V^2+\|\psi_n-r^h\|_V^2\\
&\quad+\|(\varphi_n-\varphi_n^{hk})_x\|_Y^2+\|\psi_n-\psi_n^{hk}\|_Y^2+\|(\varphi_n-\varphi_n^{hk})_{xx}\|_Y^2+\|(u_n-u_n^{hk})_x\|_Y^2\\
&\quad+\|(u_n-u_n^{hk})_{xx}\|_Y^2+((\psi_n-\psi_{n-1}-(\psi_n^{hk}-\psi_{n-1}^{hk}))/k,\psi_n-r^h)\Big)\quad\forall r^h\in V^h.
\end{aligned}
\tag{15}
$$

Combining the estimates (14) and (15) we find that

$$
\begin{aligned}
&\frac{\rho}{2k}\Big\{\|v_n-v_n^{hk}\|_Y^2-\|v_{n-1}-v_{n-1}^{hk}\|_Y^2\Big\}\\
&\quad+\frac{a}{2k}\Big\{\|(u_n-u_n^{hk})_x\|_Y^2-\|(u_{n-1}-u_{n-1}^{hk})_x\|_Y^2\Big\}\\
&\quad+\frac{\kappa_1}{k}\Big\{\|(u_n-u_n^{hk})_{xx}\|_Y^2-\|(u_{n-1}-u_{n-1}^{hk})_{xx}\|_Y^2+\|(u_n-u_n^{hk}-(u_{n-1}-u_{n-1}^{hk}))_{xx}\|_Y^2\Big\}\\
&\quad+\frac{J}{2k}\Big\{\|\psi_n-\psi_n^{hk}\|_Y^2-\|\psi_{n-1}-\psi_{n-1}^{hk}\|_Y^2\Big\}\\
&\quad+\frac{\delta}{2k}\Big\{\|(\varphi_n-\varphi_n^{hk})_x\|_Y^2-\|(\varphi_{n-1}-\varphi_{n-1}^{hk})_x\|_Y^2+\|(\varphi_n-\varphi_n^{hk}-(\varphi_{n-1}-\varphi_{n-1}^{hk}))_x\|_Y^2\Big\}\\
&\quad+\frac{\kappa_2}{k}\Big\{\|(\varphi_n-\varphi_n^{hk})_{xx}\|_Y^2-\|(\varphi_{n-1}-\varphi_{n-1}^{hk})_{xx}\|_Y^2\Big\}\\
&\quad+\eta((\varphi_n-\varphi_n^{hk})_x,(v_n-v_n^{hk})_{xx})+\eta((\psi_n-\psi_n^{hk})_x,(u_n-u_n^{hk})_{xx})
\end{aligned}
$$

$$\leq C\Big(\|\dot{v}_n-(v_n-v_{n-1})/k\|_Y^2+\|\dot{u}_n-(u_n-u_{n-1})/k\|_V^2+\|v_n-w^h\|_V^2$$
$$+\|(u_n-u_n^{hk})_x\|_Y^2+\|v_n-v_n^{hk}\|_Y^2+\|(u_n-u_n^{hk})_{xx}\|_Y^2+\|(\varphi_n-\varphi_n^{hk})_x\|_Y^2$$
$$+\|(\varphi_n-\varphi_n^{hk})_{xx}\|_Y^2+((v_n-v_{n-1}-(v_n^{hk}-v_{n-1}^{hk}))/k,v_n-w^h)$$
$$+\|\dot{\psi}_n-(\psi_n-\psi_{n-1})/k\|_Y^2+\|\dot{\varphi}_n-(\varphi_n-\varphi_{n-1})/k\|_V^2+\|\psi_n-r^h\|_V^2$$
$$+\|\psi_n-\psi_n^{hk}\|_Y^2+((\psi_n-\psi_{n-1}-(\psi_n^{hk}-\psi_{n-1}^{hk}))/k,\psi_n-r^h)\Big)\quad\forall w^h,\,r^h\in V^h.$$

Now, observing that

$$\begin{aligned}\eta((\varphi_n-\varphi_n^{hk})_x,(v_n-v_n^{hk})_{xx})+\eta((\psi_n-\psi_n^{hk})_x,(u_n-u_n^{hk})_{xx})\\
=\eta((\varphi_n-\varphi_n^{hk})_x,(\dot{u}_n-(u_n-u_{n-1})/k)_{xx})+\eta((\dot{\varphi}_n-(\varphi_n-\varphi_{n-1})/k)_x,(u_n-u_n^{hk})_{xx})\\
+\frac{\eta}{k}((\varphi_n-\varphi_n^{hk})_x,(u_n-u_{n-1}-(u_n^{hk}-u_{n-1}^{hk}))_{xx})\\
+\frac{\eta}{k}((\varphi_n-\varphi_{n-1}-(\varphi_n^{hk}-\varphi_{n-1}^{hk}))_x,(u_n-u_n^{hk})_{xx}),\end{aligned}$$

$$\begin{aligned}\frac{\eta}{k}((\varphi_n-\varphi_n^{hk})_x,(u_n-u_{n-1}-(u_n^{hk}-u_{n-1}^{hk}))_{xx})\\
+\frac{\eta}{k}((\varphi_n-\varphi_{n-1}-(\varphi_n^{hk}-\varphi_{n-1}^{hk}))_x,(u_n-u_n^{hk})_{xx})\\
=\frac{\eta}{k}\Big\{((\varphi_n-\varphi_n^{hk})_x,(u_n-u_n^{hk})_{xx})-((\varphi_{n-1}-\varphi_{n-1}^{hk})_x,(u_{n-1}-u_{n-1}^{hk})_{xx})\\
+((\varphi_n-\varphi_n^{hk}-(\varphi_{n-1}-\varphi_{n-1}^{hk}))_x,(u_n-u_n^{hk}-(u_{n-1}-u_{n-1}^{hk}))_{xx})\Big\},\end{aligned}$$

and taking into account that (thanks again to the condition $\delta\kappa_1>\eta^2$)

$$\begin{aligned}\frac{\kappa_1}{2k}\|(u_n-u_n^{hk}-(u_{n-1}-u_{n-1}^{hk}))_{xx}\|_Y^2+\frac{\delta}{2k}\|(\varphi_n-\varphi_n^{hk}-(\varphi_{n-1}-\varphi_{n-1}^{hk}))_x\|_Y^2\\
+\frac{\eta}{k}((\varphi_n-\varphi_n^{hk}-(\varphi_{n-1}-\varphi_{n-1}^{hk}))_x,(u_n-u_n^{hk}-(u_{n-1}-u_{n-1}^{hk}))_{xx})\geq 0,\end{aligned}$$

multiplying the above estimates by k and summing up to n it follows that

$$\begin{aligned}&\rho\|v_n-v_n^{hk}\|_Y^2+a\|(u_n-u_n^{hk})_x\|_Y^2+\kappa_1\|(u_n-u_n^{hk})_{xx}\|_Y^2+J\|\psi_n-\psi_n^{hk}\|_Y^2\\
&+\delta\|(\varphi_n-\varphi_n^{hk})_x\|_Y^2+\kappa_2\|(\varphi_n-\varphi_n^{hk})_{xx}\|_Y^2+2\eta((\varphi_n-\varphi_n^{hk})_x,(u_n-u_n^{hk})_{xx})\\
&\leq Ck\sum_{j=1}^{n}\Big(\|\dot{v}_j-(v_j-v_{j-1})/k\|_Y^2+\|\dot{u}_j-(u_j-u_{j-1})/k\|_V^2+\|v_j-w_j^h\|_V^2\\
&+\|(u_j-u_j^{hk})_x\|_Y^2+\|v_j-v_j^{hk}\|_Y^2+\|(u_j-u_j^{hk})_{xx}\|_Y^2+\|(\varphi_j-\varphi_j^{hk})_x\|_Y^2\\
&+\|(\varphi_j-\varphi_j^{hk})_{xx}\|_Y^2+((v_j-v_{j-1}-(v_j^{hk}-v_{j-1}^{hk}))/k,v_j-w_j^h)\\
&+\|\dot{\psi}_j-(\psi_j-\psi_{j-1})/k\|_Y^2+\|\dot{\varphi}_j-(\varphi_j-\varphi_{j-1})/k\|_V^2+\|\psi_j-r_j^h\|_V^2\\
&+\|\psi_j-\psi_j^{hk}\|_Y^2+((\psi_j-\psi_{j-1}-(\psi_j^{hk}-\psi_{j-1}^{hk}))/k,\psi_j-r_j^h)\Big)\\
&+C\Big(\|v_0-v_0^h\|_Y^2+\|u_0-u_0^h\|_V^2+\|\psi_0-\psi_0^h\|_Y^2+\|\varphi_0-\varphi_0^h\|_V^2\Big)\quad\forall w^h,\,r^h\in V^h.\end{aligned}$$

Using again the condition $\delta\kappa_1>\eta^2$ we find that there exist two positive constants C_1 and C_2 such that

$$\begin{aligned}\kappa_1\|(u_n-u_n^{hk})_{xx}\|_Y^2+\delta\|(\varphi_n-\varphi_n^{hk})_x\|_Y^2+2\eta((\varphi_n-\varphi_n^{hk})_x,(u_n-u_n^{hk})_{xx})\\
\geq C_1\|(u_n-u_n^{hk})_{xx}\|_Y^2+C_2\|(\varphi_n-\varphi_n^{hk})_x\|_Y^2.\end{aligned}$$

Keeping in mind that

$$k\sum_{j=1}^{n}(v_j - v_j^{hk} - (v_{j-1} - v_{j-1}^{hk}), v_j - w_j^h) = (v_n - v_n^{hk}, v_n - w_n^h) + (v_0^h - v_0, v_1 - w_1^h)$$
$$+\sum_{j=1}^{n-1}(v_j - v_j^{hk}, v_j - w_j^h - (v_{j+1} - w_{j+1}^h)),$$
$$k\sum_{j=1}^{n}(\psi_j - \psi_j^{hk} - (\psi_{j-1} - \psi_{j-1}^{hk}), \psi_j - r_j^h) = (\psi_n - \psi_n^{hk}, \psi_n - r_n^h) + (\psi_0^h - \psi_0, \psi_1 - r_1^h)$$
$$+\sum_{j=1}^{n-1}(\psi_j - \psi_j^{hk}, \psi_j - r_j^h - (\psi_{j+1} - r_{j+1}^h)),$$

and applying a discrete version of Gronwall's inequality (see, again, [2]) we conclude the desired a priori error estimates. □

Remark 3.3 The estimates provided in the above theorem can be used to obtain the convergence order of the approximations given by discrete problem (8)-(10). Hence, as an example, if we assume the additional regularity:

$$u, \varphi \in H^3(0,T;Y) \cap H^2(0,T;V) \cap \mathcal{C}^1([0,T]; H^3(0,\ell)), \tag{16}$$

we obtain the linear convergence of the algorithm applying some results on the approximation by finite elements (see [3]) and previous estimates already derived in [2]. That is, we can prove that there exists a positive constant $C > 0$, independent of the discretization parameters h and k, such that

$$\max_{0\le n\le N}\left\{\|v_n - v_n^{hk}\|_Y + \|u_n - u_n^{hk}\|_V + \|\psi_n - \psi_n^{hk}\|_Y + \|\varphi_n - \varphi_n^{hk}\|_V\right\} \le C(h+k).$$

4 Numerical results

In this section, we present some numerical experiments to show the accuracy of the algorithm proved before and the behavior of the solution in two distinct cases.

4.1 Numerical convergence

We start by showing the numerical convergence of the algorithm. For this purpose we solve the fully discrete problem implemented in Matlab. In order to be able to compare the results with a known analytical solution, we manufacture one of the form:

$$u(x,t) = (1-x)^3 x^3\, e^t, \quad \varphi(x,t) = (1-x)^3 x^3\, e^t \quad \forall (x,t) \in (0,1)\times(0,1).$$

Then, we add to system (1) supply terms F and G in both variables (only for this particular study), such that

$$F = a u_{xx} + b\varphi_x - \kappa_1 u_{xxxx} - \eta\varphi_{xxx} - \kappa_1^* \dot{u}_{xxxx} + a^* \dot{u}_{xx} - \rho\ddot{u},$$
$$G = \eta u_{xxx} - b u_x + \delta\varphi_{xx} - \xi\varphi - \kappa_2\varphi_{xxxx} - \xi^*\varphi - J\ddot{\varphi},$$

and thus the established solution is verified. We choose the following parameters for this example:

$$\ell = 1, \quad \rho = 1, \quad \kappa_1 = 1, \quad a = 1, \quad b = 1, \quad \eta = 1, \quad \kappa_1^* = 1,$$
$$a^* = 1, \quad J = 1, \quad \kappa_2 = 1, \quad \delta = 2, \quad \xi = 2.$$

We note that the variational formulation of this modified problem is rather similar to problem (5)-(6).

The numerical errors are then estimated as

$$\max_{0\le n\le N}\Big\{\|v_n - v_n^{hk}\|_Y + \|(u_n - u_n^{hk})_x\|_Y + \|(u_n - u_n^{hk})_{xx}\|_Y + \|\psi_n - \psi_n^{hk}\|_Y$$
$$+\|(\varphi_n - \varphi_n^{hk})_x\|_Y + \|(\varphi_n - \varphi_n^{hk})_{xx}\|_Y\Big\},$$

and they are shown in Table 1 for some values of the discretization parameters h and k. The main diagonal of these errors is plotted in Fig. 1, with $h+k$ in the horizontal axis. It can be seen that the linear convergence of the algorithm that was proved before appears in the numerical example.

$h\downarrow\ k\rightarrow$	$\mathbf{1}\times10^{-1}$	$\mathbf{5}\times10^{-2}$	$\mathbf{1}\times10^{-2}$	$\mathbf{5}\times10^{-3}$	$\mathbf{1}\times10^{-3}$	$\mathbf{1}\times10^{-4}$
$\mathbf{2}\times10^{-2}$	0.33338	8.71117×10^{-2}	6.25846×10^{-3}	3.69119×10^{-3}	2.86547×10^{-3}	2.83284×10^{-3}
$\mathbf{1}\times10^{-2}$	0.330723	8.44552×10^{-2}	3.60285×10^{-3}	1.03572×10^{-3}	2.10128×10^{-4}	1.77524×10^{-4}
$\mathbf{5}\times10^{-3}$	0.330557	8.4289×10^{-2}	3.43667×10^{-3}	8.69558×10^{-4}	4.40456×10^{-5}	1.13862×10^{-5}
$\mathbf{2}\times10^{-3}$	0.330546	8.42775×10^{-2}	3.42835×10^{-3}	8.56855×10^{-4}	3.34037×10^{-5}	1.21044×10^{-6}
$\mathbf{1.4}\times10^{-3}$	0.330552	8.42705×10^{-2}	3.42487×10^{-3}	8.64312×10^{-4}	3.38834×10^{-5}	9.93172×10^{-9}

Table 1 Numerical errors ($\times1000$) for some values of the discretization parameters.

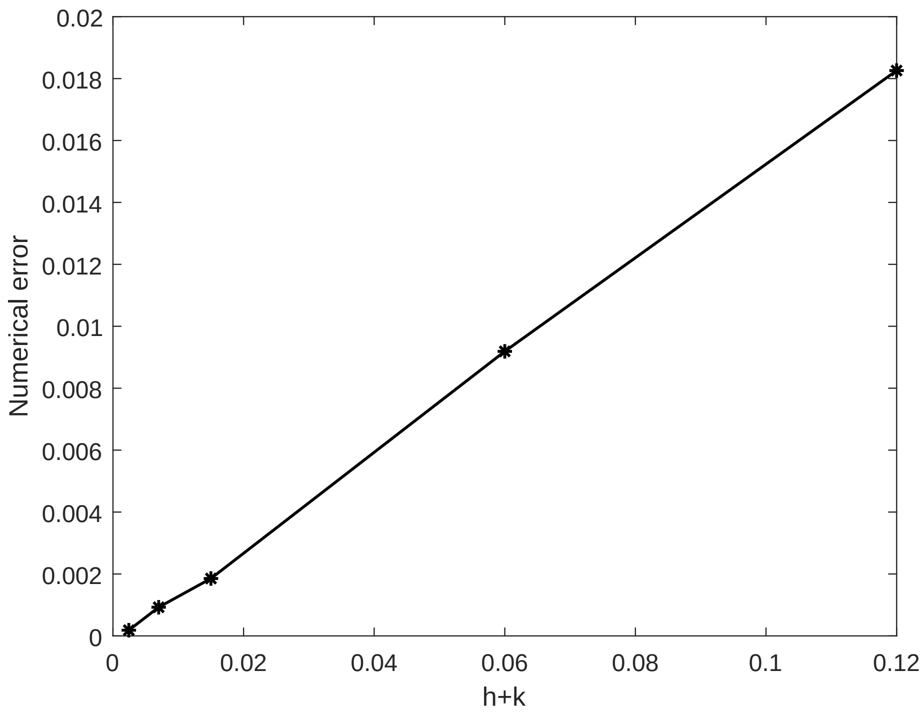

Fig. 1 Linear convergence of the algorithm.

4.2 Damping behavior in a viscoelastic case

In this section, we study differences in the behavior that appears for high and low values of the dissipation parameter. In particular, we consider the porous-viscoelastic case (since this phenomenon is similar in the porous-hyperviscoelastic one). We note that the viscoporosity case would lead to similar results.

For this study we consider the following parameters:

$$\ell=1,\quad \rho=1,\quad \kappa_1=200,\quad a=1,\quad b=1,\quad \eta=1,\quad a^*=1,\quad J=1,$$
$$\kappa_2=1,\quad \delta=2,\quad \xi=2,\quad \xi^*=0,\quad \delta^*=0.$$

The initial conditions are given by

$$u(x,0)=(1-x)^3x^3,\quad \varphi(x,0)=(1-x)^3x^3\quad \forall x\in(0,1).$$

Both cases are solved with a mesh size of $h=0.025$ and a time step of $k=10^{-6}$.

In the numerical results shown below we plot the discrete energy of the system given by

$$E_n^{hk}=\frac{1}{2}\int_0^\ell \rho(v_n^{hk})^2+J(\psi_n^{hk})^2+a((u_n^{hk})_x)^2+\xi(\varphi_n^{hk})^2+\kappa_1((u_n^{hk})_{xx})^2$$
$$+\delta((\varphi_n^{hk})_x)^2+\kappa_2((\varphi_n^{hk})_{xx})^2+2b(u_n^{hk})_x\varphi_n^{hk}+2\eta(u_n^{hk})_{xx}(\varphi_n^{hk})_x\,dx.$$

The first case we show is the one with $\kappa_1^* = 10$. In Fig. 2 we show the evolution of both variables at the center of the domain ($x = 0.5$) with time (left axis), along with the energy evolution (right axis). We can see that the evolution of both the energy and the displacements are smooth and monotonically decreasing because of the effect of dissipation. The absence of oscillations with time indicates that the dissipation is high enough to obtain an overdamped system. The porosity shows a more erratic behavior due to the coupling between both variables.

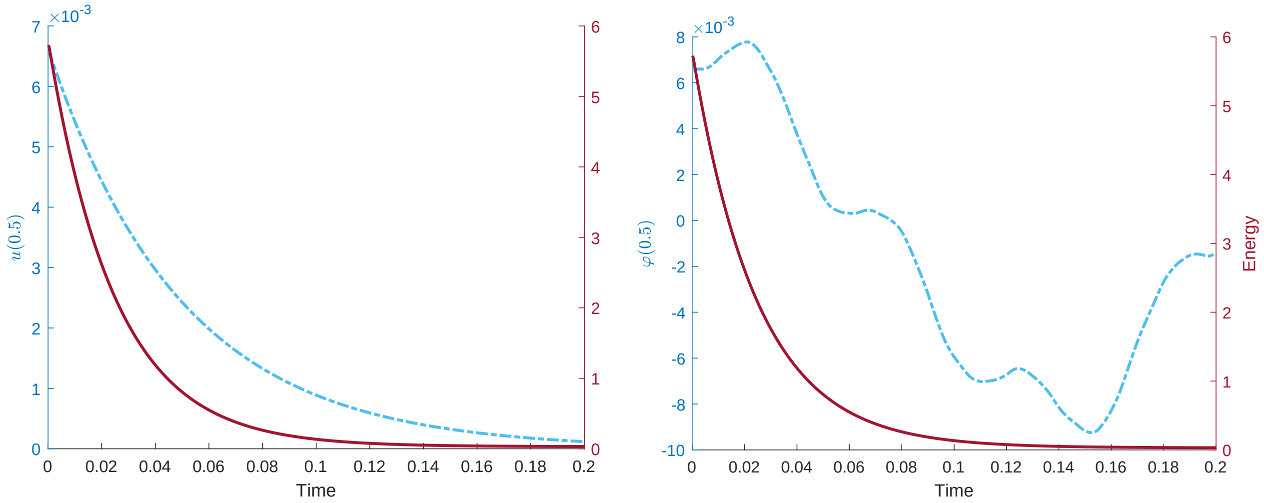

Fig. 2 On the left: evolution of $u(0.5, t)$ (dashed in blue, left axis) and energy (solid red, right axis). On the right: $\varphi(0.5, t)$ variable (dashed in blue, left axis) and energy (solid red, right axis).

Secondly, we change the value of the dissipation to $\kappa_1^* = 0.1$. Fig. 3 shows the evolution of both variables at the center of the domain, as before. Now, the behavior of both displacement field u and the energy is different. Regarding the displacements, we can see oscillations with time. This is caused by the smaller dissipation value. These oscillations in the displacements cause the energy to present some flat regions, when the displacement is reaching a maximum or a minimum. The qualitative behavior of the porosity is similar to the previous case.

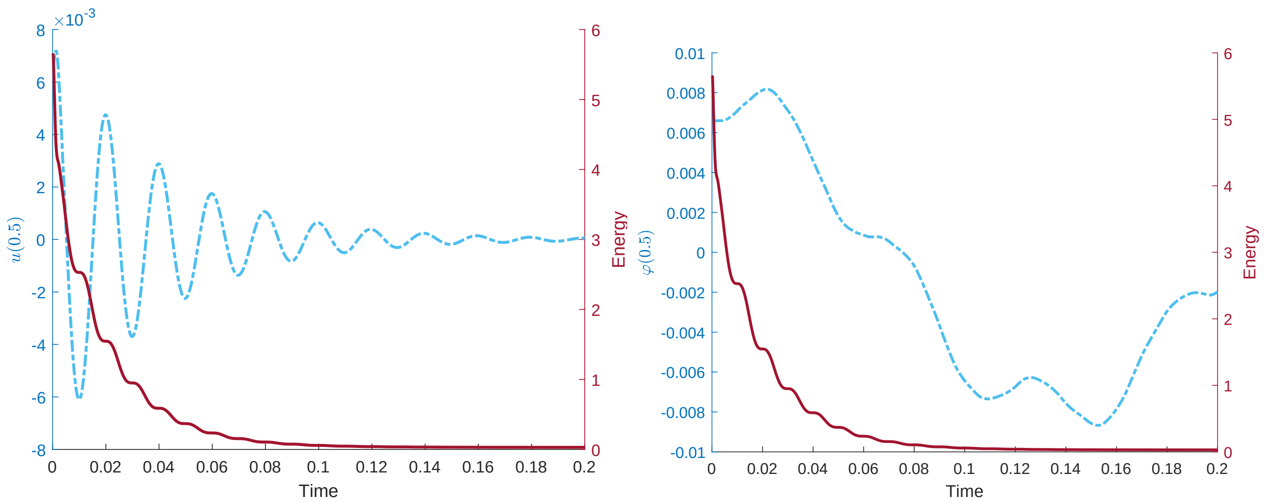

Fig. 3 On the left: evolution of $u(0.5, t)$ (dashed in blue, left axis) and energy (solid red, right axis). On the right: $\varphi(0.5, t)$ variable (dashed in blue, left axis) and energy (solid red, right axis).

Acknowledgements

The authors thank the two anonymous reviewers whose comments have improved the final quality of the article.

The work of J.R. Fernández has been partially supported by Ministerio de Ciencia, Innovación y Universidades under the research project PGC2018-096696-B-I00 (FEDER, UE). The work of A. Magaña and R. Quintanilla was supported by project *Análisis Matemático Aplicado a la Termomecánica* (PID2019-105118GB-I00) of the Spanish Ministry of Science, Innovation and Universities.

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