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On the theory of chiral plates and associated system of Timoshenko-Ehrenfest type

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Abstract

The chiral effects cannot be described by means of the classical theory of elasticity. In this paper we study the thermoelastic deformation of chiral plates in the context of the strain gradient theory of thermoelasticity. The work is motivated by the interest in using chiral continuum as model for some carbon nanotubes, auxetic materials and bones. First, we derive the basic equations which govern the deformation of thin thermoelastic plates. In contrast with the theory of achiral plates, the stretching and flexure cannot be treated independently of each other. A system of Timoshenko-Ehrenfest type is presented and an existence result is established. Then, we consider the dynamic theory of plates and present a uniqueness result with no definiteness assumption on the elastic constitutive coefficients. The effects of a concentrated heat source are investigated.

Key words: Chiral materials; Strain gradient theory of thermoelasticity; Thermal stresses in plates; Uniqueness results; Timoshenko-Ehrenfest system; Concentrated heat source.

1 Introduction

The behaviour of chiral materials has received in recent years a widespread attention. The chiral elastic solid was used to model the behaviour of carbon nanotubes (see, e.g., Chandraseker et al. 2009; Zhang et al. 2010; Askes and Aifantis, 2011), auxetic materials (see, e.g., Spadoni and Ruzzene, 2012; Ha et al. 2016, Reasa and Lakes, 2020) , bones (Lakes 1982; Lakes et al. 1983; Park et al. 1986) and piezoelectric materials (Lakes, 2015). In chiral materials qualitative new phenomena are predicted. Lakes and Benedict (1982) studied the deformation of an elastic rod of circular cross section, made of

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an isotropic chiral material. The rod is stretched by an axial force and the lateral surface is free of tractions. It is shown that the rod is predicted to undergo torsional deformation when is subjected to tensile load. Unlike the case of achiral solids, the flexure of isotropic chiral cylinders with arbitrary cross section is accompanied by extension and bending of terminal couples. It is shown that a uniform pressure acting on the lateral surface of a chiral circular elastic cylinder produces a twist around its axis (Ieşan, 2010). In the case of a chiral hollow cylinder the torsion produces a radial deformation (Papanicolopulos, 2010). Within the thermoelasticity theory it has been established that, in contrast to the case of achiral materials, a uniform thermal field in an isotropic chiral cylinder produces torsional effects (Ieşan, 2013). The chiral effects cannot be described within classical elasticity (Lakes, 2001). The strain gradient theory of elasticity is an adequate tool to describe the deformation of chiral elastic solids (Marangati and Sharma, 2007; Papanicolopulos, 2011 and references therein). In the linear theory the chirality behavior is controlled by a single material parameter, in contrast to the three additional material parameters required in Cosserat theory. The equations and the boundary conditions of the strain gradient theory of elastic solids were first established by Toupin (1962, 1964). The linear theory has been developed by Mindlin (1964) and Mindlin and Eshel (1968). The interest in the gradient theory of elasticity is stimulated by the fact that this theory is adequate to investigate important problems related to size effects and nanotechnology (Askes and Aifantis, 2011).

Deformation of thermoelastic plates is of interest both from a mathematical and a technical point of view (Nowacki, 1962; Lagnese, 1989). The gradient theories of thermomechanics have been studied in various papers (see, e.g., Ahmadi and Firoozbakhsh, 1975; Ieşan, 2004; Gurtin and Anand, 2009; Forest and Aifantis, 2010; Ieşan and Quintanilla, 2018). The theory proposed by Altan and Aifantis (1997) has been used to investigate the deformation of achiral elastic plates (Lazopoulos, 2004; Papargyri-Beskou et al., 2010). Ramezani (2012) developed a first order shear deformation micro-plate model which is based on the general form of the strain gradient elasticity established by Mindlin (1964). The plate is assumed to be made of an isotropic and homogeneous achiral material. In recent years there has been an interest for the investigation of chiral thermoelastic materials. Deformation of a cylinder subjected to a prescribed thermal field presents new chiral effects. It is shown that a temperature field which is independent of the axial coordinate produces axial extension, bending and torsion (Ieşan, 2013). A plane temperature field in an achiral cylinder does not produce torsional effects.

In this paper we establish a theory of isotropic chiral plates in the framework of the strain gradient thermoelasticity. We assume that on the upper and lower faces of the plate there are prescribed the surface tractions and the heat flux. In contrast with the theory of achiral plates, the stretching and flexure cannot be treated independently of each other. First, we present the basic equations of homogenenous and isotropic chiral thermoelastic solids in the context of the strain gradient theory. Then, we derive a theory of thermoelastic thin plates. As a special case of the considered model we present a generalization of the Timoshenko-Ehrenfest system from the classical theory of elasticity. Existence and uniqueness results are established. The effects of a concentrated heat source are also investigated.

2 Preliminaries

In this section we present the basic equations of the gradient thermoelasticity for an isotropic chiral continua . Mindlin (1964) presented three forms of the linear theory of gradient elasticity. The relations among the three forms have been established by Mindlin and Eshel (1968). In what follows we will use the first form of the gradient elasticity. We note that the three forms of the theory lead to the same displacement equations of motion for isotropic elastic solids. The constitutive equations of isotropic chiral elastic solids in linear gradient elasticity have been established by Papanicolopulos (2011).

Let us consider a body that in the undeformed state occupies the region B of Euclidean three-dimensional space and is bounded by the surface ∂B. We refer the deformation of the body to a fixed system of rectangular axes Ox_k, $(k = 1, 2, 3)$. Let $\mathbf{n}$ be the outward unit normal of ∂B. Letters in boldface stand for tensors of an order $p \geq 1$, and if $\mathbf{v}$ has the order p, we write $v_{ij\ldots k}$ (p subscripts) for the components of $\mathbf{v}$ in the Cartesian coordinate system. We shall employ the usual summation and differentiation conventions: Latin subscripts (unless otherwise specified) are understood to range over the integer $(1, 2, 3)$, whereas Greek subscripts to the range $(1, 2)$, summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. The partial derivative with respect to time t is denoted by a superposed dot. We assume that B is a bounded region with Lipschitz boundary ∂B. The boundary ∂B consists in the union of a finite number of smooth surfaces, smooth curves (edges) and points (corners). Let C_k be the union of the edges. We assume that B is occupied by a homogeneous and isotropic chiral elastic solid. Let u_j be the components of the displacement vector field on $B \times \mathcal{T}$, where $\mathcal{T}$ is a given interval of time. The strain measures are given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \kappa_{ijk} = u_{k,ij}. \tag{1}$$

The equations of motion are

$$t_{ji,j} + f_i = \rho \ddot{u}_i, \tag{2}$$

where t_{ij} is the Cauchy stress tensor, f_i is the body force and ρ is the reference mass density. In the strain gradient theory of elasticity the tensor t_{ij} has the form

$$t_{ji} = \tau_{ji} - \mu_{sji,s}, \tag{3}$$

where τ_{ij} is the partial stress tensor and μ_{sij} is the double stress tensor. The energy equation can be expressed as

$$\rho T_0 \dot{\eta} = q_{i,i} + s, \tag{4}$$

where η is the entropy, q_i is the heat flux vector, s is the heat supply, and T_0 is the constant absolute temperature of the body in the reference configuration.

The constitutive equations for isotropic chiral thermoelastic solids are (Mindlin and Eshel, 1968; Papanicolopulos, 2011)

$$\begin{aligned}
\tau_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + f(\varepsilon_{ikm}\kappa_{jkm} + \varepsilon_{jkm}\kappa_{ikm}) - \beta T\delta_{ij},\\
\mu_{ijk} &= \frac{1}{2}\alpha_1(\kappa_{rri}\delta_{jk} + 2\kappa_{krr}\delta_{ij} + \kappa_{rrj}\delta_{ik})\\
&\quad + \alpha_2(\kappa_{irr}\delta_{jk} + \kappa_{jrr}\delta_{ik}) + 2\alpha_3\kappa_{rrk}\delta_{ij} +\\
&\quad + 2\alpha_4\kappa_{ijk} + \alpha_5(\kappa_{kji} + \kappa_{kij}) + f(\varepsilon_{iks}e_{js} + \varepsilon_{jks}e_{is}),\\
\rho\eta &= \beta e_{jj} + aT, \quad q_i = kT_{,i},
\end{aligned} \tag{5}$$

where T is the temperature measured from the constant absolute temperature T_0 of the reference state, δ_{ij} is Kronecker delta, ε_{ijk} is the alternating symbol and λ, μ, α_s $(s = 1, 2, \ldots, 5)$, β, a, k and f are constitutive constants. For a centrosymmetric material the coefficient f is equal to zero.

The equations of motion can be expressed in the form

$$\tau_{ji,j} - \mu_{sji,sj} + f_i = \rho\ddot{u}_i. \tag{6}$$

The basic equations of the strain gradient thermoelasticity consist of the geometrical equations (1), the equation of motion (6), the equation of energy (4) and the constitutive equations (5) on $B \times \mathcal{T}$. To the field equations we must adjoin boundary conditions and initial conditions.

Following Toupin (1962) and Mindlin (1964), we introduce the functions P_i, R_i and Q_i^* by

$$\begin{aligned}
P_i &= (\tau_{ki} - \mu_{ski,s})n_k - D_j(n_r\mu_{rji}) + (D_k n_k)n_s n_p \mu_{spi},\\
R_i &= \mu_{rsi}n_r n_s, \quad Q_i^* = \langle \mu_{pji}n_p n_q \rangle \varepsilon_{jrq}s_r,
\end{aligned} \tag{7}$$

where D_i are the components of the surface gradient, $D_i = (\delta_{ik} - n_i n_k)\partial/\partial x_k$, s_j are the components of the unit vector tangent to C, and $< g >$ denotes the difference of limits of g from both sides of C.

If the boundary ∂B is smooth, then $Q_i^* = 0$ and the second boundary-initial-value problem is characterized by the following boundary conditions

$$P_i = \widetilde{P}_i, \; R_i = \widetilde{R}_i, \;\; q_j n_j = \widetilde{q} \;\text{ on }\; \partial B \times \mathcal{T},$$

where $\widetilde{P}_i, \widetilde{R}_i$ and $\widetilde{q}$ are prescribed functions.

The initial conditions are

$$u_i(x_k, 0) = u_i^0(x_k), \; \dot{u}_i(x_k, 0) = v_i^0(x_k), T(x_k, 0) = T^0(x_k), (x_k) \in \overline{B},$$

where u_i^0, v_i^0 and T^0 are given. We assume that: (i) f_i and s are continuous on $\overline{B} \times \mathcal{T}$; (ii) ρ is a given positive constant; (iii) $\widetilde{P}_i, \widetilde{R}_i$ and $\widetilde{q}$ are continuous on $\partial B \times \mathcal{T}$;

(iv) u_i^0, v_i^0 and T^0 are continuous on $\overline{B}$.

3 Chiral plates

We assume now that the region B refers to the interior of a right cylinder of length 2h with open cross-section Σ and the lateral boundary Π (Fig. 1).

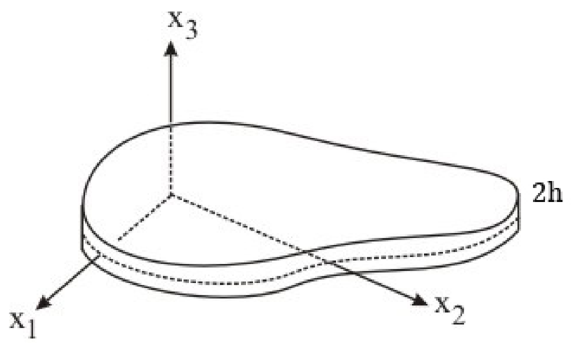

Fig. 1. A plate of thickness 2h

We assume that the surface Π is smooth. The Cartesian coordinate frame consists of the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and the origin O. The system Ox_j is supposed to be chosen in such a way that the plane x_1Ox_2 is middle plane. We denote by Γ the boundary of Σ. In what follows we derive a theory of thin plates of uniform thickness where the displacements and the temperature have the form (Eringen, 1999; Nowacki, 1962)

$$\begin{aligned} u_\alpha &= w_\alpha(x_1, x_2, t) + x_3 v_\alpha(x_1, x_2, t), \; u_3 = w_3(x_1, x_2, t), \\ T &= T_1(x_1, x_2, t) + x_3 T_2(x_1, x_2, t), \end{aligned} \tag{8}$$

$(x_1, x_2) \in \Sigma$, $-h < x_3 < h, t \in \mathcal{T}$, where w_j, v_α and T_α are unknown functions.

Following Eringen (1999) and Nowacki (1962), to obtain a plate theory we have to perform the following integrations: (i) we integrate equations of balance of momenta with respect to x_3 over the thickness of the plate; (ii) we take the cross product of the equations of the balance of momenta with $x_3\mathbf{e}_3$ and integrate over the thickness of the plate; (iii) we integrate the equation of energy over x_3 between the limits $-h$ and h; (iv) we multiply the equation of energy by x_3 and integrate over the thickness of the plate. Let us introduce the notations

$$s_{ki} = \frac{1}{2h}\int_{-h}^{h} t_{ki}dx_3, \quad F_i^* = \frac{1}{2h}\int_{-h}^{h} f_i dx_3. \tag{9}$$

The results of (i) are

$$s_{\beta i,\beta} + \frac{1}{2h}[t_{3i}]_{-h}^{h} + F_i^* = \rho\ddot{w}_i. \tag{10}$$

To the equations (10) we add the results of (ii),

$$\Gamma_{\beta\alpha,\beta} - 2hs_{3\alpha} + H_\alpha^* + [x_3 t_{3\alpha}]_{-h}^{h} = \rho I\ddot{v}_\alpha, \tag{11}$$

where

$$\Gamma_{\beta\alpha} = \int_{-h}^{h} x_3 t_{\beta\alpha}dx_3, \ H_\alpha^* = \int_{-h}^{h} x_3 f_\alpha dx_3, I = \frac{2}{3}h^3. \tag{12}$$

We denote

$$\sigma_{\beta i} = \frac{1}{2h}\int_{-h}^{h} \tau_{\beta i}dx_3, \ m_{k\beta i} = \frac{1}{2h}\int_{-h}^{h} \mu_{k\beta i}dx_3. \tag{13}$$

In view of (3), (9) and (13) we obtain

$$s_{\beta i} = \sigma_{\beta i} - m_{\rho\beta i,\rho} - \frac{1}{2h}[\mu_{3\beta i}]_{-h}^{h}. \tag{14}$$

From (7) we get

$$\begin{aligned} &P_i(x_1,x_2,h) = \tau_{3i} - 2\mu_{\rho 3i,\rho} - \mu_{33i,3}, \ R_i(x_1,x_2,h) = \mu_{33i}, \quad \text{on} \quad x_3 = h, \\ &P_i(x_1,x_2,-h) = -\tau_{3i} + 2\mu_{\rho\beta i,\rho} + \mu_{33i,3}, R_i(x_1,x_2,-h) = \mu_{33i}, \quad \text{on } x_3 = -h. \end{aligned} \tag{15}$$

We have

$$[t_{3i} - \mu_{3\beta i,\beta}]_{-h}^{h} = [P_i]_{-h}^{h}. \tag{16}$$

In view of (14) and (16) the equations (10) become

$$\sigma_{\beta i,\beta} - m_{\rho\beta i,\rho\beta} + F_i = \rho\ddot{w}_i, \tag{17}$$

where

$$F_i = F_i^* + \frac{1}{2h}[P_i]_{-h}^{h}. \tag{18}$$

We assume that P_i, R_i and $q_j n_j$ are prescribed on the surfaces $x_3 = \pm h$.

By using (9) and (13) we obtain

$$2hs_{3\alpha} = 2h\sigma_{3\alpha} - 2hm_{\rho 3\alpha,\rho} - [\mu_{33\alpha}]_{-h}^{h}. \tag{19}$$

From (12) and (3) we get

$$\Gamma_{\beta\alpha} = M_{\beta\alpha} - \Lambda_{\rho\beta\alpha,\rho} - [x_3\mu_{3\beta\alpha}]_{-h}^{h} + 2hm_{3\beta\alpha}, \tag{20}$$

where we have used the notations

$$M_{\beta\alpha} = \int_{-h}^{h} x_3\tau_{\beta\alpha}dx_3, \;\; \Lambda_{\rho\beta\alpha} = \int_{-h}^{h} x_3\mu_{\rho\beta\alpha}dx_3. \tag{21}$$

Clearly, we have

$$[x_3t_{3\alpha} + \mu_{33\alpha} - x_3\mu_{3\beta\alpha,\beta}]_{-h}^{h} = [hP_\alpha + R_\alpha]_{-h}^{h}. \tag{22}$$

It follows from (19), (20) and (22) that the equations (11) can be expressed in the form

$$M_{\beta\alpha,\beta} - \Lambda_{\rho\beta\alpha,\rho\beta} - 2h\sigma_{3\alpha} + 4hm_{3\beta\alpha,\beta} + H_\alpha = \rho I\ddot{v}_\alpha, \tag{23}$$

where

$$H_\alpha = H_\alpha^* + [hP_\alpha + R_\alpha]_{-h}^{h}. \tag{24}$$

If we integrate the equation (4) with respect to x_3 between the limits $-h$ and h, then we obtain the equation

$$\rho T_0\dot{\zeta} = \chi_{\alpha,\alpha} + S_1, \tag{25}$$

where the functions ζ, χ_j and S_1 are introduced by

$$\zeta = \frac{1}{2h}\int_{-h}^{h}\eta dx_3, \chi_j = \frac{1}{2h}\int_{-h}^{h} q_j dx_3, S_1 = \frac{1}{2h}\int_{-h}^{h} sdx_3 + \frac{1}{2h}[q_3]_{-h}^{h}. \tag{26}$$

The equation which results from the multiplication of the equation (4) by x_3 and integration over x_3 between the limits $-h$ and h can be written as

$$\rho T_0\dot{\sigma} = Q_{\alpha,\alpha} - 2h\chi_3 + S_2, \tag{27}$$

where

$$\sigma = \int_{-h}^{h} x_3\eta dx_3, Q_\alpha = \int_{-h}^{h} x_3 q_\alpha dx_3, S_2 = \int_{-h}^{h} x_3 sdx_3 + [x_3q_3]_{-h}^{h}. \tag{28}$$

We note that the functions F_j, H_α and S_α are prescribed.

It follows from (1) and (8) that e_{ij} and κ_{ijk} are given by

$$\begin{aligned} &e_{\alpha\beta} = \gamma_{\alpha\beta} + x_3\xi_{\alpha\beta}, \; 2e_{\alpha 3} = \varphi_\alpha, \; e_{33} = 0, \\ &\kappa_{\alpha\beta\rho} = \eta_{\alpha\beta\rho} + x_3\zeta_{\alpha\beta\rho}, \; \kappa_{\alpha\beta 3} = \eta_{\alpha\beta 3}, \; \kappa_{\alpha 3\rho} = \psi_{\alpha\rho}, \; \kappa_{33j} = 0, \; \kappa_{3\alpha 3} = 0, \end{aligned} \tag{29}$$

where

$$\begin{aligned} &\gamma_{\alpha\beta} = \frac{1}{2}(w_{\alpha,\beta} + w_{\beta,\alpha}), \; \xi_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}), \\ &\eta_{\alpha\beta k} = w_{k,\alpha\beta}, \; \zeta_{\alpha\beta\rho} = v_{\rho,\beta\alpha}, \; \varphi_\alpha = w_{3,\alpha} + v_\alpha, \; \psi_{\alpha\beta} = v_{\beta,\alpha}. \end{aligned} \tag{30}$$

Thus, we find

$$\begin{aligned}&e_{rr}=\gamma_{\rho\rho}+x_3\xi_{\rho\rho},\ \kappa_{\alpha rr}=\eta_{\alpha\nu\nu}+x_3\zeta_{\alpha\nu\nu},\ \kappa_{3rr}=\psi_{\rho\rho},\\ &\kappa_{rr\alpha}=\eta_{\rho\rho\alpha}+x_3\zeta_{\rho\rho\alpha},\ \kappa_{rr3}=\eta_{\rho\rho3}=\Delta w_3,\end{aligned} \tag{31}$$

where Δ is the two-dimensional Laplacian.

It follows from (5), (12), (13), (26), (28) and (29) that

$$\begin{aligned}
\sigma_{\beta\alpha}&=\lambda\gamma_{\rho\rho}\delta_{\alpha\beta}+2\mu\gamma_{\alpha\beta}+f[\varepsilon_{3\rho\beta}(v_{\rho,\alpha}-w_{3,\rho\alpha})+\varepsilon_{3\rho\alpha}(v_{\rho,\beta}-w_{3,\beta\rho})]\\
&\quad-\beta T_1\delta_{\alpha\beta},\\
\sigma_{\beta3}&=\mu\varphi_\beta+f\varepsilon_{3\rho\nu}\eta_{\beta\rho\nu},\\
m_{\rho\beta\alpha}&=\frac{1}{2}\alpha_1(\eta_{\nu\nu\rho}\delta_{\beta\alpha}+2\eta_{\alpha\nu\nu}\delta_{\rho\beta}+\eta_{\nu\nu\beta}\delta_{\rho\alpha})+\alpha_2(\eta_{\rho\nu\nu}\delta_{\beta\alpha}+\eta_{\beta\nu\nu}\delta_{\rho\alpha})+\\
&\quad+2\alpha_3\eta_{\nu\nu\alpha}\delta_{\rho\beta}+2\alpha_4\eta_{\rho\beta\alpha}+\alpha_5(\eta_{\alpha\beta\rho}+\eta_{\alpha\rho\beta})\\
&\quad+\frac{1}{2}f(\varepsilon_{3\rho\alpha}\varphi_\beta+\varepsilon_{3\beta\alpha}\varphi_\rho),\\
m_{\rho\beta3}&=\alpha_1\psi_{\eta\eta}\delta_{\rho\beta}+2\alpha_3\zeta_{\nu\nu3}\delta_{\rho\beta}+2\alpha_4\eta_{\rho\beta3}+2\alpha_5\xi_{\rho\beta}\\
&\quad+f(\varepsilon_{3\eta\rho}\gamma_{\beta\eta}+\varepsilon_{3\eta\beta}\gamma_{\rho\eta}),\\
m_{3\beta\alpha}&=\frac{1}{2}\alpha_1\eta_{\rho\rho3}\delta_{\beta\alpha}+\alpha_2\psi_{\rho\rho}\delta_{\beta\alpha}+2\alpha_4\psi_{\beta\alpha}+\alpha_5(\eta_{\alpha\beta3}+\psi_{\alpha\beta})+f\varepsilon_{3\alpha\nu}\gamma_{\beta\nu},\\
M_{\alpha\beta}&=I(\lambda\xi_{\rho\rho}\delta_{\alpha\beta}+2\mu\xi_{\alpha\beta}-\beta T_2\delta_{\alpha\beta}),\\
\Lambda_{\rho\beta\alpha}&=I[\frac{1}{2}\alpha_1(\zeta_{\nu\nu\rho}\delta_{\beta\alpha}+2\zeta_{\alpha\nu\nu}\delta_{\rho\beta}+\zeta_{\nu\nu\beta}\delta_{\rho\alpha})+\\
&\quad+\alpha_2(\zeta_{\rho\nu\nu}\delta_{\beta\alpha}+\zeta_{\beta\nu\nu}\delta_{\rho\alpha})+2\alpha_3\zeta_{\nu\nu\alpha}\delta_{\rho\beta}+2\alpha_4\zeta_{\rho\beta\alpha}+\alpha_5(\zeta_{\alpha\beta\rho}+\zeta_{\alpha\rho\beta})],\\
\rho\zeta&=\beta\gamma_{\rho\rho}+aT_1,\ \chi_\alpha=kT_{1,\alpha},\chi_3=kT_2,\\
\rho\sigma&=I(\beta\xi_{\rho\rho}+aT_2),\ Q_\alpha=kIT_{2,\alpha}.
\end{aligned} \tag{32}$$

The theory of thermoelastic chiral plates is governed by the equations of motion (17) and (23), the equations of the energy (25) and (27), the constitutive equations (32), and the geometrical equations (30). The equations (17), (23), (25) and (27) lead to the following equations for the unknown functions w_j, v_α and T_α,

$$\begin{aligned}
&\mu\Delta w_\alpha+(\lambda+\mu)w_{\rho,\rho\alpha}-2(\alpha_1+\alpha_2+\alpha_5)\Delta w_{\rho,\rho\alpha}-2(\alpha_3+\alpha_4)\Delta\Delta w_\alpha-\\
&-2f\varepsilon_{3\rho\alpha}\Delta w_{3,\rho}+f\varepsilon_{3\rho\alpha}\Delta v_\rho+f\varepsilon_{3\rho\beta}v_{\rho,\alpha\beta}-f\varepsilon_{3\rho\alpha}v_{\beta,\beta\rho}-\beta T_{1,\alpha}+F_\alpha=\rho\ddot{w}_\alpha,\\
&\mu\Delta w_3+\mu v_{\rho,\rho}-2(\alpha_3+\alpha_4)\Delta\Delta w_3-(\alpha_1+2\alpha_5)\Delta v_{\rho,\rho}+\\
&\qquad+2f\varepsilon_{3\rho\nu}\Delta w_{\nu,\rho}+F_3=\rho\ddot{w}_3,\\
&I[\mu\Delta v_\alpha+(\lambda+\mu)v_{\rho,\rho\alpha}-2(\alpha_3+\alpha_4)\Delta\Delta v_\alpha-2(\alpha_1+\alpha_2+\alpha_5)\Delta v_{\rho,\rho\alpha}-\beta T_{2,\alpha}]-\\
&\qquad-2h[\mu(w_{3,\alpha}+v_\alpha)-(\alpha_1+2\alpha_5)\Delta w_{3,\alpha}-4\alpha_4\Delta v_\alpha-\\
&\qquad-2(\alpha_2+\alpha_5)v_{\rho,\rho\alpha}+f\varepsilon_{3\rho\nu}w_{\nu,\rho\alpha}-f\varepsilon_{3\alpha\nu}(w_{\rho,\rho\nu}+\Delta w_\nu)]+H_\alpha=\rho I\ddot{v}_\alpha,\\
&k\Delta T_1-c\dot{T}_1-\beta T_0\dot{w}_{\alpha,\alpha}=-S_1,\\
&I(k\Delta T_2-c\dot{T}_2-\beta T_0\dot{v}_{\alpha,\alpha})-2hkT_2=-S_2,
\end{aligned} \tag{33}$$

on $\Sigma \times \mathcal{T}$, where we have used the notation $c = aT_0$. In the case of centrosymmetric materials the equations (33) reduce to two uncoupled systems: one for the functions w_α and T_1 and the other for the functions w_3, v_α and T_2.

The initial conditions are

$$\begin{aligned} &w_j(x_1,x_2,0) = w_j^0(x_1,x_2),\ v_\alpha(x_1,x_2,0) = v_\alpha^0(x_1,x_2), T_\alpha(x_1,x_2) = T_\alpha^0(x_1,x_2),\\ &\dot{w}_j(x_1,x_2,0) = \omega_j^0(x_1,x_2), \dot{v}_\alpha(x_1,x_2,0) = \xi_\alpha^0(x_2,x_2),\ (x_1,x_2) \in \Sigma, \end{aligned} \tag{34}$$

where $w_j^0, v_\alpha^0, \omega_j^0$ and ξ_α^0 are prescribed functions. The first boundary-initial-value problem is characterized by the conditions

$$w_i = \widetilde{w}_i,\ Dw_i = \widetilde{\sigma}_i, v_\alpha = \widetilde{v}_\alpha, Dv_\alpha = \widetilde{\gamma}_\alpha, T_\alpha = \widetilde{T}_\alpha \text{ on } \Gamma \times \mathcal{T}, \tag{35}$$

where $D\varphi = \varphi_{,\alpha} n_\alpha$ and $\widetilde{w}_i, \widetilde{\sigma}_i, \widetilde{v}_i, \gamma_\alpha$ and $\widetilde{T}_\alpha$ are given functions. We introduce the notations

$$\begin{aligned} &\Pi_i = (\sigma_{\beta i} - m_{\rho\beta i,\rho})n_\beta + (D_\nu n_\nu) m_{\rho\beta i} n_\rho n_\beta - D_\beta(m_{\rho\beta i} n_\rho),\\ &M_i = m_{\rho\beta i} n_\rho n_\beta,\quad N_\alpha = \Lambda_{\rho\beta\alpha} n_\rho n_\beta,\\ &\Sigma_\alpha = (M_{\beta\alpha} - \Lambda_{\rho\beta\alpha,\rho} + 4h m_{\beta 3\alpha})n_\beta + (D_\nu n_\nu)\Lambda_{\rho\beta\alpha} n_\rho n_\beta - D_\beta(\Lambda_{\rho\beta\alpha} n_\rho). \end{aligned} \tag{36}$$

In the second boundary-initial-value problem the boundary conditions are

$$\Pi_i = \widetilde{\Pi}_i, M_i = \widetilde{M}_i, \Sigma_\alpha = \widetilde{\Sigma}_\alpha, N_\alpha = \widetilde{N}_\alpha, \chi_\alpha n_\alpha = \widetilde{\chi},\ Q_\alpha n_\alpha = \widetilde{Q} \text{ on } \Gamma \times \mathcal{T}, \tag{37}$$

where the functions $\widetilde{\Pi}_i, \widetilde{M}_i, \widetilde{\Sigma}_\alpha, \widetilde{N}_\alpha, \widetilde{\chi}$ and $\widetilde{Q}$ are continuous in time and piecewise regular on $\Gamma \times \mathcal{T}$.

4 A system of Timoshenko-Ehrenfest type

In this section we use the theory of thermoelastic plates presented in Section 3 to derive a generalization of the Timoshenko-Ehrenfest system from the classical theory of elasticity (Elishakoff, 2019). Let us investigate the deformation of the one-dimensional elastic fiber characterized by $x_1 \in (0,l)$, $x_2 = c_0$, $x_3 = 0$, where l and c_0 are constants. In this case we have $w_j = w_j(x,t), v_\alpha = v_\alpha(x,t)$, $T_\alpha = T_\alpha(x,t)$, where we have denoted x_1 by x. In what follows we shall use the notations

$$\begin{aligned} &w_1 = u, w_2 = v, w_3 = \varphi, v_1 = \psi, v_2 = \chi,\\ &T_1 = \tau, T_2 = \theta, \varphi_{,1} = \varphi_x, \dot{\varphi} = \varphi_t, \Delta\varphi = \varphi_{xx},\\ &a_1 = \alpha_1 + 2\alpha_5, a_2 = \alpha_2 + 2\alpha_4 + \alpha_5, a_3 = 2(\alpha_3 + \alpha_4),\\ &a_4 = \mu I + 8h\alpha_4, l_1^2 = \sum_{s=1}^{5} \alpha_s. \end{aligned} \tag{38}$$

The equations (33) reduce to two uncoupled systems: one for the unknown functions u, χ and τ and the other for the functions v, φ, ψ and θ. The first system is characterized by the following equations

$$\begin{aligned}
&(\lambda + 2\mu - 2l_1^2\Delta)u_{xx} - 2f\chi_{xx} - \beta\tau_x + F_1 = \rho u_{tt},\\
&(a_4\Delta - 2h\mu - Ia_3\Delta\Delta)\chi - 4hfu_{xx} + H_2 = \rho I\chi_{tt},\\
&k\tau_{xx} - c\tau_t - \beta T_0 u_{xt} = -S_1.
\end{aligned} \tag{39}$$

on $\mathcal{F}$, where $\mathcal{F} = (0, l) \times \mathcal{T}$. In the case of achiral materials ($f = 0$) this system describes an extensional motion. The second system can be written as

$$\begin{aligned}
&(\mu - a_3\Delta)v_{xx} - 2f\Delta\varphi_x + F_2 = \rho v_{tt},\\
&(\mu - a_3\Delta)\varphi_{xx} + (\mu - a_1\Delta)\psi_x + 2f\Delta v_x + F_3 = \rho\varphi_{tt},\\
&2h(a_1\Delta - \mu)\varphi_x + [I(\lambda + 2\mu - 2l_1^2\Delta)\Delta - 2h(\mu - 2a_2\Delta)]\psi - I\beta_2\theta_x = \rho I\psi_{tt},\\
&I(k\theta_{xx} - c\theta_t - \beta T_0\psi_{xt}) - 2hk\theta = -S_2,
\end{aligned} \tag{40}$$

on $\mathcal{F}$. The system (40) generalizes the Timoshenko-Ehrenfest system from the classical theory. If $f = 0$, then the system describes the behavior of achiral nonsimple beams. The equations corresponding to simple materials can be found by taking $a_m = 0$, $(m = 1, 2, \ldots, 5)$. We consider the first boundary value problem and assume that the boundary data are equal to zero. Thus we have the following conditions associated to the system (39)

$$u = 0, Du = 0, \chi = 0, D\chi = 0, \tau = 0, \tag{41}$$

and the corresponding conditions for the system (40)

$$v = 0, Dv = 0, \varphi = 0, D\varphi = 0, \psi = 0, D\psi = 0, \theta = 0, \tag{42}$$

for $x = 0$ and $x = l$, and $t \in \mathcal{T}$. The initial conditions are

$$\begin{aligned}
&u(x, 0) = u^0(x), u_t(x, 0) = z^0(x), \chi(x, 0) = \chi^0(x),\\
&\chi_t(x, 0) = \eta^0(x), \tau(x, 0) = \tau^0(x),
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
&v(x, 0) = v^0(x), v_t(x, 0) = v_1^0(x), \varphi(x, 0) = \varphi^0(x), \varphi_t(x, 0) = \varphi_1^0(x),\\
&\psi(x, 0) = \psi^0(x), \psi_t(x, 0) = \psi_1^0(x), \theta(x, 0) = \theta^0(x),
\end{aligned} \tag{44}$$

for $x \in (0, l)$. As in the classical theory of thermoelastic plates (see Nowacki, 1962; Lagnese, 1989), equations (40) contain a term that depends linearly on temperature

In what follow we give an existence theorem (see Goldstein, 1985; Iesan and Quintanilla 1994; Magana and Quintanilla 2018) for the solutions of the problem determined by the system (39) with the boundary conditions (41) and the initial conditions (43). We assume that:

(i) $\rho, l_1^2, 3\lambda+2\mu, I, a_3, a_4, h, \mu, k$ and c are positive.

(ii) $ha_4(\lambda+2\mu) > 2h^2f^2$;

To study this problem we consider the Hilbert space

$$\mathcal{H} = W_0^{1,2}(0,l)\times L^2(0,l)\times W_0^{1,2}(0,l)\times L^2(0,l)\times L^2(0,l).$$

and we define the inner product

$$\langle (u,z,\chi,\eta,\tau),(u^*,z^*,\chi^*,\eta^*,\tau^*)\rangle = T_0 h\int_0^l (\rho z z^* + 2l_1^2 u_{xx}u_{xx}^* + (\lambda+2\mu)u_x u_x^*)dx$$

$$+\frac{T_0}{2}\int_0^l (\rho I\eta\eta^* + Ia_3\chi_{xx}\chi_{xx}^* + a_4\chi_x\chi_x^* + 2h\mu\chi\chi^* + 2hf(u_x\chi^* + u_x^*\chi)dx + \frac{c}{2}\int_0^l \tau\tau^* dx.$$

It is clear that in view of the our assumptions, this inner product determines a norm which is equivalent to the usual one in the Hilbert space.

We can also define the matrix operator

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 \\ \frac{1}{\rho}((\lambda+2\mu)D_x^2 - 2l_1^2 D_x^4) & 0 & -\frac{2}{\rho}D_x^2 & 0 & -\frac{\beta}{\rho}D_x \\ 0 & 0 & 0 & I & 0 \\ -\frac{4hf}{\rho I}D_x^2 & 0 & \frac{1}{\rho I}(a_4 D_x^2 - 2h\mu - Ia_3 D_x^4) & 0 & 0 \\ 0 & -\frac{\beta T_0 D_x}{c} & 0 & 0 & -\frac{k}{c} \end{pmatrix}$$

where $D_x = d/dx$.

Our problem can be written as a Cauchy problem

$$\frac{d\mathcal{U}}{dt} = \mathcal{A}\mathcal{U}(t) + \mathcal{G}(t), \quad \mathcal{U}(0) = \mathcal{U}^0,$$

where

$$\mathcal{G}(t) = (0, F_1, 0, H_2, S_1), \quad \mathcal{U}^0 = (u^0, z^0, \chi^0, \eta^0, \tau^0).$$

It is worth noting that the domain of the operator $\mathcal{A}$ is given by the elements in the Hilbert space such that

$$\tau \in W_0^{1,2}\cup W^{2,2}, \quad \eta, z \in W_0^{2,2}, \quad u_{xxxx}, \chi_{xxxx} \in L^2.$$

It is clear that

$$\langle \mathcal{A}\mathcal{U}, \mathcal{U}\rangle = -\frac{k}{2}\int_0^l |\tau_x|^2 dx \le 0.$$

To prove the existence of solutions, the new step we need to prove is that zero belongs to the resolvent of the operator $\mathcal{A}$. To this end we need to see that if

$F = (f_1, f_2, f_3, f_4, f_5)$ belongs to the Hilbert space $\mathcal{H}$ there exists an element $\mathcal{U}$ in the domain of the operator such that

$$\mathcal{A}\mathcal{U} = F.$$

That is

$$z = f_1, \ \ \eta = f_3, \ \ (\lambda + 2\mu - 2l_1^2 D_x^2) D_x^2 u - 2f D_x^2 \chi - \beta D_x \tau = \rho f_2,$$

$$(a_4 D_x^2 - 2h\mu - I a_3 D_x^4)\chi - 4hf D_x^2 u = \rho I f_4, \ \ k D_x^2 \tau - \beta T_0 D_x z = c f_5.$$

It is clear the existence of $z, \eta \in W_0^{2,2}$. Then we have the equation

$$k D_x^2 \tau = c f_5 + \beta T_0 D_x f_1.$$

As the right hand side belongs to L^2, we obtain the existence of a solution τ that belongs to $W_0^{1,2} \cap W^{2,2}$. Now, we can consider the system

$$(\lambda + 2\mu - 2l_1^2 D_x^2) D_x^2 u - 2f D_x^2 \chi = \rho f_2 + \beta D_x \tau,$$

$$(a_4 D_x^2 - 2h\mu - I a_3 D_x^4)\chi - 4hf D_x^2 u = \rho I f_4.$$

To study this system we can consider the bilinear form

$$\mathcal{B}[(u,\chi),(u^*,\chi^*)] = T_0 h \int_0^l [2l_1^2 u_{xx} u_{xx}^* + (\lambda + 2\mu) u_x u_x^*] dx$$

$$+ \frac{T_0}{2} \int_0^l [I a_3 \chi_{xx} \chi_{xx}^* + a_4 \chi_x \chi_x^* + 2h\mu \chi \chi^* + 2hf(u_x \chi^* + u_x^* \chi)] dx.$$

In view of the asumptions (i) and (ii), this is a bilinear coercive and bounded form on $W_0^{2,2} \times W_0^{2,2}$. As the $\rho f_2 + \beta D\tau$ and $\rho I f_4$ belong to L^2 we can obtain the existence of (u, χ) satisfying the above system. It is also clear that $\mathcal{U} = (u, z, \chi, \eta, \tau)$ belongs to the domain of the operator and even more there exists a positive constant M such that

$$||\mathcal{U}|| \leq MF.$$

In view of the fact that $\mathcal{A}$ is defined in a dense subspace, it is dissipative and zero belongs to the resolvent of the operator. We can conclude that:

Theorem 1. *Under the assumptions (i), (ii) the operator $\mathcal{A}$ generates a contractive semigroup.*

A consequence of the previous theorem is the well posedness of the problem. That is:

Theorem 2. *Let us assume that (F_1, H_2, S_1) is C^1 in $[L^2]^3$ and continuous on $W_0^{2,2} \times W_0^{2,2} \times W_0^{1,2} \cap W^{2,2}$ and the conditions (i) and (ii) hold. Then for* *every initial data (43) belonging to the domain of the operator $\mathcal{A}$ there exists a unique solution to boundary-initial-value problem (39),(41),(42).*

It is worth noting that, as the operator $\mathcal{A}$ generates a contractive semigroup, we can obtain the estimate

$$||\mathcal{U}(t)|| \leq ||\mathcal{U}^0|| + \int_0^t ||\mathcal{G}(s)||ds, \tag{45}$$

which is a result of continuous dependence of the solutions with respect initial data and supply terms. Therefore, we can conclude that our problem is well posed in the sense of Hadamard.

It is also worth saying that we could also study the problem determined by the system (40) with the boundary conditions (42) and the initial conditions (44). Again under suitable conditions on the constitutive parameters, we could also prove that the boundary-initial-value problem (33)-(35) is well posed in the sense of Hadamard. We note that the value of the chirality coefficient f plays a relevant role to define the energy of the system and therefore the inner product in the Hilbert space. Limits on the possible values of f are imposed by the requirement for positive definiteness of the potential energy. The results about existence and stability only hold in the case that the parameter f can be controlled by means of the other constitutive coefficients.

5 Uniqueness

In this section we consider the the second boundary-initial-value problem formulated in Section 3. We derive a uniqueness result in the dynamic theory by using the method of Brun (1969). By an admissible process on $\Sigma\times\mathcal{T}$ we mean an ordered array $\vartheta = \{w_i, v_\alpha, T_\alpha, \gamma_{\alpha\beta}, \varphi_\alpha, \xi_{\alpha\beta}, \psi_{\alpha\beta}, \eta_{\alpha\beta k}, \zeta_{\alpha\beta\rho}, \sigma_{\alpha j}, m_{i\beta j}, M_{\alpha\beta}, \Lambda_{\alpha\beta\gamma},$ $\zeta, \sigma, \chi_j, Q_\alpha\}$ with the property that w_i and v_α are of class $C^{4,2}$ on $\Sigma\times\mathcal{T}$ and of class $C^{2,0}$ on $\overline{\Sigma}\times\mathcal{T}$; T_α are of class $C^{2,1}$ on $\Sigma\times\mathcal{T}$; $\gamma_{\alpha\beta}, \xi_{\alpha\beta}, \varphi_\alpha, \psi_{\alpha\beta}, \eta_{\alpha\beta j}, \zeta_{\alpha\beta\rho}$ and $\sigma_{3\alpha}$ are continuous on $\Sigma\times\mathcal{T}$; $\sigma_{\alpha j}, M_{\beta\alpha}, m_{3\beta\alpha}, \chi_j$ and Q_α are of class $C^{1,0}$ on $\Sigma\times\mathcal{T}$; $m_{\rho\beta i}$ and $\Lambda_{\alpha\beta\gamma}$ are of class $C^{2,0}$ on $\Sigma\times\mathcal{T}$, and ζ and σ are of class $C^{0,1}$ on $\Sigma\times\mathcal{T}$. By a solution of the second boundary-initial-value problem we mean an admissible process that satisfies the equations (17), (23), (25) and (30) on $\Sigma\times\mathcal{T}$, the initial conditions (34) and the boundary conditions (37). In this section we restrict our attention to the second boundary-initial-value problem but the results presented hold also for the first boundary-initial-value problem.

If G is a continuous function on $\Sigma\times\mathcal{T}$, then we denote by $\widehat{G}$ the function defined by

$$\widehat{G}(x,t) = \int_0^t G(x,\tau)d\tau, \quad x\in\Sigma, t\in\mathcal{T}.$$

We introduce the functions Φ_1 and Φ_2 by

$$\Phi_1 = \widehat{S}_1 + \rho T_0 \zeta^0, \quad \Phi_2 = \widehat{S}_2 + \rho T_0 \sigma^0, \tag{46}$$

where

$$\rho\zeta^0 = \beta w^0_{\rho,\rho} + aT^0_1, \quad \rho\sigma^0 = I(\beta v^0_{\alpha,\alpha} + aT^0_2). \tag{47}$$

By using (25), (27), (46) and (47) we obtain the following result.

Lemma 1 *The functions ζ, σ, χ_j and Q_α satisfy the equations (25), (27) and the initial conditions $\zeta(x,0) = \zeta^0(x)$, $\sigma(x,0) = \sigma^0(x)$, $x \in \Sigma$, if and only if*

$$\rho T_0 \zeta = \widehat{\chi}_{\alpha,\alpha} + \Phi_1, \quad \rho T_0 \sigma = \widehat{Q}_{\alpha,\alpha} - 2h\widehat{\chi}_3 + \Phi_2, \tag{48}$$

on $\Sigma \times [0,\infty)$.

Let us consider two external data systems $L^{(\alpha)} = \{F^{(\alpha)}_j, H^{(\alpha)}_\beta, S^{(\alpha)}_\beta, \widetilde{\Pi}^{(\alpha)}_i, \widetilde{M}^{(\alpha)}_i, \widetilde{\Sigma}^{(\alpha)}_\beta, \widetilde{N}^{(\alpha)}_\beta, \widetilde{\chi}^{(\alpha)}, \widetilde{Q}^{(\alpha)}, w^{0(\alpha)}_j, v^{0(\alpha)}_\alpha, T^{0(\alpha)}_\beta, \omega^{0(\alpha)}_j, \xi^{0(\alpha)}_\beta\}$. We denote by $A^{(\alpha)} = \{w^{(\alpha)}_j, v^{(\alpha)}_\beta, T^{(\alpha)}_\beta, \gamma^{(\alpha)}_{\rho\nu}, \xi^{(\alpha)}_{\rho\nu}, \varphi^{(\alpha)}_\beta, \psi^{(\alpha)}_{\beta\nu}, \eta^{(\alpha)}_{\rho\beta\nu}, \zeta^{(\alpha)}_{\rho\beta\nu}, \sigma^{(\alpha)}_{\beta j}, m^{(\alpha)}_{i\beta j}, M^{(\alpha)}_{\rho\nu}, \Lambda^{(\alpha)}_{\rho\beta\nu}, \zeta^{(\alpha)}, \sigma^{(\alpha)}, \chi^{(\alpha)}_j, Q^{(\alpha)}_\beta\}$ a solution of the second boundary-initial-value problem corresponding to $L^{(\alpha)}$, $(\alpha = 1,2)$. We denote by $\Pi^{(\alpha)}_i, M^{(\alpha)}_i, \Sigma^{(\alpha)}_\beta$ and $N^{(\alpha)}_\beta$ the functions Π_i, M_i, Σ_β and N_β from (36) associated to $A^{(\alpha)}$, and introduce the notations

$$\chi^{(\alpha)} = \chi^{(\alpha)}_\beta n_\beta, Q^{(\alpha)} = Q^{(\alpha)}_\beta n_\beta, \Phi^{(\alpha)}_1 = \widehat{S}^{(\alpha)}_1 + \rho T_0 \zeta^{0(\alpha)}, \Phi^{(\alpha)}_2 = \widehat{S}^{(\alpha)}_2 + \rho T_0 \sigma^{0(\alpha)}. \tag{49}$$

We denote

$$\begin{aligned}
K_{\kappa\nu}(r,s) = &\int_\Sigma \{2h\rho \ddot{w}^{(\kappa)}_j(r) w^{(\nu)}_j(s) + \rho I \ddot{v}^{(\kappa)}_\alpha(r) v^{(\nu)}_\alpha(s) - \\
&- \frac{k}{T_0}[2h\widehat{T}^{(\kappa)}_{1,\alpha}(r) T^{(\nu)}_{1,\alpha}(s) + I\widehat{T}^{(\kappa)}_{2,\alpha}(r) T^{(\nu)}_{2,\alpha}(s) + 2h\widehat{T}^{(\kappa)}_2(r) T^{(\nu)}_2(s)]\} da, \\
\Pi_{\kappa\nu}(r,s) = &\int_\Sigma [2hF^{(\kappa)}_i(r) w^{(\nu)}_i(s) + H^{(\kappa)}_\alpha(r) v^{(\nu)}_\alpha(s) - \\
&- \frac{2h}{T_0}\Phi^{(\kappa)}_1(r) T^{(\nu)}_1(s) - \frac{1}{T_0}\Phi^{(\kappa)}_2(r) T^{(\nu)}_2(s)] da + \\
&+ \int_\Gamma [2h\Pi^{(\kappa)}_i(r) w^{(\nu)}_i(s) + 2hM^{(\kappa)}_i(r) Dw^{(\nu)}_i(s) + \\
&+ \Sigma^{(\kappa)}_\alpha(r) v^{(\nu)}_\alpha(s) + N^{(\kappa)}_\alpha(r) Dv^{(\nu)}_\alpha(s) - \frac{2h}{T_0}\widehat{\chi}^{(\kappa)}(r) T^{(\nu)}_1(s) \\
&- \frac{1}{T_0}\widehat{Q}^{(\kappa)}(r) T^{(\nu)}_2(s)] dl,
\end{aligned} \tag{50}$$

for all $r, s \in \mathcal{T}$. For convenience, in (50) we have supressed the argument x.

Theorem 3. *Let*

$$E_{\alpha\beta}(r,s) = \Pi_{\alpha\beta}(r,s) - K_{\alpha\beta}(r,s), \tag{51}$$

for all $r, s \in \mathcal{T}$. *Then*

$$E_{\alpha\beta}(r,s) = E_{\beta\alpha}(r,s), \quad (\alpha,\beta = 1,2). \tag{52}$$

Proof. We introduce the notation

$$\begin{aligned} W_{\kappa\nu}(r,s) = 2h[&\sigma^{(\kappa)}_{\beta\alpha}(r)\gamma^{(\nu)}_{\beta\alpha}(s) + \sigma^{(\kappa)}_{\alpha 3}(r)\varphi^{(\nu)}_{\alpha}(s) + m^{(\kappa)}_{\rho\beta j}(r)\eta^{(\nu)}_{\rho\beta j}(s) + \\ &+ 2m^{(\kappa)}_{3\beta\alpha}(r)\psi^{(\nu)}_{\beta\alpha}(s) - \rho\zeta^{(\kappa)}(r)T^{(\nu)}_1(s)] + M^{(\kappa)}_{\beta\alpha}(r)\xi^{(\nu)}_{\beta\alpha}(s) + \\ &+ \Lambda^{(\kappa)}_{\rho\beta\alpha}(r)\zeta^{(\nu)}_{\rho\beta\alpha}(s) - \rho\sigma^{(\kappa)}(r)T^{(\nu)}_2(s). \end{aligned} \tag{53}$$

By using the constitutive equations (32) we find that

$$W_{\kappa\nu}(r,s) = 2hJ_{\kappa\nu}(r,s) + IN_{\kappa\nu}(r,s), \tag{54}$$

where

$$\begin{aligned} J_{\kappa\nu}(r,s) &= \lambda\gamma^{(\kappa)}_{\rho\rho}(r)\gamma^{(\nu)}_{\eta\eta}(s) + 2\mu\gamma^{(\kappa)}_{\beta\alpha}(r)\gamma^{(\nu)}_{\beta\alpha}(s) + \mu\varphi^{(\kappa)}_{\alpha}(r)\varphi^{(\nu)}_{\alpha}(s) + \\ &+ \alpha_1[\psi^{(\kappa)}_{\rho\rho}(r)\eta^{(\nu)}_{\alpha\alpha 3}(s) + \psi^{(\nu)}_{\rho\rho}(s)\eta^{(\kappa)}_{\alpha\alpha 3}(r) + \eta^{(\kappa)}_{\beta\beta\rho}(r)\eta^{(\nu)}_{\rho\alpha\alpha}(s) + \\ &+ \eta^{(\kappa)}_{\alpha\beta\beta}(r)\eta^{(\nu)}_{\rho\rho\alpha}(s)] + 2\alpha_2[\psi^{(\kappa)}_{\rho\rho}(s)\psi^{(\nu)}_{\alpha\alpha}(s) + \eta^{(\kappa)}_{\rho\beta\beta}(r)\eta^{(\nu)}_{\rho\alpha\alpha}(s)] + \\ &+ 2\alpha_3\eta^{(\kappa)}_{\beta\beta j}(r)\eta^{(\nu)}_{\rho\rho j}(s) + 2\alpha_4[2\psi^{(\kappa)}_{\beta\alpha}(r)\psi^{(\nu)}_{\beta\alpha}(s) + \eta^{(\kappa)}_{\rho\beta j}(r)\eta^{(\nu)}_{\rho\beta j}(s)] + \\ &+ 2\alpha_5[\psi^{(\kappa)}_{\beta\alpha}(r)\eta^{(\nu)}_{\alpha\beta 3}(s) + \psi^{(\nu)}_{\beta\alpha}(s)\eta^{(\kappa)}_{\alpha\beta 3}(r)] + \alpha_5[\eta^{(\kappa)}_{\alpha\beta\rho}(r)\eta^{(\nu)}_{\rho\beta\alpha}(s) + \\ &+ \eta^{(\nu)}_{\alpha\beta\rho}(s)\eta^{(\kappa)}_{\rho\beta\alpha}(r) + 2\psi^{(\kappa)}_{\alpha\beta}(r)\psi^{(\nu)}_{\beta\alpha}(s)] + f\varepsilon_{3\rho\beta}\{2[\psi^{(\kappa)}_{\alpha\rho}(r)\gamma^{(\nu)}_{\beta\alpha}(s) + \\ &+ \psi^{(\nu)}_{a\rho}(s)\gamma^{(\kappa)}_{\beta\alpha}(r)] + 2[\eta^{(\kappa)}_{\beta\alpha 3}\gamma^{(\nu)}_{\rho\alpha}(s) + \eta^{(\nu)}_{\beta\alpha 3}(s)\gamma^{(\kappa)}_{\rho\alpha}(r)] + \\ &+ \eta^{(\kappa)}_{\alpha\rho\beta}(r)\varphi^{(\nu)}_{\alpha}(s) + \eta^{(\nu)}_{\alpha\rho\beta}(s)\varphi^{(\kappa)}_{\alpha}(r)\} - \beta[T^{(\kappa)}_1(r)\gamma^{(\nu)}_{\rho\rho}(s) \\ &+ T^{(\nu)}_1(s)\gamma^{(\kappa)}_{\rho\rho}(r)] - aT^{(\kappa)}_1(r)T^{(\nu)}_1(s), \\ N_{\kappa\nu}(r,s) &= \lambda\xi^{(\kappa)}_{\rho\rho}\xi^{(\nu)}_{\alpha\alpha}(s) + 2\mu\xi^{(\kappa)}_{\alpha\beta}(r)\xi^{(\nu)}_{\alpha\beta}(s) + \\ &+ \alpha_1[\zeta^{(\kappa)}_{\eta\eta\rho}(r)\zeta^{(\nu)}_{\rho\alpha\alpha}(s) + \zeta^{(\kappa)}_{\alpha\eta\eta}(r)\zeta^{(\nu)}_{\rho\rho\alpha}(s)] + 2\alpha_2\zeta^{(\kappa)}_{\rho\alpha\alpha}(r)\zeta^{(\nu)}_{\rho\beta\beta}(s) + \\ &+ 2\alpha_3\zeta^{(\kappa)}_{\beta\beta\alpha}(r)\zeta^{(\nu)}_{\rho\rho\alpha}(s) + 2\alpha_4\zeta^{(\kappa)}_{\rho\beta\alpha}(r)\zeta^{(\nu)}_{\rho\beta\alpha}(s) + \\ &+ \alpha_5[\zeta^{(\kappa)}_{\alpha\beta\rho}(r)\zeta^{(\nu)}_{\rho\beta\alpha}(s) + \zeta^{(\nu)}_{\alpha\beta\rho}(s)\zeta^{(\kappa)}_{\rho\beta\alpha}(r)] - \beta[T^{(\kappa)}_2(r)\xi^{(\nu)}_{\rho\rho}(s) + \\ &+ T^{(\nu)}_2(s)\xi^{(\kappa)}_{\rho\rho}(r)] - aT^{(\kappa)}_2(r)T^{(\nu)}_2(s). \end{aligned} \tag{55}$$

From (53), (54) and (55) we obtain

$$W_{\kappa\nu}(r,s) = W_{\nu\kappa}(s,r), \quad s,r \in \mathcal{T}. \tag{56}$$

On the other hand if we use the relations (17), (23), (30), (32) and (48), we find that

$$W_{\kappa\nu}(r,s)=2h\{[\sigma^{(\kappa)}_{\beta i}(r)-m^{(\kappa)}_{\rho\beta i,\rho}(r)]w^{(\nu)}_i(s)+[M^{(\kappa)}_{\beta\alpha}(r)-$$
$$-\Lambda^{(\kappa)}_{\rho\beta\alpha,\rho}(r)+2m^{(\kappa)}_{\beta 3\alpha}(r)]v^{(\nu)}_\alpha(s)-\frac{1}{T_0}\widehat{\chi}^{(\kappa)}_\beta(r)T^{(\nu)}_1(s)\}_{,\beta}+[2hm^{(\kappa)}_{\rho\beta i}(r)w^{(\nu)}_i(s)+$$
$$+\Lambda^{(\kappa)}_{\rho\beta\alpha}(r)v^{(\nu)}_{\alpha,\beta}(s)-\frac{1}{T_0}\widehat{Q}^{(\kappa)}_\rho(r)T^{(\nu)}_2(s)]_{,\rho}+2h[F^{(\kappa)}_i(r)w^{(\nu)}_i(s)-\frac{1}{T_0}\Phi^{(\kappa)}_1(r)T^{(\nu)}_1(s)]+$$
$$+H^{(\kappa)}_\alpha(r)v^{(\nu)}_\alpha(s)-\frac{1}{T_0}\Phi^{(\kappa)}_2(r)T^{(\nu)}_2(s)-2h\rho\ddot{w}^{(\kappa)}_j(r)w^{(\nu)}_j(s)- \quad (57)$$
$$-\rho I\ddot{v}^{(\kappa)}_\alpha(r)v^{(\nu)}_\alpha(s)+\frac{k}{T_0}[2h\widehat{T}^{(\kappa)}_{1,\alpha}(r)T^{(\nu)}_{1,\alpha}(s)+I\widehat{T}^{(\kappa)}_{2,\alpha}(r)T^{(\nu)}_{2,\alpha}(s)+$$
$$+2h\widehat{T}^{(\kappa)}_2(r)T^{(\nu)}_2(s)].$$

If we integrate (57) over Σ and use the divergence theorem and the relations (49), (50) and (51) then we get

$$\int_\Sigma W_{\kappa\nu}(r,s)da=E_{\kappa\nu}(r,s). \quad (58)$$

From (56) and (58) we obtain the desired result. □

Theorem 3 implies the next result.

Theorem 4. *Let* $A=\{w_j,v_\alpha,T_\alpha,\gamma_{\rho\nu},\xi_{\rho\nu},\varphi_\alpha,\psi_{\alpha\beta},\eta_{\rho\alpha\beta},\ \zeta_{\rho\alpha\beta},\sigma_{\beta i},m_{i\beta j},M_{\alpha\beta},\Lambda_{\rho\beta\alpha},\zeta,\sigma,\chi_i,Q_\alpha\}$ *be a solution corresponding to the external data system* $L=\{F_i,H_\alpha,S_\alpha,\widetilde{\Pi}_i,\widetilde{M}_i,\widetilde{\Sigma}_\alpha,\widetilde{N}_\alpha,\widetilde{\chi},\widetilde{Q},w^0_i,T^0_\alpha,\omega^0_i,\xi^0_\alpha\}$ *and let*

$$\Lambda(r,s)=\int_\Sigma\{2h[F_i(r)w_i(s)-\frac{1}{T_0}\Phi_1(r)T_1(s)]+$$
$$+H_\alpha(r)v_\alpha(s)-\frac{1}{T_0}\Phi_2(r)T_2(s)\}da+\int_\Gamma\{2h[\Pi_i(r)w_i(s)+$$
$$+M_i(r)Dw_i(s)-\frac{1}{T_0}\widehat{\chi}(r)T_1(s)]+\Sigma_\alpha(r)v_\alpha(s)+N_\alpha(r)Dv_\alpha(s)- \quad (59)$$
$$-\frac{1}{T_0}\widehat{Q}(r)T_2(s)\}dl,$$

for all $r,s\in\mathcal{T}$. *Then*

$$\frac{d}{dt}\{\int_\Sigma\rho(2hw_iw_i+Iv_\alpha v_\alpha)da+\frac{k}{T_0}\int_0^t\int_\Sigma(2h\widehat{T}_{1,\alpha}\widehat{T}_{1,\alpha}+$$
$$+I\widehat{T}_{2,\alpha}\widehat{T}_{2,\alpha}+2h\widehat{T}^2_2)dtda\}=\int_0^t[\Lambda(t-s,t+s)-\Lambda(t+s,t-s)]ds+ \quad (60)$$
$$+\int_\Sigma\{2h\rho[\dot{w}_j(0)w_j(2t)+\dot{w}_j(2t)w_j(0)]+\rho I[\dot{v}_\alpha(0)v_\alpha(2t)+\dot{v}(2t)v_\alpha(0)]\}da.$$

Proof. It follows from (52) that

$$\int_0^t E_{11}(t+s,t-s)ds = \int_0^t E_{11}(t-s,t+s)ds. \tag{61}$$

We apply this relation to the solution $A^{(1)} = A$. From (50), (51) and (59) we get

$$\begin{aligned}
&\int_0^t E_{11}(t+s,t-s)ds = \int_0^t \Lambda(t+s,t-s)ds - \int_0^t \int_\Sigma \{2h\rho\ddot{w}_j(t+s)w_i(t-s)+\\
&+\rho I\ddot{v}_\alpha(t+s)v_\alpha(t-s) - \frac{k}{T_0}[2h\widehat{T}_{1,\alpha}(t+s)T_{1,\alpha}(t-s)+\\
&+I\widehat{T}_{2,\alpha}(t+s)T_{2,\alpha}(t-s) + 2h\widehat{T}_2(t+s)T_2(t-s)]\}dsda,
\end{aligned} \tag{62}$$

and

$$\begin{aligned}
&\int_0^t E_{11}(t-s,t+s)ds = \int_0^t \Lambda(t-s,t+s)ds - \int_0^t \int_\Sigma \{2h\rho\ddot{w}_j(t-s)w_j(t+s)+\\
&+\rho I\ddot{v}_\alpha(t-s)v_\alpha(t+s) - \frac{k}{T_0}[2h\widehat{T}_{1,\alpha}(t-s)T_{1,\alpha}(t+s)+\\
&+I\widehat{T}_{2,\alpha}(t-s)T_{2,\alpha}(t+s) + 2h\widehat{T}_2(t-s)T_2(t+s)]\}dsda.
\end{aligned} \tag{63}$$

If g_1 and g_2 are functions of class C^2 on $[0,\infty)$, then we have

$$\begin{aligned}
&\int_0^t g_1(t+s)\dot{g}_2(t-s)ds = -g_2(0)g_1(2t) + g_1(t)g_2(t) + \int_0^t \dot{g}_1(t+s)g_2(t-s)ds,\\
&\int_0^t \ddot{g}_1(t+s)g_2(t-s)ds = \dot{g}_1(2t)g_2(0) - \dot{g}_1(t)g_2(t) + \int_0^t \dot{g}_2(t-s)\dot{g}_1(t+s)ds,\\
&\int_0^t \ddot{g}_2(t-s)g_1(t+s)ds = \dot{g}_2(t)g_1(t) - \dot{g}_2(0)g_1(2t) + \int_0^t \dot{g}_2(t-s)\dot{g}_1(t+s)ds.
\end{aligned}$$

With the help of these relations we can express (62) and (63) in a different way. Thus, the relation (62) can be written in the form

$$\begin{aligned}
&\int_0^t E_{11}(t+s,t-s)ds = \int_0^t \Lambda(t+s,t-s)ds - \int_\Sigma \{2h\rho[\dot{w}_i(2t)w_i(0)-\\
&-\dot{w}_i(t)w_i(t) + \int_0^t \dot{w}_j(t+s)\dot{w}_j(t-s)ds] + \rho I[\dot{v}_\alpha(2t)v_\alpha(0) - \dot{v}_\alpha(t)v_\alpha(t)+\\
&+\int_0^t \dot{v}_\alpha(t+s)\dot{v}_\alpha(t-s)ds] - \frac{2hk}{T_0}[\widehat{T}_{1,\alpha}\widehat{T}_{1,\alpha} + \int_0^t T_{1,\alpha}(t+s)\widehat{T}_{1,\alpha}(t-s)ds]-\\
&-\frac{kI}{T_0}[\widehat{T}_{2,\alpha}\widehat{T}_{2,\alpha} + \int_0^t T_{2,\alpha}(t+s)\widehat{T}_{2,\alpha}(t-s)ds]-\\
&-\frac{2hk}{T_0}[\widehat{T}_2^2 + \int_0^t T_2(t+s)\widehat{T}_2(t-s)ds]\}da.
\end{aligned} \tag{64}$$

In a similar way we obtain

$$\int_0^t E_{11}(t-s,t+s)ds = \int_0^t \Lambda(t-s,t+s)ds - \int_\Sigma \{2h\rho[\dot{w}_i(t)w_i(t)- \\ - \dot{w}_i(0)w_i(2t) + \int_0^t \dot{w}_i(t-s)\dot{w}_i(t+s)ds] + \rho I[\dot{v}_\alpha(t)v_\alpha(t) - \dot{v}_\alpha(0)v_\alpha(2t)+ \\ + \int_0^t \dot{v}_\alpha(t-s)\dot{v}_\alpha(t+s)ds]\}da + \frac{k}{T_0}\int_0^t \int_\Sigma [2h\widehat{T}_{1,\alpha}(t-s)T_{1,\alpha}(t+s)+ \\ + I\widehat{T}_{2,\alpha}(t-s)T_{2,\alpha}(t+s) + 2h\widehat{T}_2(t-s)T_2(t+s)]dsda. \tag{65}$$

From (63), (64) and (65) we obtain the desired result. □

The following uniqueness result is a consequence of Theorem 4.

Theorem 5. *Assume that ρ and k are strictly positive and a is different from zero. Then the boundary-initial-value problem has at most one solution.*

Proof. Assume that there are two solutions. Then their difference A corresponds to null data. From (60) and the initial conditions we find that

$$\int_\Sigma \rho(2hw_iw_i + Iv_\alpha v_\alpha)da + \frac{k}{T_0}\int_0^t \int_\Sigma (2h\widehat{T}_{1,\alpha}\widehat{T}_{1,\alpha}+ \\ + I\widehat{T}_{2,\alpha}\widehat{T}_{2,\alpha} + 2h\widehat{T}_2^2)dtda = 0. \tag{66}$$

If we use (66) and the hypotheses of theorem we obtain

$$w_i = 0, v_\alpha = 0, \widehat{T}_{1,\alpha} = 0, \widehat{T}_2 = 0 \text{ on } \Sigma \times \mathcal{T}. \tag{67}$$

Thus, we have $T_{1,\alpha} = 0$ on $\Sigma \times \mathcal{T}$ so that $T_1 = \vartheta(t)$, $t \in \mathcal{T}$. From (32) we obtain $\chi_\alpha = 0$ on $\Sigma \times \mathcal{T}$ and $\rho\zeta = a\vartheta$. The energy equation (25) implies that $aT_0\dot{\vartheta} = 0$. Since a and T_0 are different from zero we find that ϑ is a constant on $\Sigma \times \mathcal{T}$. The initial conditions lead to $\vartheta(0) = 0$ so that $T_1 = 0$ on $\Sigma \times \mathcal{T}$. From (67) we obtain that $T_2 = 0$ on $\Sigma \times \mathcal{T}$. The proof is complete. □

6 Effects of a concentrated heat source

In this section we study the deformation of an infinite chiral plate subjected to a concentrated heat source. We consider the equilibrium theory and assume that

$$F_j = 0, \quad H_\alpha = 0, \quad S_1 = 0. \tag{68}$$

We suppose that the concentrated heat source S_2 acts in the point (y_1, y_2), and introduce the notation $r = [(x_1-y_1)^2 + (x_2-y_2)^2]^{1/2}$. We suppose that

$S_2 = S\delta(x-y)$, where δ is the Dirac measure and S is a given constant. In this case from (33) we obtain

$$T_1 = 0,\ \ T_2 = \frac{S}{2\pi k}K_0(dr),\ d = (2h/I)^{1/2}, \tag{69}$$

where K_0 denotes the modified Bessel functions of the third kind and order zero. We try to solve the problem assuming that

$$w_\alpha = \varepsilon_{3\alpha\gamma}U_{,\gamma},\ \ w_3 = W,\ \ v_\alpha = V_{,\alpha}, \tag{70}$$

where U, W and V are unknown functions which depend on the variable r. In the case of equilibrium the equations (33) are satisfied if the functions U, V and W satisfy the following equations

$$\begin{aligned}&\mu U - 2(\alpha_3+\alpha_4)\Delta U + 2fW = 0,\\&[\mu - 2(\alpha_3+\alpha_4)\Delta]W + [\mu - (\alpha_1+2\alpha_5)\Delta]V = 2f\Delta U,\\&(e\Delta - l_1^2\Delta\Delta - \mu d^2)V - d^2[\mu - (\alpha_1+2\alpha_5)\Delta]W = \beta_2 T_2,\end{aligned} \tag{71}$$

where

$$e = \lambda + 2\mu + 2d^2(\alpha_2 + 2\alpha_4 + \alpha_5),\ \ l_1^2 = \sum_{j=1}^{5}\alpha_j. \tag{72}$$

We introduce the notations

$$\begin{aligned}&a_1^* = \alpha_1 + 2\alpha_5,\ a_2^* = 4(\alpha_3+\alpha_4)^2,\ a_3^* = 4[(\alpha_3+\alpha_4)\mu - f^2],\\&a_4^* = \mu(\alpha_1 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5),\ a_5^* = 2(\alpha_1+2\alpha_5)(\alpha_3+\alpha_4),\end{aligned} \tag{73}$$

and the operators

$$\begin{aligned}&\mathcal{D}_1 = 2f(\mu - a_1^*\Delta),\ \mathcal{D}_2 = a_3^*\Delta - a_2\Delta\Delta - \mu^2,\\&\mathcal{D}_3 = 2f(e\Delta - l_1^2\Delta\Delta - d^2\mu),\ \mathcal{D}_4 = d^2(\mu^2 - a_4^*\Delta + a_5^*\Delta\Delta).\end{aligned} \tag{74}$$

The system (71) can be presented in the form

$$2fW = 2(\alpha_3+\alpha_4)\Delta U - \mu U, \tag{75}$$

$$\mathcal{D}_1 V + \mathcal{D}_2 U = 0,\ \ \mathcal{D}_3 V + \mathcal{D}_4 U = 2f\beta_2 T_2. \tag{76}$$

We consider the representation

$$U = \mathcal{D}_1\Omega,\ V = -\mathcal{D}_2\Omega, \tag{77}$$

where Ω is a function of class C^8. It is easy to see that the functions U and V given by (77) satisfy the system (76) if the functions Ω satisfies the equation

$$(\mathcal{D}_1\mathcal{D}_4 - \mathcal{D}_2\mathcal{D}_3)\Omega = 2f\beta_2 T_2. \tag{78}$$

In what follows we shall use the notations

$$b_1 = a_3^* l_1^2 + e a_2^* - a_1^* a_5^* d^2, b_2 = d^2 a_1^* a_4^* + \mu d^2 a_5^* - l_1^2 \mu^2 - e a_3^* - d^2 \mu a_2^*, \quad (79)$$
$$b_3 = d^2 \mu (a_3^* - a_4^*) - \mu^2 (a_1^* d^2 - e).$$

We note that

$$\mathcal{D}_1 \mathcal{D}_4 - \mathcal{D}_2 \mathcal{D}_3 = -2 f l_1^2 a_2^* (\Delta - k_1^2)(\Delta - k_2^2)(\Delta - k_3^2)\Delta,$$

where k_j^2 are the roots of the equation

$$l_1^2 a_2^* z^3 - b_1 z^2 - b_2 z - b_3 = 0. \quad (80)$$

We assume that k_1, k_2, k_3 and d are distinct positive constants. The function Ω satisfies the equation

$$(\Delta - k_1^2)(\Delta - k_2^2)(\Delta - k_3^2)\Delta\Omega = P K_0(dr), \quad (81)$$

where

$$P = -\beta_2 S/(2\pi l_1^2 k a_2^*). \quad (82)$$

The function Ω that satisfies the equation (82) and vanishes at infinity is given by

$$\Omega = A K_0(dr),\ A = P[(d^2 - k_1^2)(d^2 - k_2^2)(d^2 - k_3^2)d^2]^{-1}.$$

It follows from (75) and (76) that the functions U, V and W have the form

$$U = 2fA(\mu - d^2 a_1^*)K_0(dr),\ V = (a_3^* d^2 - a_2^* d^4 - \mu^2) A K_0(dr),$$
$$W = A(\mu - d^2 a_1^*)[2(\alpha_3 + \alpha_4)d^2 - \mu]K_0(dr).$$

The displacements w_j and v_α can be determined from (70). In contrast with the theory of centrosymmetric plates, the stretching and flexure of the plate cannot be treated independently of each other.

7 Concluding remarks

In this paper we establish a theory of thermoelastic plates which is able to describe the chiral behavior. We considered the strain gradient theory of thermoelasticity since the behavior of chirality in linear theory is controlled by a single material parameter. The results presented in this paper can be summarized as follows:

a) We derive the basic equations which govern the deformation of chiral thermoelastic thin plates. In contrast with the theory of achiral plates, the stretching and flexure cannot be treated independently of each other.

b) As a special case of the equations of chiral plates, we deduce a generalization of the Timoshenko-Ehrenfest system from the classical theory of elasticity.

c) In the case of the dynamic theory we establish a uniqueness result with no definiteness assumption on the elastic constitutive coefficients as well as the existence of solutions under suitable conditions on the constitutive coefficients.

d) We investigate the effects of a heat source in an unbounded chiral plate.

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References

Ahmadi, G., Firoozbakhsh, K., 1975. First strain-gradient theory of thermoelasticity. Int. J. Solids Struct.11, 339-345.

Altan, S.B., Aifantis, E.C., 1997. On some aspects in the special theory of gradient elasticity. J. Mech. Behaviour of Materials 8, 231-282.

Askes, H., Aifantis, E.C., 2011. Gradient elasticity in statics and dynamics: An overview of formulations, length scale identification procedures, finite element implementations and new results. Int. J. Solids Struct. 48, 1962-1990.

Brun, L. 1969. Méthods énergétiques dans les systèmes évolutifs linéaires. J. Mécanique, 8, 125-192.

Elishakoff, I., 2019. Who developed the so-called Timoshenko beam theory?. Mathematics and Mechanics of Solids, 25 (1), 97-116.

Eringen, A.C. 1999 Microcontinuum field theories. I. Foundations of solids. New York, NY: Springer-Verlag.

Forest, S., Aifantis, E.C., 2010. Some links between recent gradient thermo-elasto-plasticity theories and the thermomechanics of generalized continua. Int.J. Solid.Struct. 47, 3367-3376.

Goldstein, J., 1985. Semigroups of linear operators and applications. Oxford University Press, New York.

Gurtin, M.E., Anand, L., 2009. Thermodynamics applied to gradient theories involving the accumulated plastic strain: The theories of Aifantis and Fleck and Hutchinson and their generalization. J. Mech. Phys. Solids 57, 405-421.

Ha, C.S., Plesha, M.E., Lakes, R.S., 2016. Chiral three dimensional lattices with tunable Poissons ratio. Smart Mater. Struct. 25, 054005

Ieşan, D. 2004. Thermoelastic models of continua. Dordrecht, The Netherlands: Kluwer Academic Publishers.
Ieşan, D., 2010. Chiral effects in uniformly loaded rods.J. Mech. Phys. Solids 58 ,12721285.
Ieşan, D., 2013. Deformation of chiral rods in the strain gradient theory of thermoelasticity. Eur.J. Mech. A/Solids 37, 351-360 .
Ieşan, D., Quintanilla, R., 1994. Existence and continuous dependence results in the theory of interacting continua, J. Elasticity, 36, 85-98
Ieşan, D., Quintanilla, R., 2018. Qualitative properties in strain gradient thermoelasticity with microtemperatures, Mathematics and Mechanics of Solids, 23(2), 240-258.
Lagnese J.E. (1989) Uniform boundary stabilization of thermoelastic plates. In: Simon J. (eds) Control of Boundaries and Stabilization. Lecture Notes in Control and Information Sciences, vol 125. Springer, Berlin, Heidelberg. https://doi.org/10.1007/BFb0043359
Lakes, R.S., 1982. Dynamical study of couple stress effects in human compact bone. J. Biomech. Eng. 104,611 .
Lakes, R.S., Benedict, R.L., 1982. Noncentrosymmetry in micropolar elasticity. Int. J. Eng. Sci. 29, 11611167.
Lakes, R.S., Yoon, H.S., Katz, J.L., 1983. Slow compressional wave propagation in wet human and bovine cortical bone. Science 200, 513e515.
Lakes, R., 2001. Elastic and viscoelastic behaviour of chiral materials. Int. J. Mech. Sci. 43, 1579-1589.
Lakes, R., 2015. Third-rank piezoelectricity in isotropic chiral solids, Appl. Phys. Lett., 106, 212905
Lazopoulos, K.A., 2004. On the gradient strain elasticity theory of plates. Eur. J. Mech. A/Solids 23, 843-853.
Maranganti, R., Sharma, P., 2007. A novel atomistic approach to determine strain-gradient elasticity constants: Tabulation and comparison for various metals, semiconductors, silica, polymers and the (ir)relevance for nanotechnologies. J. Mech. Phys. Solids 55, 1823-1852.
Magana, A., Quintanilla, R. 2018. Exponential stability in type III thermoelasticity with microtemperatures. J. Appl. Math. Physics (ZAMP). 69, Article number: 129(1)-129(8).
Mindlin, R. D. 1964. Microstructure in linear elasticity. Arch. Rat. Mech. Anal. 16, 51-77.
Mindlin, R.D., Eshel, N.N., 1968. On first strain gradient theories in linear elasticity. Int. J. Solids Struct. 4, 109-124.
Nowacki, W., 1962. Thermoelasticity, Pergamon Press, Oxford, London, New York, Paris.
Papanicolopulos, S.A., 2011. Chirality in isotropic linear gradient elasticity. Int. J. Solids Struct. 48, 745-752.
Papargyri-Beskou, S., Giannakopoulos, A.E., Beskos, D.E., 2010. Variational analysis of gradient elastic flexural plates under static loading. Int. J. Solids Struct. 47, 2755-2766.

Ramezani, S., 2012. A shear deformation micro-plate model based on the most general form of strain gradient elasticity. Int. J. Mech. Sci. 57, 34-42.

Reasa, D.R. and R. S. Lakes, R.S., 2020. Nonclassical chiral elasticity of the gyroid lattice. Phys. Rev.Lett. 125, 205502.

Spadoni, A., Ruzzene, M., 2012. Elasto-static micropolar behaviour of a chiral auxetic lattice, J. Mech. Phys. Solids 60, 156-171.

Toupin, R.A., 1962. Elastic materials with couple stresses. Arch. Ration. Mech. Anal. 11, 385-414.

Toupin, R.A., 1964. Theories of elasticity with couple-stress. Arch. Ration. Mech. Anal. 17, 85-112.

Zhang, Y., Wang, C., Xiang, Y., 2010. A molecular dynamics investigation of the torsional responses of defective single-walled carbon nanotubes. Carbon 48, 4100-4108