Overdetermined partial resolvent kernel for generalized cylinders

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Abstract

Overdetermined partial boundary value problems are those where no data are prescribed on part of the boundary, whereas both the values of the function and of its normal derivative are given on another part of the boundary. The study of the existence and uniqueness of its solution for finite networks involves Schrödinger operators.

In the present work, we study the resolvent kernels associated with overdetermined partial boundary value problems and we express them in terms of the well-known Green operator and the Dirichlet-to-Robin map. Moreover, we analyze their main properties and we compute them in the case of a generalized cylinder. The obtained expression involves polynomials that can be seen as a generalization of Chebyshev polynomials, and indeed, when the conductances along axes are constant, the expressions for the overdetermined partial resolvent kernels are given in terms of second kind Chebyshev polynomials.

1 Introduction

The *Inverse Boundary Value Problem* arised for the first time around 1950 due to Alberto Calderón's work. However, it was not until 1980 that he published "On an Inverse Boundary Value Problem" [10] detailing his work on the subject. This problem appeared as a consequence of an engineering problem on geophysical electrical prospection in which the objective is to deduce some internal terrain properties from surface electrical measurements.

These works have motivated several developments in the inverse problem field until nowadays. More recently, this problem has been also considered for medical purposes on *Electrical Impedance Tomography* (EIT) [11], which is a medical imaging technique where an image containing visual information of internal parts of the body is obtained from electrical measurements on the boundary.

The mathematical corresponding problem that Calderón proposed is whether it is possible to determine the conductivity of a body by means of current and voltage measurements at its boundary. This problem of recovering conductances from boundary or surface current and potential measurements is a non–linear inverse problem and it is exponentially ill–posed [1, 17], since its solution is highly sensitive to changes in the boundary data.

Since its appearance, Calderón's Inverse Problems have been treated in many ways. For instance, Sylvester and Uhlmann treated in [9, 18] the uniqueness of solution; Curtis, Ingerman and Morrow have worked on critical circular planar networks conductivity reconstruction [12, 13, 14, 16]; Borcea, Druskin, Guevara and Mamonov have gone into EIT problems in depth and their last works on the subject treat numerical conductivity reconstruction [6, 7, 8].

Inverse boundary value problems have been considered both over the continuum and the discrete fields. In this work we define a new class of boundary value problems on finite networks associated with Schrödinger operators. The novelty lies on the fact that on a part of the boundary no data is prescribed, whereas in another part of the boundary both the values of the function as of its normal derivative are given. These problems are not self–adjoint, and hence we worry about the study of existence and uniqueness through the adjoint problem.

We show that overdetermined partial boundary value problems are the key in the framework of inverse boundary value problems on finite networks, since they provide the theoretical foundations of the recovery algorithm. In fact, this type of problems were implicitly considered in some previous works, but only for specific networks and boundary data, see [13, 14]. We analyze the uniqueness and existence of solution of overdetermined partial boundary value problems through the non–singularity of the partial Dirichlet–to–Neumann maps. These maps allow us to determined the value of the solution in the part of the boundary with no prescribed data. Afterwards, we give explicit formulae for the acquirement of boundary spike conductances on critical planar networks and execute a full conductance recovery for spider networks. This algorithm is an adaptation of the one proposed in [14] for the Combinatorial Laplacian and when the corresponding Dirichlet–to–Neumann map is singular.

2 Overdetermined Partial boundary value problems

We fix a proper and connected subset $F \subset V$ and $A, B \subset \delta(F)$ non–empty subsets such that $A \cap B = \emptyset$. Moreover we denote by R the set $R = \delta(F) \setminus (A \cup B)$, so $\delta(F) = A \cup B \cup R$ is a partition of $\delta(F)$. We remark that R can be an empty set. We consider a new type of boundary value problems in which the values of the functions and their normal derivatives are known at the same part of the boundary, which represents an overdetermined problem, and there exists another part of the boundary where no data is known. The limit case when $B = R = \emptyset$, the value of the function on the boundary is null and the value of the normal derivative is constant, can be seen as an extension of the so–called discrete Serrin's Problem. The analysis of this problem was carried out by the authors in [3]. For $B = \emptyset$, this kind of problem has been considered in the continuous case as an extension of Serrin's problem, see [15].

For any $f \in \mathcal{C}(F)$, $g \in \mathcal{C}(A \cup R)$ and $h \in \mathcal{C}(A)$, the *overdetermined partial Dirichlet–Neumann boundary value problem on F with data f, g, h* consists in finding $u \in \mathcal{C}(\bar{F})$ such that

$$\mathscr{L}_q(u) = f \;\text{ on }\; F, \quad \frac{\partial u}{\partial \mathsf{n}_F} = h \;\text{ on }\; A \quad\text{and}\quad u = g \;\text{ on }\; A \cup R. \tag{1}$$

Notice that as the values of u are known in A, the boundary condition $\frac{\partial u}{\partial \mathsf{n}_F} = h$ is equivalent $\frac{\partial u}{\partial \mathsf{n}_F} + q\,u = h + q\,g$.

The *homogeneous overdetermined partial Dirichlet–Neumann boundary value problem on F* consists in finding $u \in \mathcal{C}(\bar{F})$ such that

$$\mathscr{L}_q(u) = 0 \;\text{ on }\; F, \quad \frac{\partial u}{\partial \mathsf{n}_F} = u = 0 \;\text{ on }\; A \quad\text{and}\quad u = 0 \;\text{ on }\; R. \tag{2}$$

It is clear that the set of solutions of the homogeneous boundary value problem is a subspace of $\mathcal{C}(F \cup B)$ that we will denote by $\mathcal{V}_B$. Moreover, if Problem (1) has solutions and u is a particular one, then $u+\mathcal{V}_B$ describes the set of all its solutions. In addition, if u is a solution of Problem (1), then for any $x\in A$ we get that

$$\int_F c(x,y)u(y)dy = g(x)\kappa_F(x) - h(x).$$

Therefore, if u is a solution of Problem (2), then for any $x\in A$ we get that

$$\int_F c(x,y)u(y)dy = 0.$$

The *adjoint problem of the overdetermined partial Dirichlet–Neumann boundary value problem (2) on F* is given by

$$\mathscr{L}_q(v)=0 \ \text{ on } \ F, \quad \frac{\partial v}{\partial \mathsf{n}_F} = v = 0 \ \text{ on } \ B \quad \text{and} \quad v=0 \ \text{ on } \ R. \tag{3}$$

The subspace of solutions of the above problem will be denoted by $\mathcal{V}_A$. It is clear that $\mathcal{V}_A\subset\mathcal{C}(F\cup A)$.

The Second Green Identity leads to the following result.

Proposition 2.1. *Problems (2) and (3) are mutually adjoint; that is*

$$\int_F v(x)\mathscr{L}_q(u)(x)\,dx = \int_F u(x)\mathscr{L}_q(v)(x)\,dx,$$

for any $u,v\in\mathcal{C}(\bar F)$ such that $\dfrac{\partial u}{\partial \mathsf{n}_F}=u=0$ on A, $\dfrac{\partial v}{\partial \mathsf{n}_F}=v=0$ on B and $u=v=0$ on R.

Proof. By the Second Green Identity we get that

$$\begin{aligned}\int_F \Big(v(x)\mathscr{L}_q(u)(x) - u(x)\mathscr{L}_q(v)(x)\Big)\,dx &= \int_{\delta(F)} \Big(u(x)\frac{\partial v}{\partial \mathsf{n}_F}(x) - v(x)\frac{\partial u}{\partial \mathsf{n}_F}(x)\Big)\,dx \\ &= \int_B u(x)\frac{\partial v}{\partial \mathsf{n}_F}(x)\,dx - \int_A v(x)\frac{\partial u}{\partial \mathsf{n}_F}(x)\,dx = 0,\end{aligned}$$

obtaining the result. □

Proposition 2.2 (Fredholm Alternative)**.** *Given $f\in\mathcal{C}(F)$, $g\in\mathcal{C}(A\cup R)$, $h\in\mathcal{C}(A)$, the boundary value problem*

$$\mathscr{L}_q(u)=f, \ \text{ on } \ F, \quad \frac{\partial u}{\partial \mathsf{n}_F} = h \ \text{ on } \ A \ \text{ and } \ u=g \ \text{ on } \ A\cup R$$

has solution if and only if

$$\int_F f(x)v(x)\,dx + \int_A h(x)v(x)\,dx = \int_{A\cup R} g(x)\frac{\partial v}{\partial \mathsf{n}_F}(x)\,dx, \quad \text{for each } v\in\mathcal{V}_A.$$

In addition, when the above condition holds, then there exists a unique solution of the boundary value problem in $\mathcal{V}_B^\perp$, i.e. a unique solution u, such that

$$\int_{F\cup B} u(x)z(x)\,dx = 0, \quad \text{for any } z\in\mathcal{V}_B.$$

Proof. First observe that Problem (1) is equivalent to the boundary value problem

$$\mathscr{L}_q(u) = f - \mathscr{L}(g) \;\text{ on }\; F,\;\; \frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = h - g\kappa_{_F} \;\text{ on }\; A \;\text{ and }\; u = 0, \;\text{ on }\; A \cup R \tag{4}$$

in the sense that u is a solution of this problem if and only if $u+g$ is a solution of Problem (1). Notice that $\mathscr{L}(g) = \mathscr{L}_q(g)$ since $g = 0$ on F. Consider now the linear operators $\mathcal{F}\colon \mathcal{C}(F \cup B) \longrightarrow \mathcal{C}(F \cup A)$ and $\mathcal{F}^*\colon \mathcal{C}(F \cup A) \longrightarrow \mathcal{C}(F \cup B)$ defined as

$$\mathcal{F}(u) = \begin{cases} \mathscr{L}_q(u), & \text{on } F, \\ \dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}}, & \text{on } A, \end{cases} \quad \text{and} \quad \mathcal{F}^*(v) = \begin{cases} \mathscr{L}_q(v), & \text{on } F, \\ \dfrac{\partial v}{\partial \mathsf{n}_\mathsf{F}}, & \text{on } B, \end{cases}$$

respectively. Then, for any $u \in \mathcal{C}(F \cup B)$ satisfying Problem (4) and for any $v \in \mathcal{C}(F \cup A)$,

$$\begin{aligned}\int_{F\cup A} v(x)\mathcal{F}(u)(x)\,dx &= \int_F v(x)\mathscr{L}_q(u)(x)\,dx + \int_{\delta(F)} v(x)\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx = \\ &= \int_F u(x)\mathscr{L}_q(v)(x)\,dx + \int_{\delta(F)} u(x)\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx = \int_{F\cup B} u(x)\mathcal{F}^*(v)(x)\,dx.\end{aligned}$$

Clearly, $\ker\mathcal{F}^* = \mathcal{V}_{_A}$. Moreover, Problem (1) has a solution if and only if the function $\tilde{f} \in \mathcal{C}(F \cup A)$ given by $\tilde{f} = f - \mathscr{L}(g)$ on F and $\tilde{f} = h - g\kappa_{_F}$ on A satisfies that $\tilde{f} \in \mathrm{Img}\mathcal{F}$. Therefore, the Fredlhom Alternative for linear operators implies that Problem (1) has solution if and only if for any $v \in \mathcal{V}_{_A}$

$$\begin{aligned}0 = \int_{F\cup A} \tilde{f}(x)v(x)\,dx &= \int_F f(x)v(x)\,dx + \int_A h(x)v(x)\,dx - \int_F v(x)\mathscr{L}(g)(x)\,dx \\ - \int_A v(x)g(x)\kappa_{_F}(x)\,dx &= \int_F f(x)v(x)\,dx + \int_A h(x)v(x)\,dx - \int_{R\cup A} g(x)\frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}(x)\,dx.\end{aligned}$$

Finally, the Fredholm Alternative also establishes that when the necessary and sufficient condition holds there exists a unique $w \in (\ker\mathcal{F})^\perp$ such that $\mathcal{F}(w) = \tilde{f}$. Therefore, $u = w + g$ is the unique solution of Problem (1) such that for any $z \in \ker\mathcal{F} = \mathcal{V}_{_B}$ satisfies

$$\int_{F\cup B} u(x)z(x)\,dx = 0.$$

□

Observation 2.3. *The Fredholm Alternative establishes the following formula*

$$\dim\mathcal{V}_{_A} - \dim\mathcal{V}_{_B} = |A| - |B|.$$

On the other hand, the existence of solution for any data is equivalent to be $\mathcal{V}_{_A} = \{0\}$*; that is,* $|B| - |A| = \dim\mathcal{V}_{_B} \geq 0$*. Moreover, uniqueness of solutions is equivalent to be* $|A| - |B| = \dim\mathcal{V}_{_A} \geq 0$*. In particular, if* $|A| = |B|$*, the existence of solution of Problem (1) for any data* f, g *and* h *is equivalent to the uniqueness of solution and hence it is equivalent to the fact that the homogeneous problem has* $v = 0$ *as its unique solution.*

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