BACHELOR'S DEGREE THESIS

# Degree in Mathematics

**Title: Optimal Stopping Problems as Free Boundary Problems and applications to finance**

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# Optimal Stopping Problems as Free Boundary Problems and applications to finance

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Thanks to my advisor Argimiro Arratia for introducing me to this interesting field, for all his dedication, for his constant orientation, and for sharing his passion.

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## Abstract

In this project, we present a methodology to transform Optimal Stopping Problems into Free Boundary Problems. The theory of Optimal Stopping can be found in fields such as statistics, theory of probability and mathematical finance. First of all, we include all the necessary concepts in order to understand this strategy, from the most basic definitions such as stochastic processes and Brownian motion to the most sophisticated results such as Dynkin's formula and the High Contact Principle. We also give three interesting applications, two of them from the area of mathematical finance. The third one is the most elaborated and it is about predicting resistance and support levels of an asset price. In this one, we also give an algorithm to calculate numerically the solution of the problem: the optimal stopping boundary.

### Keywords

Optimal stopping problem, free boundary problem, stochastic process, diffusion process, stopping time, Brownian motion, High Contact Principle, Dynkin's formula, financial asset modelling, resistance and support levels.

# Contents

# 1. Introduction

## 1.1 Motivation of the thesis

The main motivation for this project was the paper written by T. De Angelis and G. Peskir called *Optimal prediction of resistance and support levels* [2]. In order to understand all the important concepts which appear in this paper, I studied the book *Stochastic Differential Equations: An Introduction with Applications* of B. Øksendal [6] and I solved some exercises from each chapter. Several of them appear in this thesis since they are relevant. I also studied some parts from the book *Optimal Stopping and Free-Boundary Problems* of G. Peskir and A. Shiryaev [5].

## 1.2 Main goal

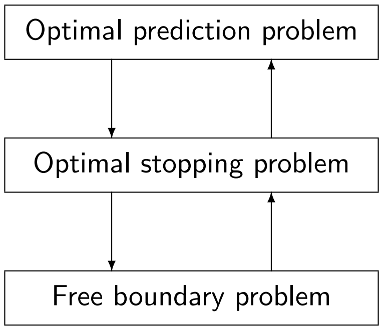
The principal tool which is explained in the project is transforming Optimal Stopping Problems into Free Boundary Problems. Solving the last ones gives us the solution of the first ones. The theory of optimal stopping treats the issue of deciding when to take a specific action to maximize an expected gain or minimize an expected cost. These type of problems can be found in fields of statistics, theory of probability and mathematical finance. In sections 9 and 10 we present three interesting applications of this methodology, two of them from the area of mathematical finance. The third one, which appears in section 10, is the most elaborated and difficult, it is about characterizing resistance and support levels of asset prices and it is based on the above mentioned paper by T. De Angelis and G. Peskir [2]. We study this article and we fill in some details that are not in the original paper. We also give an algorithm to compute numerically the solution to the problem: the optimal stopping boundary, and we attach the code on the Appendix.

## 1.3 Organization of the thesis

In section 2 we start by introducing the general scheme of solving optimal prediction problems. In section 3 we provide the definitions of stochastic process and stochastic differential equation. In section 4 we give the basic concepts of filtration, martingale and semimartingale. In section 5 we explain the concept of Brownian motion, Arithmetic Brownian motion and Geometric Brownian motion, and we present results from Itô calculus such as the Itô formula. In section 6 we introduce important theorems from stochastic analysis, for instance, the Itô-Tanaka-Meyer formula and the Optional Sampling theorem, which we will need in the third application mentioned previously. In section 7 we give some important properties of diffusion processes. In section 8 we present the main result to transform optimal stopping problems into free boundary problems. Finally, as we mentioned above, in sections 9 and 10 we include three applications of optimal stopping problems.

# 2. General scheme of solution for optimal prediction problems

We have a *optimal prediction problem* of stochastic equilibria and the method to solve this problem is based on reformulation to an *optimal stopping problem* and then reduction to a *free boundary problem*. The following scheme found in [5] illustrates this approach:




Downward is the way of reformulation and reduction. Upward is the way of finding a solution to the initial problem.

We need some results to explain the methodology of transforming an optimal stopping problem into a free boundary problem.

We consider a strong Markov process $X = (X_t)_{t\geq 0}$ and we assume that it starts at $x \in \mathbb{R}^d$.
Given a measurable function $G : \mathbb{R}^d \to \mathbb{R}$ such that

$$\mathbb{E}_x \left( \sup_{0\leq t\leq T} |G(X_t)| \right) < \infty \tag{1}$$

for all $x \in \mathbb{R}^d$, the optimal stopping problem of interest is

$$V(x) = \sup_{0\leq \tau\leq T} \mathbb{E}_x[G(X_\tau)] \tag{2}$$

where $x \in \mathbb{R}^d$ and the supremum is taken over all stopping times $\tau$ of $X$. Here we note that if $T = \infty$ we interpret $G(X_T) = 0$. In addition, note $\mathbb{E}_x[G(X_t)] = \mathbb{E}[G(X_t)|X_0 = x]$.

$V$ is called the *value function* and $G$ is known as the *gain function*. To solve the optimal stopping problem (OSP) (2) means two things:

1. We need to exhibit an optimal stopping time $\tau_*$ such that $V(x) = \mathbb{E}_x[G(X_{\tau_*})]$ when the supremum is attained.

2. We want an expression for $V(x)$ with $x \in \mathbb{R}^d$ that allows to compute its value as explicitly as possible.

We may think of $X_t$ as the state of a game at time $t$ and each $\omega \in \Omega$ corresponds to one sample of the game. At each time $t$ we have to decide either to stop the game, obtaining the gain $G(X_t)$, or to continue the game and stop later in order to try to get a bigger gain. The problem is that we do not know what state has the game in the future. So, among all possible stopping times $\tau$ we want to find the optimal one, $\tau_*$, which gives the biggest expected gain.

Since $X$ is Markovian, $Dom(V(x)) \subseteq \mathbb{R}^d$. Hence, following the previous idea, the state space $E := \mathbb{R}^d$ can be split into the *continuation set* $C$ and the *stopping set* $D = E \setminus C$. So, as soon as the observed value $X_t(\omega)$ enters $D$, the observation should be stopped and the optimal stopping time is attained. Therefore, the key issue is to define the sets $C$ and $D$.

Let us define the continuation set as

$$C = \{x \in E; V(x) > G(x)\} \tag{3}$$

and the stopping set as

$$D = \{x \in E; V(x) = G(x)\} \tag{4}$$

Let $\tau_D$ be the first entry time of $X$ into $D$

$$\tau_D = \inf\{t \geq 0; X_t \in D\} \tag{5}$$

Note that $\tau_D$ is a stopping time.

The following definition is fundamental in order to solve the optimal stopping problem (2).

**Definition 2.1.** A measurable function $F : E \to \mathbb{R}$ is *superharmonic* if

$$\mathbb{E}_x[F(X_\sigma)] \leq F(x) \tag{6}$$

for all stopping times $\sigma$ and all $x \in E$.

It is assumed in (6) that the left-hand side is well-defined (and finite).

In the following theorem, we give **necessary conditions** for the existence of an optimal stopping time.

**Theorem 2.2.** *Assume that there exists an optimal stopping time $\tau_*$ for the OSP (2), that is*

$$V(x) = \mathbb{E}_x[G(X_{\tau_*})] \tag{7}$$

*for all $x \in E$. Then:*

*The value function $V$ is the smallest superharmonic function which dominates $G$: $V(x) > G(x)$ $\forall x \in E$.*
*The stopping time $\tau_D$ satisfies $\tau_D \leq \tau_*$ a.s. for all $x \in E$ and is optimal in (2).*

Let us present another theorem that provides **sufficient conditions** for the existence of an optimal stopping time.

**Theorem 2.3.** *Consider the optimal stopping problem (2) and assume that the condition (1) is satisfied. Let us assume that there exists the smallest superharmonic function $\widehat{V}$ which dominates $G$ on $E$. We define $D = \{x \in E; \widehat{V}(x) = G(x)\}$ and $\tau_D$ by (5). Then:*

*i) If $\mathbb{P}_x(\tau_D < \infty) = 1$ for all $x \in E$, then $\widehat{V} = V$ and $\tau_D$ is an optimal stopping time in (2).*

*ii) If $\mathbb{P}_x(\tau_D < \infty) < 1$ for some $x \in E$, then there is no optimal stopping time (with probability 1) in (2).*

Theorem 2.2 and its reciprocal 2.3 show that the OSP (2) is equivalent to find the smallest superharmonic function $\widehat{V}$ which dominates the gain function $G$. Once $\widehat{V}$ is found we have that $V = \widehat{V}$ and $\tau_D = \inf\{t \geq 0 : X_t \in D\}$ is optimal, where $D = \{x \in E : \widehat{V}(x) = G(x)\}$.

To find $\widehat{V}$ we use the method of free boundary problem. The basic idea is that $\widehat{V}$ and $C$ (or $D$) solve the following free boundary problem (FBP):

$$\mathbb{L}_X \widehat{V} \leq 0 \tag{8}$$

$$\widehat{V} \geq G \quad (\widehat{V} > G \text{ on } C \ \& \ \widehat{V} = G \text{ on } D) \tag{9}$$

where $\mathbb{L}_x$ is the *infinitesimal generator* of $X$ (e.g. for a diffusion process we have $E = \mathbb{R}$ and $\mathbb{L}_X V = \mu(x)\frac{\partial V}{\partial x} + \frac{\sigma(x)^2}{2}\frac{\partial^2 V}{\partial x^2}$ ).

If $G$ is smooth in a neighbourhood of $\partial G$ (boundary of $G$), the FBP (8, 9) can be expressed as

$$\mathbb{L}_X \widehat{V} = 0 \text{ on } C \tag{10}$$

$$\frac{\partial \widehat{V}}{\partial x}\Big|_{\partial C} = \frac{\partial G}{\partial x}\Big|_{\partial C} \tag{11}$$

$$\widehat{V} > G \text{ on } C \ \& \ \widehat{V} = G \text{ on } D \tag{12}$$

These are sufficient conditions for solving the Optimal Stopping problem and they are known as High Contact Principle (or smooth fit principle). We will give more details in section 8. More aspects of this methodology are explained in references [5] and [6].

Let us see briefly this strategy applied to the problem from [2] which motivated this thesis. The financial problem is to determine when the price of a stock, represented by random variable $X_t$, reaches or hits a certain level (support or resistance), represented by random variable $l$. The **optimal prediction problem** of interest will be

$$V_*(x) = \inf_{0 \leq \tau \leq T} \mathbb{E}_x[|X_\tau - l|] \tag{13}$$

where $X$ is a geometric Brownian motion with $X_0 = x > 0$, $l > 0$ is a random variable independent from $X$ and $T > 0$ is the given horizon.

We will reformulate this problem to an optimal stopping problem using the following lemma (which will be proved later):

**Lemma 2.4.** *For all $x > 0$, we have:*

$$\mathbb{E}[|x - l|] = 2\int_0^x \left(F(y) - \frac{1}{2}\right) dy + \mathbb{E}[l] \tag{14}$$

*where $F$ is the distribution function of $l$, which is fixed and given.*

This lemma allows us to rewrite the optimal prediction problem (13) to the following **optimal stopping problem**

$$V(x) = \inf_{0\le\tau\le T} \mathbb{E}_x[G(X_\tau)] \tag{15}$$

where the gain function $G$ is defined for $x > 0$ as $G(x) = \int_0^x \left(F(y) - \frac{1}{2}\right) dy$ and we will have assumed that $\mathbb{E}[I] < \infty$.

We will write the previous problem as an extended optimal stopping problem in the time and space domain.

$$V(t,x) = \inf_{0\le\tau\le T-t} \mathbb{E}_{t,x}\left[G(X_{t+\tau})\right] \tag{16}$$

Finally, applying the results explained above and other important ones which we will see later on, we will obtain the following **free boundary problem**

$$\frac{\partial V}{\partial t}(t,x) + \mu x \frac{\partial V}{\partial x}(t,x) + \frac{\sigma^2}{2}x^2\frac{\partial^2 V}{\partial x^2}(t,x) = 0 \quad \text{for } x \in (0, b(t)) \text{ and } t \in [0, T) \tag{17}$$

$$V(t,x) = G(x) \quad \text{for } x \in [b(t), \infty) \text{ and } t \in [0, T) \tag{18}$$

$$\frac{\partial V}{\partial x}(t,x) = G'(x) \quad \text{for } x = b(t) \text{ and } t \in [0, T) \tag{19}$$

$$V(t,x) < G(x) \quad \text{for } x \in (0, b(t)) \text{ and } t \in [0, T) \tag{20}$$

$$V(T,x) = G(x) \quad \text{for } x \in (0, \infty). \tag{21}$$

where $b(t) = \min\{x \in (0,\infty); V(t,x) = G(x)\}$.

In the next sections, we will present some definitions and results that are necessary to understand the problem which we will be working with.

# 3. Stochastic Processes

**Definition 3.1.** A *stochastic process* is a collection of random variables $\{X_t; t \in T\}$, each one referred to an instant in time $t$, known as the *time-parameter*, which runs over an index set $T$, known as the *time-parameter set*. It is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega$ is a sample space, $\mathcal{F}$ is a $\sigma$-algebra on this sample space and $\mathbb{P}$ is a probability measure on $\mathcal{F}$.

*Remark* 3.2. We will abuse notation and write $X$ instead of $\{X_t; t \in T\}$. Other notations for $X$ are $\{X(t); t \in T\}$, $\{X_t\}_{t\in T}$, $(X_t)_{t\in T}$ or $X(t)$.

**Definition 3.3.** Let $X$ be a stochastic process. $X$ is called a *continuous-time stochastic process* when $T$ is a continuous set, for example when $T$ is an interval in the real line such as $T = [0,\infty)$. By contrast, $X$ is called a *discrete-time stochastic process* when $T$ is a discrete set (i.e. finite or countable), for instance when $T = \mathbb{Z}^+$ or $T = \mathbb{Z}$.

**Definition 3.4.** Let $X$ be a stochastic process. For any given $\omega \in \Omega$ fixed, as time passes, we get a sequence $\{X_t(\omega); t \in T\}$ which is called a *realization*, *sample path* or *trajectory* of $X$ corresponding to $\omega$.

Note that for each $t \in T$ fixed we have a random variable

$$\omega \mapsto X_t(\omega); \quad \omega \in \Omega.$$

To understand better the concept of a stochastic process intuitively, it may be useful to think of $t$ as "time" and each $\omega$ as an individual "particle" or "experiment". So, $X_t(\omega)$ represent the position (or result) at time $t$ of the particle (or experiment) $\omega$. Observe that we can also write $X(t,\omega)$ instead of $X_t(\omega)$, thus we may think of the process as a function of two variables

$$\begin{aligned} X : T \times \Omega &\to \mathbb{R}^n \\ (t,\omega) &\mapsto X(t,\omega) \end{aligned}$$

**Definition 3.5.** A *stochastic differential equation* is a differential equation in which we allow randomness in the coefficients, i.e., with one or more terms that are stochastic processes. So, its solution is also a stochastic process.

We will consider stochastic differential equations of the form

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \tag{22}$$

or in integral form

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s \tag{23}$$

where $B_t$ is a Brownian motion (we give the definition in section 5), and $b$ is called the *drift coefficient* and $\sigma$ the *diffusion coefficient*. A solution of these equations is called a *diffusion process* or *diffusion*.

# 4. Filtrations and martingales

**Definition 4.1.** A *filtration* on $(\Omega, \mathcal{F}, \mathbb{P})$ is an increasing family $(\mathcal{M}_t)_{t\geq 0}$ of $\sigma$-algebras of $\mathcal{F}$ such that for $0 \leq s \leq t$

$$\mathcal{M}_s \subseteq \mathcal{M}_t \subseteq \mathcal{F}$$

Intuitively, we can think of $\mathcal{M}_t$ as the historical information that we have up to time $t$.

**Definition 4.2.** The space $(\Omega, \mathcal{F}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P})$ is known as a *filtered probability space*.

We present the concept of martingale which is a stochastic process such that the conditional expected value of an observation at a future time $s$, given all the past information up to time $t$, is equal to the value of the observation at time $t$. More formally:

**Definition 4.3.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *martingale* with respect to the a filtration $(\mathcal{M}_t)_{t\geq 0}$ is a stochastic process $X$ such that

i) $X_t$ is $\mathcal{M}_t$-measurable $\forall t$

ii) $\mathbb{E}[|X_t|] < \infty \ \forall t$

iii) $\mathbb{E}[X_s|\mathcal{M}_t] = X_t \ \forall s \geq t$

Similarly, if i) and ii) hold and we have iii) $\mathbb{E}[X_s|\mathcal{M}_t] \leq X_t \ \forall s \geq t$, then $X$ is called a *supermartingale*. And if we have iii) $\mathbb{E}[X_s|\mathcal{M}_t] \geq X_t \ \forall s \geq t$, then $X$ is called a *submartingale*.

**Definition 4.4.** Let $(\mathcal{M}_t)_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$. A stochastic process $X$ on $\mathbb{R}^n$ is called $\mathcal{M}_t$-*adapted* if for each $t \geq 0$ the function $\omega \mapsto X_t(\omega)$ is $\mathcal{M}_t$-measurable.

**Definition 4.5.** Let $(\mathcal{M}_t)_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ and $X$ an $\mathcal{M}_t$-adapted stochastic process on $\mathbb{R}^n$. $X$ is called *local martingale* with respect to $(\mathcal{M}_t)_{t\geq 0}$ if there exists an increasing sequence of $\mathcal{M}_t$-stopping times $\tau_k$ such that

i) $\mathbb{P}(\lim_{k\to\infty} \tau_k = \infty) = 1$

ii) $X_{t\wedge\tau_k}$ is a martingale with respect to $(\mathcal{M}_t)_{t\geq 0}$ for all $k$

where $t \wedge \tau = \min(t, \tau)$ (we give the definition of a stopping time in section 7.1).

**Definition 4.6.** Let $X$ be a stochastic process. The *total variation* of $X$ on $[0, t]$ is defined by

$$V_t(X) = \lim_{\Delta t_j \to 0} \sum_{j=0}^{n-1} |X_{t_{j+1}} - X_{t_j}|$$

where $0 = t_0 < t_1 < ... < t_{n-1} < t_n = t$ is a partition and $\Delta t_j = t_{j+1} - t_j$. We say that $X$ is a process of *bounded variation* if the total variation of $X$ is finite.

**Definition 4.7.** Let $X$ be a stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P})$ with càdlàg trajectories, i.e. right-continuous with left limits. Then $X$ is called *semimartingale* with respect to $(\mathcal{M}_t)_{t\geq 0}$ if it admits a decomposition of the form

$$X = x + M + A \tag{24}$$

where $X_0 = x$ is finite, $M$ is a local martingale with respect to $(\mathcal{M}_t)_{t\geq 0}$ and $A$ is a process of bounded variation.

# 5. Brownian motion or Wiener process

The botanist Robert Brown was the first person who observed and documented the physical phenomenon of the erratic movement of particles suspended in a fluid, specifically he observed that pollen grains suspended in liquid performed an irregular motion. This motion takes the name of Brownian motion and to describe it mathematically it is natural to use the concept of a stochastic process. The mathematical formalization was due to Norbert Wiener and for this reason, it is also known as Wiener process.

**Definition 5.1.** A *Gaussian process* is a stochastic process $\{X_t; t \in T\}$ where $(X_{t_1}, ..., X_{t_k})$ follows a multivariate normal distribution for any choice of distinct values $t_1, ..., t_k \in T$,

**Definition 5.2.** A *(1-dimensional or standard) Wiener process* or *Brownian motion* $B_t$, $t \in \mathbb{R}$ is continuous-time Gaussian process such that

i) $B_0 = 0$

ii) the trajectories of $B_t$ are continuous functions

iii) it has independent and stationary increments $\{B_t - B_s; t < s\}$

iv) the increments have $\mathbb{E}[B_t - B_s] = 0$ and $\mathbb{V}\text{ar}[B_t - B_s] = |t - s|$, for $s, t \in \mathbb{R}$

One important consequence of the previous properties is that $B_t - B_s \sim N(0, |t - s|)$.

*Remark* 5.3. The Wiener process or Brownian motion can also have an initial value $B_0 = x$ with $x \in \mathbb{R}$.

**Definition 5.4.** A *n-dimensional Wiener process* or *Brownian motion* $B_t$, $t \in \mathbb{R}$ is a vector-valued stochastic process of the form

$$B_t = (B_t^{(1)}, ..., B_t^{(n)}) \tag{25}$$

where the components $B_t^{(i)}$, $1 \leq i \leq n$, are independent, 1-dimensional Brownian motions.

**Lemma 5.5.** *Let $B_t$ be a n-dimensional Brownian motion starting at a point $x \in \mathbb{R}^n$. Then, $B_t$ satisfies the next properties:*

*i)* $\mathbb{E}_x[B_t] = x$

*ii)* $\mathbb{E}_x[(B_t - x)^2] = nt$

*iii)* $\mathbb{E}_x[(B_t - x)(B_s - x)] = n \min(s, t)$

*Hence,*

*iv)* $\mathbb{E}_x[(B_t - B_s)^2] = n(t - s) \quad \text{if} \quad t \geq s$

*v)* $B_t - B_s \sim N(0, n(t - s)) \quad \text{if} \quad t \geq s$

*Proof.* Let us prove the last two properties which can be deduced from the three first ones.

iv) For $t \geq s$:

$$\begin{aligned}\mathbb{E}_x[(B_t - B_s)^2] &= \mathbb{E}_x[(B_t - x)^2 - 2(B_t - x)(B_s - x) + (B_s - x)^2] \\ &= \mathbb{E}_x[(B_t - x)^2] - 2\mathbb{E}_x[(B_t - x)(B_s - x)] + \mathbb{E}_x[(B_s - x)^2] \\ &= nt - 2ns + ns = n(t - 2s + s) = n(t - s)\end{aligned}$$

v) For $t \geq s$:

$$\begin{aligned}&\mathbb{E}_x[B_t - B_s] = \mathbb{E}_x[B_t] - \mathbb{E}_x[B_s] = x - x = 0 \\ &\mathbb{V}\text{ar}_x[B_t - B_s] = \mathbb{E}_x[(B_t - B_s)^2] - \mathbb{E}_x[B_t - B_s]^2 = n(t - s)\end{aligned}$$

Recall that a Brownian motion is a Gaussian process and hence $B_t - B_s \sim N(0, n(t - s))$. □

**Lemma 5.6.** *A n-dimensional Brownian motion $B_t = (B_t^{(1)}, ..., B_t^{(n)})$, $t \geq 0$ with initial value $x \in \mathbb{R}^n$ is a martingale with respect to the $\sigma$-algebras $\mathcal{F}_t$ generated by $\{B_s; s \leq t\}$.*

*Proof.* We need to see that i), ii) and iii) from Definition 4.3 hold. Note that i) is clearly satisfied, let us see the other two points:

ii) $\mathbb{E}_x[|B_t|]^2 \leq \mathbb{E}_x[|B_t|^2] = \mathbb{E}_x[|B_t - x|^2 - |x|^2 + 2|xB_t|] = nt - |x|^2 + 2|x|^2 = nt + |x|^2 < \infty \; \forall t \geq 0$
$\Rightarrow \mathbb{E}_x[|B_t|] < \infty \; \forall t \geq 0$
where we have used that $\mathbb{E}_x[|B_t - x|^2] = nt$ and $\mathbb{E}_x[B_t] = x$.

iii) $\mathbb{E}_x[B_s|\mathcal{F}_t] = \mathbb{E}_x[B_s - B_t + B_t|\mathcal{F}_t] = \mathbb{E}_x[B_s - B_t|\mathcal{F}_t] + \mathbb{E}_x[B_t|\mathcal{F}_t] \overset{(*)}{=} \mathbb{E}_x[B_s - B_t] + B_t = B_t \; \forall s \geq t$
where in $(*)$ we have used that $B_s - B_t$ is independent of $\mathcal{F}_t$ and that $B_t$ is $\mathcal{F}_t$-measurable. □

## 5.1 Arithmetic Brownian motion

**Definition 5.7.** Let $f$ be a function with continuous derivative $f'$ over the interval $[a, b]$. We define the *Wiener integral* of $f$ by

$$\int_a^b f(s)dB_s = [f(s)B_s]_a^b - \int_a^b B_s f'(s)ds \tag{26}$$

where $B_s$ is a Brownian motion.

**Definition 5.8.** The *generalized Wiener process* or *arithmetic Brownian motion* is a stochastic process $X$ which satisfies the following differential equation

$$dX_t = \alpha dt + \sigma dB_t \tag{27}$$

where $\alpha$ and $\sigma$ are constants and $B_t$ is Brownian motion with $B_0 = 0$.

*Remark* 5.9. Note that the previous equation is a stochastic differential equation (SDE).

**Lemma 5.10.** *The solution of the SDE (27) can be obtained taking Wiener integrals and it is*

$$X_t = X_0 + \alpha t + \sigma B_t \tag{28}$$

*Proof.* Taking Wiener integrals in (27) and using (26) with $f \equiv \sigma$ we obtain the claimed solution:

$$\int_0^t X_s = \int_0^t \alpha ds + \int_0^t \sigma dB_t \;\Leftrightarrow\; X_t - X_0 = \alpha t + [\sigma B_s]_0^t - \int_0^t 0 B_s ds \;\Leftrightarrow\; X_t = X_0 + \sigma B_t$$

□

**Lemma 5.11.** *The previous process satisfies the following properties:*

*i) For any choice of $s$ and $t$, with $s < t$,*

$$X_t - X_s \sim N(\alpha(t-s), \sigma^2(t-s)) \tag{29}$$

*ii) It has stationary and statistically independent increments $\{X_t - X_s; s < t\}$.*

*iii) The sample paths of the process are everywhere continuous, but nowhere differentiable with probability 1.*

*Proof.* We are going to prove the first point. Consider $s < t$ and use the expression (28) for an arithmetic Brownian motion:

$$X_t - X_s = (X_0 + \alpha t + \sigma B_t) - (X_0 + \alpha s + \sigma B_s) = \alpha(t-s) - \sigma(B_t - B_s)$$

Recall that $B_t - B_s \sim N(0, t-s)$, so we have

$$\mathbb{E}_x[X_t - X_s] = \mathbb{E}_x[\alpha(t-s) - \sigma(B_t - B_s)] = \alpha(t-s) - \sigma\mathbb{E}_x[B_t - B_s] = \alpha(t-s)$$
$$\mathbb{V}\text{ar}_x[X_t - X_s] = \mathbb{V}\text{ar}_x[\alpha(t-s) - \sigma(B_t - B_s)] = \sigma^2\mathbb{V}\text{ar}_x[B_t - B_s] = \sigma^2(t-s)$$

Hence, $X_t - X_s \sim N(\alpha(t-s), \sigma^2(t-s))$.

□

## 5.2 The Itô Integral and the Itô Formula

We now present the concept of *Itô integral*, very important in stochastic calculus. It allows us to integrate stochastic processes with respect to the increments of a Brownian motion.

**Definition 5.12.** Let $\{B_t\}_{t\geq 0}$ be a Brownian motion and $(\mathcal{F}_t)_{t\geq 0}$ a filtration generated by $\{B_s; s \leq t\}$. Consider a function $f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ such that

i) $f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ is the Borel $\sigma$-algebra on $[0,\infty)$

ii) $f(t,\omega)$ is $\mathcal{F}_t$-adapted

iii) $\mathbb{E}\left[\int_S^T f(t,\omega)^2 dt\right] < \infty$, for $0 \leq S < T$

Consider a partition $S = t_0 < t_1 < ... < t_{n-1} < t_n = T$ of the interval $[S, T]$. Then the *Itô integral* of $f$ is defined by

$$I[f](\omega) = \int_S^T f(t,\omega)dB_t(\omega) = \lim_{\Delta t_j \to 0} \sum_{j=0}^{n-1} f(t_j,\omega)[B_{t_{j+1}} - B_{t_j}](\omega) \tag{30}$$

where $\Delta t_j = t_{j+1} - t_j$ and the limit is meant in $\mathcal{L}^2$ sense.

**Lemma 5.13.** *The Itô integral has the following properties for* $0 \leq S < U < T$*:*

*i)* $\int_S^T f_t dB_t = \int_S^U f_t dB_t + \int_U^T f_t dB_t$

*ii)* $\int_S^T (cf_t + g_t) dB_t = c\int_S^T f_t dB_t + \int_S^T g_t dB_t$ *for* $c$ *constant*

*iii)* $\mathbb{E}\left[\int_S^T f_t dB_t\right] = 0$

*iv)* $M = \int_S^T f_t dB_t$ *is a martingale*

The Itô integral can also be defined more generally with respect to a semimartingale.

**Definition 5.14.** Let $X$ be a semimartingale and $(\mathcal{M}_t)_{t\geq 0}$ be a filtration generated by $\{X_s; s \leq t\}$. Consider a function $f(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ satisfying the same conditions of Definition 5.12 with the filtration $\mathcal{M}_t$ and consider a partition $S = t_0 < t_1 < ... < t_{n-1} < t_n = T$ of the interval $[S, T]$. Then the *Itô integral of $f$ with respect to the semimartingale $X$* is defined by

$$I[f](\omega) = \int_S^T f(t,\omega)dX_t(\omega) = \lim_{\Delta t_j \to 0} \sum_{j=0}^{n-1} f(t_j,\omega)[X_{t_{j+1}} - X_{t_j}](\omega) \tag{31}$$

where $\Delta t_j = t_{j+1} - t_j$ and the limit is meant in $\mathcal{L}^2$ sense.

**Definition 5.15.** Let $B_t$ be a 1-dimensional Brownian motion. A (1-dimensional) *Itô process* (or *stochastic integral*) is a stochastic process $X$ on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$X_t = X_0 + \int_0^t a(s,X_s)ds + \int_0^t b(s,X_s)dB_s \tag{32}$$

or, equivalently,

$$dX_t = a(t,X_t)dt + b(t,X_t)dB_t \tag{33}$$

where $a$ is (Lebesgue) integrable and $b$ is $B$-integrable, i.e. they satisfy

$$\mathbb{P}\left[\int_0^t b(s,X_s)^2 ds < \infty;\ \forall t \geq 0\right] = 1$$
$$\mathbb{P}\left[\int_0^t |a(s,X_s)| ds < \infty;\ \forall t \geq 0\right] = 1$$

**Definition 5.16.** An *n-dimensional Itô process* is a stochastic process $X$ in $\mathbb{R}^n$ of the form

$$dX(t) = adt + bdB(t)$$

where

$$X(t) = \begin{pmatrix} X_1(t) \\ \vdots \\ X_n(t) \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad b = \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}, \quad B(t) = \begin{pmatrix} B_1(t) \\ \vdots \\ B_m(t) \end{pmatrix}$$

with $a_i(t, X_i(t))$ and $b_{i,j}(t, X_i(t))$ satisfying the conditions given in Definition 5.15 for $1 \leq i \leq n$, $1 \leq j \leq m$.

**Theorem 5.17.** *(The 1-dimensional Itô formula). Let $X$ be an Itô process in $\mathbb{R}$ given by*

$$dX_t = a(t,X_t)dt + b(t,X_t)dB_t$$

*where $B_t$ is a Brownian motion. Let $g = g(t,x)$ be a differentiable function on $[0,\infty) \times \mathbb{R}$ and consider $Y_t = g(t,X_t)$. Then, $Y_t$ is again an Itô process and the following identity holds*

$$dY_t = \left(a(t,X_t)\frac{\partial g}{\partial x}(t,X_t) + \frac{\partial g}{\partial t}(t,X_t) + b(t,X_t)^2 \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)\right)dt + b(t,X_t)\frac{\partial g}{\partial x}(t,X_t)dB_t \tag{34}$$

**Theorem 5.18.** *(The general Itô formula). Let $X$ be an n-dimensional Itô process given by*

$$dX(t) = adt + bdB(t)$$

*where $B(t)$ is an m-dimensional Brownian motion. Let $g = g(t,x) = (g_1(t,x), ..., g_p(t,x))$ be a differentiable map from $[0,\infty) \times \mathbb{R}^n$ into $\mathbb{R}^p$ and consider $Y(t,\omega) = g(t, X(t))$. Then, $Y$ is again an Itô process and the following identity holds for every component $Y_k$ of $Y$:*

$$dY_k = \frac{\partial g_k}{\partial t}(t,X)dt + \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i}(t,X)dX_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,X)dX_i dX_j \tag{35}$$

*where $dB_i dB_j = \delta_{ij}dt$, $dB_i dt = dtdB_i = 0$.*

**Lemma 5.19.** *Let $X$ be an Itô process in $\mathbb{R}$. Then, $X$ is a semimartingale.*

*Proof.* Recall that $X$ has the form

$$X_t = X_0 + \int_0^t a(s,X_s)ds + \int_0^t b(s,X_s)dB_s$$

Note that $M_t = \int_0^t b(s,X_s)dB_s$ is a (local) martingale by the properties of Itô integral (see Lemma 5.13). Define $A_t := \int_0^t a(s,X_s)ds$, then we need to see that the total variation of $A$ is finite:

$$\begin{aligned}\sum_{j=0}^{n-1} |A_{t_{j+1}} - A_{t_j}| &\leq \sum_{j=0}^{n-1} \left| \int_0^{t_{j+1}} a(s,X_s)ds - \int_0^{t_j} a(s,X_s)ds \right| = \sum_{j=0}^{n-1} \left| \int_{t_j}^{t_{j+1}} a(s,X_s)ds \right| \\ &\leq \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} |a(s,X_s)|ds = \int_0^t |a(s,X_s)|ds < \infty \text{ with probability 1}\end{aligned}$$

for all partitions $0 = t_0 < t_1 < ... < t_{n-1} < t_n = t$. Hence, we have

$$X_t = X_0 + A_t + M_t$$

with $A$ a process of bounded variation and $M$ a (local) martingale. Therefore, $X$ is a semimartingale. □

## 5.3 Geometric Brownian Motion

**Definition 5.20.** A *geometric Brownian motion* is the continuous model of the form $P_t = e^{X_t}$, where $X_t$ is an arithmetic Brownian motion.

Note that the arithmetic Brownian motion can take negative and positive values because it follows a normal distribution. But, we may want to allow only positive values, for example, to model the price of a stock, and therefore it is a better option to consider a geometric Brownian motion that we have just defined.

**Lemma 5.21.** *The geometric Brownian motion satisfies the following stochastic differential equation*

$$dP_t = \mu P_t dt + \sigma P_t dB_t \tag{36}$$

*where $\mu = \alpha + \frac{1}{2}\sigma^2$ and $B_t$ is a Brownian motion.*

*Proof.* Consider an arithmetic Brownian motion $X$ satisfying the SDE $dX_t = a(t, X_t)dt + b(t, X_t)dB_t$ with $a \equiv \alpha$ and $b \equiv \sigma$, hence it is an Itô process. We choose $g(t, x) = e^x$ and we set $Y_t = g(t, X_t) = e^{X_t} = P_t$, which is again an Itô process by Theorem 5.17. Note that

$$\frac{\partial g}{\partial t}(t, x) = 0, \quad \frac{\partial g}{\partial x}(t, x) = \frac{\partial^2 g}{\partial x^2}(t, x) = e^x$$

Then by Itô's formula (34),

$$\begin{aligned}
dY_t &= \left(\alpha e^{X_t} + 0 + \sigma^2 \frac{1}{2} e^{X_t}\right) dt + \sigma e^{X_t} dB_t \\
\Leftrightarrow\ dP_t &= \left(\alpha P_t + \sigma^2 \frac{1}{2} P_t\right) dt + \sigma P_t dB_t \\
\Leftrightarrow\ dP_t &= \left(\alpha + \frac{1}{2}\sigma^2\right) P_t dt + \sigma P_t dB_t \\
\Leftrightarrow\ dP_t &= \mu P_t dt + \sigma P_t dB_t
\end{aligned}$$

□

**Lemma 5.22.** *The strong solution of the SDE (36) is the following one:*

$$P_t = P_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t} \tag{37}$$

*Proof.* We can prove this fact directly using the expression for an arithmetic Brownian motion $X$ (28):

$$P_t = e^{X_t} = e^{X_0 + \alpha t + \sigma B_t} = P_0 e^{\alpha t + \sigma B_t} = P_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t}$$

□

*Remark* 5.23. Let $X$ be a geometric Brownian motion starting at a point $x \in \mathbb{R}$, i.e. $X_0 = x$. So, in this starting point we have the following expression for $X_t$:

$$X_0 = x e^{(\mu - \sigma^2/2)0 + \sigma B_0} = x e^{\sigma B_0}$$

Then, since $X_0 = x$ we need to have $B_0 = 0$, hence the Brownian motion $B_t$ starts at point $x = 0$.

**Lemma 5.24.** *Consider a Brownian motion $\{B_t\}_{t \geq 0}$ starting at a point $x \in \mathbb{R}$. Then, it follows the next property:*

$$\mathbb{E}_x[e^{\sigma B_t}] = e^{\sigma x + \frac{1}{2}\sigma^2 t} \tag{38}$$

*Proof.* We present two methods for proving this equality:

1) Recall that if $Z$ is a random variable which follows a normal distribution, $Z \sim N(\mu, \sigma^2)$, then its moment-generating function is

$$M_Z(t) := \mathbb{E}[e^{tZ}] = e^{t\mu + \frac{1}{2}\sigma^2 t^2}$$

We know that if $B_t$ is a Brownian motion in $\mathbb{R}$ starting at a point $x$ then

$$\begin{aligned}
\mathbb{E}_x[B_t] &= x \\
\mathbb{E}_x[(B_t - x)^2] &= t
\end{aligned}$$

and $B_t$ follows a normal distribution. So we have

$$\mathbb{E}_x[B_t^2] = \mathbb{E}_x[(B_t - x)^2 - x^2 + 2xB_t] = t - x^2 + 2x^2 = t + x^2$$
$$\Rightarrow \mathbb{V}\text{ar}_x[B_t] = \mathbb{E}_x[B_t^2] - \mathbb{E}_x[B_t]^2 = t + x^2 - x^2 = t$$

Hence, $B_t \sim N(x, t)$. Therefore, the moment-generating function of $B_t$ is

$$\mathbb{E}[e^{\sigma B_t}] = e^{\sigma x + \frac{1}{2}\sigma^2 t}$$

2) In this second method we are going to use Itô's formula. We choose $X_t = B_t$, i.e. $dX_t = dB_t$ where $B_t$ is a Brownian motion in $\mathbb{R}$ starting at a point $x$. So if we think of $X$ satisfying $dX_t = a(t, X_t)dt + b(t, X_t)dB_t$ we have $a \equiv 0$ and $b \equiv 1$.

Set $g(t, x) = e^{\sigma x}$, then

$$Y_t = g(t, X_t) = g(t, B_t) = e^{\sigma B_t}$$

Note that

$$\frac{\partial g}{\partial t}(t, x) = 0, \quad \frac{\partial g}{\partial x}(t, x) = \sigma e^{\sigma x}, \quad \frac{\partial^2 g}{\partial x^2}(t, x) = \sigma^2 e^{\sigma x}$$

Hence, applying Itô's formula (34) we obtain

$$dY_t = \frac{1}{2}\sigma^2 e^{\sigma B_t} dt + \sigma e^{\sigma B_t} dB_t$$

which is equivalent to

$$Y_t = Y_0 + \frac{1}{2}\sigma^2 \int_0^t e^{\sigma B_s} ds + \sigma \int_0^t e^{\sigma B_s} dB_s$$

Taking the expected value in the previous expression we get

$$\mathbb{E}_x[Y_t] = \mathbb{E}_x[Y_0] + \frac{1}{2}\sigma^2 \mathbb{E}_x\left[\int_0^t e^{\sigma B_s} ds\right] + \sigma \mathbb{E}_x\left[\int_0^t e^{\sigma B_s} dB_s\right]$$

Recall that a property of the Itô integral (see Lemma 5.13) is that the expected value of the integral is 0, so we have

$$\mathbb{E}_x\left[\int_0^t e^{\sigma B_s} dB_s\right] = 0$$

Hence,

$$\mathbb{E}_x[Y_t] = \mathbb{E}_x[Y_0] + \frac{1}{2}\sigma^2 \mathbb{E}_x\left[\int_0^t Y_s ds\right] = \mathbb{E}_x[Y_0] + \frac{1}{2}\sigma^2 \int_0^t \mathbb{E}_x[Y_s] ds$$

which yields

$$\frac{d}{dt}\mathbb{E}_x[Y_t] = \frac{1}{2}\sigma^2 \mathbb{E}_x[Y_t]$$
$$\mathbb{E}_x[Y_0] = \mathbb{E}_x[e^{\sigma B_0}] = e^{\sigma x}$$

If we solve this ODE we obtain:

$$\mathbb{E}_x[Y_t] = Ce^{\frac{1}{2}\sigma^2 t}, C \in \mathbb{R}$$
$$\mathbb{E}_x[Y_0] = Ce^{\frac{1}{2}\sigma^2 0} = C = e^{\sigma x} \Rightarrow \mathbb{E}_x[Y_t] = e^{\sigma x + \frac{1}{2}\sigma^2 t}$$

Therefore

$$\mathbb{E}_x[e^{\sigma B_t}] = e^{\sigma x + \frac{1}{2}\sigma^2 t}$$

which is the same result obtained with the first method. □

**Lemma 5.25.** *Let $X$ be a geometric Brownian motion starting at a point $x \in \mathbb{R}$. Then, for $t \in [0, \infty)$*

*i)* $\mathbb{E}_x[X_t] = xe^{\mu t}$

*ii)* $\mathbb{E}_x[X_t^k] = x^k e^{k(\mu-\sigma^2/2)t} e^{\frac{1}{2}k^2\sigma^2 t}$ *for* $k \in \mathbb{N}$

*iii)* $\mathbb{V}ar_x[X_t] = x^2 e^{2\mu t}(e^{\sigma^2 t} - 1)$

*Proof.* We are going to use that, from Lemma 5.22, $X_t$ can be written as

$$X_t = x\exp((\mu - \sigma^2/2)t + \sigma B_t)$$

i) For the first expected value we have

$$\mathbb{E}_x[X_t] = \mathbb{E}[xe^{(\mu-\sigma^2/2)t+\sigma B_t}] = xe^{(\mu-\sigma^2/2)t}\mathbb{E}[e^{\sigma B_t}]$$

where $\mathbb{E} = \mathbb{E}_0$. Note that $B_t$ starts at point $x = 0$ (see Remark 5.23), so using Lemma 5.24 we get

$$\mathbb{E}[e^{\sigma B_t}] = e^{\frac{1}{2}\sigma^2 t}$$

Hence,

$$\mathbb{E}_x[X_t] = xe^{(\mu-\sigma^2/2)t}e^{\frac{1}{2}\sigma^2 t} = xe^{\mu t}$$

as it was claimed.

ii) We need to calculate $\mathbb{E}_x[X_t^k]$ for $k \in \mathbb{N}$:

$$\mathbb{E}_x[X_t^k] = \mathbb{E}[(xe^{(\mu-\sigma^2/2)t+\sigma B_t})^k] = \mathbb{E}[x^k e^{k(\mu-\sigma^2/2)t+k\sigma B_t}] = x^k e^{k(\mu-\sigma^2/2)t}\mathbb{E}[e^{k\sigma B_t}]$$

Using again equation (38) applied to $\mathbb{E}[e^{k\sigma B_t}]$ we obtain

$$\mathbb{E}_x[X_t^k] = x^k e^{k(\mu-\sigma^2/2)t} e^{\frac{1}{2}k^2\sigma^2 t}$$

iii) To compute the variance we can use points i) and ii):

$$\begin{aligned}\mathbb{V}\text{ar}_x[X_t] &= \mathbb{E}_x[X_t^2] - \mathbb{E}_x[X_t]^2 = x^2 e^{2(\mu-\sigma^2/2)t} e^{\frac{1}{2}2^2\sigma^2 t} - x^2 e^{2\mu t} = x^2 e^{2\mu t - \sigma^2 t + 2\sigma^2 t} - x^2 e^{2\mu t} \\ &= x^2 e^{2\mu t}(e^{\sigma^2 t} - 1)\end{aligned}$$

□

# 6. Other important results of Stochastic Analysis

## 6.1 Integration by parts formula for Itô processes

**Lemma 6.1.** *Let $X$ and $Y$ be Itô processes in $\mathbb{R}$. Then*

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t$$

*and*

$$\int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int dX_s \cdot dY_s \tag{39}$$

*which is known as the general* integration by parts formula *for Itô processes.*

*Proof.* Note that $(X_t, Y_t)$ is a 2-dimensional Itô process and let $g(t, x, y) = xy$. Then

$$Z_t = g(t, X_t, Y_t) = X_tY_t$$

Note that

$$\frac{\partial g}{\partial t}(t,x,y) = 0, \quad \frac{\partial g}{\partial x}(t,x,y) = y, \quad \frac{\partial g}{\partial y}(t,x,y) = x,$$
$$\frac{\partial^2 g}{\partial x \partial y}(t,x,y) = \frac{\partial^2 g}{\partial y \partial x}(t,x,y) = 1, \quad \frac{\partial^2 g}{\partial x^2}(t,x,y) = \frac{\partial^2 g}{\partial y^2}(t,x,y) = 0$$

Then by the general Itô formula (35),

$$\begin{aligned} dZ_t =& \frac{\partial g}{\partial t}(t, X_t, Y_t)dt + \frac{\partial g}{\partial x}(t, X_t, Y_t)dX_t + \frac{\partial g}{\partial y}(t, X_t, Y_t)dY_t + \frac{1}{2}\frac{\partial^2 g}{\partial x \partial y}(t, X_t, Y_t)dX_t \cdot dY_t + \\ & \frac{1}{2}\frac{\partial^2 g}{\partial y \partial x}(t, X_t, Y_t)dY_t \cdot dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t, Y_t)(dX_t)^2 + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t, X_t, Y_t)(dY_t)^2 \\ =& Y_t dX_t + X_t dY_t + \frac{1}{2}dX_t dY_t + \frac{1}{2}dY_t \cdot dX_t \end{aligned}$$

Therefore,

$$d(X_tY_t) = Y_t dX_t + X_t dY_t + dX_t \cdot dY_t$$

as it was claimed.

Finally, taking integrals

$$\begin{aligned} & \int_0^t d(X_sY_s) = \int_0^t Y_s dX_s + \int_0^t X_s dY_s + \int_0^t dX_s \cdot dY_s \\ \Leftrightarrow\ & X_tY_t - X_0Y_0 = \int_0^t Y_s dX_s + \int_0^t X_s dY_s + \int_0^t dX_s \cdot dY_s \\ \Leftrightarrow\ & \int_0^t X_s dY_s = X_tY_t - X_0Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s \cdot dY_s \end{aligned}$$

and we have obtained the general integration by parts formula as we wanted. □

## 6.2 Itô-Tanaka-Meyer formula

**Definition 6.2.** Let $X$ be a stochastic process. The *quadratic variation* of $X$ is defined by

$$[X]_t = \lim_{\Delta t_j \to 0} \sum_{j=0}^{n-1} (X_{t_{j+1}} - X_{t_j})^2$$

where $0 = t_0 < t_1 < ... < t_{n-1} < t_n = t$ is a partition and $\Delta t_j = t_{j+1} - t_j$.

**Definition 6.3.** Let $X$ and $Y$ be two stochastic processes. The *quadratic covariation* of $X$ and $Y$ is defined by

$$[X, Y]_t = \lim_{\Delta t_j \to 0} \sum_{j=0}^{n-1} (X_{t_{j+1}} - X_{t_j})(Y_{t_{j+1}} - Y_{t_j})$$

where $0 = t_0 < t_1 < ... < t_{n-1} < t_n = t$ is a partition and $\Delta t_j = t_{j+1} - t_j$.

**Lemma 6.4.** *If $X$ and $Y$ are semimartingales then*

$$d[X, Y]_t = dX_t \cdot dY_t \tag{40}$$

*Proof.* Consider a partition $0 = t_0 < t_1 < ... < t_{n-1} < t_n = t$ and $\Delta t_j = t_{j+1} - t_j$. Then, we can write

$$\begin{aligned} X_t Y_t &= X_0 Y_0 + \sum_{j=0}^{n-1} (X_{t_{j+1}} Y_{t_{j+1}} - X_{t_j} Y_{t_j}) \\ &= X_0 Y_0 + \sum_{j=0}^{n-1} X_{t_j}(Y_{t_{j+1}} - Y_{t_j}) + \sum_{j=0}^{n-1} Y_{t_j}(X_{t_{j+1}} - X_{t_j}) + \sum_{j=0}^{n-1} (X_{t_{j+1}} - X_{t_j})(Y_{t_{j+1}} - Y_{t_j}) \end{aligned}$$

Since $X$ and $Y$ are semimartingales, the Itô integral exists. So taking limits as $\Delta t_j \to 0$, we obtain

$$X_t Y_t = X_0 Y_0 + \int_0^t X_{s-} dY_s + \int_0^t Y_{s-} dX_s + [X, Y]_t$$

which is the *integration by parts formula for semimartingales*. This can be written in differential form as

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + d[X, Y]_t \tag{41}$$

On the other hand, we know that Itô's formula works well for semimartingales and hence applying the same argument as we did in proof from Lemma 6.1 we get

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + dX_t \cdot dY_t \tag{42}$$

Therefore, from (41) and (42) we finally obtain

$$d[X, Y]_t = dX_t \cdot dY_t$$

□

**Theorem 6.5.** *(Itô-Tanaka-Meyer formula). If $B = \{B_t\}_{t\geq 0}$ is a Brownian motion and $F = F(x)$ is a function such that its derivative $F'(x)$ has bounded variation then*

$$F(B_t) = F(B_0) + \int_0^t F'(B_s)dB_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a F''(da)$$

*where $L_t^a$ is the* local time *that the Brownian motion $B$ "spends" at level $a$ up to time $t$:*

$$L_t^a = \lim_{\epsilon\to 0^+} \frac{1}{2\epsilon}\int_0^t \mathbb{I}_{\{|B_s-a|\leq\epsilon\}} ds$$

*with $\mathbb{I}$ denoting the characteristic (indicator) function.*

**Corollary 6.6.** *If $X$ is a continuous semimartingale and $F = F(x)$ is a concave (convex or the difference of the two) function, then the Itô-Tanaka-Meyer formula takes the form*

$$F(X_t) = F(X_0) + \int_0^t \frac{1}{2}(F'_+(X_s) + F_-(X_s))dX_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a(X)F''(da) \tag{43}$$

*where $L_t^a(X)$ is the local time at level $a$ of $X$ over $[0, t]$ (i.e. the amount of time $X$ has "spent" at a given level $a$) defined as*

$$L_t^a(X) = \lim_{\epsilon\to 0^+} \frac{1}{2\epsilon}\int_0^t \mathbb{I}_{\{|X_s-a|\leq\epsilon\}} d[X]_s \tag{44}$$

*where $[X]$ is the quadratic variation of $X$.*

**Corollary 6.7.** *(Occupation times formula). Let $X$ be a continuous semimartingale and $\Phi = \Phi(x)$ a positive Borel function, then*

$$\int_0^t \Phi(X_s)d[X]_s = \int_{\mathbb{R}} \Phi(a)d_aL_t^a(X) \tag{45}$$

*where $L_t^a(X)$ is the local time at $a$ of $X$ over $[0, t]$.*

*Remark* 6.8. Notation:

$$\int_{\mathbb{R}} L_s^a(X)\Phi(da) = \int_{\mathbb{R}} \Phi(a)d_aL_s^a(X) \tag{46}$$

## 6.3 The Optional Sampling theorem

**Theorem 6.9.** *(The optional sampling theorem). Let $X$ be a submartingale (or martingale) with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ and consider two stopping times $\tau$ and $\sigma$ such that*

- $\mathbb{E}[|X_\tau|] < \infty$
- $\mathbb{E}[|X_\sigma|] < \infty$
- $\liminf_{t\to\infty} \mathbb{E}[\mathbb{I}_{\tau>t}|X_t] = 0$

*Then*

*i)* $\mathbb{E}[X_\tau|\mathcal{F}_\sigma] \geq (=)\ X_\sigma$ *a.s. on the set* $\{\tau \geq \sigma\}$

*ii)* $\mathbb{E}[X_\tau] \geq (=)\ \mathbb{E}[X_\sigma]$ *if* $\mathbb{P}(\tau \geq \sigma) = 1$

# 7. Properties of diffusions

## 7.1 Markov time and stopping time

**Definition 7.1.** Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P})$ and let $\tau : \Omega \to [0, \infty]$ be a random variable. Then $\tau$ is called *stopping time* with respect to the filtration $(\mathcal{M}_t)_{t\geq 0}$ if

$$\{\omega; \tau(\omega) \leq t\} \in \mathcal{M}_t, \quad \text{for all } t \geq 0 \tag{47}$$

Intuitively, this means that we should be able to determine whether or not the event $\tau \leq t$ has occurred based only on the knowledge of $\mathcal{M}_t$, not on any future information.

**Definition 7.2.** Consider a stochastic process $X$ on a filtered space $(\Omega, \mathcal{F}, (\mathcal{M}_t)_{t\geq 0}, \mathbb{P})$. We define a *discrete Markov time* for $X$ as a non-negative integer-valued random variable $\tau$ satisfying that for each $t \geq 0$, the event $\{\tau = t\}$ depends only on $\{X_0, X_1, ..., X_t\}$ and not on $\{X_{t+s}; s \geq 1\}$.

A *discrete stopping time* is a (discrete) Markov time $\tau$ such that it has probability one of being finite, i.e. $\mathbb{P}(\tau < \infty) = 1$.

**Example 7.3.** Let $X$ be a stochastic process and let $U \in \mathbb{R}^n$ be a measurable set. Then we define the *first exit time from $U$* as follows

$$\tau_U = \inf\{t > 0; X_t \notin U\} \tag{48}$$

And $\tau_U$ is a stopping time.

## 7.2 Markov property and strong Markov property

**Theorem 7.4.** *(The Markov property for diffusions). Consider a diffusion process $X$ on $\mathbb{R}^n$ such that for $t \geq 0$*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t; \quad X_0 = x \tag{49}$$

*where $B_t$ is an m-dimensional Brownian motion, $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded Borel function and consider $\mathcal{F}_t$ as the $\sigma$-algebra generated by $\{B_s; s \leq t\}$. Then, for $t, h \geq 0$*

$$\mathbb{E}_x[f(X_{t+h})|\mathcal{F}_t] = \mathbb{E}_{X_t}[f(X_h)] \tag{50}$$

Note that we can think of $\mathcal{F}_t$ as the history of $B_s$ up to time $t$. Then, the Markov property states that the future behaviour of the process given what has happened up to time $t$ is the same as the behaviour obtained when starting the process at $X_t$. So, the process is memoryless, i.e. the future state does not depend on the past.

We say that $X$ is a *Markov process* with respect to the family of $\sigma$-algebras $(\mathcal{F}_t)_{t\geq 0}$.

**Theorem 7.5.** *(The strong Markov property for diffusions). Let $X$ be a diffusion process on $\mathbb{R}^n$ starting at $X_0 = x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ a bounded Borel function. Consider $\mathcal{F}_t$ as the $\sigma$-algebra generated by $\{B_s; s \leq t\}$ where $B_s$ is an m-dimensional Brownian motion and let $\tau$ be a stopping time with respect to $\mathcal{F}_t$ such that $\tau < \infty$ a.s. Then, for $h \geq 0$*

$$\mathbb{E}_x[f(X_{\tau+h})|\mathcal{F}_\tau] = \mathbb{E}_{X_\tau}[f(X_h)] \tag{51}$$

*where $\mathcal{F}_\tau$ is defined as the $\sigma$-algebra generated by $\{B_{s\wedge\tau}; s \geq 0\}$.*

Note that the strong Markov property is the same as the Markov property but considering stopping times.

We say that $X$ is a *strong Markov process* with respect to the family of $\sigma$-algebras $(\mathcal{F}_t)_{t\geq 0}$.

## 7.3 Hitting distribution

**Definition 7.6.** Let $U$ and $V$ be subsets of $\mathbb{R}^n$. Then, $U$ is *compactly embedded* in $V$, $U \subset\subset V$, if $U \subseteq \overline{U} \subseteq V^\circ$ and $\overline{U}$ is compact.

**Definition 7.7.** Let $H \in \mathbb{R}^n$ be a measurable set and let $X$ be a diffusion on $\mathbb{R}^n$ starting at a point $x \in \mathbb{R}^n$ in the interior of $H$. Consider $\tau_H$ as the first exit time from $H$. If we consider $G \subset\subset H$ measurable, then we define *the harmonic measure* (or *hitting distribution*) of $X$ on $\partial G$, $\mu_G^x$, by

$$\mu_G^x(F) = Q_x[X_{\tau_G} \in F] \tag{52}$$

for $F \subset \partial G$ and $x \in G$, where $\tau_G$ is the first exit time from $G$ and $Q_x$ is the probability law of $\{X_t\}_{t\geq 0}$ with $X_0 = x$.

Intuitively, the harmonic measure of $X$ on $\partial G$ is the probability that the first contact of $X$ with $\partial G$ is somewhere in the set $F$. So, it describes the distribution of $X$ as it hits the boundary of $G$.

**Lemma 7.8.** *With the previous definition, we have that if $f$ is a bounded measurable function, then the function $\phi(x) = \mathbb{E}_x[f(X_{\tau_H})]$ satisfies* the mean value property*:*

$$\phi(x) = \int_{\partial G} \phi(y) d\mu_G^x(y) \tag{53}$$

*for all $x \in G$ and for all Borel sets $G \subset\subset H$.*

Note that the mean value property is an analogous result in stochastic analysis to the mean value theorem for integrals in calculus.

We will also make use of the probabilistic analogue of the general mean value theorem.

*Remark* 7.9. (Mean value theorem). Consider two non-negative random variables $X$, $Y$ such that $\mathbb{E}[X] < \mathbb{E}[Y] < \infty$. Let $f(x)$ be a measurable and differentiable function with its derivative $f'(x)$ being also measurable and satisfying $\mathbb{E}[f(X)], \mathbb{E}[f(Y)] < \infty$. Then, there exists a non-negative random variable $Z$ such that

$$\mathbb{E}[f(Y)] - \mathbb{E}[f(X)] = \mathbb{E}[f'(Z)](\mathbb{E}[Y] - \mathbb{E}[X]) \tag{54}$$

## 7.4 Diffusion process generator and Dynkin's formula

We present an important concept in stochastic analysis: a partial differential operator that encodes a large amount of information about a diffusion.

**Definition 7.10.** Let $X$ be a diffusion process in $\mathbb{R}^n$. The *infinitesimal generator* (or simply *generator*) of $X$ is the operator $A$ defined as

$$Af(x) = \lim_{t\to 0^+} \frac{\mathbb{E}_x[f(X_t)] - f(x)}{t} \tag{55}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$.

The relation between $A$ and the coefficients $\sigma$, $b$ in the SDE (56) of a diffusion process $X$ is given in the next Theorem.

**Theorem 7.11.** *Consider a diffusion process $X$ in $\mathbb{R}^n$ such that*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t \tag{56}$$

*Let $f \in C_0^2(\mathbb{R}^n)$, i.e. $f \in C^2(\mathbb{R}^n)$ and has compact support. Then, the infinitesimal generator of $X$ is*

$$Af(x) = \sum_{i=1}^{n} b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j} \tag{57}$$

*where $a = \sigma\sigma^T$.*

*Remark* 7.12. We will denote the differential operator which appears on the right-hand side of equation (57) as $\mathbb{L}$ or $\mathbb{L}_X$.

**Example 7.13.** The $n$-dimensional Brownian motion $B_t$ is the solution of the SDE

$$dX_t = dB_t$$

So, it satisfies (56) with $b = 0$ and $\sigma = I$ where $I$ the identity matrix in $\mathbb{R}^n$. Then, applying Theorem 7.11, we have that the infinitesimal generator of $B_t$ is

$$Af = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 f}{\partial x_i^2} = \frac{1}{2}\Delta f$$

where $f \in C_0^2(\mathbb{R}^n)$ and $\Delta$ is the Laplace operator or Laplacian.

**Example 7.14.** Consider a geometric Brownian motion $X$ given by the SDE

$$dX_t = \mu X_t dt + \sigma X_t dB_t$$

where $B_t$ is a (1-dimensional) Brownian motion and $\mu, \sigma$ are constants. We have that the process satisfies (56) with $b(x) = \mu x$ and $\sigma(x) = \sigma x$. Then, using formula (57) with $n = 1$ we obtain that the generator of $X_t$ is

$$Af(x) = \mu x f'(x) + \frac{1}{2}\sigma^2 x^2 f''(x)$$

for $f \in C_0^2(\mathbb{R})$.

Now we present an important theorem in stochastic analysis known as Dynkin's formula, which gives the expected value of a diffusion process at a stopping time. We can interpret it as a stochastic generalization of the (second) fundamental theorem of calculus.

**Theorem 7.15.** *(Dynkin's formula). Let $X$ be a diffusion on $\mathbb{R}^n$ and $f \in C_0^2(\mathbb{R}^n)$. Assume that $\tau$ is a stopping time with $\mathbb{E}_x[\tau] < \infty$. Then Dynkin's formula holds:*

$$\mathbb{E}_x[f(X_\tau)] = f(x) + \mathbb{E}_x\left[\int_0^\tau Af(X_s)ds\right] \tag{58}$$

*where $A$ is the infinitesimal generator of $X$.*

This formula is one of the keys to solve optimal stopping problems and in section 8, we will use it to transform an optimal stopping problem into a free boundary problem.

Now we introduce the following operator which is very related to the infinitesimal generator $A$.

**Definition 7.16.** Let $X$ be a diffusion process in $\mathbb{R}^n$. Consider open sets $U_k$ such that $U_{k+1} \subset U_k$ and $\bigcap_k U_k = \{x\}$ and let $\tau_U$ be the first exit from $U$ for $X$: $\tau_U = \inf\{t > 0; X_t \notin U\}$. Then the *characteristic operator* $\mathcal{A}$ of $X$ is defined as

$$\mathcal{A}f(x) = \lim_{U \to x^+} \frac{\mathbb{E}_x[f(X_{\tau_U})] - f(x)}{\tau_U} \tag{59}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$.

The next theorem states that $A$ and $\mathcal{A}$ coincide in $C^2$.

**Theorem 7.17.** *Consider a diffusion process $X$ in $\mathbb{R}^n$. Let $f \in C^2(\mathbb{R}^n)$, then*

$$\mathcal{A}f = Af = \mathbb{L}f = \sum_{i=1}^{n} b_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} \tag{60}$$

Finally, we present a useful criterion that we will use to solve the optimal stopping problem from section 9.2.

**Lemma 7.18.** *Let $X$ be a diffusion on $\mathbb{R}^n$ and $f \in C^2(\mathbb{R}^n)$. Then $f$ is superharmonic with respect to $X$ if and only if $\mathcal{A}f \leq 0$.*

*Proof.* Recall that $f$ is superharmonic if

$$f(x) \geq \mathbb{E}_x[f(X_\tau)]$$

for all stopping times $\tau$ and all $x \in \mathbb{R}^n$. Then using Dynkin's formula (58) and Theorem 7.17 we have

$$\mathbb{E}_x[f(X_\tau)] = f(x) + \mathbb{E}_x\left[\int_0^\tau Af(X_s)ds\right] = f(x) + \mathbb{E}_x\left[\int_0^\tau \mathcal{A}f(X_s)ds\right]$$

Therefore,

$$\mathbb{E}_x[f(X_\tau)] \leq f(x) \Leftrightarrow \mathcal{A}f \leq 0$$

□

# 8. The High Contact Principle

In this section, we will study a methodology to transform an optimal stopping problem into a free boundary problem. We will see that, under some conditions, a possible solution to an OSP satisfying the *High Contact Principle* is an optimal solution to the problem. [7]

Let us start considering a diffusion process $X$ in $\mathbb{R}^n$ such that

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t; \quad X_0 = x$$

where $x \in \mathbb{R}^n$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are Lipschitz continuous functions. Let $G : \mathbb{R}^n \to \mathbb{R}$ be the gain function such that it is real bounded continuous. Consider the optimal stopping problem

$$V(x) = \sup_{\tau} \mathbb{E}_x[G(X_\tau)] = \mathbb{E}_x[G(X_{\tau^*})]$$

where the supremum is taken over all stopping times $\tau$ of $X$ and $\tau^*$ is the optimal stopping time.

Note that if $V$ is known then we can find $\tau^*$ easily because if $G(X_t) < V(X_t)$ we have to continue since we have obtained a reward smaller than the optimum, whereas if $G(X_t) \geq V(X_t)$ we should stop as we have attained the optimal gain. Recall that the continuation set is defined as

$$C = \{x \in E; V(x) > G(x)\}$$

and the stopping set as

$$D = \{x \in E; V(x) = G(x)\}$$

Recall also that $\tau_D$ is the first entry time of $X$ into $D$

$$\tau_D = \inf\{t \geq 0; X_t \in D\}$$

Then $\tau^* = \tau_D$.

On the other hand, if we know C, we can transform our problem of finding $V$ into a Dirichlet problem. Let us consider the infinitesimal generator of $X$ given by Theorem 7.11

$$\mathbb{L} = \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} \tag{61}$$

where $a = \sigma\sigma^T$. By the mean value property (see Lemma 7.8), we have for $\overline{H} \subset C$

$$V(x) = \int_{\partial H} V(y)d\mu_H^x(y) = \int_{\partial H} V(y)Q_x[X_{\tau_H} \in dy] = \mathbb{E}_x[V(X_{\tau_D})]$$

where $\tau_H$ is the first exit time from $H$ and $\mu_H^x(F) = Q_x[X_{\tau_H} \in F]$ is the hitting distribution (or harmonic measure) of $X$ on $\partial H$.

Hence, we have that $V(x) = \mathbb{E}_x[V(X_{\tau_D})]$ and by Dynkin's formula (Theorem 7.15)

$$\mathbb{L}V = 0$$

Thus, we have that $V$ is the solution of the differential equation

$$\sum_{i=1}^{n} b_i(x)\frac{\partial V}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(x)\frac{\partial^2 V}{\partial x_i \partial x_j} = 0 \tag{62}$$

on $C$ with boundary conditions

$$\lim_{x\to y} V(x) = G(y) \tag{63}$$

for all regular $y \in \partial C$.

But, this is a free boundary problem because we do not know $C$. Hence, we need to impose an additional boundary condition and this is given by the *High Contact principle*. This principle states that

$$\nabla V = \nabla G \quad \text{on } \partial C \tag{64}$$

and so we obtain the extra condition that we wanted.

This principle is important because it was proved that, under certain conditions, the High Contact property is a sufficient condition for a solution of the OSP to be optimal. More formally, if there exists an open set $C \subset \mathbb{R}^n$ with boundary $C^1$ and a function $f$ on $C$ satisfying

$$f \geq G \quad \text{on } C \tag{65}$$

$$\mathbb{L}G \leq 0 \quad \text{outside } \overline{C} \tag{66}$$

and such that $f$ and $C$ are a solution of the free boundary problem

$$\mathbb{L}f = 0 \quad \text{on } C \tag{67}$$

$$f = G \quad \text{on } \partial C \tag{68}$$

$$\nabla f = \nabla G \quad \text{on } \partial C \tag{69}$$

then $f = V$ on $C$.

All this methodology is summarized in the following Theorem where we consider a more general problem [6, Chapter 10]:

**Theorem 8.1.** *(Variational inequalities for optimal stopping). Let $W$ be a domain in $\mathbb{R}^n$ and $X$ a diffusion process in $\mathbb{R}^n$ starting at a point $X_0 = x$. Consider*

$$T = \inf\{t > 0; X_t \notin W\} \tag{70}$$

*Let $G : \mathbb{R}^n \to \mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}$ be continuous functions such that*

- $\mathbb{E}_x \left[\int_0^T |F(X_t)| dt\right] < \infty \quad \forall x \in \mathbb{R}^n$
- $\{G^-(X_\tau); \tau$ *stopping time,* $\tau \leq T\}$ *is uniformly integrable for all* $x \in \mathbb{R}^n$

*Consider the problem of finding $\Phi(x)$ and $\tau^* \leq T$ such that*

$$\Phi(x) = \sup_{\tau \leq T} J_\tau(x) = J_{\tau^*}(x) \tag{71}$$

*where*

$$J_\tau(x) = \mathbb{E}_x \left[\int_0^\tau F(X_t) dt + G(X_\tau)\right] \tag{72}$$

*Assume that we are able to find a function $\phi : \overline{W} \to \mathbb{R}$ satisfying:*

*i)* $\phi \in C^1(W) \cap C(\overline{W})$

*ii)* $\phi \geq G$ *on* $W$ *and* $\phi = G$ *on* $\partial W$

*iii)* $\mathbb{E}_x\left[\int_0^T \mathbb{I}_{\partial C}(X_t)dt\right] = 0 \quad \forall x \in W$, *where*

$$C = \{x \in W; \phi(x) > G(x)\}$$

*This means that* $X$ *spends 0 time on* $\partial C$ *with probability 1.*

*iv)* $\partial C$ *is a Lipschitz surface*

*v)* $\phi \in C^2(W \setminus \partial C)$ *and* $\phi$ *has locally bounded second order derivatives near* $\partial C$

*vi)* $\mathbb{L}\phi + F \leq 0$ *on* $W \setminus \overline{C}$

*vii)* $\mathbb{L}\phi + F = 0$ *on* $C$

*viii)* $\tau_D := \inf\{t \geq 0; X_t \in D\} < \infty$ *with probability 1, where* $D = \{x \in W; \phi(x) = G(x)\}$

*ix)* $\{\phi(X_\tau); \tau \leq \tau_D\}$ *is uniformly integrable*

*Then* $\phi(x) = \Phi(x)$ *for* $x \in W$ *and* $\tau^* = \tau_D$ *is an optimal stopping time.*

# 9. Applications to optimal stopping problems

## 9.1 Brownian motion recurrence and transience

We want to know what is the likelihood of a particle's trajectory described by Brownian motion beginning in some state to return to that particular state. States for which there is some non-zero probability that a stochastic process beginning in a state will never return to that state are called *transient*, while those states for which there is a guarantee (probability 1) that the process will return to them are called *recurrent*. We will show

**Theorem 9.1.** *For an n-dimensional Brownian motion* $B = (B_1, \ldots, B_n)$*, when* $n = 1$ *all states are recurrent; for* $n = 2$ *all non-zero states are recurrent; for* $n > 2$ *all states are transient. We thus say that n-dimensional Brownian motion is recurrent for* $n \leq 2$ *and transient for* $n > 2$.

We will prove this by applying the High Contact Principle scheme for solving optimal stopping problems. The steps are:

1. State the probabilistic problem as an Optimal Stopping Problem (OSP)
2. Convert to a Free Boundary Problem (FBP)
3. Solve the differential equations
4. Reinterpret solutions probabilistically

**Step 1.** The $n$-dimensional Brownian motion starts at a point $x$ in the annulus

$$K = \{x \in \mathbb{R}^n; R_1 < ||x|| < R_2\}$$

where $R_1, R_2 > 0$ and $||x|| = \sqrt{x_1^2 + ... + x_n^2}$.

Our problem is to determine the probability that $x$ will reach the outer circle ($||x|| = R_2$) before reaching the inner circle, and for $R_2$ (resp. $R_1$) arbitrary large (resp. small).

Consider the stopping time $\tau_K = \inf\{t > 0; B_t \notin K\}$, that is the first time that $B_t$ hits $\partial K$ given that $B_0 = x \in K$. Then our problem is to determine the following probability

$$f(x) = \mathbb{P}_x(||B_{\tau_K}|| = R_2) \tag{73}$$

This is the same as the OSP

$$f(x) = \mathbb{E}_x[G(B_{\tau_K})] \tag{74}$$

where the gain function is given by

$$G(x) = \begin{cases} 1 & \text{if } ||x|| = R_2 \\ 0 & \text{if } ||x|| = R_1 \end{cases}$$

**Step 2.** By the mean value property (see Lemma 7.8), we have for $\overline{V} \subset K$

$$f(x) = \int_{\partial V} f(y) d\mu_V^x(y) = \int_{\partial V} f(y) Q_x[B_{\tau_V} \in dy] = \mathbb{E}_x[f(B_{\tau_K})]$$

where $\tau_V$ is the first exit time from $V$ and $\mu_V^x(F) = Q_x[B_{\tau_V} \in F]$ is the hitting distribution (or harmonic measure) of $B$ on $\partial V$.

Thus, we have that $f(x) = \mathbb{E}_x[f(B_{\tau_K})]$, and by Dynkin's formula (Theorem 7.15) and Example 7.13 we obtain

$$Af = \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = 0$$

Then, $f(x)$ is the solution of the differential equation

$$\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = 0 \tag{75}$$

on $K$ with boundary conditions

$$f(x) = \begin{cases} 1 & \text{if } ||x|| = R_2 \\ 0 & \text{if } ||x|| = R_1 \end{cases} \tag{76}$$

By rotational symmetry the solution is a function of $||x||$, but it is also a function of $z = ||x||^2 = \sum x_i^2$, which is best to consider to avoid working with square roots. So, we seek a solution of eq. (75) of the form

$$f(x) = \phi(z) = \phi\left(\sum x_i^2\right)$$

And we have

$$\frac{\partial f}{\partial x_i} = 2x_i\phi'(z) \quad \text{and} \quad \frac{\partial^2 f}{\partial x_i^2} = 2\phi'(z) + 4x_i^2\phi''(z)$$

and summing over all $i = 1, 2, \dots, n$ to get

$$2Af(x) = 2n\phi'(z) + 4z\phi''(z) = 0 \tag{77}$$

**Step 3.** We now solve the differential equation (77) with the boundary conditions (76). Put $\psi(z) = \phi'(z)$ and this simplifies (77) to the ODE

$$2n\psi(z) + 4z\psi'(z) = 0$$

Note that we can use the method of separation of variables. We have

$$2n\psi(z) + 4z\psi'(z) = 0 \;\Rightarrow\; \frac{\psi'(z)}{\psi(z)} = -\frac{n}{2z}$$

Integrating both sides with respect to $z$ we get

$$\int \frac{\psi'(z)}{\psi(z)}dz = \int -\frac{n}{2z}dz \;\Rightarrow\; \ln(\psi(z)) = -\frac{n}{2}\ln(z) + C$$

where $C$ is a constant. Then, applying the exponential function, we obtain

$$\psi(z) = e^{-\frac{n}{2}\ln(z)+C} = e^{\ln(z^{-n/2})}e^{C} = c_1 z^{-n/2}$$

where $c_1 = e^C$ is a constant. So, we just have to integrate with respect to $z$ to get a general expression for $\phi$:

$$\psi(z) = \phi'(z) \;\Rightarrow\; \phi(z) = c_2 + \int \psi(z)dz = c_2 + \int c_1 z^{-n/2}dz = c_2 + 2c_1\frac{z^{-\frac{n}{2}+1}}{2-n}$$

where $c_2$ is another constant. We are going to use the boundary conditions to find the constants $c_1$ and $c_2$. Note that equation (76) is equivalent to

$$\phi(z) = \begin{cases} 1 & \text{if } z = R_2^2 \\ 0 & \text{if } z = R_1^2 \end{cases}$$

So, using that $\phi(R_1^2) = 0$, we get the following equation

$$\phi(R_1^2) = c_2 + 2c_1\frac{R_1^{2-n}}{2-n} = 0 \;\Leftrightarrow\; c_2 = -2c_1\frac{R_1^{2-n}}{2-n}$$

Taking the other condition, $\phi(R_2^2) = 1$, and using the previous expression we obtain

$$\phi(R_2^2) = 1 \;\Leftrightarrow\; c_2 + 2c_1\frac{R_2^{2-n}}{2-n} = 1 \;\Leftrightarrow\; -2c_1\frac{R_1^{2-n}}{2-n} + 2c_1\frac{R_2^{2-n}}{2-n} = 1$$
$$\Leftrightarrow\; c_1\frac{2}{2-n}\left(R_2^{2-n} - R_1^{2-n}\right) = 1 \;\Leftrightarrow\; c_1 = \frac{2-n}{2(R_2^{2-n} - R_1^{2-n})}$$

Then

$$c_2 = -2c_1 \frac{R_1^{2-n}}{2-n} = -\frac{R_1^{2-n}}{R_2^{2-n} - R_1^{2-n}}$$

Therefore, the solution for $\phi(z)$ is

$$\phi(z) = \frac{z^{-\frac{n}{2}+1} - R_1^{2-n}}{R_2^{2-n} - R_1^{2-n}}$$

and hence

$$f(x) = \phi(z) = \phi(||x||^2) = \frac{||x||^{2-n} - R_1^{2-n}}{R_2^{2-n} - R_1^{2-n}} \tag{78}$$

This expression can also be rewritten as

$$f(x) = \frac{R_2^{n-2}(R_1^{n-2} - ||x||^{n-2})}{(R_1^{n-2} - R_2^{n-2})||x||^{n-2}}$$

**Step 4.** Interpret solutions. We are going to fix the values $n = 1$, $n = 2$, $n > 2$ and set $R_2 \to \infty$ and $R_1 \to 0$ to see if $B$ is recurrent or transient in each case.

**Case n = 1**: We have that

$$f(x) = \frac{||x|| - R_1}{R_2 - R_1}$$

hence

$$\lim_{\substack{R_1 \to 0 \\ R_2 \to \infty}} f(x) = \lim_{\substack{R_1 \to 0 \\ R_2 \to \infty}} \frac{||x|| - R_1}{R_2 - R_1} = 0$$

Then

$$\lim_{\substack{R_1 \to 0 \\ R_2 \to \infty}} f(x) = \lim_{\substack{R_1 \to 0 \\ R_2 \to \infty}} \mathbb{P}_x(||B_{\tau_K}|| = R_2) = \mathbb{P}_x(\tau_K = \infty) = 0$$

and therefore

$$\mathbb{P}_x(\tau_K < \infty) = 1$$

which means that 1-dimensional Brownian motion is recurrent.

**Case n = 2**: We have that

$$\lim_{n\to 2} f(x) = \lim_{n\to 2} \frac{||x||^{2-n} - R_1^{2-n}}{R_2^{2-n} - R_1^{2-n}} \overset{(*)}{=} \lim_{n\to 2} \frac{||x||^{2-n}\ln(||x||) - R_1^{2-n}\ln(R_1)}{R_2^{2-n}\ln(R_2) - R_1^{2-n}\ln(R_1)} = \frac{\ln(||x||) - \ln(R_1)}{\ln(R_2) - \ln(R_1)}$$

where in $(*)$ we have applied l'Hôpital's rule since we had an indetermination (0/0). Then we fix $R_1 = \epsilon$ with $\epsilon > 0$ very small and set $R_2 \to \infty$. Assuming that $x \neq 0$ we get

$$\lim_{R_2\to\infty} \frac{\ln(||x||) - \ln(\epsilon)}{\ln(R_2) - \ln(\epsilon)} = 0$$

On the other hand, we fix $R_2 = M$ with $M > 0$ huge and set $R_1 \to 0$. Assuming again that $x \neq 0$ we obtain

$$\lim_{R_1\to 0} \frac{\ln(||x||) - \ln(R_1)}{\ln(M) - \ln(R_1)} = \lim_{R_1\to 0} \frac{1/R_1}{1/R_1} = 1$$

where we have used again l'Hôpital's rule in the first equality. Hence

$$\mathbb{P}_x(\tau_K < \infty) = 1$$

and 2-dimensional Brownian motion is recurrent for $x \neq 0$.

**Case n $>$ 2**: We have that

$$\lim_{\substack{R_1 \to 0 \\ R_2 \to \infty}} f(x) = \lim_{\substack{R_1 \to 0 \\ R_2 \to \infty}} \frac{||x||^{2-n} - R_1^{2-n}}{R_2^{2-n} - R_1^{2-n}} = \lim_{\substack{R_1 \to 0 \\ R_2 \to \infty}} \frac{\frac{1}{||x||^{n-2}} - \frac{1}{R_1^{n-2}}}{\frac{1}{R_2^{n-2}} - \frac{1}{R_1^{n-2}}} = \frac{-1}{-1} = 1$$

Therefore

$$\mathbb{P}_x(\tau_K = \infty) = 1$$

and we conclude that $n$-dimensional Brownian motion is transient for $n > 2$.

## 9.2 Optimal time to sell a warrant

Paul A. Samuelson published a paper called *Rational Theory of Warrant Pricing* (1965) [8] where the High Contact Principle was formulated for the first time. He presented a theory of rationally evaluating a warrant, taking into account the right to sell or buy the warrant at any time and deducing the value of the stock which an investor will pay to exercise the warrant.

A *warrant* is a derivative which gives the right, but not the obligation, to buy or sell an asset at a fixed price before an expiration date. The *exercise price* or *strike price* is the guaranteed price at which the investor has the right to buy or sell the underlying asset. *Exercising* the warrant means that the trade specified on the warrant is to be carried out.

He considered that the reward obtained by selling the asset at time $t$ and when the price is $\xi$ is given by the gain function

$$G(t, \xi) = e^{-\rho t}(\xi - 1)^+ \tag{79}$$

where $x^+ = \max\{x, 0\}$, $\rho$ is a discounting factor and, in this case, the exercise price is 1.

The price $X_t$ at time $t$ of a person's asset is assumed to be a geometric Brownian motion given by the next SDE

$$dX_t = rX_t dt + \alpha X_t dB_t, \quad X_0 = x > 0 \tag{80}$$

where $B_t$ is a 1-dimensional Brownian motion and $r$, $\alpha$ are known constants with $r < \rho$.

We will apply the High Contact Principle scheme to find the optimal time for selling the asset and the optimal reward. As in the previous section, the steps are:

1. State the Optimal Stopping Problem (OSP)
2. Convert to a Free Boundary Problem (FBP)
3. Solve the differential equations
4. Reinterpret solutions

First of all, we need to present a lemma which we will use in the first step.

**Lemma 9.2.** *Consider $G \in C^2(\mathbb{R}^n)$ and define the set*

$$U := \{x; \mathcal{A}G(x) > 0\} \tag{81}$$

*where $\mathcal{A}$ is the characteristic operator of $X$. Then*

$$U \subset C \tag{82}$$

*where $C$ is the continuation region.*

This tells us that stopping the process, e.g. selling an asset, before it exits from $U$ is not optimal.

*Proof.* Consider $x \in U$ and a bounded open set $W$ such that $W \subset U$ and $x \in W$. Let $\tau_W$ be the first exit time from $W$ defined by $\tau_W = \inf\{t > 0; X_t \notin W\}$. Then, using the Dynkin's formula and Theorem 7.17, we have for $t > 0$

$$\mathbb{E}_x[G(X_{\tau_W \wedge t})] = G(x) + \mathbb{E}_x\left[\int_0^{\tau_W \wedge t} AG(X_s)ds\right] = G(x) + \mathbb{E}_x\left[\int_0^{\tau_W \wedge t} \mathcal{A}G(X_s)ds\right] > G(x)$$

where $\tau_W \wedge t = \min(\tau_W, t)$. Hence,

$$V(x) = \sup_\sigma \mathbb{E}_x[G(X_\sigma)] \geq \mathbb{E}_x[G(X_{\tau_W \wedge t})] > G(x)$$

where the supremum is taken over all stopping times $\sigma$ of $X$. Then

$$V(x) > G(x)$$

and therefore $x \in C = \{x \in \mathbb{R}^n; V(x) > G(x)\}$. □

Let us start with the steps defined above:

**Step 1.** We consider the next optimal stopping problem:

$$V(s, x) = \sup_\tau \mathbb{E}_{(s,x)}\left[e^{-\rho(s+\tau)}(X_\tau - 1)^+\right] = \mathbb{E}_{(s,x)}\left[e^{-\rho(s+\tau^*)}(X_{\tau^*} - 1)^+\right] \tag{83}$$

where we denote $\mathbb{E}_{(s,x)}[G(X_t)] = \mathbb{E}[G(X_t)|X_s = x]$ for $t \geq s$.

We define $Y_t = Y_t^{(s,x)} = \begin{pmatrix} s+t \\ X_t \end{pmatrix} \in \mathbb{R}^2$ with $t \geq 0$ and so we have that our optimal stopping problem is

$$V(s, x) = \sup_\tau \mathbb{E}_{(s,x)}[G(Y_\tau)]$$

**Step 2.** First, we are going to prove that the continuation region $C$ has the form

$$C = \{(s, x); 0 < x < x_0\}$$

for some $x_0 > \frac{\rho}{\rho - r}$.

Note that the characteristic operator $\widehat{\mathcal{A}}$ of $Y_t$ is given by

$$\widehat{\mathcal{A}}f(s,x) = \frac{\partial f}{\partial s}(s,x) + \mathcal{A}f(s,x); \quad f \in C^2(\mathbb{R}^2) \tag{84}$$

where $\mathcal{A}$ is the characteristic operator of $X_t$. Then in our case, we have (see Example 7.14)

$$\widehat{\mathcal{A}}f(s,x) = \frac{\partial f}{\partial s} + rx\frac{\partial f}{\partial x} + \frac{1}{2}\alpha^2 x^2 \frac{\partial^2 f}{\partial x^2}; \quad f \in C^2(\mathbb{R}^2)$$

Note that

$$\frac{\partial G}{\partial x}(s,x) = \begin{cases} 0 & x \leq 1 \\ e^{-\rho s} & x > 1 \end{cases}$$

and

$$\frac{\partial G}{\partial s}(s,x) = -\rho e^{-\rho s}(x-1)^+$$

Thus

$$\widehat{\mathcal{A}}G(s,x) = \begin{cases} 0 & x \leq 1 \\ -\rho e^{-\rho s}(x-1) + rxe^{-\rho s} & x > 1 \end{cases} = \begin{cases} 0 & x \leq 1 \\ e^{-\rho s}((-\rho + r)x + \rho) & x > 1 \end{cases}$$

Recall that we have $r < \rho$ from the statement and note that

$$e^{-\rho s}((-\rho + r)x + \rho) > 0 \iff (-\rho + r)x + \rho > 0 \iff x > \frac{\rho}{\rho - r}$$

So, using Lemma 9.2 we can define

$$U := \{(s,x); \widehat{\mathcal{A}}G(s,x) > 0\} = \left\{(s,x); x > \frac{\rho}{\rho - r}\right\}$$

satisfying $U \subset C$.

Let us establish that the region $C$ must be invariant with respect to $t$, in the sense that

$$C + (t_0, 0) = C \quad \forall t_0$$

To prove this, we consider

$$\begin{aligned} C + (t_0, 0) &= \{(t + t_0, x); (t,x) \in C\} = \{(s,x); (s - t_0, x) \in C\} \\ &= \{(s,x); G(s - t_0, x) < V(s - t_0, x)\} \end{aligned}$$

Note that

$$\begin{aligned} V(s - t_0, x) &= \sup_\tau \mathbb{E}_{(s-t_0,x)}[G(Y_\tau)] = \sup_\tau \mathbb{E}_{(s-t_0,x)}[e^{-\rho(s+\tau)}(X_\tau - 1)^+] = \sup_\tau \mathbb{E}_{(s,x)}[e^{-\rho(s+\tau-t_0)}(X_\tau - 1)^+] \\ &= e^{\rho t_0} \sup_\tau \mathbb{E}_{(s,x)}[e^{-\rho(s+\tau)}(X_\tau - 1)^+] = e^{\rho t_0} \sup_\tau \mathbb{E}_{(s,x)}[G(Y_\tau)] = e^{\rho t_0} V(s,x) \\ G(s - t_0, x) &= e^{-\rho(s-t_0)}(x-1)^+ = e^{\rho t_0} e^{-\rho s}(x-1)^+ = e^{\rho t_0} G(s,x) \end{aligned}$$

Therefore

$$C + (t_0, 0) = \{(s,x); e^{\rho t_0} G(s,x) < e^{\rho t_0} V(s,x)\} = \{(s,x); G(s,x) < V(s,x)\} = C$$

Hence, $C$ must have the form

$$C = \{(s,x); 0 < x < x_0\} \quad \text{for some } x_0 > \frac{\rho}{\rho - r}$$

Now, following the High Contact Principle scheme, we need to solve the following free boundary problem

$$\begin{cases} \widehat{\mathcal{A}} f = 0 & \text{on } C \\ f = G & \text{on } \partial C \\ \nabla f = \nabla G & \text{on } \partial C \end{cases} \tag{85}$$

which corresponds to solving

$$\begin{cases} \frac{\partial f}{\partial s} + rx \frac{\partial f}{\partial x} + \frac{1}{2}\alpha^2 x^2 \frac{\partial^2 f}{\partial x^2} = 0 & \text{for } 0 < x < x_0 \\ f(s,0) = 0 & \\ f(s,x_0) = e^{-\rho s}(x_0 - 1)^+ & \\ \frac{\partial f}{\partial x} = \frac{\partial G}{\partial x} & \text{when } x \in \{x_0, 0\} \end{cases} \tag{86}$$

**Step 3.** Let us try a solution of the form $f(s,x) = e^{-\rho s}\phi(x)$, so we get

$$\begin{cases} -\rho e^{-\rho s}\phi(x) + rxe^{-\rho s}\phi'(x) + \frac{1}{2}\alpha^2 x^2 e^{-\rho s}\phi''(x) = 0 & \text{for } 0 < x < x_0 \\ f(s,0) = e^{-\rho s}\phi(0) = 0 & \\ f(s,x_0) = e^{-\rho s}\phi(x_0) = e^{-\rho s}(x_0 - 1)^+ & \\ \frac{\partial f}{\partial x} = e^{-\rho s}\phi'(x) = \frac{\partial G}{\partial x} & \text{when } x \in \{x_0, 0\} \end{cases}$$

that is

$$\begin{cases} -\rho\phi(x) + rx\phi'(x) + \frac{1}{2}\alpha^2 x^2 \phi''(x) = 0 & \text{for } 0 < x < x_0 \\ \phi(0) = 0 & \\ \phi(x_0) = (x_0 - 1)^+ & \\ \phi'(0) = 0 & \\ e^{-\rho s}\phi'(x_0) = \frac{\partial G}{\partial x}(s, x_0) & \end{cases}$$

Note that we have a Cauchy-Euler differential equation which has the form

$$ax^2 y''(x) + bxy'(x) + cy(x) = 0$$

with $a = \frac{1}{2}\alpha^2$, $b = r$ and $c = -\rho$. If we put $x = e^t$ and $z(t) = y(e^t)$, we get the ODE

$$az'' + (b - a)z' + cz = 0$$

Let $\lambda_1, \lambda_2$ be the roots of the characteristic polynomial

$$a\lambda^2 + (b - a)\lambda + c = 0$$

Then, the general solution of the Cauchy-Euler equation is

$$y(x) = c_1 x^{\lambda_1} + c_2 x^{\lambda_2}$$

with $c_1, c_2$ constants.

So, in our case we can find that

$$\lambda_i = \frac{1}{\alpha^2}\left[\frac{1}{2}\alpha^2 - r \pm \sqrt{(r-\alpha^2/2)^2 + 2\rho\alpha^2}\right], \quad \text{for } i = 1,2, \quad \lambda_2 < 0 < \lambda_1$$

and hence we have that the general solution of $\phi$ is

$$\phi(x) = c_1 x^{\lambda_1} + c_2 x^{\lambda_2}$$

with $c_1, c_2$ constants. Let us find now these constants. Note that using the first boundary condition we get

$$\phi(0) = c_1 \cdot 0 + c_2 \cdot 0 = 0$$

so this condition does not give any new information. Let us try with the second one

$$\phi(x_0) = c_1 x_0^{\lambda_1} + c_2 x_0^{\lambda_2} = (x_0 - 1)^+ \;\Rightarrow\; c_1 = \frac{(x_0-1)^+ - c_2 x_0^{\lambda_2}}{x_0^{\lambda_1}}$$

If we use the third condition, which is the one from the High Contact Principle evaluated at $x = 0$, we obtain

$$\phi'(0) = 0 \;\Rightarrow\; \lambda_1 c_1 \cdot 0 + \lambda_2 c_2 \cdot 0 = 0$$

but this does not provide any new information. We can use that $\phi(x)$ is bounded as $x \to 0$ and so we must have $c_2 = 0$. Then $c_1 = \frac{(x_0-1)^+}{x_0^{\lambda_1}}$ and hence

$$\phi(x) = \left(\frac{x}{x_0}\right)^{\lambda_1} (x_0 - 1)^+$$

Therefore

$$f(s,x) = e^{-\rho s}\phi(x) = e^{-\rho s}(x_0 - 1)^+ \left(\frac{x}{x_0}\right)^{\lambda_1}$$

Finally, let us determine $x_0$ using the last condition for $x = x_0$ which is the one from the High Contact Principle:

$$\frac{\partial f}{\partial x} = \frac{\partial G}{\partial x} \quad \text{when } x = x_0$$

Note that

$$\frac{\partial f}{\partial x}(s,x) = e^{-\rho s}(x_0-1)^+ \lambda_1 \left(\frac{x}{x_0}\right)^{\lambda_1 - 1} \frac{1}{x_0} \;\Rightarrow\; \frac{\partial f}{\partial x}(s,x_0) = \frac{e^{-\rho s}\lambda_1}{x_0}(x_0-1)^+$$

that is

$$\frac{\partial f}{\partial x}(s,x_0) = \begin{cases} 0 & x_0 \le 1 \\ \frac{e^{-\rho s}\lambda_1 (x_0-1)}{x_0} & x_0 > 1 \end{cases}$$

Note also that

$$\frac{\partial G}{\partial x}(s,x_0) = \begin{cases} 0 & x_0 \le 1 \\ e^{-\rho s} & x_0 > 1 \end{cases}$$

Then, imposing the High Contact Principle, we get

$$\frac{\partial f}{\partial x}(s,x_0)=\frac{\partial G}{\partial x}(s,x_0)\ \Rightarrow\ \frac{e^{-\rho s}\lambda_1(x_0-1)}{x_0}=e^{-\rho s}\ \Rightarrow\ \lambda_1(x_0-1)=x_0\ \Rightarrow\ x_0=\frac{\lambda_1}{\lambda_1-1}$$

with $\lambda_1>1$ (one can see that $\lambda_1>1\Leftrightarrow r<\rho$).

Finally, the solution to our optimal stopping problem is the optimal expected reward

$$V(s,x)=f(s,x)=e^{-\rho s}(\lambda_1-1)^{\lambda_1-1}\left(\frac{x}{\lambda_1}\right)^{\lambda_1} \tag{87}$$

and the optimal stopping time

$$\tau^*=\tau_D=\inf\{t>0;X_t\geq x_0\}=\inf\{t>0;X_t\geq\frac{\lambda_1}{\lambda_1-1}\} \tag{88}$$

**Step 4.** The conclusion from the obtained results is that one should sell the assets the first time the price of them reaches the value $x_0=\frac{\lambda_1}{\lambda_1-1}$ and the expected profit would be $V(s,x)$ from equation (87).

# 10. Optimal Prediction of Resistance and Support Levels

## 10.1 Introduction

We are going to study the paper of T. De Angelis and G. Peskir [2] and complete the missing details which are taken for granted. In this paper, the authors developed a method that provides the strategies for optimal trading with the aim of predicting the *resistance* and *support* levels of asset prices. A resistance level is the price at which the majority of traders wish to sell the asset and when this level is reached the price goes down during an interval of time. Whereas, a support level is the price at which the majority of traders wish to buy the asset and when this level is reached the price goes up during an interval of time. The principal problem is that these levels are not directly observable and they are seen as *hidden targets*.

We will assume that traders already have a goal price in mind at which they want to sell or buy an asset and this approach is called *aspiration level hypothesis* [11]. We will consider a representative trader and suppose that his or her aspiration level follows a random variable which we will represent by $l$. So, we wish to detect when the asset price reaches this level. We will also consider that the trader selects a horizon $T>0$ for making the decision of buying (or selling) the asset. Then, the problem of interest is minimising the expected distance from the asset price process to $l$ and it is formulated as the following optimal prediction problem

$$\inf_{0\leq\tau\leq T}\mathbb{E}_x[|X_\tau-l|] \tag{89}$$

where the infimum is taken over all stopping times $\tau$ of $X$ and $X$ is the observed asset price process. Note that we will assume that $l$ is independent from $X$.

### Hidden Targets: The Median Rule

The aspiration level $l$ is not directly observable and we do not have any information about $l$ in terms of the observed path of the asset price $t \mapsto X_t$. Then, $l$ is called a hidden target.

In the paper by G. Peskir, [10] it is presented a rule called *median rule* that states that there exists a stopping time of $X$ that is optimal for (89). In particular, it states that when the asset price process $X$ enters in the set of all medians of $l$, $M_l = [m, M]$, then it is optimal to stop. In other words, if $X$ has a positive trend then it will be optimal to stop when the price reaches the lowest median $m$, which represents the resistance level. Whereas, if $X$ has a negative trend then it will be optimal to stop when the price reaches the highest median $M$, which represents the support level.

## 10.2 Formulation of the problem

We assume that the asset price $X$ follows a geometric Brownian motion which satisfies the following SDE

$$dX_t = \mu X_t dt + \sigma X_t dB_t \tag{90}$$

with $X_0 = x$ for $x > 0$, where $\mu \in \mathbb{R}$ is the drift, $\sigma > 0$ is the volatility and $B_t$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that if $\mu > 0$ the price has a positive trend and the goal of the trader is to predict when is the optimal time to sell the asset, i.e. when the price will reach the resistance level. Whereas, if $\mu < 0$ the price has a negative trend and the goal of the trader is to detect when is the optimal time to buy the asset, i.e. when the price will reach the support level.

We know from Lemma (5.22) that the SDE (90) has a strong solution given by

$$X_t = xe^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t} \tag{91}$$

for $t \geq 0$.

We consider the **optimal prediction problem**

$$V_*(x) = \inf_{0 \leq \tau \leq T} \mathbb{E}_x[|X_\tau - l|] \tag{92}$$

where the infimum is taken over all stopping times $\tau$ of $X$ which are bounded by the horizon $T > 0$ and $l > 0$ is a random variable independent from $X$. Let us denote the distribution function of $l$ by $F$, which is given. Our aim is to solve (92) applying the methodology explained at the beginning and thus find the optimal trading strategy.

The following lemma will be useful to reformulate our optimal prediction problem to an optimal stopping problem.

**Lemma 10.1.** *For all $x > 0$, we have:*

$$\mathbb{E}[|x - l|] = 2\int_0^x \left(F(y) - \frac{1}{2}\right) dy + \mathbb{E}[l] \tag{93}$$

*Proof.* Note that for all $x > 0$

$$\mathbb{E}[|x - l|] = \int_0^\infty |x - y| f(y) dy \tag{94}$$

where $f$ is the density function of the random variable $l$. We do not know $f$, but the distribution function $F$ of $l$ is given and fixed. So, we can write $f$ as $dF/dy$ and we have:

$$\mathbb{E}[|x-l|] = \int_0^\infty |x-y| \frac{dF}{dy}(y)dy \tag{95}$$

Let us remove the absolute value separating the previous integral in two.

$$\int_0^\infty |x-y| \frac{dF}{dy}(y)dy = \int_0^x (x-y)\frac{dF}{dy}(y)dy + \int_x^\infty (y-x)\frac{dF}{dy}(y)dy \tag{96}$$

Adding and subtracting $\int_0^x (x-y)\frac{dF}{dy}(y)dy$, we get that the previous expression is equal to

$$\begin{aligned}
&\int_0^x (x-y)\frac{dF}{dy}(y)dy + \int_0^x (x-y)\frac{dF}{dy}(y)dy - \int_0^x (x-y)\frac{dF}{dy}(y)dy + \int_x^\infty (y-x)\frac{dF}{dy}(y)dy \\
&= 2\int_0^x (x-y)\frac{dF}{dy}(y)dy + \int_0^x (y-x)\frac{dF}{dy}(y)dy + \int_x^\infty (y-x)\frac{dF}{dy}(y)dy \\
&= 2\int_0^x (x-y)\frac{dF}{dy}(y)dy + \int_0^\infty (y-x)\frac{dF}{dy}(y)dy
\end{aligned}$$

Note that

$$\int_0^\infty (y-x)\frac{dF}{dy}(y)dy = \mathbb{E}[l-x] = \mathbb{E}[l]-x \tag{97}$$

So we get that

$$\mathbb{E}[|x-l|] = 2\int_0^x (x-y)\frac{dF}{dy}(y)dy + \mathbb{E}[l]-x \tag{98}$$

Let us integrate by parts the previous integral setting $u = x-y$ and $dv = \frac{dF}{dy}(y)dy$, so we have $du = -dy$ and $v = F(y)$. We obtain that

$$\begin{aligned}
\int_0^x (x-y)\frac{dF}{dy}(y)dy &= [(x-y)F(y)]_0^x - \int_0^x -F(y)dy \\
&= (x-x)F(x) - xF(0) + \int_0^x F(y)dy \\
&= \int_0^x F(y)dy
\end{aligned}$$

where we have used that $F(0) = 0$. This last expression yields

$$\begin{aligned}\mathbb{E}[|x - I|] &= 2\int_0^x (x-y)\frac{dF}{dy}(y)dy + \mathbb{E}[I] - x \\ &= 2\int_0^x F(y)dy + \mathbb{E}[I] - x \\ &= 2\int_0^x F(y)dy + \mathbb{E}[I] - \int_0^x dy \\ &= 2\int_0^x \left(F(y) - \frac{1}{2}\right) dy + \mathbb{E}[I]\end{aligned}$$

for all $x > 0$, as we wanted to obtain. □

Let us assume that $\mathbb{E}[I] < \infty$ and define a function $G$ for $x > 0$ by

$$G(x) = \int_0^x \left(F(y) - \frac{1}{2}\right) dy \tag{99}$$

As $I$ and $X$ are independent, we see from (93) that the optimal prediction problem (92) reduces to the following **optimal stopping problem**

$$V(x) = \inf_{0\le\tau\le T} \mathbb{E}_x[G(X_\tau)] \tag{100}$$

where the infimum is taken over all stopping times $\tau$ of $X$ which are bounded by the horizon $T > 0$. Note that $V_*(x) = 2V(x) + \mathbb{E}[I]$ for $x > 0$. From Eq. (99) observe that $G'(x) = F(x) - 1/2$ is increasing for $x \in (0, \infty)$ and thus $G(x)$ is convex on $(0, \infty)$. We need the following definition:

**Definition 10.2.** Let $X$ be a random variable and $F$ its distribution function. Then the number $m$ is called a *median* of $X$ if

$$\mathbb{P}(X \le m) \ge \frac{1}{2} \text{ and } \mathbb{P}(X \ge m) \ge \frac{1}{2} \tag{101}$$

which is equivalent to

$$F(m-) \le \frac{1}{2} \le F(m) \tag{102}$$

where $F(m-) = \mathbb{P}(X < m)$.

The set of all medians of $X$ is a bounded and closed interval $[m, M]$ where $m$ is the *lowest median* of $X$ and $M$ is the *highest median* of $X$. If $X$ has a unique median then $m = M$.

Using this definition, observe that $G' = F - \frac{1}{2} < 0$ on $(0, m)$ and $G' = F - \frac{1}{2} > 0$ on $(M, \infty)$. Hence, $G$ is strictly decreasing on $(0, m)$, strictly increasing on $(M, \infty)$, constant on the set of all medians $[m, M]$ and it satisfies $G(0) = 0$.

Note that we will treat the finite horizon formulation ($T < \infty$) and the infinite horizon formulation ($T = \infty$) of the optimal stopping problem (100) at the same time. In the case of $T < \infty$ we have to substitute the process $X_t$ for the process $Z_t = (t, X_t)$ for $t \ge 0$ so that we enable the process $X$ to start at arbitrary points at any allowable time, and we consider the extended optimal stopping problem in the time and space domain

$$V(t,x) = \inf_{0\le\tau\le T-t} \mathbb{E}_{t,x}\left[G(X_{t+\tau})\right] \tag{103}$$

where $X_t = x$. Note that the results obtained for the problem (100) with $T = \infty$ will automatically hold for the problem (103) if we just think of $X$ to be $Z$. It is also a technical advantage to work with a two variable function $V(t,x)$ for formulating the free boundary problem associated with the OSP.

We define the continuation set $C$ as

$$C = \{(t,x) \in [0,T] \times (0,\infty); V(t,x) < G(x)\} \tag{104}$$

and the stopping set $D$ as

$$D = \{(t,x) \in [0,T] \times (0,\infty); V(t,x) = G(x)\} \tag{105}$$

Then the first entry time of $X$ into $D$ is given by

$$\tau_D = \inf\{s \in [0, T-t]; (t+s, X_{t+s}) \in D\} \tag{106}$$

Before solving the optimal stopping problem, we are going to see some observations and definitions. First of all, we are going to prove that if $\mu = 0$ then it is optimal to stop immediately in (103).

**Theorem 10.3.** *(Jensen's inequality). If $\phi$ is a convex function and $X$ is a random variable such that $\mathbb{E}[|\phi(X)|] < \infty$ then*

$$\phi(\mathbb{E}[X]) \le \mathbb{E}[\phi(X)] \tag{107}$$

**Lemma 10.4.** *$M_t := e^{\sigma B_t - \frac{\sigma^2}{2}t}$, $t \ge 0$, is a martingale.*

*Proof.* Note that using Lemma 5.25 we have

$$\mathbb{E}[|M_s|] = \mathbb{E}[\exp(\sigma B_t - (\sigma^2/2)t)] = e^{0t} = 1 < \infty \quad \forall t \ge 0$$

So we just need to see that $\mathbb{E}[M_s|\mathcal{F}_t] = M_t \ \forall s \ge t$. Hence, consider now $s \ge t$ and note that we can write

$$M_s = \exp(\sigma B_s - (\sigma^2/2)s) = \exp(\sigma B_t + \sigma(B_s - B_t) - (\sigma^2/2)s)$$

Then

$$\begin{aligned}\mathbb{E}[M_s|\mathcal{F}_t] &= \mathbb{E}[\exp(\sigma B_t + \sigma(B_s - B_t) - (\sigma^2/2)s)|\mathcal{F}_t] \\ &= e^{-(\sigma^2/2)s} \cdot \mathbb{E}[e^{\sigma B_t}|\mathcal{F}_t] \cdot \mathbb{E}[e^{\sigma(B_s-B_t)}|\mathcal{F}_t] \overset{(*)}{=} e^{-(\sigma^2/2)s} \cdot e^{\sigma B_t} \cdot \mathbb{E}[e^{\sigma(B_s-B_t)}] \\ &\overset{(**)}{=} e^{-(\sigma^2/2)s} \cdot e^{\sigma B_t} \cdot e^{(\sigma^2/2)(s-t)} = e^{\sigma B_t - (\sigma^2/2)t} = M_t\end{aligned}$$

where in $(*)$ we have used that $B_t$ is measurable with respect to $\mathcal{F}_t$ and that $B_s - B_t$ is independent from $\mathcal{F}_t$, and in $(**)$ we have applied Lemma 5.24. □

Since $G$ is convex, using Jensen's inequality, the expression (91) and the fact that $d\tilde{P} = M_\tau dP$ defines a probability measure because $M_s$ is a martingale, we obtain

$$\mathbb{E}_x[G(X_\tau)] \geq G(\mathbb{E}_x[X_\tau]) = G(x\tilde{\mathbb{E}}[e^{\mu\tau}]) \overset{\mu=0}{=} G(x) \tag{108}$$

So, we have that the optimal stopping time is $\tau^* = 0$ as claimed. Since the problems (103) and (92) are equivalent, we also have that it is optimal to stop at once in (92).

Using the previous argument in (108) with a small change, applying that $G$ is increasing on $[m, \infty)$ and decreasing on $(0, M]$, we can easily show that $[0, T] \times [m, \infty) \subseteq D$ if $\mu > 0$ and $[0, T] \times (0, M] \subseteq D$ if $\mu < 0$.

The form of $D$ below $m$ when $\mu < 0$ or above $M$ when $\mu > 0$ can be complicated and so we are going to present a class of distribution functions $F$ of $I$ that give a simple structure of $D$. But first, we need a definition:

**Definition 10.5.** Let $X$ be a random variable with distribution function $F$ on $\mathbb{R}$ satisfying $F(0) = 0$. Then $F$ is *piecewise* $C^1$ if there exists a partition $[x_{i-1}, x_i)$ of $\mathbb{R}_+$ for $i \geq 1$ such that $F$ restricted on each interval $[x_{i-1}, x_i]$ is $C^1$.

With this definition and Definition 10.2, we can define the admissible aspiration level laws:

**Definition 10.6.** (*Admissible aspiration level laws*). For $\mu > 0$ and $\sigma > 0$ consider $\mathcal{F}(\mu, \sigma)$ as the family of piecewise $C^1$ probability distribution functions $F$ on $\mathbb{R}$ satisfying $F(0) = 0$ for which there exists $\alpha \in (0, m)$ such that

$$xF'(x) < \frac{\mu}{\sigma^2/2}\left(\frac{1}{2} - F(x)\right) \text{ for } x \in (0, \alpha) \quad \text{and} \quad xF'(x) > \frac{\mu}{\sigma^2/2}\left(\frac{1}{2} - F(x)\right) \text{ for } x \in (\alpha, m) \tag{109}$$

where $m$ is the lowest median of $F$.
For $\mu < 0$ and $\sigma > 0$ consider $\mathcal{F}(\mu, \sigma)$ as the family of piecewise $C^1$ probability distribution functions $F$ on $\mathbb{R}$ satisfying $F(0) = 0$ for which there exists $\beta \in (M, \infty)$ such that

$$xF'(x) > \frac{\mu}{\sigma^2/2}\left(\frac{1}{2} - F(x)\right) \text{ for } x \in (M, \beta) \quad \text{and} \quad xF'(x) < \frac{\mu}{\sigma^2/2}\left(\frac{1}{2} - F(x)\right) \text{ for } x \in (\beta, \infty) \tag{110}$$

where $M$ is the highest median of $F$.

## 10.3 Solution to the problem

The next step is to transform the optimal stopping problem (103) into a free boundary problem and solve it. First, we need to define some functions that are used in the paper [2]:

$$J(t, x) = \mathbb{E}_x[G(X_{T-t})] = \int_0^\infty G(z)f(T - t, x, z)dz \tag{111}$$

$$H(x) = \mathbb{L}_X G(x) = \mu x\left(F(x) - \frac{1}{2}\right) + \frac{\sigma^2}{2}x^2 F'(x) \tag{112}$$

$$K(s, x, y) = \mathbb{E}_x[H(X_s)\mathbb{I}_{X_s > y}] = \int_y^\infty H(z)f(s, x, z)dz \tag{113}$$

$$L(s, x, y) = \mathbb{E}_x[H(X_s)\mathbb{I}_{X_s < y}] = \int_0^y H(z)f(s, x, z)dz \tag{114}$$

for $t \in [0,T]$, $x \in (0,\infty)$, $s \in (0, T-t]$ and $y \in (0,\infty)$. Note that $\mathbb{L}_X$ is the infinitesimal generator of $X$ (see Example 7.14) and $z \mapsto f(s,x,z)$ is the probability density function of $X_s$ under $\mathbb{P}_x$ defined as

$$f(s,x,z) = \frac{1}{\sigma\sqrt{s}z}\varphi\left(\frac{1}{\sigma\sqrt{s}}\left[\log\left(\frac{z}{x}\right) + \left(\frac{\sigma^2}{2} - \mu\right)s\right]\right) \tag{115}$$

for $s > 0$, $x > 0$ and $z > 0$, and $\varphi$ is the standard normal density function given by $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ for $x \in \mathbb{R}$.

The solution of our problem is given in Theorem 3 of the paper [2], which states the following:

**Theorem 10.7.** *Let $X$ be a geometric Brownian motion solving (90) and $l > 0$ a random variable independent from $X$ such that $\mathbb{E}[l] < \infty$. Consider the optimal prediction problem (92) and the extended optimal stopping problem (103). Assume that the distribution function $F$ of $l$ belongs to $\mathcal{A}(\mu,\sigma)$ with $\mu \neq 0$ (because when $\mu = 0$ it is optimal to stop immediately in both (103) and (92)). We have two cases:*

**Case 1**: *$\mu > 0$. The stopping set $D$ in problem (92) has the form $D = \{(t,x) \in [0,T) \times (0,\infty); x \geq b(t)\} \cup (\{T\} \times (0,\infty))$ where the optimal stopping boundary $b : [0,T) \to \mathbb{R}$ is the unique solution to the nonlinear integral equation*

$$J(t,b(t)) = G(b(t)) + \int_t^T K(s-t, b(t), b(s))ds \tag{116}$$

*with $b$ continuous decreasing and satisfying $\alpha \leq b(t) \leq m$ for $t \in [0,T)$ and $b(T-) = \alpha$. We have that the stopping time $\tau_D$ defined in (106) is optimal in problem (103) and the stopping time given by*

$$\tau_b = \inf\{t \in [0,T); X_t \geq b(t)\} \tag{117}$$

*is optimal in problem (92). The value function $V$ from (103) is given by*

$$V(t,x) = J(t,x) - \int_t^T K(s-t, x, b(s))ds \tag{118}$$

*for $(t,x) \in [0,T] \times (0,\infty)$ and so the value function $V_*(x)$ from (92) is $2V(0,x) + \mathbb{E}[l]$ for $x > 0$.*

**Case 2**: *$\mu < 0$. The stopping set $D$ in problem (103) has the form $D = \{(t,x) \in [0,T) \times (0,\infty); x \leq b(t)\} \cup (\{T\} \times (0,\infty))$ where the optimal stopping boundary $b : [0,T) \to \mathbb{R}$ is the unique solution to the nonlinear integral equation*

$$J(t,b(t)) = G(b(t)) + \int_t^T L(s-t, b(t), b(s))ds \tag{119}$$

*with $b$ continuous increasing and satisfying $M \leq b(t) \leq \beta$ for $t \in [0,T)$ and $b(T-) = \beta$. We have that the stopping time $\tau_D$ defined in (106) is optimal in problem (103) and the stopping time given by*

$$\tau_b = \inf\{t \in [0,T); X_t \leq b(t)\} \tag{120}$$

*is optimal in problem (92). The value function $V$ from (103) is given by*

$$V(t,x) = J(t,x) - \int_t^T L(s-t, x, b(s))ds \tag{121}$$

*for $(t,x) \in [0,T] \times (0,\infty)$ and so the value function $V_*(x)$ from (92) is $2V(0,x) + \mathbb{E}[l]$ for $x > 0$.*

To prove this theorem the authors follow several steps. Note that they only treat the case $\mu > 0$ in detail because the case $\mu < 0$ is analogous. We give a sketch of the main steps of the proof and fill in some details of some of the steps that are not in the original paper.

## Steps of the proof

1. Show that the value function $V(t,x)$ is continuous on $[0,T] \times (0,\infty)$. This is achieved using the mean value theorem, the definition of continuous function and the dominated convergence theorem. In addition, the authors use that $X_t^1 = \exp((\mu - \sigma^2/2)t + \sigma B_t)$ is a submartingale, so let us prove this statement.

**Lemma 10.8.** *$X^1$ is a submartingale.*

*Proof.* Note that using Lemma 5.25 we have

$$\mathbb{E}[|X_t^1|] = \mathbb{E}[\exp((\mu - \sigma^2/2)t + \sigma B_t)] = e^{\mu t} < \infty \quad \forall t \geq 0$$

So we just need to see that $\mathbb{E}[X_s^1|\mathcal{F}_t] \geq X_t^1 \ \forall s \geq t$. Hence, consider now $s \geq t$ and note that we can write

$$X_s^1 = \exp((\mu - \sigma^2/2)s + \sigma B_s) = \exp((\mu - \sigma^2/2)s + \sigma B_t + \sigma(B_s - B_t))$$

Then

$$\begin{aligned}\mathbb{E}[X_s^1|\mathcal{F}_t] &= \mathbb{E}[\exp((\mu - \sigma^2/2)s + \sigma B_t + \sigma(B_s - B_t))|\mathcal{F}_t] \\ &= e^{(\mu-\sigma^2/2)s} \cdot \mathbb{E}[e^{\sigma B_t}|\mathcal{F}_t] \cdot \mathbb{E}[e^{\sigma(B_s-B_t)}|\mathcal{F}_t] \overset{(*)}{=} e^{(\mu-\sigma^2/2)s} \cdot e^{\sigma B_t} \cdot \mathbb{E}[e^{\sigma(B_s-B_t)}] \\ &\overset{(**)}{=} e^{(\mu-\sigma^2/2)s} \cdot e^{\sigma B_t} \cdot e^{(\sigma^2/2)(s-t)} = e^{\sigma B_t + (\mu s - (\sigma^2/2)t)} \overset{s \geq t}{\geq} e^{\sigma B_t + (\mu - \sigma^2/2)t} = X_t^1\end{aligned}$$

where in $(*)$ we have used that $B_t$ is measurable with respect to $\mathcal{F}_t$ and that $B_s - B_t$ is independent from $\mathcal{F}_t$, and in $(**)$ we have applied Lemma 5.24. □

2. Show that the stopping set in problem (103) is given by

$$D = \{(t,x) \in [0,T) \times (0,\infty); x \geq b(t)\} \cup (\{T\} \times (0,\infty)) \tag{122}$$

where $b : [0,T) \to \mathbb{R}$ is a decreasing function such that $b(t) \in [\alpha, m]$ for $t \in [0,T)$. To prove this, we need the next statement:

**Lemma 10.9.** *The following equality holds:*

$$G(X_{t+s}) = G(x) + \int_0^s (\mathbb{L}_X G)(X_{t+u})du + \int_0^s \sigma X_{t+u} G'(X_{t+u})dB_u \tag{123}$$

*where $\mathbb{L}_X = \mu x(d/dx) + (\sigma^2 x^2/2)(d^2/dx^2)$ is the infinitesimal generator of $X$ (see Example 7.14).*

*Proof.* Recall that the gain function $G(x)$ is convex on $(0,\infty)$ and we have that $X$ is a semimartingale (see Lemma 5.19), hence we can apply Itô-Tanaka-Meyer formula (Corollary 6.6) to $G$ yielding

$$\begin{aligned}G(X_{t+s}) &= G(X_t) + \int_0^s \frac{1}{2}(G'_+(X_{t+u}) + G'_-(X_{t+u}))dX_{t+u} + \frac{1}{2}\int_{\mathbb{R}} L_s^a(X)G''(da) \\ &= G(x) + \int_0^s G'(X_{t+u})dX_{t+u} + \frac{1}{2}\int_{\mathbb{R}} L_s^a(X)G''(da)\end{aligned} \tag{124}$$

Recall that $X$ is a geometric Brownian motion solving

$$dX_s = \mu X_s ds + \sigma X_s dB_s$$

So, we have

$$\begin{aligned}\int_0^s G'(X_{t+u})dX_{t+u} &= \int_0^s G'(X_{t+u})(\mu X_{t+u}d(t+u) + \sigma X_{t+u}dB_{t+u}) \\ &= \int_0^s \mu X_{t+u}G'(X_{t+u})du + \int_0^s \sigma X_{t+u}G'(X_{t+u})dB_{t+u} \end{aligned} \tag{125}$$

Note that, using the Occupation times formula (Corollary 6.7) with $\Phi = G''$, we obtain

$$\int_{\mathbb{R}} L_s^a(X)G''(da) = \int_0^s G''(X_{t+u})d[X]_{t+u} \tag{126}$$

Applying lemma 6.4, we have

$$\begin{aligned}d[X]_s = d[X,X]_s = dX_s \cdot dX_s &= (\mu X_s ds + \sigma X_s dB_s)\cdot(\mu X_s ds + \sigma X_s dB_s) \\ &= \mu^2 X_s^2 ds ds + 2\sigma\mu X_s^2 ds dB_s + \sigma^2 X_s^2 dB_s \cdot dB_s = \sigma^2 X_s^2 ds\end{aligned}$$

Hence, substituting this in the expression (126) we get

$$\begin{aligned}\int_{\mathbb{R}} L_s^a(X)G''(da) &= \int_0^s G''(X_{t+u})d[X]_{t+u} = \int_0^s G''(X_{t+u})\sigma^2 X_{t+u}^2 d(t+u) \\ &= \int_0^s \sigma^2 X_{t+u}^2 G''(X_{t+u})du \end{aligned} \tag{127}$$

Then, replacing Eq. (125) and (127) into Eq. (124), we obtain

$$\begin{aligned}G(X_{t+s}) &= G(x) + \int_0^s \mu X_{t+u}G'(X_{t+u})du + \int_0^s \sigma X_{t+u}G'(X_{t+u})dB_{t+u} + \frac{1}{2}\int_0^s \sigma^2 X_{t+u}^2 G''(X_{t+u})du \\ &= G(x) + \int_0^s (\mu X_{t+u}G'(X_{t+u}) + \frac{1}{2}\sigma^2 X_{t+u}^2 G''(X_{t+u}))du + \int_0^s \sigma X_{t+u}G'(X_{t+u})dB_{t+u} \\ &= G(x) + \int_0^s (\mathbb{L}_X G)(X_{t+u})du + \int_0^s \sigma X_{t+u}G'(X_{t+u})dB_u\end{aligned}$$

as it was claimed. □

This result is equivalent to

$$G(X_{t+s}) = G(x) + \int_0^s H(X_{t+u})du + M_s \tag{128}$$

where $H$ is given by (112) and $M_s := \int_0^s \sigma X_{t+u}G'(X_{t+u})dB_u$ is a (continuous local) martingale (see Lemma 5.13). We can use this to see that the set $[0,T)\times(0,\alpha)$ is contained in the continuation set $C$. So, define the stopping time $\sigma_\alpha = \inf\{u \in [0, T-t]; X_{t+u} \geq \alpha\}$ and note that from (109) we have $H(x) < 0$ for $x \in (0,\alpha)$. Then, taking the expected value in (128), replacing $s$ by $\sigma_\alpha$ and using the optional sampling theorem with $\sigma = 0$ and $\tau = \sigma_\alpha$ (see Theorem 6.9 ii)) we obtain

$$\mathbb{E}_{t,x}[G(X_{t+\sigma_\alpha})] = G(x) + \mathbb{E}_{t,x}\left[\int_0^{\sigma_\alpha} H(X_{t+u})du\right] + \mathbb{E}_{t,x}[M_{\sigma_\alpha}] < G(x) + \mathbb{E}_{t,x}[M_0] = G(x)$$

for $(t,x) \in [0,T) \times (0,\alpha)$. Hence, taking the infimum we get

$$V(t,x) = \inf_{0 \le \tau \le T-t} \mathbb{E}_{t,x}[G(X_{t+\tau})] \le \mathbb{E}_{t,x}[G(X_{t+\sigma_\alpha})] < G(x)$$

for $(t,x) \in [0,T) \times (0,\alpha)$. Therefore, we conclude that $[0,T) \times (0,\alpha) \subseteq C$. It is easy to see that

i) $[0,T) \times [m,\infty) \subseteq D$ (we have proved this in the previous section),

ii) if a point $(t,x) \in [0,T) \times [\alpha,m)$ belongs to $D$, then $(s,x)$ with $s \in (t,T]$ belongs to $D$ (it is evident since $G$ does not depend on time),

iii) if a point $(t,x) \in [0,T) \times [\alpha,m)$ belongs to $D$, then $(t,y)$ with $y > x$ belongs to $D$ (it is evident for $y \ge m$ and for $y \in (x,m)$ it can be proved using the optional sampling theorem similarly as we did before)

Finally, we conclude that

$$b(t) := \min\{x \in (0,\infty); V(t,x) = G(x)\} \tag{129}$$

with $b(t) \in [\alpha,m]$ for $t \in [0,T)$.

3. Prove that $b$ is continuous on $[0,T)$ and $b(T-) = \alpha$. To see the first fact, the authors show that $b$ is right-continuous on $[0,T)$ considering a sequence $t_n \to t^+$ as $n \to \infty$ with $t \in [0,T)$ and then they make a proof by contradiction assuming that $b$ makes a jump at some $t \in (0,T)$ and using the dominated convergence theorem. Finally, they prove that $b(T-) = \alpha$ supposing that $b(T-) > \alpha$ and reaching another contradiction.

4. Show that the Hight Contact Principle holds at $b$, i.e. see that $x \mapsto V(t,x)$ is differentiable at $b(t)$ and

$$\frac{\partial V}{\partial x}(t,b(t)) = G'(b(t)) \quad \forall t \in [0,T) \tag{130}$$

On the one hand, to prove $\frac{\partial V}{\partial x}(t,b(t)) \ge G'(b(t))$ we fix $t \in [0,T)$, $x = b(t)$ and we use the definition of derivative. So, for $\epsilon > 0$ we have

$$\frac{V(t,x-\epsilon) - V(t,x)}{-\epsilon} \ge \frac{G(x-\epsilon) - G(x)}{-\epsilon}$$

and setting $\epsilon \to 0^+$ we obtain the claimed result. On the other hand, to prove the other inequality we set $\tau_\epsilon := \tau_*(t,x-\epsilon)$ for $\epsilon > 0$ and we use the mean value theorem. So, we get

$$\frac{V(t,x-\epsilon) - V(t,x)}{-\epsilon} \le \frac{\mathbb{E}[G((x-\epsilon)X^1_{\tau_\epsilon})] - \mathbb{E}[G(xX^1_{\tau_\epsilon})]}{-\epsilon} = \mathbb{E}[G'(\xi)X^1_{\tau_\epsilon}]$$

where $\xi \in [(x-\epsilon)X^1_{\tau_\epsilon}, xX^1_{\tau_\epsilon}]$. Setting $\epsilon \to 0^+$ and using the dominated convergence theorem we get what we wanted.

5. Putting all of the previous results together, the authors obtain the following free boundary problem:

$$\frac{\partial V}{\partial t}(t,x) + \mu x \frac{\partial V}{\partial x}(t,x) + \frac{\sigma^2}{2} x^2 \frac{\partial^2 V}{\partial x^2}(t,x) = 0 \quad \text{for } x \in (0, b(t)) \text{ and } t \in [0, T) \tag{131}$$

$$V(t,x) = G(x) \quad \text{for } x \in [b(t), \infty) \text{ and } t \in [0, T) \quad \text{(instantaneous stopping)} \tag{132}$$

$$\frac{\partial V}{\partial x}(t,x) = G'(x) \quad \text{for } x = b(t) \text{ and } t \in [0, T) \quad \text{(High Contact Principle)} \tag{133}$$

$$V(t,x) < G(x) \quad \text{for } x \in (0, b(t)) \text{ and } t \in [0, T) \tag{134}$$

$$V(T,x) = G(x) \quad \text{for } x \in (0, \infty). \tag{135}$$

The next steps consist of solving this problem. This is the hard part.

6. Show that $V$ is given by equation (118) and that $b$ is a solution of (116). The most important tools which the authors use are the mean value theorem and the change-of-variable formula with local time on curves, which is a theorem that G. Peskir developed [12].

7. Prove that the function $b$ is the unique solution of equation (116) which satisfies $\alpha \geq b(t) \geq m$ for $t \in [0, T)$. To show this, they consider a continuous function $c : [0, T) \to \mathbb{R}$ which solves (116) and satisfies $\alpha \leq c(t) \leq m$ for $t \in [0, T)$ and they define the function $V^c : [0, T] \times (0, \infty) \to \mathbb{R}$ by

$$V^c(t,x) = \mathbb{E}_{t,x}[G(X_T)] - \int_0^{T-t} \mathbb{E}_{t,x}[H(X_{t+u}) \mathbb{I}_{X_{t+u} > c(t+u)}] du$$

for $(t,x) \in [0, T] \times (0, \infty)$. Then:

i) They show that $V^c(t,x) = G(x)$ $\forall (t,x) \in [0, T) \times (0, \infty)$ with $x \geq c(t)$. For this they use the Markov property of $X$ and the optional sampling theorem.

ii) They prove that $V^c(t,x) \geq V(t,x)$ $\forall (t,x) \in [0, T] \times (0, \infty)$. They apply (i) and the optional sampling theorem.

iii) They show that $c(t) \leq b(t)$ $\forall t \in [0, T)$. They use an argument by contradiction, (i), (ii) and the optional sampling theorem.

iv) They prove that $b(t) = c(t)$ $\forall t \in [0, T)$. They use again an argument by contradiction, (i), (ii), (iii) and the optional sampling theorem.

## 10.4 Calculation of the optimal stopping boundary

Theorem 10.7 presents the nonlinear integral equations which characterised the optimal stopping boundary $b(t)$. For the case $\mu > 0$, $b(t)$ is the unique solution of equation (116) and for $\mu < 0$, it is the unique solution of (119). We cannot find an explicit solution for these equations generally, but we can use them to find $b(t)$ numerically. In this section, we are going to see how to do this for the case $\mu > 0$ and considering that the random variable $I$ follows an exponential distribution. In the next section, we will show examples and we will explain why we choose this distribution for $I$ in the two first examples.

## Methodology

Recall that the equation that we want to solve is the following one:

$$J(t, b(t)) = G(b(t)) + \int_t^T K(s - t, b(t), b(s))ds$$

where $\alpha \leq b(t) \leq m$ for $t \in [0, T)$ and $b(T-) = \alpha$.

Set $t_k = kh$ for $k = 0, 1, ..., n$ with $h = T/n$. Then, the previous expression can be written as

$$J(t_k, b(t_k)) = G(b(t_k)) + \sum_{l=k}^{n-1} K(t_{l+1} - t_k, b(t_k), b(t_{l+1}))h \tag{136}$$

for $k = 0, ..., n-1$. Setting $k = n-1$ and using $b(t_n) = \alpha$, we can solve equation (136) numerically and get a value for $b(t_{n-1})$. Setting $k = n - 2$ and using $b(t_{n-1})$ and $b(t_n)$ we can solve (136) and obtain a value for $b(t_{n-2})$. If we continue this backward induction we obtain the values $b(t_n), b(t_{n-1}), ..., b(t_1), b(t_0)$ as an approximation of the optimal stopping boundary $b$ at the points $T, T - h, ..., h, 0$.

## Computations

We need to calculate $J(t, x)$, $G(x)$ and $K(s, x, y)$. First, recall that in our case $I$ follows an exponential distribution with parameter $\lambda > 0$ and so its distribution function $F$ is defined by

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{137}$$

Using the definition of $G$ from equation (99) we can find an explicit expression for this function:

$$G(x) = \int_0^x \left(F(y) - \frac{1}{2}\right) dy = \int_0^x \left(\frac{1}{2} - e^{-\lambda y}\right) dy = \left[\frac{1}{2}y\right]_0^x + \left[\frac{1}{\lambda}e^{-\lambda y}\right]_0^x = \frac{1}{2}x + \frac{1}{\lambda}e^{-\lambda x} - \frac{1}{\lambda} \tag{138}$$

Let us now calculate $J$. Recall that $J$ is given by equation (111), so using the expression for $G$ in (138) we can write

$$J(t, x) = \mathbb{E}_x[G(X_{T-t})] = \frac{1}{2}\mathbb{E}_x[X_{T-t}] + \frac{1}{\lambda}\mathbb{E}_x[e^{-\lambda X_{T-t}}] - \frac{1}{\lambda} \tag{139}$$

Thus, we need to calculate $\mathbb{E}_x[X_t]$ and $\mathbb{E}_x[e^{-\lambda X_t}]$. Recall that from Lemma 5.25 we have

$$\mathbb{E}_x[X_t] = xe^{\mu t} \tag{140}$$

Hence, we only need to find an expression for $\mathbb{E}_x[e^{-\lambda X_t}]$. For this we are going to use the Taylor series expansion of the exponential function:

$$e^{-\lambda X_t} = \sum_{k=0}^{\infty} \frac{(-\lambda X_t)^k}{k!}$$

So taking the expected value

$$\mathbb{E}_x[e^{-\lambda X_t}] = \mathbb{E}_x\left[\sum_{k=0}^{\infty}\frac{(-\lambda X_t)^k}{k!}\right] = \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\mathbb{E}_x[X_t^k] \tag{141}$$

From Lemma 5.25 we know that

$$\mathbb{E}_x[X_t^k] = x^k e^{k(\mu-\sigma^2/2)t}e^{\frac{1}{2}k^2\sigma^2 t}$$

So substituting in (141) we get

$$\mathbb{E}_x[e^{-\lambda X_t}] = \sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}x^k e^{k(\mu-\sigma^2/2)t}e^{\frac{1}{2}k^2\sigma^2 t} \tag{142}$$

Finally, we replace the values of $\mathbb{E}_x[X_t]$ and $\mathbb{E}_x[e^{-\lambda X_t}]$ into the expression (139) for $J$

$$\begin{aligned} J(t,x) &= \frac{1}{2}\mathbb{E}_x[X_{T-t}] + \frac{1}{\lambda}\mathbb{E}_x[e^{-\lambda X_{T-t}}] - \frac{1}{\lambda} \\ &= \frac{1}{2}xe^{\mu(T-t)} + \frac{1}{\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}x^k e^{k(\mu-\sigma^2/2)(T-t)}e^{\frac{1}{2}k^2\sigma^2(T-t)} - \frac{1}{\lambda} \end{aligned} \tag{143}$$

Once we have an expression for $G$ and $J$, we need to find another for $K$. Recall the definition of $K$ from equation (113)

$$K(s,x,y) = \mathbb{E}_x[H(X_s)\mathbb{I}_{X_s>y}] = \int_y^{\infty} H(z)f(s,x,z)dz$$

where $H = \mathbb{L}_X G$ and $f$ is given by equation (115). The expression for $H$ is the following:

$$\begin{aligned} H(x) = \mathbb{L}_X G(x) &= \mu x\left(F(x) - \frac{1}{2}\right) + \frac{\sigma^2}{2}x^2F'(x) = \mu x\left(\frac{1}{2} - e^{-\lambda x}\right) + \frac{\sigma^2}{2}x^2\lambda e^{-\lambda x} \\ &= \frac{\mu}{2}x - \mu x e^{-\lambda x} + \frac{\sigma^2}{2}x^2\lambda e^{-\lambda x} \end{aligned} \tag{144}$$

Recall that $f$ is defined as

$$f(s,x,z) = \frac{1}{\sigma\sqrt{s}z}\varphi\left(\Phi(s,x,z)\right)$$

where

$$\Phi(s,x,z) = \frac{1}{\sigma\sqrt{s}}\left[\log\left(\frac{z}{x}\right) + \left(\frac{\sigma^2}{2} - \mu\right)s\right]$$

and $\varphi$ is the standard normal density function given by $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ for $x \in \mathbb{R}$.

Hence, we have to solve the next integral

$$\begin{aligned} K(s,x,y) &= \int_y^{\infty} H(z)f(s,x,z)dz = \int_y^{\infty}\left(\frac{\mu}{2}z - \mu z e^{-\lambda z} + \frac{\sigma^2}{2}z^2\lambda e^{-\lambda z}\right)\frac{1}{\sigma\sqrt{s}z}\varphi\left(\Phi(s,x,z)\right)dz \\ &= \frac{\mu}{2\sigma\sqrt{s}}\int_y^{\infty}\varphi\left(\Phi(s,x,z)\right)dz - \frac{\mu}{\sigma\sqrt{s}}\int_y^{\infty}e^{-\lambda z}\varphi\left(\Phi(s,x,z)\right)dz \\ &+ \frac{\lambda\sigma^2}{2\sigma\sqrt{s}}\int_y^{\infty}ze^{-\lambda z}\varphi\left(\Phi(s,x,z)\right)dz \end{aligned}$$

Let us try to solve the first integral:

$$\int_y^\infty \varphi\left(\Phi(s,x,z)\right) dz$$

We are going to do a change of variable:

$$u = \Phi(s,x,z) = \frac{1}{\sigma\sqrt{s}}[\log(z) - \log(x) + A], \quad \text{with } A = \left(\frac{\sigma^2}{2} - \mu\right) s$$

So, we have

$$\begin{aligned} z &= xe^{\sigma\sqrt{s}u - A} \\ dz &= x\sigma\sqrt{s}e^{\sigma\sqrt{s}u - A}du \end{aligned}$$

Then

$$\begin{aligned} \int_y^\infty \varphi\left(\Phi(s,x,z)\right) dz &= \frac{1}{\sqrt{2\pi}}\int_y^\infty e^{-\Phi^2(s,x,z)/2}dz = \frac{1}{\sqrt{2\pi}}\int_{y'}^\infty e^{-u^2/2}x\sigma\sqrt{s}e^{\sigma\sqrt{s}u-A}du \\ &= \frac{1}{\sqrt{2\pi}}x\sigma\sqrt{s}e^{-A}\int_{y'}^\infty e^{-u^2/2}e^{\sigma\sqrt{s}u}du \end{aligned}$$

where $y' = \frac{1}{\sigma\sqrt{s}}[\log(y) - \log(x) + A]$. Note that

$$\int_{y'}^\infty e^{-u^2/2}e^{\sigma\sqrt{s}u}du = \int_{y'}^\infty e^{-1/2(u^2-2\sigma\sqrt{s}u)}du = \int_{y'}^\infty e^{-1/2((u-\sigma\sqrt{s})^2-\sigma^2 s)}du = e^{\sigma^2 s/2}\int_{y'}^\infty e^{-\left(\frac{u-\sigma\sqrt{s}}{\sqrt{2}}\right)^2}du$$

Let us do a change of variable:

$$w = \frac{u-\sigma\sqrt{s}}{\sqrt{2}}, \quad dw = \frac{1}{\sqrt{2}}du$$

So

$$\begin{aligned} \int_{y'}^\infty e^{-\left(\frac{u-\sigma\sqrt{s}}{\sqrt{2}}\right)^2}du &= \sqrt{2}\int_{y''}^\infty e^{-w^2}dw = \sqrt{2}\left(-\int_0^{y''} e^{-w^2}dw + \int_0^{y''} e^{-w^2}dw + \int_{y''}^\infty e^{-w^2}dw\right) \\ &= \sqrt{2}\left(-\int_0^{y''} e^{-w^2}dw + \int_0^\infty e^{-w^2}dw\right) \\ &= \sqrt{2}\frac{\sqrt{\pi}}{2}\left(-\frac{2}{\sqrt{\pi}}\int_0^{y''} e^{-w^2}dw + \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-w^2}dw\right) \\ &= \frac{\sqrt{\pi}}{\sqrt{2}}\left(-\text{erf}(y'') + \text{erf}(\infty)\right) = \frac{\sqrt{\pi}}{\sqrt{2}}\left(-\text{erf}(y'') + 1\right) \end{aligned}$$

where $y'' = \frac{y'-\sigma\sqrt{s}}{\sqrt{2}}$ and erf is the error function defined as

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$$

This error function can be approximated with Taylor series around $x = 0$, which are known also Maclaurin series, and we would get:

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}$$

So regrouping terms, we have

$$\begin{aligned}
\int_y^{\infty} \varphi\left(\Phi(s,x,z)\right) dz &= \frac{1}{\sqrt{2\pi}} x\sigma\sqrt{s} e^{-A} \int_{y'}^{\infty} e^{-u^2/2} e^{\sigma\sqrt{s}u} du = \frac{1}{\sqrt{2\pi}} x\sigma\sqrt{s} e^{-A} e^{\sigma^2 s/2} \int_{y'}^{\infty} e^{-\left(\frac{u-\sigma\sqrt{s}}{\sqrt{2}}\right)^2} du \\
&= \frac{1}{\sqrt{2\pi}} x\sigma\sqrt{s} e^{-A} e^{\sigma^2 s/2} \frac{\sqrt{\pi}}{\sqrt{2}} \left(-\text{erf}(y'') + 1\right) = \frac{x\sigma\sqrt{s}}{2} e^{\sigma^2 s/2 - A} (1 - \text{erf}(y'')) \\
&= \frac{x\sigma\sqrt{s}}{2} e^{\sigma^2 s/2 - A} \left(1 - \text{erf}\left(\frac{1}{\sigma\sqrt{2s}} [\log(y) - \log(x) + A] - \frac{\sigma\sqrt{s}}{\sqrt{2}}\right)\right) \qquad (145)
\end{aligned}$$

and we will denote this last expression with $I_1 = I_1(s, x, y)$.

We are going to use a method of numeric integration to approximate the other two integrals and we will consider the Simpson's rule:

**Lemma 10.10.** *(Simpson's rule). Let $f(x)$ be a given function in an interval $[a, b] \subset \mathbb{R}$. Then, the Simpson's rule states that the integral $I = \int_a^b f(x)dx$ can be approximated by*

$$I \approx \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) \qquad (146)$$

**Lemma 10.11.** *(Composite Simpson's rule). Let $f(x)$ be a given function in an interval $[a, b] \subset \mathbb{R}$. Then, the composite Simpson's rule states that the integral $I = \int_a^b f(x)dx$ can be approximated by*

$$I \approx \frac{h}{3} \left[f(x_0) + 4 \sum_{i=1}^{m/2} f(x_{2i-1}) + 2 \sum_{i=1}^{m/2-1} f(x_{2i}) + f(x_m)\right] \qquad (147)$$

*where $h = \frac{b-a}{m}$ and we have considered a partition of the interval $[a, b]$ such that $x_i = a + ih$ for $i = 0, 1, ..., m$ with $x_0 = a$ and $x_m = b$.*

So, we need to do a change of variable to the two integrals in order to have finite integration limits. We consider the new variable $u = 1/z$ with $du = -dz/z^2 = -u^2 dz$ and we replace it into the two integrals:

$$\begin{aligned}
\int_y^{\infty} e^{-\lambda z} \varphi\left(\Phi(s,x,z)\right) dz &= \frac{1}{\sqrt{2\pi}} \int_y^{\infty} e^{-\lambda z} e^{-\Phi^2(s,x,z)/2} dz = \frac{1}{\sqrt{2\pi}} \int_y^{\infty} e^{-\lambda z} e^{-\frac{1}{2\sigma^2 s}(\log(\frac{z}{x})+A)^2} dz \\
&= -\frac{1}{\sqrt{2\pi}} \int_{1/y}^{0} \frac{1}{u^2} e^{-\lambda/u} e^{-\frac{1}{2\sigma^2 s}(\log(\frac{1}{xu})+A)^2} du \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{1/y} \frac{1}{u^2} e^{-\lambda/u} e^{-\frac{1}{2\sigma^2 s}(A-\log(xu))^2} du
\end{aligned}$$

$$\int_y^\infty z e^{-\lambda z}\varphi\left(\Phi(s,x,z)\right)dz = \frac{1}{\sqrt{2\pi}}\int_y^\infty z e^{-\lambda z} e^{-\Phi^2(s,x,z)/2}dz = \frac{1}{\sqrt{2\pi}}\int_y^\infty z e^{-\lambda z} e^{-\frac{1}{2\sigma^2 s}(\log(\frac{z}{x})+A)^2}dz$$
$$= -\frac{1}{\sqrt{2\pi}}\int_{1/y}^0 \frac{1}{u^3} e^{-\lambda/u} e^{-\frac{1}{2\sigma^2 s}(\log(\frac{1}{xu})+A)^2}du$$
$$= \frac{1}{\sqrt{2\pi}}\int_0^{1/y} \frac{1}{u^3} e^{-\lambda/u} e^{-\frac{1}{2\sigma^2 s}(A-\log(xu))^2}du$$

with $A = \left(\frac{\sigma^2}{2} - \mu\right)s$. We now define the functions

$$f_1(s,x,u) = \frac{1}{\sqrt{2\pi}u^2} e^{-\lambda/u} e^{-\frac{1}{2\sigma^2 s}(A-\log(xu))^2}$$
$$f_2(s,x,u) = \frac{1}{\sqrt{2\pi}u^3} e^{-\lambda/u} e^{-\frac{1}{2\sigma^2 s}(A-\log(xu))^2}$$

and we approximate the integrals using the composite Simpson's rule:

$$\int_y^\infty e^{-\lambda z}\varphi\left(\Phi(s,x,z)\right)dz = \int_0^{1/y} f_1(s,x,u)du$$
$$\approx \frac{h_2}{3}\left[f_1(s,x,0) + 4\sum_{i=1}^{m/2} f_1(s,x,u_{2i-1}) + 2\sum_{i=1}^{m/2-1} f_1(s,x,u_{2i}) + f_1(s,x,1/y)\right]$$

$$\int_y^\infty z e^{-\lambda z}\varphi\left(\Phi(s,x,z)\right)dz = \int_0^{1/y} f_2(s,x,u)du$$
$$\approx \frac{h_2}{3}\left[f_2(s,x,0) + 4\sum_{i=1}^{m/2} f_2(s,x,u_{2i-1}) + 2\sum_{i=1}^{m/2-1} f_2(s,x,u_{2i}) + f_2(s,x,1/y)\right]$$

where $h_2 = 1/(my)$ and $u_i = ih_2$ for $i = 0, 1, ..., m$. We will denote these two last expressions with $I_2 = I_2(s,x,y)$ and $I_3 = I_3(s,x,y)$ respectively. So the approximation of $K$ becomes

$$K(s,x,y) \approx \frac{\mu}{2\sigma\sqrt{s}} I_1(s,x,y) - \frac{\mu}{\sigma\sqrt{s}} I_2(s,x,y) + \frac{\lambda\sigma^2}{2\sigma\sqrt{s}} I_3(s,x,y) \tag{148}$$

## Algorithm

Finally, replacing the expressions and approximations (138), (143), (148) found respectively for $G$, $J$ and $K$ into equation (136) evaluating in $t = t_k$, $x = b(t_k)$, $s = t_{l+1} - t_k$ and $y = b(t_{l+1})$ we obtain the next formula for $k = 0, ..., n-1$

$$\frac{1}{2}b(t_k)(e^{\mu(T-t_k)} - 1) + \frac{1}{\lambda}\sum_{j=0}^\infty \frac{(-\lambda)^j}{j!} b(t_k)^j e^{j(\mu-\sigma^2/2)(T-t_k)} e^{\frac{1}{2}j^2\sigma^2(T-t_k)}$$
$$= \frac{1}{\lambda}e^{-\lambda b(t_k)} + \sum_{l=k}^{n-1}\left(\frac{\mu}{2\sigma\sqrt{t_{l+1}-t_k}} I_1 - \frac{\mu}{\sigma\sqrt{t_{l+1}-t_k}} I_2 + \frac{\lambda\sigma}{2\sqrt{t_{l+1}-t_k}} I_3\right) h \tag{149}$$

where $l_1, l_2, l_3$ are evaluated in $(t_{l+1} - t_k, b(t_k), b(t_{l+1}))$ and $t_k = kh$ with $h = T/n$.

**Algorithm 1:** Algorithm for finding the optimal stopping boundary

```
Input  : μ, σ, α, T, λ, n
Output: b
h = T/n
Let b be a vector of length n+1 with b[0, ..., n−1] = [0, ..., 0] and b[n] = α
Define the function G(x)  (Eq. (138))
Define the function J(t, x)  (Eq. (143))
Define the function K(s, x, y)  (Eq. (148))
Function f(x, k, b):
    t = k * h
    res = J(t, x) − G(x)
    for l = k, ..., n−1 do
        s = h * (l+1) − t
        res = res − K(s, x, b[l+1])
    return res
for k = n−1, ..., 0 do
    Find the solution x of f(x, k, b) = 0 with k, b given
    b[k] = x
```

We have programmed this algorithm in R, using packages stats and DEoptim for solving optimization problems (see Appendix A.1).

## 10.5 Examples

### Example 1: Resistance level

In this case, we consider that the asset price behaves as a geometric Brownian motion with $\mu > 0$. We choose the random variable $l$ to follow an exponential distribution with parameter $\lambda > 0$, so its distribution function $F$ is defined by

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{150}$$

We choose the exponential distribution because it is the maximum entropy probability distribution for a random variable among all continuous distributions with support $[0, \infty)$ which have a particular mean of $1/\lambda$. This means that is the one with the largest entropy and so the one that has most uncertainty, i.e. the least-informative. Recall that the entropy of a distribution is the expected amount of information in an event extracted from that distribution.

Let us verify that $l$ has an expected value of $1/\lambda$. Recall the formula to compute the mean:

$$\mathbb{E}[l] = \int_0^\infty x f(x) dx$$

where $f$ is the density function of the random variable $l$ which is $f(x) = F'(x)$ with

$$F'(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{151}$$

So, we can write:

$$\mathbb{E}[I] = \int_0^\infty x\lambda e^{-\lambda x} dx$$

Let us integrate by parts setting $u = \lambda x$ and $dv = e^{-\lambda x}dx$, so we have $du = \lambda dx$ and $v = -\frac{1}{\lambda}e^{-\lambda x}$. We obtain that

$$\begin{aligned}\mathbb{E}[I] &= \int_0^\infty x\lambda e^{-\lambda x} dx = \lim_{M\to\infty}\left[-xe^{-\lambda x}\right]_0^M - \int_0^\infty -e^{-\lambda x}dx = \lim_{M\to\infty}\frac{-M}{e^{\lambda M}} - \lim_{M\to\infty}\left[\frac{1}{\lambda}e^{-\lambda x}\right]_0^M \\ &= \lim_{M\to\infty}\frac{-1}{\lambda e^{\lambda M}} - \lim_{M\to\infty}\frac{1}{\lambda}e^{-\lambda M} + \frac{1}{\lambda} = \frac{1}{\lambda}\end{aligned}$$

Let us calculate the median of $I$. First, we need to compute the quantile function $Q(p)$ which is the inverse of the distribution function:

$$F(Q(p)) = p \Leftrightarrow 1 - e^{-\lambda Q(p)} = p \Leftrightarrow 1 - p = e^{-\lambda Q(p)} \Leftrightarrow Q(p) = \frac{-\ln(1-p)}{\lambda}$$

We know that the median is the quantil value with $p = 1/2$ so we obtain

$$m = M = \frac{-\ln(1/2)}{\lambda} = \frac{\ln(2)}{\lambda} \tag{152}$$

We need to find $\alpha \in (0, m)$ such that it satisfies the definition of admissible aspiration level laws (109). Let us develop the two conditions; the first one is the following:

$$xF'(x) < \frac{\mu}{\sigma^2/2}\left(\frac{1}{2} - F(x)\right) \text{ for } x \in (0, \alpha)$$

So, in our case we have for $x \in (0, \alpha)$

$$x\lambda e^{-\lambda x} < \frac{\mu}{\sigma^2/2}\left(\frac{1}{2} - (1 - e^{-\lambda x})\right) = \frac{\mu}{\sigma^2/2}\left(-\frac{1}{2} + e^{-\lambda x}\right) \Leftrightarrow x\lambda < \frac{\mu}{\sigma^2/2}\left(1 - \frac{1}{2}e^{\lambda x}\right) \tag{153}$$

The second condition is

$$xF'(x) > \frac{\mu}{\sigma^2/2}\left(\frac{1}{2} - F(x)\right) \text{ for } x \in (\alpha, m)$$

Hence, analogously we get

$$x\lambda > \frac{\mu}{\sigma^2/2}\left(1 - \frac{1}{2}e^{\lambda x}\right) \text{ for } x \in (\alpha, m) \tag{154}$$

Therefore, combining (153) and (154) we conclude that (109) is satisfied with $\alpha \in (0, m)$ being the unique solution of the following equation

$$\alpha\lambda = \frac{\mu}{\sigma^2/2}\left(1 - \frac{1}{2}e^{\lambda\alpha}\right) \tag{155}$$

Thus, $F$ belongs to $\mathcal{A}(\mu, \sigma)$ for all $\sigma > 0$.

Recall that by Theorem 10.7 we know that the optimal stopping time for selling is given by

$$\tau_* = \inf\{t \in [0, T]; X_t \geq b(t)\} \tag{156}$$

where $b(t)$ is the unique continuous decreasing solution to (116) satisfying $\alpha \leq b(t) \leq m$ for $t \in [0, T)$ and $b(T-) = \alpha$.

Let us see an example fixing specific values $\mu = 1$, $\lambda = 1/2$, $\sigma = 2$ and $T = 1$. To find the value of $\alpha$ we can solve equation (155) numerically and we obtain $\alpha = 0.39$. To get this result, we have programmed the next function in R

$$f(\alpha) = \frac{\mu}{\sigma^2/2}\left(1 - \frac{1}{2}e^{\lambda\alpha}\right) - \alpha\lambda \tag{157}$$

and we have used the package rootSolve to find the zero of this function (see Appendix A.2). We also have $m = \ln(2)/\lambda = 1.386$. To calculate the optimal stopping boundary we use the algorithm explained in the previous section. In our case, we have chosen $n = 50$ and we have obtained the values $b(t_{50}), ..., b(t_0)$ as an approximation of $b$ at the points $1, 0.98, ..., 0.02, 0$. If we plot these results, we get the graphic in figure 1. Recall that $b(t_n) = b(T-) = \alpha = 0.39$ and note that $b$ is a continuous decreasing function as we expected. We have obtained $b(0) = 0.695$ approximately.

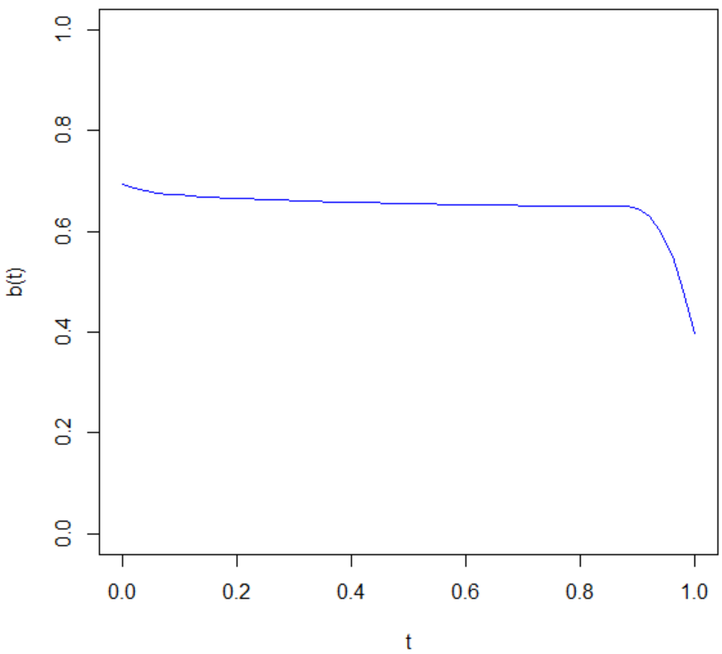



Figure 1: The optimal stopping boundary $b$ from Example 1

### Example 2: Support level

In this case, we consider that the asset price behaves as a geometric Brownian motion with $\mu < 0$. We choose the random variable $I$ to follow the same distribution as in the previous example, so $I \sim Exp(\lambda)$ with $\lambda > 0$.

Recall that the mean of $I$ is $1/\lambda$ and the median is $m = M = \ln(2)/\lambda$. We need to find $\beta \in (M, \infty)$ such that it satisfies the definition of admissible aspiration level laws (110). The first condition is

$$xF'(x) > \frac{\mu}{\sigma^2/2}\left(\frac{1}{2} - F(x)\right) \text{ for } x \in (M, \beta)$$

which yields

$$x\lambda > \frac{\mu}{\sigma^2/2}\left(1 - \frac{1}{2}e^{\lambda x}\right) \text{ for } x \in (M, \beta) \tag{158}$$

The second condition is

$$xF'(x) < \frac{\mu}{\sigma^2/2}\left(\frac{1}{2} - F(x)\right) \text{ for } x \in (\beta, \infty)$$

which gives

$$x\lambda < \frac{\mu}{\sigma^2/2}\left(1 - \frac{1}{2}e^{\lambda x}\right) \text{ for } x \in (\beta, \infty) \tag{159}$$

Therefore, combining (158) and (159) we conclude that (110) is satisfied with $\beta \in (M, \infty)$ being the unique solution of the following equation

$$\beta\lambda = \frac{\mu}{\sigma^2/2}\left(1 - \frac{1}{2}e^{\lambda\beta}\right) \tag{160}$$

Thus, $F$ belongs to $\mathcal{A}(\mu, \sigma)$ for all $\sigma > 0$.

Recall that by Theorem 10.7 we know that the optimal stopping time for buying is given by

$$\tau_* = \inf\{t \in [0, T]; X_t \leq b(t)\} \tag{161}$$

where $b(t)$ is the unique continuous increasing solution to (119) satisfying $M \leq b(t) \leq \beta$ for $t \in [0, T)$ and $b(T-) = \beta$.

Let us see an example fixing specific values $\mu = -1$, $\lambda = 4$, $\sigma = 2$ and $T = 1$. To find the value of $\beta$ we can solve equation (160) numerically, analogously to the previous example, and we obtain $\beta = 0.619$ (see Appendix A.2). We also have $M = \ln(2)/\lambda = 0.173$. To calculate the optimal stopping boundary we use a similar algorithm explained in the previous section. In this case, we have chosen $n = 15$ and we have obtained the values $b(t_{15}), ..., b(t_0)$ as an approximation of $b$. If we plot these results, we get the graphic in figure 2. Recall that $b(t_n) = b(T-) = \beta = 0.619$ and note that $b$ is a continuous increasing function as we expected. We have obtained $b(0) = 0.173$ approximately.

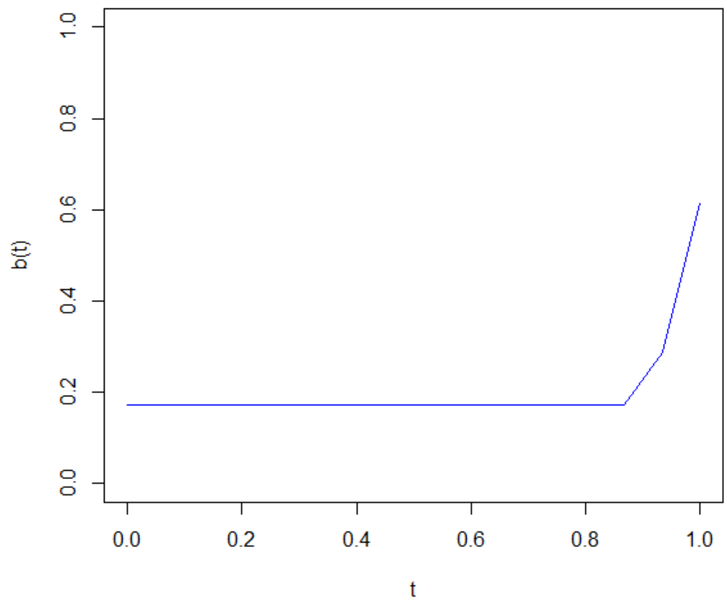



Figure 2: The optimal stopping boundary $b$ from Example 2

### Example 3: Aspiration level at the ultimate maximum

In this case, we consider that the asset price behaves as a geometric Brownian motion with $\mu \in (0, \sigma^2/2)$. We choose the random variable $I$ to follow the distribution of the ultimate maximum of $X$ which is defined as $S = \sup_{t\geq 0}\left[x_0 \exp((\mu - (\sigma^2/2))t + \sigma B_t)\right]$ where $x_0$ is known and fixed. To calculate the distribution function $F$ of $I$ we need the following formula:

**Theorem 10.12.** *(Doob's formula). Let $B_t$ be a Brownian motion. Then for $\alpha > 0$ and $\beta > 0$*

$$\mathbb{P}\left(\sup_{t\geq 0}(B_t - \alpha t) \geq \beta\right) = e^{-2\alpha\beta} \tag{162}$$

Let us compute $\mathbb{P}(S \geq x)$:

$$\begin{aligned}
\mathbb{P}(S \geq x) = \mathbb{P}(\ln(S) \geq \ln(x)) &= \mathbb{P}\left(\sup_{t\geq 0}\left(\ln(x_0 e^{\sigma B_t + (\mu - \sigma^2/2)t})\right) \geq \ln(x)\right) \\
&= \mathbb{P}\left(\sup_{t\geq 0}\left(\sigma B_t + (\mu - \sigma^2/2)t\right) \geq \ln(x) - \ln(x_0)\right) \\
&= \mathbb{P}\left(\sup_{t\geq 0}\left(B_t + \frac{(\mu - \sigma^2/2)t}{\sigma}\right) \geq \frac{\ln(x) - \ln(x_0)}{\sigma}\right)
\end{aligned}$$

Applying Doob's formula (162) with $\alpha = -(\mu - \sigma^2/2)/\sigma$ and $\beta = (\ln(x) - \ln(x_0))/\sigma$ we obtain

$$\mathbb{P}(S \geq x) = \exp\left(\frac{2(\mu - \sigma^2/2)}{\sigma} \cdot \frac{\ln(x) - \ln(x_0)}{\sigma}\right) = \exp\left(\frac{\mu - \sigma^2/2}{\sigma^2/2} \ln\left(\frac{x}{x_0}\right)\right) = \exp\left(\ln\left(\frac{x}{x_0}\right)^{\frac{\mu-\sigma^2/2}{\sigma^2/2}}\right)$$
$$= \left(\frac{x}{x_0}\right)^{\frac{\mu-\sigma^2/2}{\sigma^2/2}} = \left(\frac{x}{x_0}\right)^{\frac{\mu}{\sigma^2/2}-1} = \left(\frac{x_0}{x}\right)^{1-\frac{\mu}{\sigma^2/2}}$$

Therefore, the distribution function $F$ of $I$ is defined by

$$F(x) = \begin{cases} 1 - \left(\frac{x_0}{x}\right)^{1-\frac{\mu}{\sigma^2/2}} & x \geq x_0 \\ 0 & x < x_0 \end{cases} \tag{163}$$

Let us calculate the median of $I$. First, we need to compute the quantile function $Q(p)$ which is the inverse of the distribution function:

$$F(Q(p)) = p \Leftrightarrow 1 - \left(\frac{x_0}{Q(p)}\right)^{1-\frac{\mu}{\sigma^2/2}} = p \Leftrightarrow 1 - p = \left(\frac{x_0}{Q(p)}\right)^{1-\frac{\mu}{\sigma^2/2}} \Leftrightarrow \frac{1}{1-p} = \left(\frac{Q(p)}{x_0}\right)^{1-\frac{\mu}{\sigma^2/2}}$$
$$\Leftrightarrow \frac{x_0^{1-\mu/(\sigma^2/2)}}{1-p} = Q(p)^{1-\mu/(\sigma^2/2)} \Leftrightarrow Q(p) = \frac{x_0}{(1-p)^{1/(1-\mu/(\sigma^2/2))}}$$

We know that the median is the quantil value with $p = 1/2$ so we obtain

$$m = M = \frac{x_0}{(\frac{1}{2})^{1/(1-\mu/(\sigma^2/2))}} = 2^{1/(1-\mu/(\sigma^2/2))} x_0 \tag{164}$$

We need to find $\alpha \in (0, m)$ such that it satisfies the definition of admissible aspiration level laws (109). First, we have to calculate $F'(x)$:

$$F'(x) = \begin{cases} \left(1 - \frac{\mu}{\sigma^2/2}\right)\left(\frac{x_0}{x}\right)^{1-\frac{\mu}{\sigma^2/2}}\left(\frac{1}{x}\right) & x \geq x_0 \\ 0 & x < x_0 \end{cases} \tag{165}$$

Note that $F'$ has a discontinuity at $x_0$ and so it is only piecewise $C^1$. The first condition of (109) is

$$xF'(x) < \frac{\mu}{\sigma^2/2}\left(\frac{1}{2} - F(x)\right) \text{ for } x \in (0, \alpha)$$

which yields for $x \in (0, \alpha)$

$$
\begin{aligned}
& x\left(1-\frac{\mu}{\sigma^2/2}\right)\left(\frac{x_0}{x}\right)^{1-\frac{\mu}{\sigma^2/2}}\left(\frac{1}{x}\right) < \frac{\mu}{\sigma^2/2}\left(-\frac{1}{2}+\left(\frac{x_0}{x}\right)^{1-\frac{\mu}{\sigma^2/2}}\right) \\
\Leftrightarrow\ & \left(1-\frac{\mu}{\sigma^2/2}\right) < \frac{\mu}{\sigma^2/2}\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^{1-\frac{\mu}{\sigma^2/2}}+1\right) \\
\Leftrightarrow\ & \left(\frac{\sigma^2/2}{\mu}-2\right) < -\frac{1}{2}\left(\frac{x}{x_0}\right)^{1-\frac{\mu}{\sigma^2/2}} \\
\Leftrightarrow\ & 2\left(2-\frac{\sigma^2/2}{\mu}\right) < \left(\frac{x}{x_0}\right)^{1-\frac{\mu}{\sigma^2/2}} \\
\Leftrightarrow\ & x^{1-\mu/(\sigma^2/2)} > 2\left(2-\frac{\sigma^2/2}{\mu}\right) x_0^{1-\mu/(\sigma^2/2)} \\
\Leftrightarrow\ & x > \left(2\left(2-\frac{\sigma^2/2}{\mu}\right)\right)^{1/(1-\mu/(\sigma^2/2))} x_0
\end{aligned} \tag{166}
$$

The second condition is

$$xF'(x) > \frac{\mu}{\sigma^2/2}\left(\frac{1}{2}-F(x)\right) \text{ for } x \in (\alpha, m)$$

Hence, analogously we get

$$x < \left(2\left(2-\frac{\sigma^2/2}{\mu}\right)\right)^{1/(1-\mu/(\sigma^2/2))} x_0 \text{ for } x \in (\alpha, m) \tag{167}$$

Therefore, combining (166) and (167) we conclude that (109) is satisfied with $\alpha \in (0, m)$ given by

$$\alpha = \left(2\left(2-\frac{\sigma^2/2}{\mu}\right)\right)^{1/(1-\mu/(\sigma^2/2))} x_0 \tag{168}$$

for $\mu \in [\sigma^2/3, \sigma^2/2)$ and $\alpha = x_0$ for $\mu \in (0, \sigma^2/3)$. Thus, $F$ belongs to $\mathcal{A}(\mu, \sigma)$ for all $\sigma > 0$.

Recall that by Theorem 10.7 we know that the optimal stopping time for selling is given by

$$\tau_* = \inf\{t \in [0, T]; X_t \geq b(t)\} \tag{169}$$

where $b(t)$ is the unique continuous decreasing solution to (116) satisfying $\alpha \leq b(t) \leq m$ for $t \in [0, T)$ and $b(T-) = \alpha$.

Finally, let us mention an example fixing specific values $\mu = 1$, $x_0 = 1$, $\sigma = 2$ and $T = 1$. In this case, $\mu \in (0, \sigma^2/3)$ and so we have $\alpha = x_0 = 1$. We also have $m = 2^{1/(1-\mu/(\sigma^2/2))} x_0 = 4$. In addition, the authors computed $b(t)$ numerically and they obtained $b(0) = 1.68$ approximately.

# References

[1] A. Arratia. *Computational Finance: An Introductory Course with R*, Atlantis Press, 2014.

[2] T. De Angelis and G. Peskir. Optimal prediction of resistance and support levels, *Applied Mathematical Finance.* **23 (6)** (2017), 465–483.

[3] A. Etheridge. *C8.2: Stochastic analysis and PDEs*, 2016.

[4] D. Stirzaker. *Stochastic Processes and Models*, Oxford University Press, 2005.

[5] G. Peskir and A. Shiryaev. *Optimal Stopping and Free-Boundary Problems*, Birkhäuser Basel, 2006.

[6] B. Øksendal. *Stochastic Diferential Equations: An Introduction with Applications*, 5th ed. Springer-Verlag, 1998.

[7] K. Brekke and B. Øksendal. The High Contact Principle as a Sufficiency Condition for Optimal Stopping, *Stochastic models and Option Values*, North-Holland (1991), 187–208.

[8] Paul A. Samuelson. Rational Theory of Warrant Pricing, *Industrial Management Review.* **6** (1965), 13–32.

[9] H. P. McKean. Appendix: A Free Boundary Problem for the Heat Equation Arising from a Problem in Mathematical Economics, *Industrial Management Review.* **6** (1965), 32–39.

[10] G. Peskir. Optimal Detection of a Hidden Target: The Median Rule, *Stochastic Processes and their Applications.* **122** (2012), 2249–2263.

[11] H. A. Simon. A Behavioral Model of Rational Choice, *The Quarterly Journal of Economics.* **69** (1955), 99–118.

[12] G. Peskir. A Change-of-Variable Formula with Local Time on Curves, *Journal of Theoretical Probability.* **18** (2005), 499–535.

[13] A. Di Crescenzo. A Probabilistic Analogue of the Mean Value Theorem and Its Applications to Reliability Theory, *Journal of Applied Probability.* **36** (1999), 706–719.

# A. Programs code

## A.1 Algorithm for finding the optimal stopping boundary

```
library(stats)
library(DEoptim)
library(pracma)

## Initialization parameters
mu <- 1
lambda <- 1/2
sigma <- 2
m <- (1/lambda)*log(2)
T <- 1
alpha <- 0.3917735
n <- 50
h <- T/n

## Function G
G <- function(x) {
  res = x/2+(1/lambda)*exp(-lambda*x)-(1/lambda)
  return(res)
}

## Function J
J <- function(t,x) {
  sum = 0
  for (j in (0:10)) {
    sum = sum + (((-lambda)^j)/(factorial(j)))*(x^j)
          *exp(j*(mu-(sigma^2)/2)*(T-t))*exp((j^2)*(sigma^2)*(T-t)/2)
  }
  res = (1/2)*(x*exp(mu*(T-t)))-(1/lambda)+(1/lambda)*sum
  return(res)
}

## Auxiliary functions for K
I1 <- function(x,t,l,b) {
  s <- (l+1)*h - t
  y <- b[l+2]
  A <- ((sigma^2)/2-mu)*s
  z <- (1/(sigma*sqrt(2*s)))*(log(y/x)+A) - (sigma*sqrt(s))/sqrt(2)
  erf = 0
  for (i in (0:10)) {
    erf = erf + (2/sqrt(pi))*(((-1)^i)*(z^(2*i+1)))/(factorial(i)*(2*i+1))
  }
```

```
  res = ((x*sigma*sqrt(s)*exp((s*(sigma^2))/2-A))/2)*(1-erf)
  return(res)
}

f1 <- function(l,x,u,t) {
  s <- (l+1)*h - t
  A <- ((sigma^2)/2-mu)*s
  res = (1/(sqrt(2*pi))*u^2)*exp(-lambda/u)*exp(-(1/(2*(sigma^2)*s))
        *(A-log(x*u))^2)
  return(res)
}

f2 <- function(l,x,u,t) {
  s <- (l+1)*h - t
  A <- ((sigma^2)/2-mu)*s
  res = (1/(sqrt(2*pi))*u^3)*exp(-lambda/u)*exp(-(1/(2*(sigma^2)*s))
        *(A-log(x*u))^2)
  return(res)
}

I2 <- function(x,t,l,b) {
  y <- b[l+2]
  m2 <- 14
  h2 <- 1/(m2*y)
  sum = f1(l,x,0,t) + f1(l,x,1/y,t)
  for (i in (1:(m2/2))) {
    u <- (2*i-1)*h2
    sum = sum + 4*f1(l,x,u,t)
  }
  for (i in (1:(m2/2-1))) {
    u <- (2*i)*h2
    sum = sum + 2*f1(l,x,u,t)
  }
  res = (h2/3)*sum
  return(res)
}

I3 <- function(x,t,l,b) {
  y <- b[l+2]
  m2 <- 14
  h2 <- 1/(m2*y)
  sum = f2(l,x,0,t) + f2(l,x,1/y,t)
  for (i in (1:m2/2)) {
    u <- (2*i-1)*h2
    sum = sum + 4*f2(l,x,u,t)
  }
```

```
  for (i in (1:(m2/2-1))) {
    u <- (2*i)*h2
    sum = sum + 2*f2(l,x,u,t)
  }
  res = (h2/3)*sum
  return(res)
}

## Function K
K <- function(x,t,l,b) {
  s <- (l+1)*h - t
  res = (mu/(2*sigma*sqrt(s)))*I1(x,t,l,b)-(mu/(sigma*sqrt(s)))
        *I2(x,t,l,b) +((lambda*sigma)/(2*sqrt(s)))*I3(x,t,l,b)
  return(res)
}

## Function to optimize
f <- function(x,k,b) {
  t <- k*h
  res = J(t,x) - G(x)
  for (l in k:(n-1)) {
    res = res - K(x,t,l,b)*h
  }
  return(res)
}

## Loop which starts with k=n-1 and calculates b
b <- rep(0, n+1) # vector b = b[t0,t1,...,tn] = b[1,2,...,n+1]
b[n+1] <- alpha # b[tn] = alpha

for (k in (n-1):0) {
  print(paste("starting iteration", k))

  # Two methods to optimize:
  # Method 1:
  res1 <- optimize(function(x) f(x,k,b), c(b[k+2]+1/((k+1)*100),m),
                   maximum = FALSE)
  root <- res1$minimum
  # Method 2:
  # res2 <- DEoptim(function(x) return(f(x,k,b)),
                   lower = b[k+2]+1/((k+1)*100), upper = m,
                   control = list(storepopfrom = 1, trace = FALSE,
                   itermax=100))$optim
  # root <- res2$bestmem

  b[k+1] <- root
```

```
  print(paste("iteration", k, "finished"))
  print(paste("b after iteration", k, ":"))
  print(b)
}

## Graph of the optimal stopping boundary b
x <- linspace(0,1,n+1)
y <- b
smoothingSpline <- smooth.spline(x, y, spar=0.35)
plot(smoothingSpline, ylim=c(0,1), type='l', xlab="t", ylab="b(t)",
     col='blue')
```

## A.2 Calculation of alpha and beta from Examples 1 and 2

```
library(rootSolve)

## Example 1
mu <- 1
lambda <- 1/2
sigma <- 2
m <- (1/lambda)*log(2)

fun_a <- function (alpha) (mu/(sigma^2/2))*(1-1/2*exp(lambda*alpha))
                          -lambda*alpha
alpha <- uniroot(fun_a, c(0, m))$root

## Example 2
mu <- -1
lambda <- 4
sigma <- 2
M <- (1/lambda)*log(2)

fun_b <- function (beta) (mu/(sigma^2/2))*(1-1/2*exp(lambda*beta))
                         -lambda*beta
beta <- uniroot(fun_b, c(M,100))$root
```