THE GROUP INVERSE OF CIRCULANT MATRICES WITH FEW PARAMETERS

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Abstract. The need of solving linear systems with circulant matrices occurs in many problems related to the periodicity of that problem. This class of system appears in many applications that range from numerical solution of partial differential equations with periodic boundary conditions, until coding theory, statistics, time series analysis, image processing, or when we approximate periodic functions with splines. In spite the problem of computing the group inverse of a circulant matrix of order n can be considered solved from a theoretical or algebraic point of view, even for low dimensions the computational cost to find the solution could be very high, typically with order $\mathcal{O}(n^2)$.

Different approches to compute the inverse of a circulant matrix have focused on special classes of circulant matrices. For structured matrices with three parameters the application of the FFT, leads to algorithms to solve circulant systems with order $\mathcal{O}(n \log_2 n)$ and moreover the proper election of circulant preconditioners can reduce the computation to order $\mathcal{O}(n)$.

In this communication we present our last advances on the computation of the group inverse of a family of circulant matrices with four complex parameters. Specifically, we obtain analytical expressions for the coefficients of their group inverse. This means that by just checking some relations between the four coefficients we can explicitly compute the coefficients of the group inverse and there is no need to apply any numerical method to compute them. Therefore, we improve the computational cost of computing the group inverse of this class of matrices that, in the worst case, is now reduced to the evaluation of a polynomial. Moreover our methodology applies to both the invertible and the singular case, the latter being computationally less expensive. The techniques we use are related with the solution of boundary value problems associated with first or second order linear difference equations and hence involve the evaluation of Chebyshev polynomials.

Keywords: Circulant matrix, Group Inverse, Chebyshev polynomials, Difference Equations

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1. Introduction and Preliminaries

The problem of solving a linear system with circulant coefficient matrices appears in many practical issues related to the periodicity of the problem. This kind of system occurs in many applications: prediction, time series analysis, spline approximation, difference solution of partial differential equations, the finite difference method to approximate elliptic equations with periodic boundary conditions, etc.

Although the problem of computing the inverse, or the group inverse, of a circulant matrix can be considered solved from a theoretical point of view, the computational cost to find the solution is very high even for low dimension.

Different approches to solve the problem have focused on special classes of circulant matrix. For instance, S. R. Searle in [13], provided a method for obtaining analytic expressions for the coefficients of the

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inverse matrix of a family of three-element circulant matrices; O. Rojo in [12] gives the solution of a linear system having a symmetric circulant tridiagonal matrix and L. Fuyong, [9], obtained the elements of the inverse matrix as functions of zero points of the characteristic polynomial of the circulant matrix. With the application of the FFT, M. Chen [5] gave algorithms to solve circulant systems in $O(n\log_2 n)$ operations instead of the $O(n^2)$ arithmetic operations required; or the properly election of circulant preconditioners, see [4], reduced the problem from $O(n^2)$ to $O(n)$.

In this paper we study circulant matrices of type $\mathrm{Circ}(a,b,c,\dots,c,d)$. We give necessary and sufficient conditions for the invertibility of such matrices and we obtain analytical expressions for the coefficients of their group inverse. This means that by just checking some relations between the four coefficients we can explicitly compute the coefficients of the group inverse and there is no need to apply any numerical method to compute them. The results here obtained encompass those in [1] when the corresponding matrix is non singular. Our methodology is the same than the one used in the above mentioned paper and it is related with linear difference equations, see also [12, 13], for a similar treatment in the case of circulant matrices with few parameters.

If the inverse, or the group inverse, of a circulant matrix can be easily computed we can slightly modify it by introducing new parameters in such a way that the group inverse of the new matrix is still computable with a reasonable number of new operations, we achive to give the corresponding explicit expression according to the values of the coefficients of the matrix.

Throughout the paper, $\mathbb{N}$ denote the set of positive integers, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of rational, real and complex numbers, respectively, whereas $\mathbb{C}^*$ stands for $\mathbb{C}\setminus\{0\}$ and $\Re(z)$ denotes the real part of the complex number z. Given $a\in\mathbb{C}$, we define $a^\#$ as

$$a^\# = \begin{cases} a^{-1}, & \text{if } a\neq 0,\\ 0, & \text{if } a=0. \end{cases} \tag{1}$$

Notice that $(ab)^\# = a^\# b^\#$ for any $a,b\in\mathbb{C}$.

Given $n\in\mathbb{N}$ we denote by R_n the multiplicative group of n-th roots of unity; that is, the set

$$R_n = \Big\{ r\in\mathbb{C} : r^n = 1\Big\} = \Big\{ e^{\frac{2\pi k}{n} i} : k=0,\dots,n-1\Big\}. \tag{2}$$

Clearly $\bar{r} = r^{-1} = r^{n-1}$ for any $r\in R_n$ and $-1\in R_n$ iff n is even. We also consider the set

$$\Re_n = \Big\{ \frac{r+r^{-1}}{2} : r\in R_n\Big\} = \Big\{ \Re(r) : r\in R_n\Big\} = \Big\{ \cos\big(\tfrac{2\pi k}{n}\big) : k=0,\dots,\lceil\tfrac{n-1}{2}\rceil\Big\}. \tag{3}$$

More generally, given $\rho\in\mathbb{C}^*$ we define

$$R_n(\rho) = \Big\{ \frac{\rho r + (\rho r)^{-1}}{2} : r\in R_n\Big\} \tag{4}$$

and hence, $R_n(-\rho) = \{-z : z\in R_n(\rho)\}$. Moreover, when $|\rho|=1$, then $R_n(\rho) = \Big\{\Re(\rho r) : r\in R_n\Big\}$ and in particular, $R_n(1) = \Re_n$.

Therefore if we consider the function $\varphi\colon \mathbb{C}^* \longrightarrow \mathbb{C}$ defined as

$$\varphi(w) = \frac{1}{2}(w + w^{-1}), \tag{5}$$

then $R_n(\rho) = \big\{\varphi(\rho r) : r\in R_n\big\}$. Notice that $\varphi(w) = \cosh\big(\log(w)\big)$, but throughout the paper we will maintain the notation φ for the sake of simplicity. Function φ is odd, surjective and moreover $\varphi(\pm1)=\pm1$ and $\varphi(z)=\varphi(w)$ iff either $z=w$ or $z=w^{-1}$. Therefore, except for $z=\pm1$, φ is 2 to 1 and in fact, if $z\in\mathbb{C}$, then $w = z+\sqrt{z^2-1}$ satisfies that $w^{-1} = z-\sqrt{z^2-1}$ and moreover that $\varphi(w)=z$. In addition, $\varphi(\pm i)=0$, $\varphi(w)=\Re(w)$ when $|w|=1$, $\varphi(\mathbb{R}\setminus\{0,\pm1\}) = \mathbb{R}\setminus(-1,1)$ and $[-1,1]=\varphi(S)$, where S is the unit circle.

Lemma 1.1. *Given $\rho\in\mathbb{C}^*$, the following statements hold:*

(i) $1\in R_n(\rho)$ *iff* $\rho\in R_n$ *whereas* $-1\in R_n(\rho)$ *iff* $-\rho\in R_n$.

(ii) *When* $\rho^2 \notin R_n$ *then* $|R_n(\rho)| = n$ *and moreover* $\pm 1 \notin R_n(\rho)$.

(iii) $\rho^2 \in R_n$ *iff* $\rho = \pm e^{\frac{\pi k}{n} i}$ *for some* $k = 0, \dots, n-1$. *Moreover, for any* $k = 0, \dots, n-1$ *we have that*

$$R_n\big(e^{\frac{\pi k}{n} i}\big) = \Big\{ \cos\Big(\frac{\pi}{n}(2m+k)\Big) : m = 0, \dots, \lceil \tfrac{n-1-k}{2} \rceil \text{ and } m = n - \lfloor \tfrac{k}{2} \rfloor, \dots, n-1 \Big\}$$

and hence, $|R_n\big(e^{\frac{\pi k}{n} i}\big)| = \lceil \frac{n+1}{2} \rceil$, *except when* n *is even and* k *odd in which case* $|R_n\big(e^{\frac{\pi k}{n} i}\big)| = \lceil \frac{n-1}{2} \rceil$.

(iv) $0 \in R_n(e^{\frac{\pi k}{n} i})$ *iff* $n \equiv 2k \pmod 4$.

For fixed $n \in \mathbb{N}$, we consider the vector space $\mathbb{C}^n$. We denote the components of the vector $\mathsf{a} \in \mathbb{C}^n$ as a_j, $j = 1, \dots, n$, *i.e.*, $\mathsf{a} = (a_1, \dots, a_n)^\top$. As usual, $\bar{\mathsf{a}}$ denotes the vector whose components are the conjugates of those of a; that is, $\bar{\mathsf{a}} = (\bar{a}_1, \dots, \bar{a}_n)^\top$, whereas $\mathsf{1}$ is the all ones vector and $\mathsf{0}$ is the all zeroes vector in $\mathbb{C}^n$. Besides e is the vector with a 1 in its first coordinate and 0's elsewhere and $\hat{\mathsf{1}}$ is the vector whose components are $(-1)^j$, $j = 1, \dots, n$.

We consider $\mathbb{C}^n$ endowed with the standard inner product $\langle \cdot, \cdot \rangle$, whose associated norm is $||\cdot||$. Therefore, for any $\mathsf{a}, \mathsf{b} \in \mathbb{C}^n$ we have that $\langle \mathsf{a}, \mathsf{b} \rangle = \sum\limits_{j=1}^{n} a_j \bar{b}_j$.

Given V a subset of $\mathbb{C}^n$, $V^\perp$ denotes the subspace of $\mathbb{C}^n$ orthogonal to V. In particular if $\mathsf{a} \in \mathbb{C}^n$, $\mathsf{a}^\perp$ is the subspace of $\mathbb{C}^n$ orthogonal to a.

We denote by τ the permutation of the set $\{1, \dots, n\}$ defined as,

$$\tau(j) = 1 + (n+1-j) (\operatorname{mod} n), \quad j = 1, \dots, n. \tag{6}$$

Moreover, if $\mathsf{v} \in \mathbb{C}^n$ we define by v_τ the vector $\mathsf{v}_\tau = (v_1, v_n, v_{n-1}, \dots, v_3, v_2)^\top$. In particular, $\mathsf{1}_\tau = \mathsf{1}$, $\hat{\mathsf{1}}_\tau = \hat{\mathsf{1}}$ iff n is even and moreover $(\mathsf{v}_\tau)_\tau = \mathsf{v}$ for any $\mathsf{v} \in \mathbb{C}^n$, since $\tau^{-1} = \tau$. In addition, $\langle \mathsf{a}_\tau, \mathsf{b}_\tau \rangle = \langle \mathsf{a}, \mathsf{b} \rangle$ for any $\mathsf{a}, \mathsf{b} \in \mathbb{C}^n$ and in particular, $\langle \mathsf{a}_\tau, \mathsf{1} \rangle = \langle \mathsf{a}, \mathsf{1} \rangle$, for any $\mathsf{a} \in \mathbb{C}^n$.

If $\mathbb{K}$ is any of the fields $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, the set of square matrices of order n with coefficients in $\mathbb{K}$ is denoted $\mathcal{M}_n(\mathbb{K})$. If $\mathsf{A} \in \mathcal{M}_n(\mathbb{C})$, its *transpose* and its *conjugate transpose* are denoted by $\mathsf{A}^\top$ and A^*, respectively. Moreover, we denote by $\mathsf{O}, \mathsf{I}, \mathsf{J} \in \mathcal{M}_n(\mathbb{Q})$ the null matrix, the identity matrix and the all ones matrix, respectively.

Given $\mathsf{A} \in \mathcal{M}_n(\mathbb{K})$, a matrix $\mathsf{M} \in \mathcal{M}_n(\mathbb{K})$ is called *generalized inverse of* A if it satisfies the identity

$$\mathsf{AMA} = \mathsf{A}. \tag{7}$$

This kind of generalized inverses are also called 1-*inverses* and also *system solver inverses*, since M satisfies (7) iff given $\mathsf{b} \in \mathbb{K}^n$ in the range of A, then $\mathsf{g} = \mathsf{Mb}$ satisfies that $\mathsf{Ag} = \mathsf{b}$. Clearly, if A is invertible, then any generalized inverse coincides with A^{-1}, the inverse of A, but when A is singular it has infinite generalized inverses. However, when A has index 1 (see [**?**, Chapter 4,Theorem 4]) there exists a unique generalized inverse of A called the *group inverse of* A and denoted by $\mathsf{A}^\#$, satisfying the identities

$$\mathsf{A}\mathsf{A}^\#\mathsf{A} = \mathsf{A}, \quad \mathsf{A}^\#\mathsf{A}\mathsf{A}^\# = \mathsf{A}^\#, \quad \mathsf{A}^\#\mathsf{A} = \mathsf{A}\mathsf{A}^\#.$$

When the group inverse of A exists, for instance when A is *normal*, then $(\mathsf{A}^\#)^\# = \mathsf{A}^\#$, $\mathsf{A}^\# \in \mathcal{M}_n(\mathbb{K})$, $(a\mathsf{A})^\# = a^\#\mathsf{A}^\#$, for any $a \in \mathbb{C}$, $(\mathsf{A}^\top)^\# = (\mathsf{A}^\#)^\top$ and $(\mathsf{A}^*)^\# = (\mathsf{A}^\#)^*$. Notice that $\mathsf{O}^\# = \mathsf{O}$ and $\mathsf{J}^\# = \frac{1}{n^2}\mathsf{J}$, because $\mathsf{J}^2 = n\mathsf{J}$. On the other hand, when $n = 1$, Identity (1) coincides with the group inverse of $a \in \mathbb{C}$ understood as a matrix of order 1.

2. Circulant matrices

In this section we consider fixed $n \in \mathbb{N}$ and we show some properties of circulant matrices of order n. Many of them are well-known, for instance they form an n-dimensional vector space and also a commutative algebra. Moreover, each circulant matrix has group inverse that is also a circulant matrix, see [6] and Lemma

2.2 below. Here we use the notation in [1, 2], see also [13], that allow us to express the above mentioned properties in a simple and clear way.

A square matrix $\mathsf{A} = (a_{ij}) \in \mathcal{M}_n(\mathbb{C})$ is named *circulant with parameters* $a_1, \dots, a_n \in \mathbb{C}$ if

$$\mathsf{A} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & \cdots & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{bmatrix} \tag{8}$$

or equivalently, $a_{ij} = a_{1+(j-i)(\mathrm{mod}\, n)}$, see [14, 15].

Given $\mathsf{a} = (a_1, \dots, a_n)^\top \in \mathbb{C}^n$, we denote $\mathrm{Circ}(\mathsf{a}) = \mathrm{Circ}(a_1, \dots, a_n)$ the *circulant matrix* with parameters $a_1, \dots, a_n$. In addition, $N(\mathsf{a})$ and $R(\mathsf{a})$ denote the *null space* and the *range* of $\mathrm{Circ}(\mathsf{a})$ respectively, whereas P_{a} is the orthogonal projection onto $N(\mathsf{a})$. Therefore, given $\mathsf{v} \in \mathbb{C}^n$, $P_{\mathsf{a}}(\mathsf{v}) \in N(\mathsf{a})$ is characterized by satisfying that $\mathsf{v} - P_{\mathsf{a}}(\mathsf{v}) \in N(\mathsf{a})^\perp$.

For the sake of completeness, we enumerate the properties of circulant matrices that are relevant in our work. All the statements can be easily proved.

Lemma 2.1. *Given any* $\mathsf{a}, \mathsf{b} \in \mathbb{C}^n$*, the following properties hold:*

(i) $\mathrm{Circ}(a\,\mathsf{a} + b\,\mathsf{b}) = a\,\mathrm{Circ}(\mathsf{a}) + b\,\mathrm{Circ}(\mathsf{b})$ *for any* $a, b \in \mathbb{C}$.
(ii) $\mathrm{Circ}(\mathsf{a})^\top = \mathrm{Circ}(\mathsf{a}_\tau)$ *and* $\mathrm{Circ}(\mathsf{a})^* = \mathrm{Circ}(\bar{\mathsf{a}}_\tau)$.
(iii) *If* $\mathsf{c} = \mathrm{Circ}(\mathsf{a})\mathsf{b}$ *and* $\mathsf{d} = \mathrm{Circ}(\mathsf{b})\mathsf{a}$*, then* $\mathsf{d} = \mathsf{c}_\tau = \mathrm{Circ}(\mathsf{a}_\tau)\mathsf{b}_\tau$.
(iv) $\mathrm{Circ}(\mathsf{a})\mathrm{Circ}(\mathsf{b}) = \mathrm{Circ}(\mathsf{c})$*, where* $\mathsf{c} = \mathrm{Circ}(\mathsf{a}_\tau)\mathsf{b}$.
(v) $\mathrm{Circ}(\mathsf{a})\mathrm{Circ}(\mathsf{b}) = \mathrm{Circ}(\mathsf{b})\mathrm{Circ}(\mathsf{a})$.

One of the main problems in this setting is to determine the group inverse of a circulant matrix and moreover to know when the matrix is in fact invertible. This problem has been widely studied in the literature and solved using R_n, the multiplicative group of *n-th* roots of unity, see [10, 14]. We next give a short account of these results but expressing them in the notation of [1, 2], that we believe captures the algebraic properties of circulant matrices and it will be useful in the rest of this paper.

For any $z \in \mathbb{C}$ we consider the vector $\mathsf{f}(z)$ and the matrix $\mathsf{J}(z)$ defined respectively as

$$\mathsf{f}(z) = (1, z, \dots, z^{n-1})^\top \quad \text{and} \quad \mathsf{J}(z) = \mathrm{Circ}\big(\mathsf{f}(z)\big). \tag{9}$$

Clearly $\mathsf{f}(0) = \mathsf{e}$, $\mathsf{f}(1) = \mathsf{1}$ and $\mathsf{f}(-1) = -\hat{\mathsf{1}}$. Moreover, $\bar{\mathsf{f}}(z) = \mathsf{f}(\bar{z})$ and when $r \in R_n$, then $\mathsf{f}(r)_\tau = \bar{\mathsf{f}}(r)$ and $\big\{\frac{1}{\sqrt{n}}\,\mathsf{f}(r) : r \in R_n\big\}$ is an orthonormal basis of $\mathbb{C}^n$.

The following lemma provides a necessary and sufficient condition for the invertibility of $\mathrm{Circ}(\mathsf{a})$ and gives a formula for its group inverse, see [5, 6, 13].

Lemma 2.2. *For any* $\mathsf{a} \in \mathbb{C}^n$*, the following properties hold:*

(i) $\mathrm{Circ}(\mathsf{a})\mathsf{f}(r) = \langle \mathsf{a}, \mathsf{f}(\bar{r})\rangle\,\mathsf{f}(r)$*, for any* $r \in R_n$ *and hence* $\det \mathrm{Circ}(\mathsf{a}) = \prod\limits_{r \in R_n} \langle \mathsf{a}, \mathsf{f}(r)\rangle$.
(ii) $\mathrm{Circ}(\mathsf{a})$ *is invertible iff* $\langle \mathsf{a}, \mathsf{f}(r)\rangle \neq 0$*,* $r \in R_n$*, and moreover* $\mathrm{Circ}(\mathsf{a})^\# = \mathrm{Circ}\big(\mathsf{g}(\mathsf{a})\big)$ *where*

$$g_j(\mathsf{a}) = \frac{1}{n} \sum_{r \in R_n} r^{j-1} \langle \mathsf{a}, \mathsf{f}(r)\rangle^\#, \quad j = 1, \dots, n.$$

The above Lemma can be proved directly by checking the identities. It plays a key role in the study of circulant matrices since tell us almost everything we would like to know about this class of matrices. For this reason it is strongly used, explicitly or implicitly, in most of the papers on the subject.

First, notice that property (i) implies that all circulant matrices of order n diagonalize and have the same eigenvectors but different eigenvalues. In particular, any two circulant matrices commute each other and the conjugate transpose of a circulant matrix is also circulant. Moreover, any circulant matrix is normal, which in turns implies that it is range-hermitian with index 1. On the other hand, part (ii) in the above Lemma establishes that the problem of finding the group inverse of a circulant matrix is completely solved. Moreover, we can express the vector $\mathsf{g}(\mathsf{a})$ in a more closed and nice way: Since $\{\frac{1}{\sqrt{n}}\mathsf{f}(r)\}_{r\in R_n}$ is an orthonormal basis of $\mathbb{C}^n$, for any $s \in R_n$ we have that

$$\langle \mathsf{g}(\mathsf{a}), \mathsf{f}(s)\rangle = \frac{1}{n}\sum_{j=1}^{n} \bar{s}^{j-1} \sum_{r\in R_n} r^{j-1}\langle \mathsf{a}, \mathsf{f}(r)\rangle^{\#} = \frac{1}{n}\sum_{r\in R_n} \langle \mathsf{a}, \mathsf{f}(r)\rangle^{\#}\langle \mathsf{f}(r), \mathsf{f}(s)\rangle = \langle \mathsf{a}, \mathsf{f}(s)\rangle^{\#}. \tag{10}$$

Therefore

$$\mathsf{g}(\mathsf{a}) = \frac{1}{n}\sum_{r\in R_n} \langle \mathsf{a}, \mathsf{f}(r)\rangle^{\#}\mathsf{f}(r), \tag{11}$$

or equivalently,

$$\mathrm{Circ}(\mathsf{a})^{\#} = \mathrm{Circ}\big(\mathsf{g}(\mathsf{a})\big) = \frac{1}{n}\sum_{r\in R_n} \langle \mathsf{a}, \mathsf{f}(r)\rangle^{\#}\mathsf{J}(r). \tag{12}$$

Notice that the above identity implies that $\mathsf{g}\big(\mathsf{g}(\mathsf{a})\big) = \mathsf{a}$, equality that can be proved directly because $\big(\mathrm{Circ}(\mathsf{a})^{\#}\big)^{\#} = \mathrm{Circ}(\mathsf{a})$.

With Identities (11) and (12) at hand, it seems that the raised problem is solved, as we mentioned before. For instance, when $n = 1$, $R_1 = \{1\}$, $\mathsf{a} = a \in \mathbb{C}$, and hence $\mathsf{g}(\mathsf{a}) = a^{\#}$. Besides, when $n = 2$ we have $R_2 = \{1, -1\}$ and if $\mathsf{a} = (a, b)^{\top} \in \mathbb{C}$ then, $\mathrm{Circ}(a, b)$ is invertible iff $(a + b)(a - b) = \langle \mathsf{a}, \mathsf{f}(1)\rangle \cdot \langle \mathsf{a}, \mathsf{f}(-1)\rangle \neq 0$; that is, iff $b \neq \pm a$, and moreover

$$\mathsf{g}(\mathsf{a}) = \frac{1}{2}\Big[(a+b)^{\#}\mathsf{1} - (a-b)^{\#}\hat{\mathsf{1}}\Big] = (a-b)^{\#}\,\mathsf{e} + \frac{1}{2}\Big[\big(a+b\big)^{\#} - (a-b)^{\#}\Big]\mathsf{1};$$

that is,

$$\begin{bmatrix} a & b\\ b & a\end{bmatrix}^{\#} = \frac{1}{2}\begin{bmatrix} \big(a+b\big)^{\#} + (a-b)^{\#} & \big(a+b\big)^{\#} - (a-b)^{\#}\\ \big(a+b\big)^{\#} - (a-b)^{\#} & \big(a+b\big)^{\#} + (a-b)^{\#}\end{bmatrix} = (a-b)^{\#}\mathsf{I} + \frac{1}{2}\Big[\big(a+b\big)^{\#} - (a-b)^{\#}\Big]\mathsf{J}.$$

This two cases, $n = 1, 2$, seems to be the only ones in which the formula in (ii) can be easily computed and produced manageable expressions for the group inverse. Even in low dimensions, the formulae (11) for $\mathsf{g}(\mathsf{a})$ can involved a great number of computations, see Corollaries 3.5 and 3.6 in [2] for the cases $n = 3$ and $n = 4$ with $d = c$, respectively. So it is reasonable to focus on special classes of circulant matrices and/or to look for alternative methods to compute their group inverses. For instance, if the group inverse of a given circulant matrix can be easily computed, we can perturbe it by introducing new parameters in such a way that the group inverse of the perturbed matrix is still computable with few new operations.

Theorem 2.3. *Given $\mathsf{a} \in \mathbb{C}^n$ and $\emptyset \neq S \subset R_n$, consider $\hat{\mathsf{a}} = \mathsf{a} + \sum_{s\in S} b_s\mathsf{f}(s)$, where $b_s \in \mathbb{C}$ for any $s \in S$. Then, it is satisfied that*

$$\mathsf{g}(\hat{\mathsf{a}}) = \mathsf{g}(\mathsf{a}) + \frac{1}{n}\sum_{s\in S}\Big[\big(\langle \mathsf{a}, \mathsf{f}(s)\rangle + nb_s\big)^{\#} - \langle \mathsf{a}, \mathsf{f}(s)\rangle^{\#}\Big]\,\mathsf{f}(s),$$

or equivalently

$$\Big(\mathrm{Circ}(\mathsf{a}) + \sum_{s\in S} b_s\mathsf{J}(s)\Big)^{\#} = \mathrm{Circ}(\mathsf{a})^{\#} + \frac{1}{n}\sum_{s\in S}\Big[\big(\langle \mathsf{a}, \mathsf{f}(s)\rangle + nb_s\big)^{\#} - \langle \mathsf{a}, \mathsf{f}(s)\rangle^{\#}\Big]\,\mathsf{J}(s).$$

Moreover,

$$\det\Big(\mathrm{Circ}(\mathsf{a}) + \sum_{s\in S} b_s\mathsf{J}(s)\Big) = \prod_{r\in S}\big(\langle \mathsf{a}, \mathsf{f}(r)\rangle + nb_r\big)\prod_{r\in R_n\setminus S}\langle \mathsf{a}, \mathsf{f}(r)\rangle.$$

Proof. Since $\left\{\frac{1}{\sqrt{n}}\mathsf{f}(r)\right\}_{r\in R_n}$ is an orthonormal system, given $r\in R_n$, then $\langle\hat{\mathsf{a}},\mathsf{f}(r)\rangle=\langle\mathsf{a},\mathsf{f}(r)\rangle+\sum\limits_{s\in S} b_s\langle\mathsf{f}(s),\mathsf{f}(r)\rangle$ and hence, $\langle\hat{\mathsf{a}},\mathsf{f}(r)\rangle=\langle\mathsf{a},\mathsf{f}(r)\rangle$ when $r\notin S$, whereas $\langle\hat{\mathsf{a}},\mathsf{f}(r)\rangle=\langle\mathsf{a},\mathsf{f}(r)\rangle+b_r\langle\mathsf{f}(r),\mathsf{f}(r)\rangle=\langle\mathsf{a},\mathsf{f}(r)\rangle+nb_r$ when $r\in S$. Applying Identity (11) we get

$$\begin{aligned}\mathsf{g}(\hat{\mathsf{a}})&=\frac{1}{n}\sum_{r\in R_n}\langle\hat{\mathsf{a}},\mathsf{f}(r)\rangle^{\#}\mathsf{f}(r)=\frac{1}{n}\sum_{r\in R_n\setminus S}\langle\mathsf{a},\mathsf{f}(r)\rangle^{\#}\mathsf{f}(r)+\frac{1}{n}\sum_{r\in S}\big(\langle\mathsf{a},\mathsf{f}(r)\rangle+nb_r\big)^{\#}\mathsf{f}(r)\\&=\frac{1}{n}\sum_{r\in R_n}\langle\mathsf{a},\mathsf{f}(r)\rangle^{\#}\mathsf{f}(r)+\frac{1}{n}\sum_{r\in S}\Big[\big(\langle\mathsf{a},\mathsf{f}(r)\rangle+nb_r\big)^{\#}-\langle\mathsf{a},\mathsf{f}(r)\rangle^{\#}\Big]\mathsf{f}(r).\end{aligned}$$

The formula for the determinant is consequence of part (i) of Lemma 2.2. □

Of course, when $|S|$ is big, close to n, the computational complexity of the expression for $\mathsf{g}(\hat{\mathsf{a}})$ in the above result is similar to that in the general case. Therefore, in practice we must achieve a compromise between the computation of $\mathsf{g}(\mathsf{a})$, that we assume easy and with low cost, and the number of perturbations of a. For instance, if $S=\{s\}$ the perturbation only adds the computation of $\Big[\big(\langle\mathsf{a},\mathsf{f}(s)\rangle+nb_s\big)^{\#}-\langle\mathsf{a},\mathsf{f}(s)\rangle^{\#}\Big]\,\mathsf{f}(s)$. In particular, when $s=1$ there is nothing new to compute, since $\langle\mathsf{a},\mathsf{1}\rangle=a_1+\cdots+a_n$ consists only on sums, and hence we only need to add the value $\frac{1}{n}\Big[\big(a_1+\cdots+a_n+nb\big)^{\#}-(a_1+\cdots+a_n)^{\#}\Big]$ to the entries of $\mathrm{Circ}(\mathsf{a})^{\#}$.

On the other hand, given $r\in R_n$ we have that

$$\frac{1}{n}\Big[\big(\langle\mathsf{a},\mathsf{f}(r)\rangle+nb_r\big)^{\#}-\langle\mathsf{a},\mathsf{f}(r)\rangle^{\#}\Big]=\begin{cases}\dfrac{b_r^{\#}}{n^2}, & \text{if } \langle\mathsf{a},\mathsf{f}(r)\rangle\big(\langle\mathsf{a},\mathsf{f}(r)\rangle+nb_r\big)=0,\\[2ex] -\dfrac{b_r}{\langle\mathsf{a},\mathsf{f}(r)\rangle\big(\langle\mathsf{a},\mathsf{f}(r)\rangle+nb_r\big)}, & \text{otherwise.}\end{cases}$$

The following results are straightforward consequences of Theorem 2.3 that have interest for their own.

Corollary 2.4. *Given $\{b_r\}_{r\in R_n}\subset\mathbb{C}$, then*

$$\Big(\sum_{r\in R_n}b_r\mathsf{J}(r)\Big)^{\#}=\frac{1}{n^2}\sum_{r\in R_n}b_r^{\#}\mathsf{J}(r).$$

Corollary 2.5. *Given $a,b\in\mathbb{C}$ and $r\in R_n$, then $\det\big(a\mathsf{I}+b\mathsf{J}(r)\big)=a^{n-1}(a+nb)$. Therefore when $n\geq 2$, $a\mathsf{I}+b\mathsf{J}(r)$ is singular iff $a(a+nb)=0$ and moreover,*

$$(a\mathsf{I}+b\mathsf{J}(r))^{\#}=a^{\#}\mathsf{I}+\frac{1}{n}\Big[(a+nb)^{\#}-a^{\#}\Big]\mathsf{J}(r).$$

Sometimes, for a fixed vector a, it happens that $\mathrm{Circ}(\mathsf{a})^{\#}$ or even $a\mathrm{Circ}(\mathsf{a})^{\#}+\sum\limits_{s\in S}b_s\mathsf{J}(s)$ for some $a,b_s\in\mathbb{C}$ determines an structured family of matrices depending on the parameters a,b_s. So, we can use Theorem 2.3 to establish a sort of inverse result to compute the group inverse of this new class of matrices. We take into account that $\langle\mathsf{g}(\mathsf{a}),\mathsf{f}(r)\rangle=\langle\mathsf{a},\mathsf{f}(r)\rangle^{\#}$ for any $r\in R_n$.

Proposition 2.6. *Given $\mathsf{a}\in\mathbb{C}^n$ and $\emptyset\neq S\subset R_n$, consider $\mathsf{A}=a\mathrm{Circ}(\mathsf{a})^{\#}+\sum\limits_{s\in S}b_s\mathsf{J}(s)$, where $a\in\mathbb{C}$ and $b_s\in\mathbb{C}$ for any $s\in S$. Then,*

$$\det\mathsf{A}=a^{n-|S|}\prod_{r\in S}\big(a\langle\mathsf{a},\mathsf{f}(r)\rangle^{\#}+nb_r\big)\prod_{r\in R_n\setminus S}\langle\mathsf{a},\mathsf{f}(r)\rangle^{\#}$$

and moreover,

$$\mathsf{A}^{\#}=a^{\#}\mathrm{Circ}(\mathsf{a})+\frac{1}{n}\sum_{s\in S}\Big[\big(a\langle\mathsf{a},\mathsf{f}(s)\rangle^{\#}+nb_s\big)^{\#}-a^{\#}\langle\mathsf{a},\mathsf{f}(s)\rangle\Big]\,\mathsf{J}(s).$$

Corollary 2.7. *Given* $\mathsf{a} \in \mathbb{C}^n$ *consider* $\mathsf{A} = a\mathrm{Circ}(\mathsf{a})^{\#} + b\mathsf{J}$, *where* $a, b \in \mathbb{C}$. *Then,*

$$\det \mathsf{A} = a^{n-1}\big(a(a_1 + \cdots + a_n)^{\#} + nb\big) \prod_{r \in R_n \setminus \{1\}} \langle \mathsf{a}, \mathsf{f}(r)\rangle^{\#}$$

and moreover,

$$\mathsf{A}^{\#} = a^{\#}\mathrm{Circ}(\mathsf{a}) + \frac{1}{n}\Big[\big(a(a_1 + \cdots + a_n)^{\#} + nb\big)^{\#} - a^{\#}(a_1 + \cdots + a_n)\Big]\,\mathsf{J}.$$

The complexity of Formula (12) for the determination of the group inverse of a circulant matrix, grows with the order of the matrix, so it is not useful at all from the computational point of view, even when the matrix is invertible. Therefore, it is interesting to look for alternative expressions for the group inverse of specific classes of circulant matrices. Over the years, many papers have considered this topic specially for circulant matrices depending of few parameters. In many of this cases, the special structure of the matrix is highly used and leads to employ alternative methods as solving linear difference equations, see for instance [1, 2, 12, 13]. The main aim of this work is to provide a formulae for the group inverse of some circulant matrix with at most four parameters and having an special structure, completing the results in [1, 2]. Since we wish to use alternative methods to find $\mathsf{g}(\mathsf{a}) \in \mathbb{C}^n$ for a given $\mathsf{a} \in \mathbb{C}$, we have already obtained in [2] the algebraically characterization of the vector $\mathsf{g}(\mathsf{a})$, which we reproduce here for the sake of completeness.

Theorem 2.8. *Given* $\mathsf{a} \in \mathbb{C}^n$, *there exists a unique* $\mathsf{g}(\mathsf{a}) \in N(\bar{\mathsf{a}})^{\perp}$ *such that* $\mathrm{Circ}\big(\mathsf{g}(\mathsf{a})\big)$ *is a generalized inverse of* $\mathrm{Circ}(\mathsf{a})$. *Moreover,* $\mathrm{Circ}\big(\mathsf{g}(\mathsf{a})\big) = \mathrm{Circ}(\mathsf{a})^{\#}$ *and* $\mathsf{g}(\mathsf{a})_\tau$ *is characterized as the unique solution of the system* $\mathrm{Circ}(\mathsf{a})\mathsf{z} = \mathsf{e} - P_{\mathsf{a}}(\mathsf{e})$ *belonging to* $N(\mathsf{a})^{\perp}$. *In particular,* $\mathrm{Circ}(\mathsf{a})$ *is invertible iff the system* $\mathrm{Circ}(\mathsf{a})\mathsf{z} = \mathsf{e}$ *is compatible.*

In [1], we computed the inverse matrix of some circulant matrices of order $n \geq 3$ with three real parameters at most and in [2] we considered also complex parameters and we determined the expression for their group inverse. We reduced significantly the computational cost of applying Lemma 2.2, since the key point for finding the mentioned inverse matrix consists in solving the system that provides us $\mathsf{g}(\mathsf{a})_\tau$ by means of a difference equation of order at most two. We aim here to extend the methodology to the computation of the group inverse of the matrices $\mathrm{Circ}(a, b, c, \ldots, c, d)$.

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