On Spectra of Symmetric Jacobi Matrices

A.M. Encinas & M.J. Jiménez

Abstract

A square symmetric matrix A is said bisymmetric if AS=SA, where S is the matrix with ones along the secondary diagonal and zeros elsewhere. We denote by J(a,b) the real symmetric Jacobi matrix with main diagonal $a = (a_0, a_1, \ldots, a_n)$ and second diagonal $b = (b_0, b_1, \ldots, b_{n-1})$. Hoschtadt and Hald proved that the spectrum of a bisymmetric Jacobi matrix with nonnegative off-diagonal elements defines uniquely the matrix and they give a constructive proof of the result. We characterize the spectra of nonnegative irreducible bisymmetric Jacobi matrices of size lees or equal 5, and we give the unique entries of the matrix in terms of the eigenvalues.

We also parametrize the set of monic polynomials of degree n whose roots strictly interlace a given set Λ of $n+1$ ordered real numbers, and we use this parametrization to characterize the symmetric Jacobi matrices realizing Λ. Our development is strongly based on the work by Hoschtadt but, instead of considering principal submatrices, we use the above parametrization of monic polynomials interlacing Λ.

1 Introduction and notation

A real matrix $A = (a_{ij})_1^n$ is a Jacobi matrix if $a_{ij} = 0$ for $|i-j| > 1$. It is well known that the eigenvalues of Jacobi matrices are real, that Jacobi matrices with nonnegative off-diagonal elements are isospectral with symmetric Jacobi matrices (preserving the diagonal) and also how are the spectra of nonnegative Jacobi matrices:

Theorem 1. *The real list $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ with $\lambda_1 \geq \cdots \geq \lambda_n$ is the spectrum of an $n \times n$ nonnegative Jacobi matrix if and only if $\lambda_i + \lambda_{n+1-i} \geq 0, \quad i = 1, \ldots, n$.*

The following resul is well known, see for instance [1, Lemma 0.1.1].

Lemma 1. *If $b_i c_i > 0$ for $i = 1, \dots, n-1$, then all the eigenvalues of the Jacobi matrix*

$$\begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{n-1} & b_{n-1} \\ 0 & \cdots & 0 & c_{n-1} & a_n \end{bmatrix}$$

are real and simple.

Remark 1. *The result of the above Lemma is no longer true if the hypothesis $b_i c_i > 0$ for $i = 1, \dots, n-1$ is not fulfilled. For instance, given $a, b_1, b_2 \in \mathbb{R}$ where $b_1 b_2 \neq 0$, the eigenvalues of the irreducible matrix*

$$\begin{bmatrix} a & b_1 & 0 \\ -b_1 & a & b_2 \\ 0 & -b_2 & a \end{bmatrix}$$

are a and $\pm i\sqrt{b_1^2 + b_2^2}$.

Lemma 2 (Frobenius'theorem)**.** *For any nonnegative irreducible matrix, $\lambda_1 + \lambda_n = 0$ implies $\lambda_i + \lambda_{n+1-i} = 0$ for $i = 1, \dots, n$.*

The realization of a spectrum by a Jacobi symmetric matrix given in [**?**, Theorem 1] is highly reducible, so Friedland and Melkman also study the irreducible case for Jacobi matrices ($a_{i,i+1}, a_{i+1,i} > 0$).

Lemma 3. *Let $\Lambda = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}$ with $\lambda_1 > \cdots > \lambda_n$ and $\lambda_i + \lambda_{n+1-i} > 0$ for $i = 1, \dots, n$. Then there exists a Jacobi matrix*

$$\begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{n-1} & b_{n-1} \\ 0 & \cdots & 0 & b_{n-1} & a_n \end{bmatrix}$$

with spectrum Λ such that $a_i > 0$, for $1 \leq i \leq n$, and $b_i > 0$, for $1 \leq i \leq n-1$.

Lemma 4. *Given the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and the diagonal matrix coefficients $a_1 \geq \cdots \geq a_n$ of a symmetric and nonnegative matrix. Then,*

$$\lambda_1 \geq a_1, \quad \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i, \quad \lambda_k + \sum_{i=1}^{s} \lambda_i \geq a_{k-1} + a_k + \sum_{i=1}^{s-1} a_i, \quad 1 \leq s < k \leq n.$$

Let K be symmetric matrix with ones along the secondary diagonal and zeros elsewhere. Note that $K^2 = I$. A matrix A is said to be *persymmetric* if $AK = KA^T$. A matrix A is said to be *centrosymmetric* if $AQ = QA$. Any two of the concepts of centrosymmetric, persymmetric and symmetric imply the other one, but the reciprocal is not true. A matrix A is named *bisymmetric* when it satisfies two of the above concepts.

Lemma 5 (Cantoni & Butler 1976)**.** *Let Q be a $N \times N$ bisymmetric matrix. If $N = 2m$, then $Q = \begin{bmatrix} A & C^T \\ C & JAJ \end{bmatrix}$, where A and C are $m \times m$ matrices with $A = A^T$ and $C^T = JCJ$, and Q and $\begin{bmatrix} A - JC & 0 \\ 0 & A + JC \end{bmatrix}$ are orthogonally similar.*

with $A = A^T$ and $C^T = JCJ$, and Q and $\begin{bmatrix} A - JC & 0 & 0 \\ 0 & q & \sqrt{2}x^T \\ 0 & \sqrt{2}x & A + JC \end{bmatrix}$ are orthogonally similar.

We want to study the spectra of irreducible bisymmetric matrices A of size $n \times n$.

2 The general case

Hochstadt [3] and Hald [2] prove that the spectrum of a bisymmetric Jacobi matrix with nonnegative off-diagonal elements defines uniquely the matrix and they give a constructive proof of the result. So, under the hypothesis of nonnegativity, the unicity and the construction are applied.

We will need the following results:

Theorem 2. *Let M and N be $n \times n$ symmetric matrices and let the eigenvalues of M, N, and $M + N$ be arranged in decreasing order:*

$$\lambda_1(M) \geq \lambda_2(M) \geq \cdots \geq \lambda_n(M)$$
$$\lambda_1(N) \geq \lambda_2(N) \geq \cdots \geq \lambda_n(N)$$
$$\lambda_1(M+N) \geq \lambda_2(M+N) \geq \cdots \geq \lambda_n(M+N).$$

For each $k = 1, 2, \ldots, n$ we have

$$\lambda_k(M) + \lambda_n(N) \leq \lambda_k(M+N) \leq \lambda_k(M) + \lambda_1(N).$$

Theorem 3. *Let M be $n \times n$ symmetric matrix and let M_i be the matrix obtained deleting column and row i from M. Let the eigenvalues of M and M_i be arranged in decreasing order:*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$
$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}.$$

Then

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

Lemma 6. *Let A be a $n \times n$ nonnegative irreducible bisymmetric Jacobi matrix with spectrum $\lambda_1 > \cdots > \lambda_n$.*

(i) If $n = 2m$, then $\{\lambda_{1+2i} : 0 \leq i \leq m-1\}$ and $\{\lambda_{2i} : 1 \leq i \leq m\}$ are the spectra of $B + KC$ and $B - KC$ respectively, where B and C are the blocks of $A = \begin{bmatrix} B & C^T \\ C & KBK \end{bmatrix}$. Furthermore, A is completely defined by its spectrum:

(ii) If $n = 2m+1$, then $\{\lambda_{1+2i} : 0 \leq i \leq m\}$ and $\{\lambda_{2i} : 1 \leq i \leq m\}$ are the spectra of $\begin{bmatrix} q & \sqrt{2}x^T \\ \sqrt{2}x & B+KC \end{bmatrix}$ and $B - KC$ respectively, where B, C, q and x are the blocks of $A = \begin{bmatrix} B & x & C^T \\ x^T & q & x^T K \\ C & Kx & KBK \end{bmatrix}$. Furthermore, A is completely defined by its spectrum:

$$\text{element } (1,1) \text{ of } \begin{bmatrix} q & \sqrt{2}x^T \\ \sqrt{2}x & B+KC \end{bmatrix}^k = -\sum_{i=1}^{n} \frac{\lambda_i^k \prod_{j=1}^{m} (\lambda_i - \lambda_{2j})}{\sum_{s=0}^{m} \prod_{\substack{j=0 \\ j \neq s}}^{m} (\lambda_i - \lambda_{1+2j})} \quad (1)$$

for k=1,...,n.

References

[1] S.M. Fallat, C.R. Johnson: *Totally Nonnegative Matrices.* Princeton University Press, 2011.

[2] O.H. Hald: *Inverse Eigenvalue Problems for Jacobi Matrices.* Linear Algebra Appl., **14** (1976), 63-85.

[3] H. Hochstadt: *On the Construction of a Jacobi Matrix from Spectral Data.* Linear Algebra Appl., **8** (1974), 435-446.