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Dual-phase-lag heat conduction with microtemperatures

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Abstract

In this paper, we propose a system of equations governing the dual-phase-lag heat conduction with microtemperatures. Several conditions on the coefficients are imposed so that the energy of the system is positive definite and dissipative. On this base we prove the well-posedness and exponential stability of the system by means of the semigroup theory and frequency domain method.

Key words: dual-phase-lag heat conduction, microtemperatures, exponential stability.
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1 Introduction

Heat conduction is usually based on the Fourier law. This constitutive equation express the heat flux vector as a linear form of the gradient of temperature. Unfortunately, this assumption gives rise a paradox because the thermal waves propagates instantaneously. This fact violates *the causality principle*[1]. As a consequence, many scientists have been interested to propose an alternative law for the heat flux vector. The most famous of these alternatives is the one proposed by Cattaneo and Maxwell[3]. The introduction of a relaxation parameter allows to overcome the drawback proposed by the Fourier Law because it brings to a damped hyperbolic equation. Other propositions have been considered in recent years. We can recall the ones of Green and Naghdi[7–9] or the ones proposed by Gurtin and several co-workers[4–6]. In this paper, we want to work with the proposition suggested by Tzou[26]. Some theories can be developed from it. The introduction of two delay parameters are considered and the Cattaneo and Maxwell law can be seen as a particular case. We here pay attention to the constitutive equation[2]

$$q_i + \tau_1 \dot{q}_i + \frac{\tau_1^2}{2} \ddot{q}_i = k(\theta_{,i} + \tau_2 \dot{\theta}_{,i}), \tag{1.1}$$

where q_i is the heat flux vector, θ is the temperature, k is the thermal conductivity(usually assumed positive) and τ_1, τ_2 are two positive constants. It is worth recalling that the theory obtained by the

1. The principle of causality suggests that every effect must have a prior cause. For the Fourier's law the thermal waves propagate instantaneously which is an effect without prior causes.

2. In this paper, the sub-indices mean the components of a vector or tensor and a sub-index following a comma means the partial derivative with respect the corresponding direction.

combination of this equation with the heat equation

$$a\dot{\theta} = q_{i,i}, \quad a > 0, \tag{1.2}$$

has attracted a big interest in the recent years and a quantity of contributions for this theory became huge. We recall some of them [1, 2, 13, 19, 22, 23]. In fact, all these theories have been extended to consider the thermoelastic context [17, 20, 21, 24] (see also [11, 12, 16, 25]).

The study of materials with microstructure was considered in the past century and it has a big interest in the present days. Scientists as Eringen, Maugin or Iesan have contributed in a relevant way to clarify the knowledge of these materials. A particular class of these materials corresponds to assume that the microstructure involves *microtemperatures.* In view of the applicability of these materials many people have been involved in the study of materials with microtemperatures. But when one looks to the equations determining the evolution of the microtemperatures one finds a parabolic system of equations and therefore the causality principle is violated by the microthermal waves. Therefore if we want a theory of heat conduction with microtemperatures satisfying the causality principle the system of field equations corresponding to the microtemperatures should be modified. If we want to work in the dual-phase-lag theory a modification similar to (1.1) should be considered the microtemperatures.

Our contribution is addressed in this direction. We want to propose a theory of heat conduction with microtemperatures compatible with the equation (1.1). Sufficient conditions on the parameters defining the materials are imposed to guarantee the existence and the exponential decay for the dual-phase-lag heat conduction with microtemperatures.

In this section, we consider a three-dimensional heat conducting solid determined by a bounded domain B. The evolution equations are given by [14, 15]

$$\rho T_0 \dot{\eta} = q_{i,i}, \quad \rho \dot{\varepsilon}_i = q_{ij,j} + q_i - Q_i$$

and the constitutive equations are

$$\begin{aligned}
\rho\eta &= a\theta, \\
\rho\varepsilon_i &= -bT_i, \\
q_i &= k\theta_{,i} + k_1 T_i, \\
q_{ij} &= -k_4 T_{r,r}\delta_{ij} - k_5 T_{j,i} - k_6 T_{i,j}, \\
Q_i &= (k - k_3)Q_{ji} + (k - k_2)T_i.
\end{aligned}$$

Here, η is the entropy, ε_i is the first heat flux moment tensor, Q_i is the microheat flux average, T_0 (is given in the evolution equations) is the reference temperature, T_i are the microtemperatures and a, b, k and k_i are real numbers.

In the classical theory it is usual to assume(see [10, 14, 15])

$$k \geq 0, \quad 3k_4 + k_5 + k_6 \geq 0, \quad k_6 + k_5 \geq 0,$$
$$k_6 - k_5 \geq 0 \quad and \quad (k_1 + T_0 k_3)^2 \leq 4T_0 k k_2.$$

In order to make the mathematical analysis more transparent and the notation easier we assume from now on that the reference temperature T_0 is equal to one. We now propose the natural counterpart to the equation (1.1) for the case of the microtemperatures. We change the constitutive equations for q_i, q_{ij} and Q_i in the following way[3]

$$\begin{aligned}
(1 + \tau_1 \frac{\partial}{\partial t} + \frac{\tau_1^2}{2}\frac{\partial^2}{\partial t^2})q_i &= (1 + \tau_2 \frac{\partial}{\partial t})(k\theta_{,i} + k_1 T_i), \\
(1 + \tau_1 \frac{\partial}{\partial t} + \frac{\tau_1^2}{2}\frac{\partial^2}{\partial t^2})q_{ij} &= (1 + \tau_2 \frac{\partial}{\partial t})(-k_4 T_{r,r}\delta_{ij} - k_5 T_{j,i} - k_6 T_{i,j}), \\
(1 + \tau_1 \frac{\partial}{\partial t} + \frac{\tau_1^2}{2}\frac{\partial^2}{\partial t^2})Q_i &= (1 + \tau_2 \frac{\partial}{\partial t})\big((k - k_3)\theta_{,i} + (k_1 - k_2)T_i\big).
\end{aligned}$$

3. We could consider a more general case when the relaxation parameters for the macrostrucuture and the microstructure are different. As this is a first contribution in this line we restrict our attention to the easier case.

If we substitute our constitutive equations into the evolution equations we obtain the system

$$a(\frac{\tau_1^2}{2}\dddot{\theta}+\tau_1\ddot{\theta}+\dot{\theta})=k(\theta_{,ii}+\tau_2\dot{\theta}_{,ii})+k_1(T_{i,i}+\tau_2\dot{T}_{i,i}), \tag{1.3}$$

$$\begin{aligned} b(\frac{\tau_1^2}{2}\dddot{T}_i+\tau_1\ddot{T}_i+\dot{T}_i)&=k_6(T_{i,jj}+\tau_2\dot{T}_{i,jj})+(k_4+k_5)(T_{j,ji}+\tau_2\dot{T}_{j,ji})\\ &\quad -k_2(T_i+\tau_2\dot{T}_i)-k_3(\theta_{,i}+\tau_2\dot{\theta}_{,i}). \end{aligned} \tag{1.4}$$

We need to propose initial and boundary conditions to complete our problem. We will assume null Dirichlet boundary conditions

$$\theta(\mathbf{x},t)=T_i(\mathbf{x},t)=0,\quad \mathbf{x}\in\partial B$$

and the initial conditions

$$\theta(\mathbf{x},0)=\theta^0(\mathbf{x}),\quad \dot{\theta}(\mathbf{x},0)=\vartheta^0(\mathbf{x}),\quad \ddot{\theta}(\mathbf{x},0)=\zeta^0(\mathbf{x}),$$

$$\mathbf{T}(\mathbf{x},0)=\mathbf{T}^0(\mathbf{x}),\quad \dot{\mathbf{T}}(\mathbf{x},0)=\mathbf{S}^0(\mathbf{x}),\quad \ddot{\mathbf{T}}(\mathbf{x},0)=\mathbf{R}^0(\mathbf{x}),\quad \mathbf{x}\in B.$$

The first question we need to clarify is what are the material conditions to impose in order to guarantee the stability of the solutions determined by our problem. We first recall that in the classical case the signs of k_1 and k_3 can be different in general. We note that to study this problem seems very difficult and at this moment we relax our problem assuming that k_1 and k_3 have the same sign. In fact we will assume that they are positive. However, the case when both coefficients are negative the analysis is similar. To simplify the notation we will write

$$\tilde{f}=f+\tau_1\dot{f}+\frac{\tau_1^2}{2}\ddot{f}$$

and denote

$$\begin{aligned} \Sigma_1 \;=\;& kk_3(\tau_1+\tau_2)\theta_{,i}\theta_{,i}+kk_3\frac{\tau_1^2\tau_2}{2}\dot{\theta}_{,i}\dot{\theta}_{,i}+k_1k_2(\tau_1+\tau_2)T_iT_i+k_1k_2\frac{\tau_1^2\tau_2}{2}\dot{T}_i\dot{T}_i+kk_3\tau_1^2\theta_{,i}\dot{\theta}_{,i}+k_1k_2T_i\dot{T}_i\\ &+2k_1k_3(\tau_1+\tau_2)\theta_{,i}T_i+k_1k_3\tau_1^2\theta_{,i}\dot{T}_i+k_1k_3\tau_1^2\tau_2\dot{\theta}_{,i}T_i+k_1k_3\tau_1^2T_i\dot{\theta}_{,i}, \end{aligned}$$

$$\begin{aligned} \Sigma_2 \;=\;& k_1k_6\big((\tau_1+\tau_2)T_{i,j}T_{i,j}+\frac{\tau_1^2\tau_2}{2}\dot{T}_{i,j}\dot{T}_{i,j}+\tau_1^2T_{i,j}\dot{T}_{i,j}\big)+k_1k_4\big((\tau_1+\tau_2)T_{i,i}T_{j,j}+\frac{\tau_1^2\tau_2}{2}\dot{T}_{i,i}\dot{T}_{j,j}\\ &+\tau_1^2\dot{T}_{i,i}T_{j,j}\big)+k_1k_5\big((\tau_1+\tau_2)T_{i,j}T_{j,i}+\frac{\tau_1^2\tau_2}{2}\dot{T}_{i,j}\dot{T}_{j,i}+\tau_1^2\dot{T}_{i,j}T_{j,i}\big), \end{aligned}$$

$$D_1=kk_3\big(\theta_{,i}\theta_{,i}+(\tau_1\tau_2-\frac{\tau_1^2}{2})\dot{\theta}_{,i}\dot{\theta}_{,i}\big)+k_1k_2\big(T_iT_i+(\tau_1\tau_2-\frac{\tau_1^2}{2})\dot{T}_i\dot{T}_i\big)+2k_1k_3\big(\theta_{,i}T_i+(\tau_1\tau_2-\frac{\tau_1^2}{2})\dot{T}_i\dot{\theta}_{,i}\big),$$

$$D_2=k_1k_6\big(T_{i,j}T_{i,j}+(\tau_1\tau_2-\frac{\tau_1^2}{2})\dot{T}_{i,j}\dot{T}_{i,j}\big)+k_1k_4\big(T_{i,i}T_{j,j}+(\tau_1\tau_2-\frac{\tau_1^2}{2})\dot{T}_{i,i}\dot{T}_{j,j}\big)+k_1k_5\big(T_{i,j}T_{j,i}+(\tau_1\tau_2-\frac{\tau_1^2}{2})\dot{T}_{i,j}\dot{T}_{j,i}\big),$$

$$E(t)=\frac{1}{2}\int_B[k_3a\tilde{\theta}^2+k_3b\tilde{T}_i\tilde{T}_i+\Sigma_1+\Sigma_2]dv,$$

$$\mathcal{D}(t)=\int_B(D_1+D_2)dv,$$

we have

$$E(t)+\int_0^t\mathcal{D}(\xi)d\xi=E(0).$$

Throughout this paper, we assume the following condition on the system parameters

$$\tau_2>\frac{\tau_1}{2}>0,\quad a,b,k,k_1,k_3,k_4>0,\quad kk_2>k_1k_3,\quad k_6+k_5>0,\quad k_6-k_5>0. \tag{1.5}$$

These conditions are obtained from a mathematical point of view. We note that the conditions on the coefficients k_i are more restrictive than in the classical theory. An empirical study should clarify the suitable choice of conditions, however, as far as the authors know there are no studies of this kind yet.

In the remaining part of this section, we prove that Σ_1, Σ_2, D_1 and D_2 are positive definite whenever condition (1.5) holds. This amounts to show that the following matrices

$$A = \begin{pmatrix} kk_3(\tau_1+\tau_2) & \frac{1}{2}kk_3\tau_1^2 & k_1k_3(\tau_1+\tau_2) & \frac{1}{2}k_1k_3\tau_1^2 \\ \frac{1}{2}kk_3\tau_1^2 & \frac{1}{2}kk_3\tau_1^2\tau_2 & \frac{1}{2}k_1k_3\tau_1^2 & \frac{1}{2}k_1k_3\tau_1^2\tau_2 \\ k_1k_3(\tau_1+\tau_2) & \frac{1}{2}k_1k_3\tau_1^2 & k_1k_2(\tau_1+\tau_2) & \frac{1}{2}k_1k_2\tau_1^2 \\ \frac{1}{2}k_1k_3\tau_1^2 & \frac{1}{2}k_1k_3\tau_1^2\tau_2 & \frac{1}{2}k_1k_2\tau_1^2 & \frac{1}{2}k_1k_2\tau_1^2\tau_2 \end{pmatrix}$$

$$B = \begin{pmatrix} kk_3 & 0 & k_1k_3 & 0 \\ 0 & kk_3(\tau_1\tau_2 - \frac{\tau_1^2}{2}) & 0 & k_1k_3(\tau_1\tau_2 - \frac{\tau_1^2}{2}) \\ k_1k_3 & 0 & k_1k_2 & 0 \\ 0 & k_1k_3(\tau_1\tau_2 - \frac{\tau_1^2}{2}) & 0 & k_1k_2(\tau_1^2\tau_2 - \frac{\tau_1^2}{2}) \end{pmatrix}$$

$$C = \begin{pmatrix} k_6(\tau_1+\tau_2) & k_6\frac{\tau_1^2}{2} & k_5(\tau_1+\tau_2) & k_5\frac{\tau_1^2}{2} \\ k_6\frac{\tau_1^2}{2} & k_6\frac{\tau_1^2\tau_2}{2} & k_5\frac{\tau_1^2}{2} & k_5\frac{\tau_1^2\tau_2}{2} \\ k_5(\tau_1+\tau_2) & k_5\frac{\tau_1^2}{2} & k_6(\tau_1+\tau_2) & k_6\frac{\tau_1^2}{2} \\ k_5\frac{\tau_1^2}{2} & k_5\frac{\tau_1^2\tau_2}{2} & k_6\frac{\tau_1^2}{2} & k_6\frac{\tau_1^2\tau_2}{2} \end{pmatrix}$$

$$E = \begin{pmatrix} (k_4+k_5+k_6)(\tau_1+\tau_2) & k_4\frac{\tau_1+\tau_2}{2} & k_4\frac{\tau_1+\tau_2}{2} & \frac{k_4+k_5+k_6}{2}\tau_1^2 & k_4\frac{\tau_1^2}{2} & k_4\frac{\tau_1^2}{2} \\ k_4\frac{\tau_1+\tau_2}{2} & (k_4+k_5+k_6)(\tau_1+\tau_2) & \frac{k_4+k_5+k_6}{2}\tau_1^2 & k_4\frac{\tau_1^2}{2} & k_4\frac{\tau_1+\tau_2}{2} & k_4\frac{\tau_1^2}{2} \\ k_4\frac{\tau_1+\tau_2}{2} & k_4\frac{\tau_1+\tau_2}{2} & (k_4+k_5+k_6)(\tau_1+\tau_2) & k_4\frac{\tau_1^2}{2} & k_4\frac{\tau_1^2}{2} & \frac{k_4+k_5+k_6}{2}\tau_1^2 \\ \frac{k_4+k_5+k_6}{2}\tau_1^2 & k_4\frac{\tau_1^2}{2} & k_4\frac{\tau_1^2}{2} & \frac{k_4+k_5+k_6}{2}\tau_1^2\tau_2 & k_4\frac{\tau_1^2\tau_2}{2} & k_4\frac{\tau_1^2\tau_2}{2} \\ k_4\frac{\tau_1^2}{2} & \frac{k_4+k_5+k_6}{2}\tau_1^2 & k_4\frac{\tau_1^2}{2} & k_4\frac{\tau_1^2\tau_2}{2} & \frac{k_4+k_5+k_6}{2}\tau_1^2\tau_2 & k_4\frac{\tau_1^2\tau_2}{2} \\ k_4\frac{\tau_1^2}{2} & k_4\frac{\tau_1^2}{2} & \frac{k_4+k_5+k_6}{2}\tau_1^2 & k_4\frac{\tau_1^2\tau_2}{2} & k_4\frac{\tau_1^2\tau_2}{2} & \frac{k_4+k_5+k_6}{2}\tau_1^2\tau_2 \end{pmatrix}$$

are positive definite.

Lemma 1.1. *Under the condition (1.5), the matrices A, B, C and E are positive definite.*

Proof. To prove that these matrices are positive definite we will show that the determinate of the principal minor matrices are positive. Firstly, we consider the matrix A. We have

$$Det(A_2) = \frac{1}{4}k^2k_3^2\tau_1^2(2\tau_2^2 + 2\tau_1\tau_2 - \tau_1^2) > 0,$$

$$Det(A_3) = \frac{1}{4}(kk_2 - k_1k_3)kk_1k_3^2(2\tau_2^3 + 4\tau_1\tau_2^2 + \tau_1^2\tau_2 - \tau_1^3) > 0,$$

$$Det(A) = \frac{1}{16}k_1^2k_3^2(kk_2 - k_1k_3)^2\tau_1^4(2\tau_2^2 + 2\tau_1\tau_2 - \tau_1^2)^2 > 0.$$

Secondly, we consider the matrix B. We have

$$Det(B_3) = kk_1k_3^2(kk_2 - k_1k_3)(\tau_1\tau_2 - \frac{\tau_1^2}{2}) > 0,$$

$$Det(B) = \frac{1}{4}k_1^2k_3^2(kk_2 - k_1k_3)^2\tau_1^2(\tau_1 - 2\tau_2)^2 > 0.$$

Thirdly, we consider the matrix C. We have

$$Det(C_2) = \frac{1}{4}k_6^2\tau_1^2(2\tau_2^2 + 2\tau_1\tau_2 - \tau_1^2) > 0,$$

$$Det(C_3) = \frac{1}{4}(k_6 + k_5)(k_6 - k_5)\tau_1^2(\tau_1 + \tau_2)(2\tau_2 + 2\tau_1\tau_2 - \tau_1^2) > 0,$$

$$Det(C) = \frac{1}{16}(k_6^2 - k_5^2)^2\tau_1^4(2\tau_2^2 + 2\tau_1\tau_2 - \tau_1^2) > 0.$$

Now, we consider the matrix E. We get

$$Det(E_2) = \frac{1}{4}(k_4 + 2(k_5 + k_6))(3k_4 + 2(k_5 + k_6))(\tau_1 + \tau_2)^2 > 0,$$

$$Det(E_3) = \frac{1}{4}(2k_4 + k_6 + k_5)(k_4 + 2(k_6 + k_5))^2(\tau_1 + \tau_2)^3 > 0,$$

$$\begin{aligned} Det(E_4) \quad &= \quad -\frac{1}{16}(k_4 + k_5 + k_6)\big(k_4 + 2(k_6 + k_5)\big)\tau_1^2(\tau_1 + \tau_2)^2\Bigg(k_4^2(\tau_1 - 2\tau_2)(3\tau_1 + 2\tau_2) \\ &\quad + \big(5k_4 + 2(k_5 + k_6)\big)(k_5 + k_6)(\tau_1^2 - 2\tau_1\tau_2 - 2\tau_2^2)\Bigg) > 0, \end{aligned}$$

$$\begin{aligned} Det(E_5) \quad &= \quad \frac{1}{16}(k_5 + k_6)\tau_1^4(\tau_1 + \tau_2)\Bigg(\Big((3\tau_1 + 2\tau_2)(2\tau_2 - \tau_1)\tau_2(\tau_1 + \tau_2)k_4^4 + (k_5 + k_6)k_4^3(3\tau_1^4 - 18\tau_1^3\tau_2 \\ &\quad + 2\tau_1^2\tau_2^2 + 40\tau_1\tau_2^3 + 20\tau_2^4) + (k_5 + k_6)^2k_4^2(8\tau_1^4 - 33\tau_1^3\tau_2 + 66\tau_1\tau_2^3 + 33\tau_2^4) + (k_5 + k_6)^3k_4 \\ &\quad \times 5(\tau_1^2 - 2\tau_1\tau_2 - 2\tau_2^2)^2 + (k_5 + k_6)^4(2\tau_2^2 + 2\tau_1\tau_2 - \tau_1^2)^2\Bigg), \end{aligned}$$

To prove that this is positive it will be sufficient to show that $A_1^* = 3\tau_1^4 - 18\tau_1^3\tau_2 + 2\tau_1^2\tau_2^2 + 40\tau_1\tau_2^3 + 20\tau_2^4 > 0$ and $A_2^* = 8\tau_1^4 - 33\tau_1^3\tau_2 + 66\tau_1\tau_2^3 + 33\tau_2^4 > 0$. We divide A_1^* by τ_1^4 we obtain

$$B_1 = 3 - 18x + 2x^2 + 40x^3 + 20x^4 \quad \text{where } x = \tau_2\tau_1^{-1}.$$

As we assume that $\tau_2 > \tau_1/2$ it will be sufficient to show that $B_1(x) \geq 0$ whenever $x \geq 1/2$. But

$$B_1(1/2) = 3 - 9 + \frac{1}{2} + \frac{40}{8} + \frac{20}{16} > 0.$$

At the same time

$$B_1'(x) = -18 + 4x + 120x^2 + 80x^3.$$

In particular we see that $B_1'(1/2) > 0$ and $B_1'(x) > B_1'(1/2)$ whenever $x > 1/2$.

Let's going to do a similar thing with A_2^*. We can define $B_2(x) = 8 - 33x + 66x^3 + 33x^4$ where $x = \tau_2\tau_1^{-1}$. We have $B_2(1/2) = 8 - \frac{33}{2} + \frac{66}{8} + \frac{33}{16} > 0$. We also have $B_2'(x) = -33 + \frac{198}{4}x^2 + 132x^3$. We have that $B_2'(1/2) > 0$ and $B_2'(x) > B_2'(1/2)$ whenever $x > 1/2$. Therefore we see that $Det(E_5) > 0$.

$$\begin{aligned} Det(E) \quad = \quad & \frac{1}{64}(k_5+k_6)\tau_1^6\Big(2k_4(k_5+k_6)(5\tau_2^2+5\tau_1\tau_2-3\tau_1^2)+3k_4^2(4\tau_2^2+4\tau_1\tau_2-3\tau_1^2)+(k_5+k_6)^2 \\ & \times(2\tau_2^2+2\tau_1\tau_2-\tau_1^2)\Big)\Big(k_4\tau_2(\tau_1+\tau_2)+k_5(2\tau_2^2+2\tau_1\tau_2-\tau_1^2)+k_6(2\tau_2^2+2\tau_1\tau_2-\tau_1^2)\Big)^2. \end{aligned}$$

To prove this is positive it is enough to show that

$$2k_4(k_5+k_6)(5\tau_2^2+5\tau_1\tau_2-3\tau_1^2)+3k_4^2(4\tau_2^2+4\tau_1\tau_2-3\tau_1^2)+(k_5+k_6)^2(2\tau_2^2+2\tau_1\tau_2-\tau_1^2) > 0.$$

Define

$$\Phi(k_4) = 3k_4^2(4\tau_2^2+4\tau_1\tau_2-3\tau_1^2)+2k_4(k_5+k_6)(5\tau_2^2+5\tau_1\tau_2-3\tau_1^2)+(k_5+k_6)^2(2\tau_2^2+2\tau_1\tau_2-\tau_1^2).$$

It is clear that Φ is positive if $k_4 \geq 0$.[4] □

Theorem 1.1. *There exist positive constants C_1^* and C_2^* such that*

$$\Sigma = \Sigma_1 + \Sigma_2 \geq C_1^*(\theta_{,i}\theta_{,i} + T_iT_i + \dot{\theta}_{,i}\dot{\theta}_{,i} + \dot{T}_i\dot{T}_i + T_{i,j}T_{i,j} + \dot{T}_{i,j}\dot{T}_{i,j})$$

and

$$D^* = D_1 + D_2 \geq C_2^*(\theta_{,i}\theta_{,i} + T_iT_i + \dot{\theta}_{,i}\dot{\theta}_{,i} + \dot{T}_i\dot{T}_i + T_{i,j}T_{i,j} + \dot{T}_{i,j}\dot{T}_{i,j})$$

2 Existence and uniqueness

In this section we prove the existence and uniqueness of solutions for the problem determined by the system with homogeneous Dirichlet boundary conditions and initial conditions.

We consider the Hilbert space

$$\mathcal{H} = W_0^{1,2} \times W_0^{1,2} \times L^2 \times \mathbf{W}_0^{1,2} \times \mathbf{W}_0^{1,2} \times \mathbf{L}^2$$

where $W_0^{1,2}$ and L^2 are the usual Sobolev spaces and $\mathbf{L}^2 = [L^2]^3, \mathbf{W}_0^{1,2} = [W_0^{1,2}]^3$. We shall denote the elements by $(\theta, \vartheta, \zeta, \mathbf{T}, \mathbf{S}, \mathbf{R})$.

In this space we define an inner product

$$\langle U, U^* \rangle = \frac{1}{2}\int_B [k_3 a\tilde{\theta}\tilde{\theta}^* + k_1 b\tilde{T}_i\tilde{T}_i^* + \Xi_1 + \Xi_2]dv, \tag{2.1}$$

4. In fact it is also possible to see that this is positive for $k_4 \in (-\frac{1}{3}(k_5+k_6), 0)$, we have

$$\Phi'(k_4) = 6k_4(4\tau_2^2+4\tau_1\tau_2-3\tau_1^2)+2(k_5+k_6)(5\tau_2^2+5\tau_1\tau_2-3\tau_1^2) \geq 2(k_5+k_6)(\tau_2^2+\tau_1\tau_2) \geq 0.$$

Hence $\Phi(k_4)$ is increasing in $(-\frac{1}{3}(k_5+k_6), 0)$. On the other side, $\Phi(-\frac{1}{3}(k_5+k_6)) = 0$.

where

$$\Xi_1 = kk_3(\tau_1+\tau_2)\theta_{,i}\overline{\theta}^*_{,i} + kk_3\frac{\tau_1^2\tau_2}{2}\vartheta_{,i}\overline{\vartheta}^*_{,i} + k_1k_2(\tau_1+\tau_2)T_i\overline{T}^*_i + k_1k_2\frac{\tau_1^2\tau_2}{2}S_i\overline{S}^*_i$$
$$+\frac{kk_3\tau_1^2}{2}(\theta_{,i}\overline{\vartheta}^*_{,i}+\vartheta_{,i}\overline{\theta}^*_{,i}) + \frac{k_1k_2}{2}(T_i\overline{S}^*_i + S_i\overline{T}^*_i) + k_1k_3(\tau_1+\tau_2)(\theta_{,i}\overline{T}^*_i + \overline{\theta}^*_{,i}T_i)$$
$$+\frac{k_1k_3\tau_1^2}{2}(\theta_{,i}\overline{S}^*_i + \overline{\theta}^*_{,i}S_i) + \frac{k_1k_3\tau_1^2\tau_2}{2}(\vartheta_{,i}\overline{S}^*_i + \overline{\vartheta}^*_{,i}S_i) + \frac{k_1k_3\tau_1^2}{2}(\vartheta_{,i}\overline{T}^*_i + \overline{\vartheta}^*_{,i}T_i)$$

and

$$\Xi_2 = k_1k_6\Big((\tau_1+\tau_2)T_{i,j}\overline{T}^*_{i,j} + \frac{\tau_1^2\tau_2}{2}S_{i,j}\overline{S}^*_{i,j} + \frac{\tau_1^2}{2}(T_{i,j}\overline{S}^*_{i,j} + S_{i,j}\overline{T}^*_{i,j})\Big)$$
$$+k_1k_4\Big((\tau_1+\tau_2)T_{i,i}\overline{T}^*_{j,j} + \frac{\tau_1^2\tau_2}{2}T_{i,i}\overline{S}^*_{j,j} + \frac{\tau_1^2}{2}(\overline{S}_{i,i}T^*_{j,j} + \overline{T}_{i,i}S^*_{j,j})\Big)$$
$$+k_1k_5\Big((\tau_1+\tau_2)T_{i,j}\overline{T}^*_{j,i} + \frac{\tau_1^2\tau_2}{2}S_{i,j}\overline{S}_{j,i} + \frac{\tau_1^2}{2}(S_{i,j}\overline{T}^*_{j,i} + T_{i,j}\overline{S}^*_{j,i})\Big),$$

where $U = (\theta, \vartheta, \zeta, \mathbf{T}, \mathbf{S}, \mathbf{R})$ and $U^* = (\theta^*, \vartheta^*, \zeta^*, \mathbf{T}^*, \mathbf{S}^*, \mathbf{R}^*)$.

We can write (1.3)-(1.4) as

$$\frac{d}{dt}U = \mathcal{A}U, \quad U^0 = (\theta^0, \vartheta^0, \zeta^0, \mathbf{T}^0, \mathbf{S}^0, \mathbf{R}^0) \tag{2.2}$$

where

$$\mathcal{A}\begin{pmatrix}\theta\\ \vartheta\\ \zeta\\ \mathbf{T}\\ \mathbf{S}\\ \mathbf{R}\end{pmatrix} =$$
$$\begin{pmatrix}\vartheta\\ \zeta\\ (a\tau_1^2/2)^{-1}\Big(k(\theta_{,ii}+\tau_2\vartheta_{,ii}) + k_1(T_{i,i}+\tau_2S_{i,i}) - a\tau_1\zeta - a\vartheta\Big)\\ \mathbf{S}\\ \mathbf{R}\\ (b\tau_1^2/2)^{-1}\Big(k_6(T_{i,jj}+\tau_2S_{i,jj}) + (k_4+k_5)(T_{j,ji}+\tau_2S_{j,ji}) - k_2(T_i+\tau_2S_i) - k_3(\theta_{,i}+\tau_2\vartheta_{,i}) - b\tau_1R_i - bS_i\Big)\end{pmatrix}.$$

Note that the domain of this operator is the subset of the elements of the Hilbert space such that $\vartheta, \zeta \in W_0^{1,2}$, $\mathbf{S}, \mathbf{R} \in \mathbf{W}_0^{1,2}$, $\theta_{,ii}+\tau_2\vartheta_{,ii} \in L^2$ and $k_6(T_{i,jj}+\tau_2S_{i,jj}) + (k_4+k_5)(T_{j,ji}+\tau_2S_{j,ji}) \in \mathbf{L}^2$. It is clear that this is a dense subset of the Hilbert space $\mathcal{H}$.

On the other side, it is also clear that

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle = -\frac{1}{2}\int_B \Lambda\, dv \tag{2.3}$$

where

$$\Lambda = kk_3\Big(|\nabla\theta|^2 + (\tau_1\tau_2 - \frac{\tau_1^2}{2})|\nabla\vartheta|^2\Big) + k_1k_2\Big(|\mathbf{T}|^2 + (\tau_1\tau_2 - \frac{\tau_1^2}{2})|\mathbf{S}|^2\Big) + k_1k_3\Big((\theta_{,i}\overline{T}_i + \overline{\theta}_{,i}T_i)$$
$$+(\tau_1\tau_2 - \frac{\tau_1^2}{2})(S_i\overline{\vartheta}_{,i} + \overline{S}_i\vartheta_{,i})\Big) + k_1k_6\Big(|\nabla\mathbf{T}|^2 + (\tau_1\tau_2 - \frac{\tau_1^2}{2})|\nabla\mathbf{S}|^2\Big) + k_1k_4\Big(|\mathrm{Div}\mathbf{T}|^2$$
$$+(\tau_1\tau_2 - \frac{\tau_1^2}{2})|\mathrm{Div}\mathbf{S}|^2\Big) + k_1k_5\Big(T_{i,j}\overline{T}_{j,i} + (\tau_1\tau_2 - \frac{\tau_1^2}{2})S_{i,j}\overline{S}_{j,i}\Big). \tag{2.4}$$

In view of the assumption (1.5) on the system coefficients and the Lemma 1, we see that

$$\mathrm{Re}\langle \mathcal{A}U, U\rangle \leq 0. \tag{2.5}$$

Next, we will show that $0 \in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$. To this end, let us pick any $F = (f_1, f_2, f_3, \mathbf{f}_4, \mathbf{f}_5, \mathbf{f}_6) \in \mathcal{H}$. We want to show the existence of an unique element $U = (\theta, \vartheta, \zeta, \mathbf{T}, \mathbf{S}, \mathbf{R}) \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U = F$, which leads to

$$\vartheta = f_1, \quad \zeta = f_2, \quad \mathbf{S} = \mathbf{f}_4, \quad \mathbf{R} = \mathbf{f}_5, \tag{2.6}$$

$$k(\theta_{,ii} + \tau_2\vartheta_{,ii}) + k_1(T_{i,i} + \tau_2 S_{i,i}) - a\tau_1\zeta - a\vartheta = (a\tau_1^2/2)f_3, \tag{2.7}$$

$$k_6(T_{i,jj}+\tau_2 S_{i,jj})+(k_4+k_5)(T_{j,ji}+\tau_2 S_{j,ji})-k_2(T_i+\tau_2 S_i)-k_3(\theta_{,i}+\tau_2\vartheta_{,i})-b\tau_1 R_i-bS_i = (b\tau_1^2/2)\mathbf{f}_6. \tag{2.8}$$

Substituting (2.6) into (2.7)-(2.8), we obtain the following system about θ and $\mathbf{T}$.

$$k\theta_{,ii} + k_1 T_{i,i} = F_1, \tag{2.9}$$

$$k_6 T_{i,jj} + (k_4 + k_5)T_{j,ji} - k_2 T_i - k_3\theta_{,i} = \mathbf{F}_2, \tag{2.10}$$

where

$$F_1 = (a\tau_1^2/2)f_3 - k\tau_2 f_{1,ii} - k_1\tau_2 \mathrm{Div}\mathbf{f}_4 + a\tau_1 f_2 + af_1,$$

and

$$\mathbf{F}_2 = (b\tau_1^2/2)\mathbf{f}_6 - k_6\tau_2\mathbf{f}_{4i,jj} - (k_4 + k_5)\tau_2\mathbf{f}_{4j,ji} + k_2\tau_2\mathbf{f}_{4i} + k_3\tau_2 f_{1,i} + b\tau_1\mathbf{f}_{5i} + b\mathbf{f}_{4i}.$$

Then, define a bilinear form

$$\mathcal{B}[(\theta,\mathbf{T}),(\theta^*,\mathbf{T}^*)] = -\int_B [(k\theta_{,ii} + k_1 T_{i,i})\theta^* + (k_6 T_{i,jj} + (k_4 + k_5)T_{j,ji} - k_2 T_i - k_3\theta_{,i})T^*_{,i}]dv. \tag{2.11}$$

It is not difficult to see that it is coercive and bounded on $W_0^{1,2} \times \mathbf{W}_0^{1,2}$. On the other hand, the vector $(F_1, \mathbf{F}_2)$ belongs to $W^{-1,2} \times \mathbf{W}^{-1,2}$ which is the dual of $W_0^{1,2} \times \mathbf{W}_0^{1,2}$. Therefore the Lax-Milgram lemma implies the existence of a unique solution.

In fact we also have the existence of a constant K^* such that $||U|| \leq K^*||F||$.

By Theorem 1.2.4 in [27] we have proved that:

Theorem 2.1. *The operator $\mathcal{A}$ generates a contractive semigroup.*

3 Exponential Decay

The aim of this section is to show the exponential decay of the solutions to equation (2.2)-(2.3). Our main tool is the frequency domain characterization of exponentially stable semigroup. [28, 29].

Theorem 3.1. *Let $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if*

I. the imaginary axis is contained in $\rho(\mathcal{A})$, the resolvent of the operator $\mathcal{A}$;

II. $\overline{\lim}_{|\lambda|\to\infty} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \qquad \lambda \in R.$

Therefore, to prove the exponential decay of solutions to our problem it is sufficient to verify that our operator satisfies the conditions of this theorem.

We first verify condition II. Assume that it is not true. Then, there exist a sequence of real numbers λ_n with $|\lambda_n| \to \infty$ (without losing generality, we assume that $\lambda_n > 0$) and a sequence of unit norm vectors $U_n = (\theta_n, \vartheta_n, \zeta_n, \mathbf{T}_n, \mathbf{S}_n, \mathbf{R}_n)$ in the domain of the operator $\mathcal{A}$, such that

$$\|(i\lambda_n\mathcal{I} - \mathcal{A})U_n\| \to 0, \tag{3.1}$$

i.e.,

$$
\begin{aligned}
& i\lambda_n\theta_n - \vartheta_n \to 0 \text{ in } W^{1,2}, && (3.2)\\
& i\lambda_n\vartheta_n - \zeta_n \to 0 \text{ in } W^{1,2}, && (3.3)\\
& i\lambda_n a\tau_1^2\zeta_n - 2\Big(k(\theta_{,ii}+\tau_2\vartheta_{,ii}) + k_1(T_{i,i}+\tau_2 S_{i,i}) - a\tau_1\zeta - a\vartheta\Big) \to 0 \text{ in } L^2, && (3.4)\\
& i\lambda_n\mathbf{T}_n - \mathbf{S}_n \to 0 \text{ in } \mathbf{W}^{1,2}, && (3.5)\\
& i\lambda_n\mathbf{S}_n - \mathbf{R}_n \to 0 \text{ in } \mathbf{W}^{1,2}, && (3.6)\\
& i\lambda_n b\tau_1^2\mathbf{R}_n - 2\Big(k_6(T_{i,jj}+\tau_2 S_{i,jj}) + (k_4+k_5)(T_{j,ji}+\tau_2 S_{j,ji}) - k_2(T_i+\tau_2 S_i) \\
& \qquad -k_3(\theta_{,i}+\tau_2\vartheta_{,i}) - b\tau_1 R_i - bS_i\Big) \to 0 \text{ in } \mathbf{L}^2. && (3.7)
\end{aligned}
$$

In view of (3.1) and the dissipative terms in (2.4)-(2.5), we see that

$$\|\nabla\theta_n\|,\ \|\nabla\vartheta_n\|,\ \|\nabla\mathbf{T}_n\|,\ \|\nabla\mathbf{S}_n\| \to 0, \tag{3.8}$$

which further leads to, in view of (3.3) and (3.7) that

$$\frac{1}{\lambda_n}\|\nabla\zeta_n\|,\ \frac{1}{\lambda_n}\|\nabla\mathbf{R}_n\| \to 0. \tag{3.9}$$

Hereafter, $\|\cdot\|$ denotes the $L^2(B)$ norm. We take the inner product of (3.4) with ζ_n/λ_n in L^2. After integrating by parts, in view of (3.8)-(3.9), we obtain that

$$\|\zeta_n\| \to 0. \tag{3.10}$$

Next, we take the inner product of (3.7) with $\mathbf{R}_n/\lambda_n$ in $\mathbf{L}^2$. By a similar argument, we also obtain that

$$\|\mathbf{R}_n\| \to 0. \tag{3.11}$$

In summary, combining (3.8) and (3.10)-(3.11), we have arrived at a contradiction with $\|U_n\|_{\mathcal{H}} = 1$.

Next, we verify the condition I. If it is false, then there exist a sequence of real numbers λ_n with $|\lambda_n| \to \varpi$ and a sequence of unit norm vectors $U_n = (\theta_n, \vartheta_n, \zeta_n, \mathbf{T}_n, \mathbf{S}_n, \mathbf{R}_n)$ in the domain of the operator $\mathcal{A}$ such that (3.1) holds again. Here,

$$\varpi = \sup\{|\lambda| \ :\ \lambda \in R, \quad \lambda \in \rho(\mathcal{A})\} > 0$$

since $0 \in \rho(\mathcal{A})$. The same contradiction can be obtained by the same analysis verifying condition II.

Therefore we have proved:

Theorem 3.2. *The C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ is exponentially stable. That is, there exist two positive constants M and α such that $\|S(t)U\| \leq M\|U\|e^{-\alpha t}$ for all $U \in \mathcal{H}$.*

4 Further Comments

The analysis in this paper is strongly based on the assumption that the functions Σ_1, Σ_2, D_1 and D_2 are positive definite. In Lemma 1.1 we have seen that the condition (1.5) proposed on the coefficients are sufficient to guarantee this property. However, we believe that the condition $k_4 > 0$ could be weakened to a more relaxed condition $3k_4 + k_5 + k_6 > 0$. In fact, our preliminary calculation shows that all the determinants of the minors corresponding to the matrices A, B, C and E are still positive with one exceptional unconfirmed case of $det(E_5)$. We also believe that Σ_1, Σ_2, D_1 and D_2 are positive under the this weakened condition. However, this remains as an open question.

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