

# *Near-sphere lattices with constant nonlocal mean curvature*

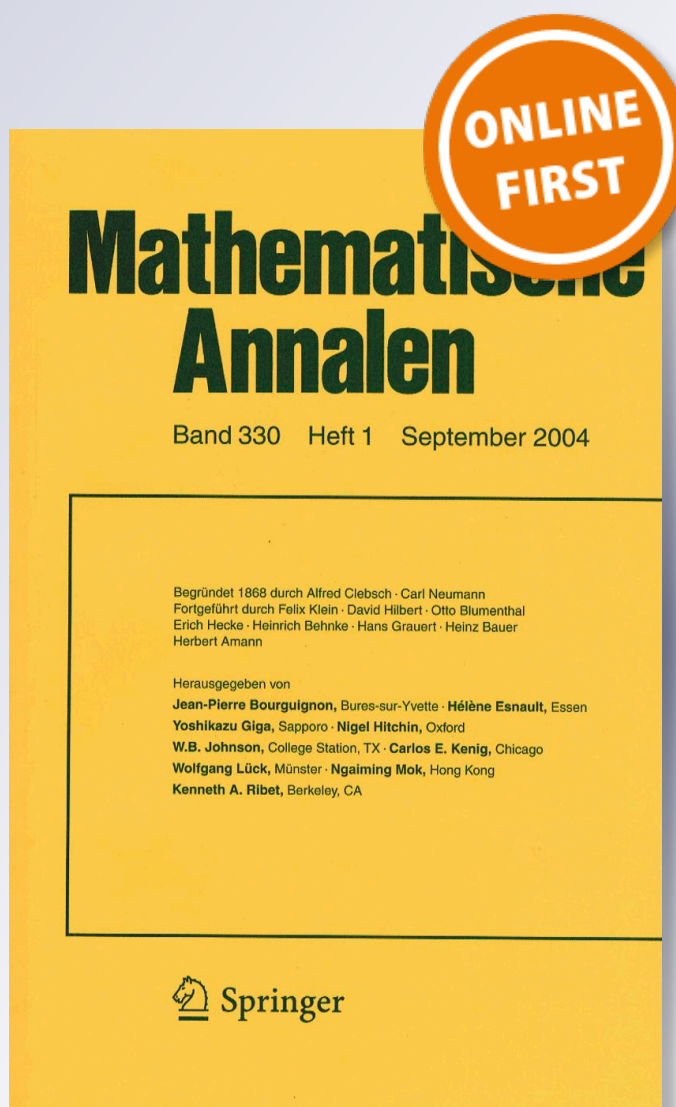
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## Near-sphere lattices with constant nonlocal mean curvature

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**Abstract** We are concerned with unbounded sets of  $\mathbb{R}^N$  whose boundary has constant nonlocal (or fractional) mean curvature, which we call CNMC sets. This is the equation associated to critical points of the fractional perimeter functional under a volume constraint. We construct CNMC sets which are the countable union of a certain bounded domain and all its translations through a periodic integer lattice of dimension  $M \leq N$ . Our CNMC sets form a  $C^2$  branch emanating from the unit ball alone and where the parameter in the branch is essentially the distance to the closest lattice

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point. Thus, the new translated near-balls (or near-spheres) appear from infinity. We find their exact asymptotic shape as the parameter tends to infinity.

### 1 Introduction

Let  $\alpha \in (0, 1)$ . If  $\mathcal{A}$  is a smooth oriented hypersurface in  $\mathbb{R}^N$  with unit normal vector field  $\nu$ , its nonlocal or fractional mean curvature (abbreviated NMC in the following) of order  $\alpha$  at a point  $x \in \mathcal{A}$  is defined as

$$H(\mathcal{A}; x) = \frac{2d_{N,\alpha}}{\alpha} \int_{\mathcal{A}} \frac{(y - x) \cdot \nu(y)}{|y - x|^{N+\alpha}} dV(y). \tag{1.1}$$

Here and in the following,  $dV$  stands for the volume element on  $\mathcal{A}$ , and

$$d_{N,\alpha} = \frac{1 - \alpha}{(N - 1)|B^{N-1}|} = \frac{(1 - \alpha)\Gamma(\frac{N+1}{2})}{(N - 1)\pi^{(N-1)/2}}, \tag{1.2}$$

where  $B^{N-1}$  is the unit ball in  $\mathbb{R}^{N-1}$ . If  $\mathcal{A}$  is of class  $C^{1,\beta}$  for some  $\beta > \alpha$  and we assume  $\int_{\mathcal{A}} (1 + |y|)^{1-N-\alpha} dV(y) < \infty$ , then the integral in (1.1) is absolutely convergent in the Lebesgue sense.

The choice of the constant  $d_{N,\alpha}$  guarantees that, if  $\mathcal{A}$  is of class  $C^2$ , the nonlocal mean curvature  $H(\mathcal{A}; \cdot)$  converges, as  $\alpha \rightarrow 1$ , locally uniformly to the classical mean curvature, i.e., the arithmetic mean of principle curvatures, see [9, Lemma A.1] and [1, Theorem 12].

There is an alternative expression for  $H(\mathcal{A}; \cdot)$  in terms of a solid integral. Suppose that  $\mathcal{A} = \partial E$  for some open set  $E \subset \mathbb{R}^N$  and  $\nu$  is the normal exterior to  $E$ . Then, for all  $x \in \mathcal{A}$ , we have

$$\begin{aligned} H(\mathcal{A}; x) &= d_{N,\alpha} PV \int_{\mathbb{R}^N} \frac{1_{E^c}(y) - 1_E(y)}{|y - x|^{N+\alpha}} dy \\ &= d_{N,\alpha} \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \frac{1_{E^c}(y) - 1_E(y)}{|y - x|^{N+\alpha}} dy, \end{aligned} \tag{1.3}$$

where  $E^c = \mathbb{R}^N \setminus \overline{E}$  and  $1_D$  denotes the characteristic function of a set  $D \subset \mathbb{R}^N$ . This can be derived using the divergence theorem and the fact that  $\nabla_y \cdot (y - x)|y - x|^{-N-\alpha} = \alpha|y - x|^{-N-\alpha}$ .

The nonlocal mean curvature is the Euler–Lagrange equation for the fractional perimeter functional. Nonlocal minimal surfaces are hypersurfaces with zero NMC and were introduced by Caffarelli et al. [5]. This paper established the first existence and regularity theorems. Later, Savin and Valdinoci [18] established that, for  $N = 2$  and any  $\alpha$ , they are  $C^\infty$ . Figalli and Valdinoci [13] have proved that if they are Lipschitz in  $\mathbb{R}^N$ , then they are  $C^\infty$ . See the paper [11] by Dipierro and Valdinoci for more details. Later, Dávila et al. [9] initiate the study of nonlocal minimal cones in any dimension, characterizing the stability or instability of  $\alpha$ -Lawson cones. Besides, they also construct surfaces of revolution with zero NMC, for instance the fractional

catenoid. On the other hand, Cinti et al. [6] show that the helicoid, besides having zero classical mean curvature, has also zero NMC.

There are less works on CNMC hypersurfaces, that is, hypersurfaces with constant nonlocal mean curvature. To our knowledge, the following articles are the existing ones. In recent independent works, a result of Alexandrov type has been proved for the nonlocal mean curvature by Ciraolo et al. [7, Theorem 1.1] and Cabré et al. [3, Theorem 1.1]. This result states that every bounded (and a priori not necessarily connected) hypersurface without boundary and with constant nonlocal mean curvature must be a sphere. This result naturally led to questions related to the existence and shape of unbounded hypersurfaces of constant NMC. Obvious examples within this class are straight cylinders. In [3] and [4] we proved the existence of periodic and connected hypersurfaces in  $\mathbb{R}^N$  with constant NMC which bifurcate from a straight cylinder. These hypersurfaces should be regarded as Delaunay type cylinders in the nonlocal setting. We point out that, unlike in the local case, straight cylinders have positive constant NMC in every space dimension  $N \geq 2$ . Thus, our result also gives periodic bands in  $\mathbb{R}^2$  with constant NMC and which bifurcate from a straight band.

Having constant nonlocal mean curvature is the equation associated to critical points of the fractional perimeter functional under a volume constraint. Thus, one would expect that CNMC sets can be constructed variationally. In this direction, the paper [8] by Dávila, del Pino, Dipierro, and Valdinoci, established variationally the existence of periodic and cylindrically symmetric hypersurfaces in  $\mathbb{R}^N$  which minimize (under the volume constraint) a certain fractional perimeter functional adapted to periodic sets. More precisely, [8] established the existence of a 1-periodic minimizer for every given volume within the slab  $\{(s, \zeta) \in \mathbb{R} \times \mathbb{R}^{N-1} : -1/2 < s < 1/2\}$ . We have realized recently that, in fact, their fractional perimeter functional adapted to periodic sets gives rise to CNMC hypersurfaces in a weak sense. They would be CNMC hypersurfaces in the classical sense defined above if one could prove that they are of class  $C^{1,\beta}$  for some  $\beta > \alpha/2$ —which is not done in [8]. The article also proves that for small volume constraints, the minimizers tend in measure (more precisely, in the so called Fraenkel asymmetry) to a periodic array of balls.

Note that sets obtained by minimizing a fractional perimeter functional under a volume constraint are expected to have Morse index 1—within a proper functional analytic framework. This will be the case for the CNMC sets constructed in the present paper—see Remark 1.2(iv). As we will see, the linearized operator at them (acting on a space of even functions) will have only one negative eigenvalue—all the rest being positive. Note that looking at the linearized operator in a space of even functions excludes the eigenfunction with zero eigenvalue produced by the invariance of the nonlinear problem under translations.

We recall that in the case of classical mean curvature, embedded Delaunay hypersurfaces vary from a cylinder to an infinite compound of tangent spheres. However, it is easy to see that an infinite compound of aligned round spheres, tangent or disconnected, does not have constant NMC. Indeed, it is an open problem to establish the existence of global continuous branches of nonlocal Delaunay cylinders and to study their limiting configurations.

In the present paper, we study nonlocal analogues of the set given by an infinite compound of aligned round spheres, tangent or disconnected. In a more general setting, we construct CNMC sets which are the countable union of a certain bounded domain and all its translations through a periodic integer lattice of dimension  $M \leq N$ . Our CNMC sets form a  $C^2$  branch emanating from the unit ball alone, where the parameter in the branch is essentially the distance to the closest lattice point. Thus, the new translated near-balls (or near-spheres) appear from infinity. We point out that it is necessary to consider infinite lattices in this problem—a finite disjoint union of two or more bounded sets cannot have constant NMC by the Alexandrov type rigidity result in [3, 7].

We expect (but we do not prove) that, when the distance from two consecutive near-spheres is large enough, our periodic CNMC set made of near-spheres is a minimizer of the fractional perimeter under the volume and periodicity constraints. Note that, after rescaling, large distance from two consecutive near-spheres turns into a fixed distance (or period) but now with a very small volume constraint—as in the result of [8] mentioned above.

To be precise, we now assume  $N \geq 2$  and let

$$S := S^{N-1} \subset \mathbb{R}^N$$

denote the unit sphere of  $\mathbb{R}^N$ . For  $M \in \mathbb{N}$  with  $1 \leq M \leq N$  we regard  $\mathbb{R}^M$  as a subspace of  $\mathbb{R}^N$  by identifying  $x' \in \mathbb{R}^M$  with  $(x', 0) \in \mathbb{R}^M \times \mathbb{R}^{N-M} = \mathbb{R}^N$ . Let  $\{\mathbf{a}_1; \dots; \mathbf{a}_M\}$  be a basis of  $\mathbb{R}^M$ . By the above identification, we then consider the  $M$ -dimensional lattice

$$\mathcal{L} = \left\{ \sum_{i=1}^M k_i \mathbf{a}_i : k = (k_1, \dots, k_M) \in \mathbb{Z}^M \right\} \tag{1.4}$$

as a subset of  $\mathbb{R}^N$ . In the case where  $\{\mathbf{a}_1; \dots; \mathbf{a}_M\}$  is an orthogonal or an orthonormal basis, we say that  $\mathcal{L}$  is a rectangular lattice or a square lattice, respectively.

We define, for  $r > 0$ ,

$$S + r\mathcal{L} := \bigcup_{p \in \mathcal{L}} (S + rp) \subset \mathbb{R}^N. \tag{1.5}$$

Then, for  $r > 2(\inf_{p \in \mathcal{L} \setminus \{0\}} |p|)^{-1}$ , the set  $S + r\mathcal{L}$  is the union of disjoint unit spheres centered at the lattice points in  $r\mathcal{L}$ . Consequently,  $S + r\mathcal{L}$  is a set of constant classical mean curvature (equal to one). In contrast, as a consequence of our main result, we shall see that the NMC  $H(S + r\mathcal{L}; \cdot)$  is in general *not* constant on this periodic set. It is therefore natural to ask if the sphere  $S$  can be perturbed smoothly to a set  $S_\varphi$ , such that  $S_\varphi + r\mathcal{L}$ , for  $r > 0$  large enough, has constant NMC.

To answer this question, we fix  $\beta \in (\alpha, 1)$  and define the set

$$\mathcal{O} := \{\varphi \in C^{1,\beta}(S) : \|\varphi\|_{L^\infty(S)} < 1\}.$$

We then consider the deformed sphere

$$S_\varphi := \{(1 + \varphi(\sigma))\sigma : \sigma \in S\}, \quad \varphi \in \mathcal{O}. \tag{1.6}$$

Provided that  $r > 0$  is large enough, the deformed sphere lattice (or near-sphere lattice)

$$S_\varphi + r\mathcal{L} := \bigcup_{p \in \mathcal{L}} (S_\varphi + rp)$$

is a noncompact hypersurface of class  $C^{1,\beta}$ , which by construction is periodic with respect to  $r\mathcal{L}$ -translations.

The main result of the present paper is the following.

**Theorem 1.1** *Let  $\alpha \in (0, 1)$ ,  $\beta \in (\alpha, 1)$ ,  $N \geq 2$ ,  $1 \leq M \leq N$  and  $\mathcal{L}$  be an  $M$ -dimensional lattice as given in (1.4). Then, there exist  $r_0 > 0$ , and a  $C^2$ -curve  $(r_0, +\infty) \rightarrow C^{1,\beta}(S)$ ,  $r \mapsto \varphi_r$ , with the following properties:*

- (i)  $\varphi_r \rightarrow 0$  in  $C^{1,\beta}(S)$  as  $r \rightarrow +\infty$ ;
- (ii) For every  $r \in (r_0, +\infty)$ , the function  $\varphi_r : S \rightarrow \mathbb{R}$  is even (with respect to reflection through the origin of  $\mathbb{R}^N$ );
- (iii) For every  $r \in (r_0, +\infty)$ , the hypersurface  $S_{\varphi_r} + r\mathcal{L}$  has constant nonlocal mean curvature given by  $H(S_{\varphi_r} + r\mathcal{L}; \cdot) \equiv H(S; \cdot)$ .
- (iv) Letting  $\mathcal{L}_* := \mathcal{L} \setminus \{0\}$ , the function  $\varphi_r$  expands as

$$\varphi_r(\theta) = r^{-N-\alpha} \left( -\kappa_0 + r^{-2} \left\{ \kappa_1 \sum_{p \in \mathcal{L}_*} \frac{(\theta \cdot p)^2}{|p|^{N+\alpha+4}} - \kappa_2 \right\} + o(r^{-2}) \right)$$

for  $\theta \in S$  as  $r \rightarrow +\infty$ ,

with positive constants  $\kappa_0$ ,  $\kappa_1$  and  $\kappa_2$  (see Remark 1.2 below for their explicit values) and with  $r^2 o(r^{-2}) \rightarrow 0$  in  $C^{1,\beta}(S)$  as  $r \rightarrow +\infty$ .

- (v) If  $1 \leq M \leq N - 1$ , then the functions  $\varphi_r$ ,  $r > r_0$ , are non-constant on  $S$ .

Moreover, if  $r_1 > r_0$  and  $(r_1, +\infty) \rightarrow C^{1,\beta}(S)$ ,  $r \mapsto \tilde{\varphi}_r$  is another (not necessarily continuous) curve satisfying (i), (ii) and (iii), then  $\tilde{\varphi}_r = \varphi_r$  for  $r$  sufficiently large.

The curve  $r \mapsto \varphi_r$  is not  $C^3$ , in general. It is not  $C^3$  for instance when  $N = 2$ . This is due to the presence of the factor  $|\tau|^{N+\alpha} = r^{-N-\alpha}$  in our functional equation (2.7).

To establish the theorem it will be essential to analyze the linearized operator for the NMC  $H$  at the unit sphere  $S$ . We will see that the linearization is given by the operator

$$\varphi \mapsto 2d_{N,\alpha}(L_\alpha \varphi - \lambda_1 \varphi), \tag{1.7}$$

where

$$L_\alpha : C^{1,\beta}(S) \rightarrow C^{\beta-\alpha}(S), \quad L_\alpha \varphi(\theta) = PV \int_S \frac{\varphi(\theta) - \varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} dV(\sigma), \tag{1.8}$$



$\lambda_1$  is defined next in (1.9), and  $\varphi$  is a deformation of  $S$  in the direction of its normal—as in (1.6). The operator  $L_\alpha$  can be seen as a spherical fractional Laplacian, and the above integral is understood in the principle value sense, i.e.,

$$PV \int_S \frac{\varphi(\theta) - \varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) := \lim_{\varepsilon \rightarrow 0} \int_{S \setminus B_\varepsilon(\theta)} \frac{\varphi(\theta) - \varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) \quad \text{for } \varepsilon \rightarrow 0.$$

The operator  $L_\alpha$  has the spherical harmonics as eigenfunctions corresponding to the increasing sequence  $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \dots$  of eigenvalues given by

$$\lambda_k = \frac{\pi^{(N-1)/2} \Gamma((1 - \alpha)/2)}{(1 + \alpha) 2^\alpha \Gamma((N + \alpha)/2)} \left( \frac{\Gamma\left(\frac{2k+N+\alpha}{2}\right)}{\Gamma\left(\frac{2k+N-\alpha-2}{2}\right)} - \frac{\Gamma\left(\frac{N+\alpha}{2}\right)}{\Gamma\left(\frac{N-\alpha-2}{2}\right)} \right), \quad (1.9)$$

see [17, Lemma 6.26] and Sect. 5 below. Here, as before,  $\Gamma$  is the Gamma function. We shall also see, as a consequence of (2.6) and (5.5), that the NMC of the unit sphere  $S = S^{N-1} \subset \mathbb{R}^N$  is given by

$$H(S; \cdot) \equiv \frac{2d_{N,\alpha}}{\alpha} \lambda_1 \quad \text{on } S. \quad (1.10)$$

Now that  $\lambda_1$  and  $\lambda_2$  have been introduced, we can give the value of the constants in Theorem 1.1(iv). In the following remark, we also comment on the size of the near-spheres depending on the parameter  $r$ , as well as on their smoothness.

*Remark 1.2* (i) The constants in Theorem 1.1(iv) are given by

$$\begin{aligned} \kappa_0 &= \frac{|S|}{N\lambda_1} \sum_{p \in \mathcal{L}_*} \frac{1}{|p|^{N+\alpha}}, \quad \kappa_1 = \frac{|S|(N + \alpha)(N + \alpha + 2)}{6N(\lambda_2 - \lambda_1)} \quad \text{and} \\ \kappa_2 &= \frac{|S|}{6} \left\{ \frac{(N + \alpha)(N + \alpha + 2)}{N^2(\lambda_2 - \lambda_1)} + \frac{2(N + \alpha)(N + 1)(\alpha + 2)}{N^2(N + 2)\lambda_1} \right\} \sum_{p \in \mathcal{L}_*} \frac{1}{|p|^{N+\alpha+2}}, \end{aligned}$$

where  $\lambda_1, \lambda_2$  are given in (1.9).

(ii) Since  $\kappa_0 > 0$ , the expansion in Theorem 1.1(iv) shows that, for large  $r$ , the perturbed spheres  $S_{\varphi_r}$  become smaller than  $S$  as the perturbation parameter  $r$  decreases. With regard to the order  $r^{-N-\alpha}$ , the shrinking process is uniform on  $S$ , whereas non-uniform deformations of the spheres may appear at the order  $r^{-N-\alpha-2}$ . In particular, we shall detect these non-uniform deformations in the case  $M \leq N - 1$ , and from this we will then deduce part (v) of Theorem 1.1. In the case  $M = N$ , it remains an open problem to characterize the lattices which give rise to non-uniform deformations. We conjecture that  $H(S + r\mathcal{L}; \cdot)$  is non-constant for any  $N$ -dimensional lattice  $\mathcal{L}$  and large  $r$ .

(iii) The smoothness (i.e., the  $C^\infty$ -character) of our  $C^{1,\beta}$  hypersurfaces  $S_{\varphi_r} + r\mathcal{L}$ , and in general of  $C^{1,\beta}$  hypersurfaces in  $\mathbb{R}^N$  with constant NMC which are, locally, Lipschitz graphs, follows (since  $\beta > \alpha$ ) from the methods and results of Barrios, Figalli, and Valdinoci [2] on nonlocal minimal graphs. This holds for all  $N \geq 2$ . More



generally, to deduce the  $C^\infty$  regularity, [2] needs to assume that the hypersurface is  $C^{1,\beta}$  for some  $\beta > \alpha/2$  and that has constant nonlocal mean curvature in the viscosity sense; this fact can be found in Section 3.3 of [2]. Here, the notion of viscosity solution is needed since the expression (1.1) for the NMC is only well defined for  $C^{1,\beta}$  sets when  $\beta > \alpha$ .

(iv) As already remarked above, the CNMC sets constructed in Theorem 1.1 have Morse index 1 within our functional analytic framework. More precisely, for  $r > r_0$  sufficiently large, the linearization of the nonlocal mean curvature operator

$$\mathcal{O} \rightarrow C^{\beta-\alpha}(S), \quad \varphi \mapsto \left[ \sigma \mapsto H(S_\varphi + r\mathcal{L}; (1 + \varphi(\sigma))\sigma) \right]$$

at  $\varphi_r$  has exactly one negative eigenvalue when restricted to even functions in  $C^{1,\beta}(S)$ , whereas all other eigenvalues are positive. This property follows from the fact that the linearization at  $\varphi_r$  converges to the operator  $2d_{N,\alpha}(L_\alpha - \lambda_1)$ , given by (1.7)–(1.8), as  $r \rightarrow \infty$ . This convergence is a mere consequence of the  $C^2$ -smoothness of the operator  $\tilde{H}$  defined in (1.15) below, and the fact that  $2d_{N,\alpha}(L_\alpha - \lambda_1)$  is the linearization at the unit sphere  $S = \lim_{r \rightarrow \infty} S_{\varphi_r}$  by Lemma 5.1. Finally, one uses the spectral decomposition of  $L_\alpha - \lambda_1$ , already mentioned previously, and sees that, among even functions, its eigenvalues are given by  $-\lambda_1 < \lambda_2 - \lambda_1 < \lambda_4 - \lambda_1 < \dots$ . The first one is negative and all others are positive.

As a corollary of Theorem 1.1, we obtain the following more explicit form of  $\varphi_r$  in the case of rectangular lattices.

**Corollary 1.3** *Assume that  $\mathcal{L}$  is a rectangular lattice of dimension  $M \in \{1, \dots, N\}$ . Then the function  $\varphi_r$  in Theorem 1.1 expands as*

$$\varphi_r(\theta) = r^{-N-\alpha} \left( -\kappa_0 + r^{-2} \left\{ \kappa_1 \sum_{j=1}^M \mu_j \theta_j^2 - \kappa_2 \right\} + o(r^{-2}) \right) \quad \text{for } \theta \in S \text{ as } r \rightarrow +\infty, \tag{1.11}$$

where  $\mu_j = \sum_{p \in \mathcal{L}_*} \frac{p_j^2}{|p|^{N+\alpha+4}}$ . If, in particular,  $\mathcal{L}$  is a square lattice then

$$\varphi_r(\theta) = r^{-N-\alpha} \left( -\kappa_0 + r^{-2} \left\{ \tilde{\kappa}_1 \sum_{j=1}^M \theta_j^2 - \kappa_2 \right\} + o(r^{-2}) \right) \quad \text{for } \theta \in S \text{ as } r \rightarrow +\infty, \tag{1.12}$$

where  $\tilde{\kappa}_1 = \frac{\kappa_1}{M} \sum_{p \in \mathcal{L}_*} \frac{1}{|p|^{N+\alpha+2}}$ .

As observed in Theorem 1.1, for  $M \leq N - 1$  the perturbation  $\varphi_r$  is nonconstant on  $S$ , i.e., the NMC of  $H(S + r\mathcal{L}; \cdot)$  is nonconstant for  $r$  large enough. On the other hand, if  $\mathcal{L}$  is a square lattice of dimension  $N$ , then by (1.12) we have

$$\varphi_r(\theta) = r^{-N-\alpha} \left( -\kappa_0 + r^{-2} (\tilde{\kappa}_1 - \kappa_2) \right) \quad \text{as } r \rightarrow \infty,$$

hence the deformation of the lattice  $S_{\varphi_r} + r\mathcal{L}$  is uniform up to the order  $r^{N-\alpha-2}$ .

In order to explain the idea of the proof of Theorem 1.1, it is convenient first to pay some attention to the linearized operator at  $S \subset \mathbb{R}^N$  for the classical mean curvature ( $\alpha = 1$ ). Since  $S$  is a CMC surface, it is well known (see for instance Section 6 of [12]) that the linearization of the mean curvature operator (recall that we take as mean curvature the arithmetic mean of the principal curvatures) agrees with  $(N - 1)^{-1}$  times the second variation of perimeter, and thus is given by the Jacobi operator

$$J\varphi := (N - 1)^{-1}\{-\Delta_S\varphi - c^2\varphi\} = (N - 1)^{-1}\{-\Delta_S\varphi - (N - 1)\varphi\} \quad \text{on } S, \tag{1.13}$$

where  $\Delta_S$  is the Laplace–Beltrami operator on  $S$  and  $c^2 = N - 1$  is the sum of the squares of the principal curvatures of  $S$ . Here  $\varphi$  is a normal deformation as in (1.6). Recall that  $\Delta_S$  has the spherical harmonics as eigenfunctions, corresponding to the increasing sequence  $k(k + N - 2)$  of eigenvalues, with  $k \geq 0$ . Thus,  $J$  has the same eigenfunctions but with eigenvalues

$$\mu_k - \mu_1 := (N - 1)^{-1}\{k(k + N - 2) - (N - 1)\}. \tag{1.14}$$

Thus, the first eigenvalue is negative and corresponds to constant functions on  $S$  (that is, to the perturbation corresponding to changing the radius of the sphere  $S$ ). The third and next eigenvalues are all positive. But the second one ( $k = 1$ ) is zero and has  $\theta_i = x_i/|x|$  for  $i = 1, \dots, N$  (the spherical harmonics of degree one) as eigenfunctions. It is simple to see that this zero eigenvalue corresponds to translations of  $S$  in  $\mathbb{R}^N$ , which do not change the mean curvature and thus provide a zero eigenvalue.

As mentioned above, the linearized operator at  $S \subset \mathbb{R}^N$  for the NMC  $H$  is given by (1.7)–(1.8). It coincides, thus, with the second variation at  $S$  of fractional perimeter. This nice formula is not immediate at all. We will derive it in Sect. 5 in the Fréchet sense of linearization, after proving the smoothness of the NMC operator in Sect. 4. In a restricted sense related to the existence of directional derivatives, this formula for the linearization also follows from results of Dávila et al. [9, Appendix B] and of Figalli et al. [12, Section 6]. These interesting papers found—at any hypersurface  $\mathcal{A}$ —a simple expression for the linearization of NMC with respect to any given normal boundary variation. Note here that the NMC as defined in these two papers agrees with our  $H/d_{N,\alpha}$ ; see (1.3).

Note that the linearization of NMC at  $S$ , (1.7)–(1.8), has also the spherical harmonics as eigenfunctions—as mentioned above. In addition, its second eigenvalue  $2d_{N,\alpha}(\lambda_k - \lambda_1)$  (which corresponds to  $k = 1$ ) vanishes—as in the local case. We will see below that, to apply the implicit function theorem, we must get rid of this zero eigenvalue. For this, we will work only with perturbations of the sphere  $S$  which are even with respect to the origin of  $\mathbb{R}^N$ .

Just for consistency, we can now check that the eigenvalues of our nonlocal linearized operator (1.7) satisfy

$$2d_{N,\alpha}(\lambda_k - \lambda_1) \rightarrow \mu_k - \mu_1 = (N - 1)^{-1}\{k(k + N - 2) - (N - 1)\} \quad \text{as } \alpha \rightarrow 1.$$

Indeed, using the fact that  $\Gamma(z) = (z - 1)\Gamma(z - 1)$ , we get

$$\frac{\Gamma\left(\frac{2k+N+1}{2}\right)}{\Gamma\left(\frac{2k+N-3}{2}\right)} - \frac{\Gamma\left(\frac{N+1}{2}\right)}{\Gamma\left(\frac{N-3}{2}\right)} = k(k + N - 2)$$

and  $((1 - \alpha)/2)\Gamma((1 - \alpha)/2) = \Gamma((3 - \alpha)/2)$ . From these identities and recalling (1.2) and (1.9), we deduce that

$$\begin{aligned} \lim_{\alpha \rightarrow 1} 2d_{N,\alpha}\lambda_k &= \lim_{\alpha \rightarrow 1} \frac{2(1 - \alpha)\Gamma((N + 1)/2)}{(N - 1)\pi^{(N-1)/2}} \\ &\quad \times \frac{\pi^{(N-1)/2} 2}{(1 - \alpha)(1 + \alpha)2^\alpha\Gamma((N + 1)/2)} k(k + N - 2) \\ &= (N - 1)^{-1}k(k + N - 2) \end{aligned}$$

for  $k \in \mathbb{N}$ .

We can now outline the idea of the proof of Theorem 1.1, which is based on the implicit function theorem. Let  $c > 0$  be sufficiently small such that the translates  $S_\varphi + rp$ ,  $p \in \mathcal{L}$  do not intersect each other for  $r > 1/c$  and  $\varphi \in \mathcal{O} = \{\varphi \in C^{1,\beta}(S) : \|\varphi\|_{L^\infty(S)} < 1\}$ . We then rewrite the problem in the variable  $\tau = 1/r$  and show that the nonlinear operator

$$\tilde{H} : (-c, c) \times \mathcal{O} \rightarrow C^{\beta-\alpha}(S)$$

given by

$$\tilde{H}(\tau, \varphi)(\theta) := \begin{cases} H(S_\varphi + \frac{1}{\tau}\mathcal{L}; (1 + \varphi(\theta))\theta) & \text{for } \tau \in (-c, c) \setminus \{0\}, \varphi \in \mathcal{O} \\ H(S_\varphi; (1 + \varphi(\theta))\theta) & \text{for } \tau = 0, \varphi \in \mathcal{O} \end{cases} \tag{1.15}$$

is of class  $C^2$  in a neighborhood of  $(\tau, \varphi) = (0, 0)$ , and that its linearization at this point is given by  $D_\varphi \tilde{H}(0, 0) = 2d_{N,\alpha}\{L_\alpha - \lambda_1\} : C^{1,\beta}(S) \rightarrow C^{\beta-\alpha}(S)$ . As mentioned earlier,  $\lambda_1$  is the first nontrivial eigenvalue of the operator  $L_\alpha$  with corresponding eigenspace spanned by the coordinate functions  $\theta_1, \dots, \theta_N$ . This yields an  $N$ -dimensional kernel for the linearized NMC operator  $D_\varphi \tilde{H}(0, 0)$ . As mentioned above, this kernel comes from the invariance of the NMC operator under translations in  $\mathbb{R}^N$ .

Thus, in order to apply the implicit function theorem, we need to introduce function subspaces contained in a complement of this kernel. We consider

$$X = \{\varphi \in C^{1,\beta}(S) : \varphi(-\theta) = \varphi(\theta) \text{ for all } \theta \in S\} \tag{1.16}$$

and

$$Y = \{\varphi \in C^{\beta-\alpha}(S) : \varphi(-\theta) = \varphi(\theta) \text{ for all } \theta \in S\}, \tag{1.17}$$

the spaces of normal deformations which are even with respect to the origin of  $\mathbb{R}^N$ . In terms of the orthogonal basis given by the spherical harmonics,  $X$  and  $Y$  are generated by the spherical harmonics of even degree. We then consider the restriction of  $\tilde{H} : (-c, c) \times (\mathcal{O} \cap X) \rightarrow Y$ , which takes values in  $Y$ —and thus is well defined—thanks to the invariance of the lattice  $\mathcal{L}$  under reflection through the origin. Moreover,  $D_\varphi \tilde{H}(0, 0) = 2d_{N,\alpha}\{L_\alpha - \lambda_1\} : X \rightarrow Y$  will be an isomorphism.

Establishing the regularity of the operator  $\tilde{H}$  turns out to be the most difficult step in the proof of Theorem 1.1. This will be done in Sect. 4.

The computation of the expansion in part (iv) of Theorem 1.1 is not straightforward and requires some care. In particular, we note that this is an expansion of order  $o(|\tau|^{N+\alpha+2})$ , whereas we shall see from (2.11) below that  $\tilde{H}$  fails to have more than  $C^N$ -regularity in the  $\tau$ -variable.

The paper is organized as follows. In Sect. 2 we set up the functional analytic formulation of the problem in order to apply the implicit function theorem. We also state Theorem 2.1 (to be proved in Sect. 4) on the smoothness of the NMC operator acting on perturbed spheres. In Sect. 3 we complete the proof of our main result, Theorem 1.1, after having stated in Theorem 3.1 the main properties of the linearized NMC operator at the unit sphere. This theorem is proved in Sect. 4, while the one on the nonlinear NMC (Theorem 2.1) is established in Sect. 4.

## 2 Preliminaries and functional analytic formulation of the problem

Throughout the remainder of the paper, we let  $N \geq 2$ , and we let  $S \subset \mathbb{R}^N$  and  $B \subset \mathbb{R}^N$  denote the unit sphere and unit ball, respectively. Let  $M \in \mathbb{N}$  with  $1 \leq M \leq N$ , and let  $\mathcal{L} \subset \mathbb{R}^N$  be an  $M$ -dimensional lattice as defined in (1.4). Throughout this paper, we put

$$\mathcal{L}_* := \mathcal{L} \setminus \{0\} \quad \text{and} \quad \mathbb{Z}_*^M := \mathbb{Z}^M \setminus \{0\}.$$

We note that

$$\inf_{p \in \mathcal{L}_*} |p| =: c_0 > 0 \tag{2.1}$$

and

$$\sum_{p \in \mathcal{L}_*} \frac{1}{|p|^{N+\alpha}} < \infty. \tag{2.2}$$

As in the introduction, we fix  $\beta \in (\alpha, 1)$  and define

$$\mathcal{O} := \{\varphi \in C^{1,\beta}(S) : \|\varphi\|_\infty < 1\}. \tag{2.3}$$

Moreover, for  $\varphi \in \mathcal{O}$ , we consider the perturbed sphere

$$S_\varphi := \{(1 + \varphi(\sigma))\sigma : \sigma \in S\}$$

and its parameterization over the standard sphere defined by

$$F_\varphi : S \rightarrow S_\varphi, \quad F_\varphi(\sigma) = (1 + \varphi(\sigma))\sigma. \tag{2.4}$$

For  $\tau \in (-c_0/4, c_0/4) \setminus \{0\}$ , we then define

$$S_\varphi^\tau := \bigcup_{p \in \mathcal{L}} \left( S_\varphi + \frac{p}{\tau} \right) = S_\varphi + \frac{1}{\tau} \mathcal{L}.$$

By (2.1) and since  $S_\varphi \subset B_2(0)$ , the set  $S_\varphi^\tau$  is a noncompact hypersurface of class  $C^{1,\beta}$ , consisting of disjoint connected perturbed spheres and periodic with respect to the lattice  $\frac{1}{\tau} \mathcal{L}$ . Due to the translation invariance properties of the lattice  $\mathcal{L}$ , the NMC of  $S_\varphi^\tau$  is completely determined by its values on  $S_\varphi$ . More precisely, we have

$$H \left( S_\varphi^\tau ; x + \frac{\bar{p}}{\tau} \right) = H(S_\varphi^\tau ; x) \quad \text{for every } \bar{p} \in \mathcal{L} \text{ and } x \in S_\varphi. \tag{2.5}$$

Thus, our aim is to solve the equation

$$H(S_\varphi^\tau ; F_\varphi(\theta)) = H(S ; \theta) = \frac{2d_{N,\alpha}}{\alpha} \int_S \frac{1 - \sigma \cdot \theta}{|\sigma - \theta|^{N+\alpha}} dV(\sigma) \quad \text{for every } \theta \in S. \tag{2.6}$$

Note that  $H(S ; \theta)$  is constant in  $\theta$ .

In the following, for  $\varphi \in \mathcal{O}$ , we also let  $B_\varphi$  denote the unique open bounded set such that  $\partial B_\varphi = S_\varphi$ , i.e.,

$$B_\varphi := \{ r F_\varphi(\sigma) = r(1 + \varphi(\sigma))\sigma : 0 \leq r < 1, \sigma \in S \}.$$

Moreover, we let  $\nu_{S_\varphi}$  denote the unit outer normal vector field on  $S_\varphi = \partial B_\varphi$ , and we let  $dV_{S_\varphi}$  denote the volume element on  $S_\varphi$ . For  $\tau \in (-c_0/4, c_0/4) \setminus \{0\}$ , and  $x \in S_\varphi \subset S_\varphi^\tau$ , we then have

$$\begin{aligned} \frac{\alpha}{2d_{N,\alpha}} H(S_\varphi^\tau ; x) &= \int_{S_\varphi^\tau} \frac{(z - x) \cdot \nu_{S_\varphi^\tau}(z)}{|z - x|^{N+\alpha}} dV_{S_\varphi^\tau}(z) \\ &= \sum_{p \in \mathcal{L}} \int_{S_\varphi} \frac{(y - x + \frac{p}{\tau}) \cdot \nu_{S_\varphi}(y)}{|y - x + \frac{p}{\tau}|^{N+\alpha}} dV_{S_\varphi}(y) \\ &= \int_{S_\varphi} \frac{(y - x) \cdot \nu_{S_\varphi}(y)}{|y - x|^{N+\alpha}} dV_{S_\varphi}(y) \\ &\quad + \frac{|\tau|^{N+\alpha}}{\tau} \sum_{p \in \mathcal{L}_*} \int_{S_\varphi} \frac{(\tau(y - x) + p) \cdot \nu_{S_\varphi}(y)}{|\tau(y - x) + p|^{N+\alpha}} dV_{S_\varphi}(y). \end{aligned}$$

It will be convenient to use an alternative expression of the integrals appearing in the sum which does not involve boundary integration and which immediately shows that the sum is well defined. For this we note that, for fixed  $\tau \in (-c_0/4, c_0/4) \setminus \{0\}$ ,  $p \in \mathcal{L}_*$  and  $x \in S_\varphi$ , the function  $y \mapsto |\tau(y - x) + p|^{-N-\alpha+2}$  is smooth in  $\overline{B_\varphi}$ , and for all  $y \in \overline{B_\varphi}$  we have

$$\nabla_y |\tau(y - x) + p|^{-N-\alpha} = (-N - \alpha)\tau(\tau(y - x) + p)|\tau(y - x) + p|^{-N-\alpha-2}.$$

Since  $S_\varphi = \partial B_\varphi$ , the divergence theorem leads to

$$\begin{aligned} \int_{S_\varphi} \frac{(\tau(y - x) + p) \cdot \nu_{S_\varphi}(y)}{|\tau(y - x) + p|^{N+\alpha}} dV_{S_\varphi}(y) &= \int_{B_\varphi} \operatorname{div}_y \frac{(\tau(y - x) + p)}{|\tau(y - x) + p|^{N+\alpha}} dy \\ &= -\alpha\tau \int_{B_\varphi} \frac{1}{|\tau(y - x) + p|^{N+\alpha}} dy. \end{aligned}$$

Consequently, writing  $x = F_\varphi(\theta)$  with  $\theta \in S$ , we have

$$\frac{\alpha}{2d_{N,\alpha}} H(S_\varphi^\tau; F_\varphi(\theta)) = h(\varphi)(\theta) + |\tau|^{N+\alpha} \sum_{p \in \mathcal{L}_*} G_p(\tau, \varphi)(\theta) \tag{2.7}$$

for  $\theta \in S$  and  $\tau \in (-c_0/4, c_0/4) \setminus \{0\}$ , where

$$h(\varphi)(\theta) := \int_{S_\varphi} \frac{(y - F_\varphi(\theta)) \cdot \nu_{S_\varphi}(y)}{|y - F_\varphi(\theta)|^{N+\alpha}} dV_{S_\varphi}(y) \tag{2.8}$$

and

$$G_p(\tau, \varphi)(\theta) := -\alpha \int_{B_\varphi} \frac{1}{|\tau(y - F_\varphi(\theta)) + p|^{N+\alpha}} dy \quad \text{for } p \in \mathcal{L}_*. \tag{2.9}$$

Note that  $h(\varphi)(\theta)$  is precisely the NMC of  $S_\varphi$  at  $F_\varphi(\theta)$ .

In the following, we will need that both  $h$  and  $G := \sum_{p \in \mathcal{L}_*} G_p$  define smooth nonlinear operators between open subsets of suitable function spaces. The following is the key result of the present paper in this regard.

**Theorem 2.1** *With  $\mathcal{O}$  defined by (2.3), expression (2.8) gives rise to a well defined map*

$$h : \mathcal{O} \rightarrow C^{\beta-\alpha}(S)$$

which is of class  $C^\infty$ .

In the following, and with some abuse due to multiplicative constants, we will also call  $h$  the nonlocal mean curvature operator over the sphere  $S$ . The proof of Theorem 2.1 is long and technically involved due to the singularity in the integrand in (2.8). Nevertheless, the result is a key step in our approach, and we believe that it

might be of independent interest. We postpone the proof of Theorem 2.1 to Sect. 4; see Theorem 4.11 below.

With regard to  $G_p$ , we have a similar result.

**Proposition 2.2** *For  $p \in \mathcal{L}_*$  and  $\mathcal{O}$  defined by (2.3), expression (2.9) gives rise to a well defined map*

$$G_p : \left(-\frac{c_0}{4}, \frac{c_0}{4}\right) \times \mathcal{O} \rightarrow C^{\beta-\alpha}(S)$$

which is of class  $C^\infty$ . Moreover, the map

$$G := \sum_{p \in \mathcal{L}_*} G_p : \left(-\frac{c_0}{4}, \frac{c_0}{4}\right) \times \mathcal{O} \rightarrow C^{\beta-\alpha}(S) \tag{2.10}$$

is well defined and of class  $C^\infty$ .

The proof of Proposition 2.2 is also lengthy if all details are carried out, but it is much easier than the proof of Theorem 2.1 since the integrand in (2.9) is not singular. We will outline the proof of Proposition 2.2 at the end of Sect. 4 below.

We conclude this section by introducing the nonlinear operator

$$\mathcal{H} : (-c_0/4, c_0/4) \times \mathcal{O} \rightarrow C^{\beta-\alpha}(S)$$

given by

$$\mathcal{H}(\tau, \varphi)(\theta) := h(\varphi)(\theta) + |\tau|^{N+\alpha} G(\tau, \varphi)(\theta) \tag{2.11}$$

for  $\tau \in (-c_0/4, c_0/4)$ ,  $\varphi \in \mathcal{O}$  and  $\theta \in S$ . By construction, we then have

$$\mathcal{H}(\tau, \varphi)(\theta) = \frac{\alpha}{2d_{N,\alpha}} H(S_\varphi^\tau; F_\varphi(\theta)) \quad \text{for } \tau \in (-c_0/4, c_0/4) \setminus \{0\}, \tag{2.12}$$

i.e., the value  $\mathcal{H}(\tau, \varphi)(\theta)$  equals the NMC of  $S_\varphi^\tau$  at the point  $F_\varphi(\theta)$  up to a multiplicative constant. We may thus formulate the parameter-dependent equation (2.6) as an operator equation in Hölder spaces. More precisely, we need to study the set of parameters  $\tau \in (-c_0/4, c_0/4)$  and functions  $\varphi \in \mathcal{O}$  satisfying

$$\mathcal{H}(\tau, \varphi) = h(0) \quad \text{in } C^{\beta-\alpha}(S). \tag{2.13}$$

Note that we will have

$$\mathcal{H} \in C^2\left((-c_0/4, c_0/4) \times \mathcal{O}, C^{\beta-\alpha}(S)\right) \tag{2.14}$$

as a consequence of Theorem 2.1, Proposition 2.2 and the fact that the map  $\tau \mapsto |\tau|^{N+\alpha}$  is of class  $C^2$  since  $N \geq 2$ . Moreover, we have

$$D_\varphi \mathcal{H}(0, 0) = Dh(0) \in \mathcal{L}(C^{1,\beta}(S), C^{\beta-\alpha}(S)). \tag{2.15}$$



In the next section we restrict our attention to even functions and use the implicit function theorem to find a locally unique solution curve  $\tau \mapsto (\tau, \varphi(\tau))$  solving Eq. (2.13) and with  $\varphi(0) = 0$ .

### 3 Completion of the proof of Theorem 1.1

For fixed  $\alpha \in (0, 1)$  and  $\beta \in (\alpha, 1)$ , we consider, as in the introduction, the spaces

$$\begin{aligned} X &= \{\varphi \in C^{1,\beta}(S) : \varphi(-\theta) = \varphi(\theta) \text{ for all } \theta \in S\}, \\ Y &= \{\varphi \in C^{\beta-\alpha}(S) : \varphi(-\theta) = \varphi(\theta) \text{ for all } \theta \in S\}. \end{aligned}$$

We claim that the operator  $\mathcal{H}$  defined in (2.11) restricts to a map

$$(-c_0/4, c_0/4) \times (\mathcal{O} \cap X) \rightarrow Y,$$

which we will also denote by  $\mathcal{H}$  in the following. Indeed, for  $\varphi \in \mathcal{O} \cap X$  and  $\theta \in S$  we have that  $-S_\varphi = S_\varphi$ ,  $F_\varphi(-\theta) = -F_\varphi(\theta)$  and  $v_{S_\varphi}(-y) = -v_{S_\varphi}(y)$  for  $y \in S$ . Thus, by a change of variables in (2.8), we have that

$$\begin{aligned} h(\varphi)(-\theta) &= \int_{S_\varphi} \frac{(y + F_\varphi(\theta)) \cdot v_{S_\varphi}(y)}{|y + F_\varphi(\theta)|^{N+\alpha}} dV_{S_\varphi}(y) \\ &= \int_{S_\varphi} \frac{(F_\varphi(\theta) - y) \cdot v_{S_\varphi}(-y)}{|y - F_\varphi(\theta)|^{N+\alpha}} dV_{S_\varphi}(y) \\ &= \int_{S_\varphi} \frac{(y - F_\varphi(\theta)) \cdot v_{S_\varphi}(y)}{|y - F_\varphi(\theta)|^{N+\alpha}} dV_{S_\varphi}(y) = h(\varphi)(\theta). \end{aligned}$$

Similarly, from (2.9), (2.10) and the fact that  $-\mathcal{L}_* = \mathcal{L}_*$ , we derive that

$$G(\tau, \varphi)(-\theta) = G(\tau, \varphi)(\theta) \quad \text{for } (\tau, \varphi) \in (-c_0/4, c_0/4) \times (\mathcal{O} \cap X) \text{ and } \theta \in S.$$

Consequently, it follows from (2.11) that  $\mathcal{H}$  maps  $(-c_0/4, c_0/4) \times (\mathcal{O} \cap X)$  into  $Y$ , as claimed.

Moreover, by (2.14) we have that

$$\mathcal{H} \in C^2(((-c_0/4, c_0/4) \times (\mathcal{O} \cap X)), Y). \tag{3.1}$$

Using the implicit function theorem within the spaces  $X$  and  $Y$ , we will derive a locally unique curve  $\tau \mapsto \varphi(\tau) \in X$  such that  $\varphi(0) = 0$  and

$$\mathcal{H}(\tau, \varphi(\tau)) = h(0) \quad \text{in } Y$$

for  $|\tau|$  sufficiently small. For this we shall need the following invertibility property of  $Dh(0) \in \mathcal{L}(X, Y)$ , which by (2.15) coincides with  $D_\varphi \mathcal{H}(0, 0) \in \mathcal{L}(X, Y)$ .

**Theorem 3.1** *The linearized operator  $Dh(0) \in \mathcal{L}(C^{1,\beta}(S), C^{\beta-\alpha}(S))$  is given by*

$$\frac{1}{\alpha} Dh(0)\varphi = L_\alpha\varphi - \lambda_1\varphi \text{ for } \varphi \in C^{1,\beta}(S),$$

where

$$L_\alpha\varphi(\theta) = PV \int_S \frac{\varphi(\theta) - \varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) \text{ for } \theta \in S$$

and  $\lambda_1$  is given in (1.9) for  $k = 1$ . Moreover,  $Dh(0)$  is an isomorphism when considered as a linear operator from  $X$  to  $Y$ .

The proof of this theorem relies, in particular, on the spectral decomposition of the operator  $L_\alpha$  and regularity estimates between Hölder spaces. It will be given in Sect. 5; see Lemma 5.1 and Theorem 5.2 below.

The following proposition is the result of applying the implicit function theorem. In its proof, we will need that

$$\mathcal{H}(-\tau, \varphi) = \mathcal{H}(\tau, \varphi) \text{ for } \tau \in (-c_0/4, c_0/4), \varphi \in \mathcal{O}, \tag{3.2}$$

which is as a consequence of (2.11) and the fact that  $-\mathcal{L}_* = \mathcal{L}_*$ .

**Proposition 3.2** *There exist  $\tau_0 > 0$  and an open neighborhood  $\mathcal{U} \subset X$  of 0 for which there exists a unique curve  $(-\tau_0, \tau_0) \rightarrow \mathcal{U}$ ,  $\tau \mapsto \varphi(\tau)$ , with  $\varphi(0) = 0$  and*

$$\mathcal{H}(\tau, \varphi(\tau)) = h(0) \text{ in } Y, \text{ for } -\tau_0 < \tau < \tau_0. \tag{3.3}$$

Moreover,  $\varphi$  is of class  $C^2$ , satisfies  $\varphi(-\tau) = \varphi(\tau)$  and the expansion

$$\varphi(\tau) = -|\tau|^{N+\alpha} \left( (Dh(0))^{-1}\Phi_0 + \frac{\tau^2}{6} (Dh(0))^{-1}\Phi_2 + o(\tau^2) \right), \tag{3.4}$$

where  $\Phi_j := \partial_\tau^j G(0, 0) \in Y$ ,  $j = 0, 2$ , and  $\frac{o(\tau^2)}{\tau^2} \rightarrow 0$  in  $C^{1,\beta}(S)$  as  $\tau \rightarrow 0$ .

*Proof* Applying the implicit function theorem to the  $C^2$ -map  $\mathcal{H} : (-c_0/4, c_0/4) \times (\mathcal{O} \cap X) \rightarrow Y$  at the point  $(0, 0) \in (-\frac{c_0}{4}, \frac{c_0}{4}) \times X$  and using Theorem 3.1, we find  $\tau_0 \in (0, c_0/4)$  and a unique  $C^2$ -regular curve  $(-\tau_0, \tau_0) \rightarrow \mathcal{U}$ ,  $\tau \mapsto \varphi(\tau)$  such that (3.3) holds. By (3.2), we also have that

$$\varphi(-\tau) = \varphi(\tau) \text{ for every } \tau \in (-\tau_0, \tau_0).$$

It thus remains to prove the expansion (3.4). For this we consider the  $C^2$ -curve

$$g : (-\tau_0, \tau_0) \rightarrow Y, \quad g(\tau) := G(\tau, \varphi(\tau)).$$

Then (3.3) can be written as

$$0 = h(\varphi(\tau)) - h(0) + |\tau|^{N+\alpha}g(\tau) = Dh(0)\varphi(\tau) + O(\|\varphi(\tau)\|_X^2) + |\tau|^{N+\alpha}g(\tau).$$

Consequently, we have

$$\varphi(\tau) = -|\tau|^{N+\alpha}(Dh(0))^{-1}g(\tau) + O(\|\varphi(\tau)\|_X^2), \tag{3.5}$$

and thus the curve

$$\tau \mapsto \psi(\tau) := |\tau|^{-N-\alpha}\varphi(\tau) \tag{3.6}$$

satisfies the expansion  $\psi(\tau) = -(Dh(0))^{-1}g(\tau) + O(|\tau|^{N+\alpha}\|\psi(\tau)\|_X^2)$  and, in particular,

$$\psi(\tau) = -(Dh(0))^{-1}g(\tau) + o(\tau^2). \tag{3.7}$$

We also note that

$$g'(\tau) = \partial_\tau G(\tau, \varphi(\tau)) + \partial_\varphi G(\tau, \varphi(\tau))\varphi'(\tau)$$

and

$$g''(\tau) = \partial_\tau^2 G(\tau, \varphi(\tau)) + 2\partial_\varphi \partial_\tau G(\tau, \varphi(\tau))\varphi'(\tau) + \partial_\varphi^2 G(\tau, \varphi(\tau))[\varphi'(\tau), \varphi'(\tau)] + \partial_\varphi G(\tau, \varphi(\tau))\varphi''(\tau)$$

for  $\tau \in (-\tau_0, \tau_0)$ . Moreover, by (3.5) we have  $\varphi(\tau) = O(|\tau|^{N+\alpha})$ , and hence  $\varphi(0) = \varphi'(0) = \varphi''(0) = 0$ . We deduce

$$g(0) = G(0, 0), \quad g'(0) = \partial_\tau G(0, 0) \quad \text{and} \quad g''(0) = \partial_\tau^2 G(0, 0).$$

We thus infer that  $g(\tau) = \sum_{j=0}^2 \frac{\tau^j}{j!} \partial_\tau^j G(0, 0) + o(\tau^2)$ , and together with (3.7) this yields the expansion

$$\psi(\tau) = - \sum_{j=0}^2 \frac{\tau^j}{j!} (Dh(0))^{-1} \partial_\tau^j G(0, 0) + o(\tau^2).$$

Using this in (3.6), and recalling that  $\varphi$  is even in  $\tau$  (and thus so is  $\psi$ ), we get the expansion (3.4), as claimed. □

In the next two lemmas we compute the precise asymptotic expansion in  $\tau$  for the perturbation  $\varphi$ .

**Lemma 3.3** *The functions  $\Phi_j := \partial_\tau^j G(0, 0) \in Y$ ,  $j = 0, 2$  are given by*

$$\Phi_0 \equiv -\frac{\alpha|S|}{N} \sum_{p \in \mathcal{L}_*} |p|^{-N-\alpha} \text{ on } S$$

and

$$\Phi_2(\theta) = a_1 \sum_{p \in \mathcal{L}_*} |p|^{-N-\alpha-2} - a_2 \sum_{p \in \mathcal{L}_*} (p \cdot \theta)^2 |p|^{-N-\alpha-4} \text{ for } \theta \in S,$$

where

$$a_1 = \alpha \frac{(N + \alpha)(N - \alpha)}{N(N + 2)} |S| \text{ and } a_2 := \alpha \frac{(N + \alpha)(N + \alpha + 2)}{N} |S|. \quad (3.8)$$

*Proof* Let  $p \in \mathcal{L}_*$  and  $\theta \in S$  be fixed. We then have

$$G_p(\tau, 0)(\theta) = -\alpha \gamma_p(\tau) \text{ for } \tau \in (-c_0/4, c_0/4) \text{ with}$$

$$\gamma_p(\tau) = \int_B |\tau(y - \theta) + p|^{-N-\alpha} dy,$$

where  $B$  is the unit ball of  $\mathbb{R}^N$ . We first note that

$$\gamma_p(0) = |p|^{-N-\alpha} |B| = |p|^{-N-\alpha} \frac{|S|}{N}.$$

Moreover, for  $\tau \in (-c_0/4, c_0/4)$  we have

$$\gamma'_p(\tau) = -(N + \alpha) \int_B (y - \theta) \cdot (\tau(y - \theta) + p) |\tau(y - \theta) + p|^{-N-\alpha-2} dy$$

and

$$\gamma''_p(\tau) = - (N + \alpha) \int_B |y - \theta|^2 |\tau(y - \theta) + p|^{-N-\alpha-2} dy$$

$$+ (N + \alpha)(N + \alpha + 2)$$

$$\int_B \{(y - \theta) \cdot (\tau(y - \theta) + p)\}^2 |\tau(y - \theta) + p|^{-N-\alpha-4} dy.$$

Consequently, using the fact that odd terms do not contribute to the integral over  $B$ , and recalling that  $\int_B y_i^2 dy = N^{-1} \int_B |y|^2 dy$  and that  $\int_B |y|^2 dy = |S|/(N + 2)$ , we find that

$$\begin{aligned}
 \gamma_p''(0) &= - (N + \alpha)|p|^{-N-\alpha-2} \int_B (|y|^2 + 1)dy \\
 &\quad + (N + \alpha)(N + \alpha + 2)|p|^{-N-\alpha-4} \int_B ((p \cdot \theta)^2 + (p \cdot y)^2)dy \\
 &= \frac{(N + \alpha)(N + \alpha + 2)}{N} |S||p|^{-N-\alpha-4} (p \cdot \theta)^2 \\
 &\quad - (N + \alpha)|S| \left( \frac{1}{N + 2} + \frac{1}{N} - \frac{(N + \alpha + 2)}{N(N + 2)} \right) |p|^{-N-\alpha-2} \\
 &= \frac{(N + \alpha)(N + \alpha + 2)}{N} |S||p|^{-N-\alpha-4} (p \cdot \theta)^2 \\
 &\quad - \frac{(N + \alpha)(N - \alpha)}{N(N + 2)} |S||p|^{-N-\alpha-2} \\
 &= \frac{a_2}{\alpha} |p|^{-N-\alpha-4} (p \cdot \theta)^2 - \frac{a_1}{\alpha} |p|^{-N-\alpha-2},
 \end{aligned}$$

with  $a_1, a_2$  defined in (3.8). We thus conclude that

$$\Phi_0(\theta) = -\alpha \sum_{p \in \mathcal{L}_*} \gamma_p(0) = -\frac{\alpha|S|}{N} \sum_{p \in \mathcal{L}_*} |p|^{-N-\alpha}$$

and

$$\Phi_2(\theta) = a_1 \sum_{p \in \mathcal{L}_*} |p|^{-N-\alpha-2} - a_2 \sum_{p \in \mathcal{L}_*} (p \cdot \theta)^2 |p|^{-N-\alpha-4}$$

for  $\theta \in S$ , as claimed. □

**Lemma 3.4** *The functions  $\Psi_j := (Dh(0))^{-1} \Phi_j \in X, j = 0, 2$ , are given by*

$$\Psi_0 \equiv \frac{|S|}{\lambda_1 N} \sum_{p \in \mathcal{L}_*} \frac{1}{|p|^{N+\alpha}} \text{ on } S$$

and

$$\begin{aligned}
 \Psi_2(\theta) &= |S| \left\{ \frac{(N + \alpha)(N + \alpha + 2)}{N^2(\lambda_2 - \lambda_1)} + \frac{2}{\lambda_1} \frac{(N + \alpha)(N + 1)(\alpha + 2)}{N^2(N + 2)} \right\} \sum_{p \in \mathcal{L}_*} \frac{1}{|p|^{N+\alpha+2}} \\
 &\quad - \frac{|S|(N + \alpha)(N + \alpha + 2)}{N(\lambda_2 - \lambda_1)} \sum_{p \in \mathcal{L}_*} \frac{(p \cdot \theta)^2}{|p|^{N+\alpha+4}} \text{ for } \theta \in S.
 \end{aligned}$$

*Proof* We recall from Theorem 3.1 that

$$(Dh(0))^{-1} = \frac{1}{\alpha} (L_\alpha - \lambda_1)^{-1} : Y \rightarrow X,$$

with  $L_\alpha$  given by (1.8). Since  $L_\alpha$  maps constant functions to zero, we find that

$$\Psi_0 \equiv \frac{1}{\alpha}(L_\alpha - \lambda_1)^{-1} \left( -\frac{\alpha|S|}{N} \sum_{p \in \mathcal{L}_*} |p|^{-N-\alpha} \right) = \frac{|S|}{\lambda_1 N} \sum_{p \in \mathcal{L}_*} |p|^{-N-\alpha} \quad \text{on } S.$$

To compute  $\Psi_2$ , we introduce the functions

$$q_e \in C^{1,\beta}(S), \quad q_e(\theta) = (e \cdot \theta)^2 - \frac{1}{N}$$

for  $e \in S$ . Since  $q_e$  is a spherical harmonic of degree two for every  $e \in S$ , we have

$$(Dh(0))^{-1} q_e = \frac{1}{\alpha}(L_\alpha - \lambda_1)^{-1} q_e = \frac{1}{\alpha(\lambda_2 - \lambda_1)} q_e \quad \text{for } e \in S.$$

Moreover, by Lemma 3.3, we have

$$\Phi_2 = \sum_{p \in \mathcal{L}_*} \left( a_1 - \frac{a_2}{N} - a_2 q_{\frac{p}{|p|}} \right) |p|^{-N-\alpha-2} \quad \text{in } Y$$

and thus

$$\begin{aligned} \Psi_2 &= \frac{1}{\alpha}(L_\alpha - \lambda_1)^{-1} \Phi_2 \\ &= - \sum_{p \in \mathcal{L}_*} \left\{ \frac{1}{\alpha \lambda_1} \left( a_1 - \frac{a_2}{N} \right) + \frac{a_2}{\alpha(\lambda_2 - \lambda_1)} q_{\frac{p}{|p|}} \right\} |p|^{-N-\alpha-2} \quad \text{in } X, \end{aligned}$$

i.e.,

$$\begin{aligned} \Psi_2(\theta) &= \left\{ \frac{a_2}{\alpha N(\lambda_2 - \lambda_1)} - \frac{1}{\alpha \lambda_1} \left( a_1 - \frac{a_2}{N} \right) \right\} \sum_{p \in \mathcal{L}_*} |p|^{-N-\alpha-2} - \frac{a_2}{\alpha(\lambda_2 - \lambda_1)} \\ &\quad \sum_{p \in \mathcal{L}_*} \frac{(p \cdot \theta)^2}{|p|^{N+\alpha+4}} = |S| \left\{ \frac{(N + \alpha)(N + \alpha + 2)}{N^2(\lambda_2 - \lambda_1)} \right. \\ &\quad \left. - \frac{1}{\lambda_1} \frac{(N + \alpha)\{N(N - \alpha) - (N + 2)(N + \alpha + 2)\}}{N^2(N + 2)} \right\} \sum_{p \in \mathcal{L}_*} \frac{1}{|p|^{N+\alpha+2}} \\ &\quad - \frac{|S|(N + \alpha)(N + \alpha + 2)}{N(\lambda_2 - \lambda_1)} \sum_{p \in \mathcal{L}_*} \frac{(p \cdot \theta)^2}{|p|^{N+\alpha+4}} \end{aligned}$$

$$\begin{aligned}
 &= |S| \left\{ \frac{(N + \alpha)(N + \alpha + 2)}{N^2(\lambda_2 - \lambda_1)} + \frac{1}{\lambda_1} \frac{(N + \alpha)(2N(\alpha + 2) + 4 + 2\alpha)}{N^2(N + 2)} \right\} \\
 &\quad \times \sum_{p \in \mathcal{L}_*} \frac{1}{|p|^{N+\alpha+2}} - \frac{|S|(N + \alpha)(N + \alpha + 2)}{N(\lambda_2 - \lambda_1)} \sum_{p \in \mathcal{L}_*} \frac{(p \cdot \theta)^2}{|p|^{N+\alpha+4}} \quad \text{for } \theta \in S,
 \end{aligned}$$

as claimed. □

We may now complete the

*Proof of Theorem 1.1* The existence and uniqueness of the curve  $r \mapsto \varphi_r$  with the properties of Theorem 1.1(i)–(iii) follow immediately from Proposition 3.2 by setting  $\varphi_r := \varphi(\frac{1}{r})$ . To obtain Theorem 1.1(iv), we note that by (3.4) and Lemma 3.4 we have the expansion

$$\begin{aligned}
 \varphi_r(\theta) &= -r^{-N-\alpha} \left( \Psi_0(\theta) + \frac{r^{-2}}{6} \Psi_2(\theta) + o(r^{-2}) \right) \\
 &= r^{-N-\alpha} \left( -\kappa_0 + r^{-2} \left\{ \kappa_1 \sum_{p \in \mathcal{L}_*} \frac{(\theta \cdot p)^2}{|p|^{N+\alpha+4}} - \kappa_2 \right\} + o(r^{-2}) \right) \\
 &\quad \text{for } \theta \in S \text{ as } r \rightarrow +\infty,
 \end{aligned}$$

where

$$\kappa_0 \equiv \Psi_0 = \frac{|S|}{\lambda_1 N} \sum_{p \in \mathcal{L}_*} \frac{1}{|p|^{N+\alpha}}, \quad \kappa_1 = \frac{|S|(N + \alpha)(N + \alpha + 2)}{6N(\lambda_2 - \lambda_1)}$$

and

$$\kappa_2 = \frac{|S|}{6} \left\{ \frac{(N + \alpha)(N + \alpha + 2)}{N^2(\lambda_2 - \lambda_1)} + \frac{2}{\lambda_1} \frac{(N + \alpha)(N + 1)(\alpha + 2)}{N^2(N + 2)} \right\} \sum_{p \in \mathcal{L}_*} \frac{1}{|p|^{N+\alpha+2}}.$$

To prove Theorem 1.1(v), it suffices to show, after making  $r_0$  larger if necessary, that the map

$$\theta \mapsto \tilde{f}(\theta) := \sum_{p \in \mathcal{L}_*} \frac{(\theta \cdot p)^2}{|p|^{N+\alpha+4}}$$

is non-constant on  $S$  if  $1 \leq M \leq N - 1$ . We readily observe that  $\tilde{f}(e_1) > 0$  and  $\tilde{f}(e_N) = 0$ . The proof of Theorem 1.1 is thus finished.

The last statement of the theorem, on uniqueness, is a direct consequence of the implicit function theorem. □

We conclude this section with the



*Proof of Corollary 1.3* By assumption and up to a rotation, we may assume that the lattice basis satisfies

$$\mathbf{a}_i = \rho_i e_i \quad \text{for } i = 1, \dots, M,$$

for some  $\rho_i \in \mathbb{R} \setminus \{0\}$ . It is convenient to define the map

$$\mathcal{J} : \mathbb{Z}^M \rightarrow \mathcal{L}, \quad \mathcal{J}(k) := \sum_{i=1}^M k_i \mathbf{a}_i = (\rho_1 k_1, \dots, \rho_M k_M, 0, \dots, 0) \in \mathbb{R}^N. \tag{3.9}$$

Then we get

$$\begin{aligned} \sum_{p \in \mathcal{L}_*} \frac{(\theta \cdot p)^2}{|p|^{N+\alpha+4}} &= \sum_{k \in \mathbb{Z}_*^M} \frac{(\theta \cdot \mathcal{J}(k))^2}{|\mathcal{J}(k)|^{N+\alpha+4}} = \sum_{k \in \mathbb{Z}_*^M} \frac{(\theta_1 \rho_1 k_1 + \dots + \theta_M \rho_M k_M)^2}{|\mathcal{J}(k)|^{N+\alpha+4}} \\ &= \sum_{i,j=1}^M \sum_{k \in \mathbb{Z}_*^M} \frac{\theta_i \theta_j \rho_i \rho_j k_i k_j}{|\mathcal{J}(k)|^{N+\alpha+4}}, \end{aligned}$$

whereas for  $i \neq j$  we have

$$\sum_{k \in \mathbb{Z}_*^M} \frac{\theta_i \theta_j \rho_i \rho_j k_i k_j}{|\mathcal{J}(k)|^{N+\alpha+4}} = 0$$

by oddness with respect to the reflection of  $k$  at the axis  $\{k_i = 0\}$ . Hence we conclude that

$$\sum_{p \in \mathcal{L}_*} \frac{(\theta \cdot p)^2}{|p|^{N+\alpha+4}} = \sum_{k \in \mathbb{Z}_*^M} \frac{\theta_1^2 \rho_1^2 k_1^2 + \dots + \theta_M^2 \rho_M^2 k_M^2}{|\mathcal{J}(k)|^{N+\alpha+4}} = \sum_{i=1}^M \mu_i \theta_i^2$$

with

$$\mu_i = \sum_{k \in \mathbb{Z}_*^M} \frac{\rho_i^2 k_i^2}{|\mathcal{J}(k)|^{N+\alpha+4}} = \sum_{p \in \mathcal{L}_*} \frac{p_i^2}{|p|^{N+\alpha+4}}.$$

Together with Theorem 1.1, this gives (1.11). To see (1.12), we note that in the case of the square lattice we have  $\rho_1 = \rho_2 = \dots = \rho_M$  and thus

$$\mu_i = \frac{1}{M} \sum_{j=1}^M \mu_j = \frac{1}{M} \sum_{p \in \mathbb{Z}_*^M} \frac{1}{|p|^{N+\alpha+2}} \quad \text{for } i = 1, \dots, M.$$

This ends the proof of Corollary 1.3. □

### 4 Regularity of the NMC operator over the unit sphere

In this section we prove the smoothness of the nonlocal mean curvature  $h$  as asserted in Theorem 2.1.

#### 4.1 Geometric preliminaries

For  $\varphi \in \mathcal{O}$ , we recall the parameterization  $F_\varphi : S \rightarrow S_\varphi$  of  $S_\varphi$ , defined in (2.4) by  $F_\varphi(\sigma) = (1 + \varphi(\sigma))\sigma$ . We shall need the following observation.

**Proposition 4.1** *Let  $\varphi \in C^1(S)$  be such that  $\|\varphi\|_{L^\infty(S)} < 1$ . Then the unit outer normal (to the set enclosed by  $S_\varphi$ ) of  $S_\varphi$  at a point  $F_\varphi(\sigma)$ ,  $\sigma \in S$  is given by*

$$\nu_{S_\varphi}(F_\varphi(\sigma)) = \frac{(1 + \varphi(\sigma))\sigma - \nabla\varphi(\sigma)}{\sqrt{(1 + \varphi(\sigma))^2 + |\nabla\varphi(\sigma)|^2}}.$$

Moreover, for every continuous function  $f$  on  $\mathbb{R}^N$ , we have

$$\int_{S_\varphi} f(y) dV_{S_\varphi}(y) = \int_S f \circ F_\varphi(\sigma) J_\varphi(\sigma) dV(\sigma) \quad \text{with}$$

$$J_\varphi = (1 + \varphi)^{N-2} \sqrt{(1 + \varphi)^2 + |\nabla\varphi|^2}. \tag{4.1}$$

Here and in the following,  $\nabla\varphi$  denotes the gradient vector field of  $\varphi$  on  $S$ .

*Proof* We fix a local parametrization  $z \mapsto \sigma(z)$  of  $S$ , which gives rise to the local parameterization  $z \rightarrow \widehat{F}(z) = (1 + \varphi(\sigma(z)))\sigma(z)$  of  $S_\varphi$ . The tangent vectors of  $S_\varphi$  at the point  $\widehat{F}(z)$  are given by

$$Z_i(z) := \partial_{z_i} \widehat{F}(z) = (1 + \varphi(\sigma(z))) \partial_{z_i} \sigma(z) + \partial_{z_i}(\varphi \circ \sigma)(z) \sigma(z) \tag{4.2}$$

with  $\partial_{z_i}(\varphi \circ \sigma)(z) = \nabla\varphi(\sigma(z)) \cdot \partial_{z_i} \sigma(z)$  for  $i = 1, \dots, N$ . Since  $\sigma \cdot \nabla\varphi(\sigma) = 0$  and  $\sigma \cdot \partial_{z_i} \sigma = 0$  (which follows from  $|\sigma|^2 = 1$ ), we thus conclude that the unit outer normal of  $S_\varphi$  at a point  $F_\varphi(\sigma)$  with  $\sigma = \sigma(z) \in S$  is given by

$$\nu_{S_\varphi}(F_\varphi(\sigma)) = \frac{(1 + \varphi(\sigma))\sigma - \nabla\varphi(\sigma)}{\sqrt{(1 + \varphi(\sigma))^2 + |\nabla\varphi(\sigma)|^2}}.$$

We now turn to the proof of (4.1). By the previous relations, the first fundamental form of  $S_\varphi$  is given by

$$g_{ij} = Z_i \cdot Z_j = (1 + \varphi \circ \sigma)^2 \partial_{z_i} \sigma \cdot \partial_{z_j} \sigma + \partial_{z_i}(\varphi \circ \sigma) \partial_{z_j}(\varphi \circ \sigma). \tag{4.3}$$

We now compute  $\sqrt{\det(g)}(z)$  at a given point  $z$  under the assumption that  $\partial_{z_i} \sigma(z) \cdot \partial_{z_j} \sigma(z) = \delta_{ij}$ . We then have that

$$(1 + \varphi(\sigma(z)))^{-2(N-1)} \det(g)(z) = \det(\text{id} + C)$$

with the matrix  $C = (C_{ij})_{ij}$  given by

$$C_{ij} = (1 + \varphi(\sigma(z)))^{-2} \partial_{z_i}(\varphi \circ \sigma)(z) \partial_{z_j}(\varphi \circ \sigma)(z).$$

Note that  $C$  has only one non-zero eigenvalue given by  $(1 + \varphi(\sigma(z)))^{-2} |\nabla \varphi(\sigma(z))|^2$  with corresponding eigenvector  $(\partial_{z_i}(\varphi \circ \sigma)(z))_i$ . We thus have

$$(1 + \varphi(\sigma(z)))^{-2(N-1)} \det(g)(z) = \det(\text{id} + C) = 1 + (1 + \varphi(\sigma(z)))^{-2} |\nabla \varphi(\sigma(z))|^2,$$

and hence

$$\sqrt{\det(g)(z)} = (1 + \varphi(\sigma(z)))^{N-2} \sqrt{(1 + \varphi(\sigma(z)))^2 + |\nabla \varphi(\sigma(z))|^2}.$$

We have thus computed the local change of the volume form when passing from  $S_\varphi$  to  $S$ , and this gives rise to the transformation rule (4.1).  $\square$

#### 4.2 Preliminary differential calculus formulas

For a finite set  $\mathcal{N}$ , we let  $|\mathcal{N}|$  denote the number of elements of  $\mathcal{N}$ . Moreover, we denote  $\mathcal{N}_\ell := \{1, \dots, \ell\}$  for  $\ell \in \mathbb{N}$ . Let  $Z$  be a Banach space and  $U$  a nonempty open subset of  $Z$ . If  $T \in \mathcal{C}^\ell(U, \mathbb{R})$  and  $u \in U$ , then  $D^\ell T(u)$  is a continuous symmetric  $\ell$ -linear form on  $Z$  whose norm is given by

$$\|D^\ell T(u)\| = \sup_{u_1, \dots, u_\ell \in Z} \frac{|D^\ell T(u)[u_1, \dots, u_\ell]|}{\prod_{j=1}^\ell \|u_j\|_Z}.$$

If  $T_1, T_2 \in \mathcal{C}^\ell(U, \mathbb{R})$ , then also  $T_1 T_2 \in \mathcal{C}^\ell(U, \mathbb{R})$ , and the  $\ell$ -th derivative of  $T_1 T_2$  at  $u$  is given by

$$D^\ell(T_1 T_2)(u)[u_1, \dots, u_\ell] = \sum_{\mathcal{N} \in \mathcal{S}_\ell} D^{|\mathcal{N}|} T_1(u)[u_n]_{n \in \mathcal{N}} D^{\ell-|\mathcal{N}|} T_2(u)[u_n]_{n \in \mathcal{N}^c}, \tag{4.4}$$

where  $\mathcal{S}_\ell$  is the set of subsets of  $\{1, \dots, \ell\}$  and  $\mathcal{N}^c = \{1, \dots, \ell\} \setminus \mathcal{N}$  for  $\mathcal{N} \in \mathcal{S}_\ell$ .

If, in particular,  $L : Z \rightarrow \mathbb{R}$  is a linear map and  $|\mathcal{N}| \geq 1$ , we have

$$D^{|\mathcal{N}|}(LT_2)(u)[u_i]_{i \in \mathcal{N}} = L(u) D^{|\mathcal{N}|} T_2(u)[u_i]_{i \in \mathcal{N}} + \sum_{j \in \mathcal{N}} L(u_j) D^{|\mathcal{N}|-1} T_2(u)[u_i]_{i \in \mathcal{N} \setminus \{j\}}. \tag{4.5}$$

Furthermore, let  $B : Z \times Z \mapsto \mathbb{R}$  be a bilinear map and let  $Q : Z \mapsto \mathbb{R}$  be its associated quadratic form (namely  $Q(\varphi) = B(\varphi, \varphi)$ ). Then

$$\begin{aligned}
 D^{|\mathcal{N}|}(QT_2)(u)[u_i]_{i \in \mathcal{N}} &= B(u, u)D^{|\mathcal{N}|}T_2(u)[u_i]_{i \in \mathcal{N}} \\
 &+ \sum_{j \in \mathcal{N}} (B(u, u_j) + B(u_j, u))D^{|\mathcal{N}|-1}T_2(u)[u_i]_{i \in \mathcal{N} \setminus \{j\}} \\
 &+ \sum_{i, j \in \mathcal{N}} (B(u_i, u_j) + B(u_j, u_i))D^{|\mathcal{N}|-2}T_2(u)[u_r]_{r \in \mathcal{N} \setminus \{i, j\}}.
 \end{aligned}
 \tag{4.6}$$

We close this section by the well known *Faá de Bruno formula*, see e.g. [15]. We let  $T$  be as above and  $g : \text{Im}(T) \rightarrow \mathbb{R}$  be a  $k$ -times differentiable map. The Faá de Bruno formula states that

$$D^k(g \circ T)(u)[u_1, \dots, u_k] = \sum_{\Pi \in \mathcal{P}_k} g^{(|\Pi|)}(T(u)) \prod_{P \in \Pi} D^{|P|}T(u)[u_j]_{j \in P}, \tag{4.7}$$

for  $u, u_1, \dots, u_k \in U$ , where  $\mathcal{P}_k$  denotes the set of all partitions of  $\{1, \dots, k\}$ .

### 4.3 Regularity of the nonlocal mean curvature operator over the sphere

For every  $a, b \in S, b \neq -a$  we consider the regular curve

$$\gamma_{a,b} : [0, 1] \rightarrow S, \quad \gamma_{a,b}(t) = \frac{ta + (1-t)b}{|ta + (1-t)b|}, \tag{4.8}$$

which clearly satisfies  $\gamma_{a,b}(0) = b$  and  $\gamma_{a,b}(1) = a$ .

**Lemma 4.2** *Consider the compact subset*

$$S_* := \{(a, b) \in S \times S : |a - b| \leq 1\} \subset S \times S.$$

*Then, there exists a constant  $C > 0$  depending only on  $N$  with the property that for  $(a, b), (a_1, b_1), (a_2, b_2) \in S_*$  and  $t \in [0, 1]$  we have*

$$|\dot{\gamma}_{a,b}(t)| \leq C|a - b|, \tag{4.9}$$

$$|\gamma_{a_1, b_1}(t) - \gamma_{a_2, b_2}(t)| \leq C(|a_1 - a_2| + |b_1 - b_2|) \quad \text{and} \tag{4.10}$$

$$|\dot{\gamma}_{a_1, b_1}(t) - \dot{\gamma}_{a_2, b_2}(t)| \leq C(|a_1 - a_2| + |b_1 - b_2|). \tag{4.11}$$

*Proof* For  $t \in [0, 1]$ , consider the function

$$\Upsilon : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}, \quad \Upsilon(t, a, b) = |b + t(a - b)| \tag{4.12}$$

Since  $|a - b| \leq 1$  on  $S_*$  and  $t(1 - t) \in [0, \frac{1}{4}]$  for  $t \in [0, 1]$ , we see that

$$\frac{\sqrt{3}}{2} \leq \Upsilon(t, a, b) = \sqrt{1 - t(1 - t)|a - b|^2} \leq 1 \quad \text{for } (t, a, b) \in [0, 1] \times S_*. \tag{4.13}$$

By direct computations, we also see that

$$\dot{\gamma}_{a,b}(t) = \frac{a - b}{\Upsilon(t, a, b)} + \frac{(1 - 2t)|a - b|^2}{2} \frac{b + t(a - b)}{\Upsilon(t, a, b)^3} \quad \text{for } (t, a, b) \in [0, 1] \times S_*. \tag{4.14}$$

Hence (4.13) yields (4.9) with a suitable constant  $C > 0$ .

Next we consider the function

$$\mathcal{V} : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad \mathcal{V}(t, a, b) := \frac{(1 - 2t)|a - b|^2}{2} (b + t(a - b)).$$

Then we may write

$$\gamma_{a,b}(t) = \frac{ta + (1 - t)b}{\Upsilon(t, a, b)} \quad \text{and} \quad \dot{\gamma}_{a,b}(t) = \frac{a - b}{\Upsilon(t, a, b)} + \frac{\mathcal{V}(t, a, b)}{\Upsilon(t, a, b)^3} \quad \text{for } (t, a, b) \in [0, 1] \times S_*.$$

By (4.13), we see that the right hand sides of these equalities define  $C^1$ -functions in an open neighborhood of the compact set  $[0, 1] \times S_*$  in  $\mathbb{R}^{2N+1} = \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ . Therefore, a standard argument shows that these functions are Lipschitz continuous on  $[0, 1] \times S_*$  with respect to the Euclidean distance of  $\mathbb{R}^{2N+1}$ , and from this (4.10) and (4.11) follow.  $\square$

The following is an expression for  $h$ , as defined in (2.8), where we remove the dependence on  $\varphi$  in the domain of integration.

**Proposition 4.3** *Let  $\varphi \in \mathcal{O}$ . Then, we have*

$$\begin{aligned} h(\varphi)(\theta) &= -(1 + \varphi(\theta)) \int_S \frac{\varphi(\theta) - \varphi(\sigma) - (\theta - \sigma) \cdot \nabla \varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} \\ &\quad \times (1 + \varphi(\sigma))^{N-2} \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma) \\ &+ \int_S \frac{(\varphi(\theta) - \varphi(\sigma))^2}{|\theta - \sigma|^{N+\alpha}} (1 + \varphi(\sigma))^{N-2} \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma) \\ &+ \frac{1 + \varphi(\theta)}{2} \int_S \frac{1}{|\theta - \sigma|^{N+\alpha-2}} (1 + \varphi(\sigma))^{N-1} \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma), \end{aligned}$$

where  $\mathcal{K}_\alpha : \mathcal{O} \times S \times S$  is given by

$$\mathcal{K}_\alpha(\varphi, \sigma, \theta) := \frac{1}{\left(\frac{(\varphi(\theta) - \varphi(\sigma))^2}{|\theta - \sigma|^2} + (1 + \varphi(\sigma))(1 + \varphi(\theta))\right)^{(N+\alpha)/2}}.$$

Moreover, all integrals above converge absolutely.

*Proof* Let  $\varphi \in \mathcal{O}$ . By Proposition 4.1, for every  $\theta \in S$ , we have

$$-h(\varphi)(\theta) = \int_S \frac{(F_\varphi(\theta) - F_\varphi(\sigma)) \cdot \nu_{S_\varphi}(F_\varphi(\sigma))}{|F_\varphi(\theta) - F_\varphi(\sigma)|^{N+\alpha}} J_\varphi(\sigma) dV(\sigma) \tag{4.15}$$

and thus

$$\begin{aligned} -h(\varphi)(\theta) &= \int_S \frac{(\varphi(\theta) - \varphi(\sigma)) \sigma \cdot \nu_{S_\varphi}(F_\varphi(\sigma)) J_\varphi(\sigma)}{((\varphi(\theta) - \varphi(\sigma))^2 + (1 + \varphi(\sigma))(1 + \varphi(\theta))|\theta - \sigma|^2)^{(N+\alpha)/2}} dV(\sigma) \\ &\quad + (1 + \varphi(\theta)) \int_S \frac{(\theta - \sigma) \cdot \nu_{S_\varphi}(F_\varphi(\sigma)) J_\varphi(\sigma)}{((\varphi(\theta) - \varphi(\sigma))^2 + (1 + \varphi(\sigma))(1 + \varphi(\theta))|\theta - \sigma|^2)^{(N+\alpha)/2}} dV(\sigma), \end{aligned}$$

where we used that  $2(1 - \theta \cdot \sigma) = |\theta - \sigma|^2$ . It follows that

$$\begin{aligned} -h(\varphi)(\theta) &= \int_S \frac{\varphi(\theta) - \varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} \mathcal{K}_\alpha(\varphi, \sigma, \theta) \sigma \cdot \nu_{S_\varphi}(F_\varphi(\sigma)) J_\varphi(\sigma) dV(\sigma) \\ &\quad + (1 + \varphi(\theta)) \int_S \frac{\theta - \sigma}{|\theta - \sigma|^{N+\alpha}} \cdot \nu_{S_\varphi}(F_\varphi(\sigma)) J_\varphi(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma). \end{aligned} \tag{4.16}$$

Letting  $\psi = 1 + \varphi$ , we get

$$\begin{aligned} J_\varphi(\sigma) &= \psi^{N-2}(\sigma) \sqrt{\psi^2(\sigma) + |\nabla \psi(\sigma)|^2} \quad \text{and} \\ \nu_{S_\varphi}(F_\varphi(\sigma)) &= \frac{\sigma \psi(\sigma) - \nabla \psi(\sigma)}{\sqrt{\psi^2(\sigma) + |\nabla \psi(\sigma)|^2}} \end{aligned}$$

from Proposition 4.1. Consequently,

$$\sigma \cdot \nu_{S_\varphi}(F_\varphi(\sigma)) = \frac{\psi(\sigma)}{\sqrt{\psi^2(\sigma) + |\nabla \psi(\sigma)|^2}}$$

and thus

$$J_\varphi(\sigma) \sigma \cdot \nu_{S_\varphi}(F_\varphi(\sigma)) = \psi^{N-1}(\sigma).$$

Furthermore

$$(\theta - \sigma) \cdot \nu_{S_\varphi}(F_\varphi(\sigma)) J_\varphi(\sigma) = -(\theta - \sigma) \cdot \nabla \psi(\sigma) \psi^{N-2}(\sigma) + (\theta - \sigma) \cdot \sigma \psi^{N-1}(\sigma).$$

Using the latter two identities in (4.16), we find that

$$\begin{aligned}
 -h(\varphi)(\theta) &= \int_S \frac{\psi(\theta) - \psi(\sigma)}{|\theta - \sigma|^{N+\alpha}} \psi^{N-1}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma) \\
 &\quad - \psi(\theta) \int_S \frac{(\theta - \sigma) \cdot \nabla \psi(\sigma)}{|\theta - \sigma|^{N+\alpha}} \psi^{N-2}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma) \\
 &\quad + \psi(\theta) \int_S \frac{(\theta - \sigma) \cdot \sigma}{|\theta - \sigma|^{N+\alpha}} \psi^{N-1}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 -h(\varphi)(\theta) &= \int_S \frac{(\psi(\theta) - \psi(\sigma))\psi(\sigma) - \psi(\theta)(\theta - \sigma) \cdot \nabla \psi(\sigma)}{|\theta - \sigma|^{N+\alpha}} \\
 &\quad \times \psi^{N-2}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma) \\
 &\quad + \psi(\theta) \int_S \frac{(\theta - \sigma) \cdot \sigma}{|\theta - \sigma|^{N+\alpha}} \psi^{N-1}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma).
 \end{aligned}$$

We add and subtract  $(\psi(\theta) - \psi(\sigma))\psi(\theta)$  to get

$$\begin{aligned}
 -h(\varphi)(\theta) &= - \int_S \frac{(\psi(\theta) - \psi(\sigma))^2}{|\theta - \sigma|^{N+\alpha}} \psi^{N-2}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma) \\
 &\quad + \psi(\theta) \int_S \frac{\psi(\theta) - \psi(\sigma) - (\theta - \sigma) \cdot \nabla \psi(\sigma)}{|\theta - \sigma|^{N+\alpha}} \\
 &\quad \times \psi^{N-2}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma) \\
 &\quad + \psi(\theta) \int_S \frac{(\theta - \sigma) \cdot \sigma}{|\theta - \sigma|^{N+\alpha}} \psi^{N-1}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma).
 \end{aligned}$$

We then conclude that

$$\begin{aligned}
 -h(\varphi)(\theta) &= - \int_S \frac{(\psi(\theta) - \psi(\sigma))^2}{|\theta - \sigma|^{N+\alpha}} \\
 &\quad \psi^{N-2}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma) \\
 &\quad + \psi(\theta) \int_S \frac{\psi(\theta) - \psi(\sigma) - (\theta - \sigma) \cdot \nabla \psi(\sigma)}{|\theta - \sigma|^{N+\alpha}} \\
 &\quad \times \psi^{N-2}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma) \\
 &\quad - \frac{\psi(\theta)}{2} \int_S \frac{1}{|\theta - \sigma|^{N+\alpha-2}} \psi^{N-1}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma). \quad (4.17)
 \end{aligned}$$

Let us now check that all integrals above converge absolutely. Indeed, it is clear that



$$\int_S \frac{1}{|\theta - \sigma|^{N+\alpha-2}} \psi^{N-1}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma) \leq (1 - \|\varphi\|_\infty)^{-N-\alpha} \int_S \frac{1}{|\theta - \sigma|^{N+\alpha-2}} \psi^{N-1}(\sigma) dV(\sigma) < \infty$$

and, since  $(\psi(\theta) - \psi(\sigma))^2 \leq \|\psi\|_{C^1(S)}^2 |\theta - \sigma|^2$ , we also get

$$\int_S \frac{(\psi(\theta) - \psi(\sigma))^2}{|\theta - \sigma|^{N+\alpha}} \psi^{N-2}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma) < \infty.$$

Next, if  $|\theta - \sigma| < 1$ , we can write

$$\psi(\theta) - \psi(\sigma) - (\theta - \sigma) \cdot \nabla \psi(\sigma) = \int_0^1 \{ \nabla \psi(\gamma_{\theta,\sigma}(t)) - \nabla \psi(\gamma_{\theta,\sigma}(0)) \} \cdot \dot{\gamma}_{\theta,\sigma}(t) dt$$

with  $\gamma_{\theta,\sigma}$  defined in (4.8). By (4.9) we thus have

$$|\psi(\theta) - \psi(\sigma) - (\theta - \sigma) \cdot \nabla \psi(\sigma)| \leq C \|\psi\|_{C^{1,\beta}(S)} |\theta - \sigma|^{1+\beta},$$

and this obviously also holds, by enlargening  $C > 0$  if necessary, for  $\theta, \sigma \in S$  with  $|\theta - \sigma| \geq 1$ . From this and the fact that  $\beta \in (\alpha, 1)$ , we obtain

$$\int_S \frac{|\psi(\theta) - \psi(\sigma) - (\theta - \sigma) \cdot \nabla \psi(\sigma)|}{|\theta - \sigma|^{N+\alpha}} \psi^{N-2}(\sigma) \mathcal{K}_\alpha(\varphi, \sigma, \theta) dV(\sigma) \tag{4.18}$$

$$\leq C(1 - \|\varphi\|_\infty)^{-N-\alpha} \|\psi\|_{C^{1,\beta}(S)}^{N-1} \int_S \frac{1}{|\theta - \sigma|^{N+\alpha-1-\beta}} dV(\sigma) < \infty. \tag{4.19}$$

We then have that the integrals in the expression of  $h$  converge absolutely. □

For  $0 < r < 2$ , we now put

$$\mathcal{B}(r) = \left\{ \psi \in C^{1,\beta}(S) : r < \psi < 2 \text{ in } S \right\}. \tag{4.20}$$

We consider the map  $\overline{\mathcal{K}}_\alpha : \mathcal{B}(r) \times S \times S \rightarrow \mathbb{R}$  defined by

$$\overline{\mathcal{K}}_\alpha(\psi, \sigma, \theta) := \frac{1}{\left( \frac{(\psi(\theta) - \psi(\sigma))^2}{|\theta - \sigma|^2} + \psi(\sigma)\psi(\theta) \right)^{(N+\alpha)/2}}. \tag{4.21}$$

We also define  $\tilde{h} : \mathcal{B}(r) \rightarrow L^\infty(S)$  by  $\tilde{h}(\psi) := h(\psi - 1)$ . Then, by Proposition 4.3, we have

$$\begin{aligned} \tilde{h}(\psi)(\theta) = h(\psi - 1)(\theta) &= \int_S \frac{(\psi(\theta) - \psi(\sigma))^2}{|\theta - \sigma|^{N+\alpha}} \psi^{N-2}(\sigma) \bar{\mathcal{K}}_\alpha(\psi, \sigma, \theta) dV(\sigma) \\ &\quad - \psi(\theta) \int_S \frac{\psi(\theta) - \psi(\sigma) - (\theta - \sigma) \cdot \nabla \psi(\sigma)}{|\theta - \sigma|^{N+\alpha}} \\ &\quad \times \psi^{N-2}(\sigma) \bar{\mathcal{K}}_\alpha(\psi, \sigma, \theta) dV(\sigma) \\ &\quad + \frac{\psi(\theta)}{2} \int_S \frac{1}{|\theta - \sigma|^{N+\alpha-2}} \psi^{N-1}(\sigma) \bar{\mathcal{K}}_\alpha(\psi, \sigma, \theta) dV(\sigma). \end{aligned} \tag{4.22}$$

The proof of Theorem 2.1 will be completed once we prove that  $\tilde{h} : \mathcal{B}(r) \rightarrow C^{\beta-\alpha}(S)$  is smooth for every  $r > 0$ .

We define  $\Lambda_1 : C^{1,\beta}(S) \times S \times S \rightarrow \mathbb{R}$  by

$$\begin{aligned} \Lambda_1(\psi, \sigma, \theta) &= \psi(\theta) - \psi(\sigma) - (\theta - \sigma) \cdot \nabla \psi(\sigma) \\ &= \int_0^1 \{ \nabla \psi(\gamma_{\theta,\sigma}(t)) - \nabla \psi(\gamma_{\theta,\sigma}(0)) \} \cdot \dot{\gamma}_{\theta,\sigma}(t) dt \end{aligned}$$

and  $\Lambda_2 : C^{1,\beta}(S) \times C^{1,\beta}(S) \times S \times S \rightarrow \mathbb{R}$  by

$$\Lambda_2(\psi_1, \psi_2, \sigma, \theta) = (\psi_1(\theta) - \psi_1(\sigma))(\psi_2(\theta) - \psi_2(\sigma)).$$

With this notation, we have

$$\begin{aligned} \tilde{h}(\psi)(\theta) = h(\psi - 1)(\theta) &= \int_S \frac{\Lambda_2(\psi, \psi, \sigma, \theta)}{|\theta - \sigma|^{N+\alpha}} \psi^{N-2}(\sigma) \bar{\mathcal{K}}_\alpha(\psi, \sigma, \theta) dV(\sigma) \\ &\quad - \psi(\theta) \int_S \frac{\Lambda_1(\psi, \sigma, \theta)}{|\theta - \sigma|^{N+\alpha}} \psi^{N-2}(\sigma) \bar{\mathcal{K}}_\alpha(\psi, \sigma, \theta) dV(\sigma) \\ &\quad + \frac{\psi(\theta)}{2} \int_S \frac{1}{|\theta - \sigma|^{N+\alpha-2}} \psi^{N-1}(\sigma) \bar{\mathcal{K}}_\alpha(\psi, \sigma, \theta) dV(\sigma). \end{aligned} \tag{4.23}$$

*Remark 4.4* It will be convenient to modify this representation further such that the singularity of the integrand does not depend on  $\theta$ . For this we fix  $e \in S$  and a Lipschitz continuous map of rotations  $S \mapsto SO(N)$ ,  $\theta \mapsto R_\theta$  with the property that

$$S_e := \{ \theta \in S : \theta \cdot e \geq 0 \} \subset \{ \theta \in S : R_\theta e = \theta \}. \tag{4.24}$$

The following is a possible way to construct  $R$ . For fixed  $e \in S$ , consider the map  $\theta \mapsto R_\theta$  defined as follows. For  $\theta \in S$  with  $\theta \cdot e \geq 0$ , we let  $R_\theta$  be the rotation of the angle  $\arccos \theta \cdot e$  which maps  $e$  to  $\theta$  and keeps all vectors perpendicular to  $\theta$  and  $e$  fixed. We then extend the map  $\theta \mapsto R_\theta$  to all of  $S$  as an even map with respect to reflection at the hyperplane  $\{ \theta \in \mathbb{R}^N : \theta \cdot e = 0 \}$ .

By construction, it is clear that

$$|R_\theta \sigma - \theta| = |\sigma - e| \quad \text{for all } \theta \in S_e \text{ and } \sigma \in S. \tag{4.25}$$

Moreover, the Lipschitz property of the map  $\theta \mapsto R_\theta$  implies that there is a constant  $C > 0$  with

$$\|R_{\theta_1} - R_{\theta_2}\| \leq C|\theta_1 - \theta_2| \quad \text{for all } \theta_1, \theta_2 \in S, \tag{4.26}$$

where, here and in the following,  $\|\cdot\|$  denotes the usual operator norm with respect to the Euclidean norm on  $\mathbb{R}^N$ .

Thanks to (4.25), a change of variable gives

$$\begin{aligned} \tilde{h}(\psi)(\theta) &= \int_S \frac{\Lambda_2(\psi, \psi, R_\theta \sigma, \theta)}{|e - \sigma|^{N+\alpha}} \psi^{N-2}(R_\theta \sigma) \bar{\mathcal{K}}_\alpha(\psi, R_\theta \sigma, \theta) dV(\sigma) \\ &\quad - \psi(\theta) \int_S \frac{\Lambda_1(\psi, R_\theta \sigma, \theta)}{|e - \sigma|^{N+\alpha}} \psi^{N-2}(R_\theta \sigma) \bar{\mathcal{K}}_\alpha(\psi, R_\theta \sigma, \theta) dV(\sigma) \\ &\quad + \frac{\psi(\theta)}{2} \int_S \frac{\psi^{N-1}(R_\theta \sigma)}{|e - \sigma|^{N+\alpha-2}} \bar{\mathcal{K}}_\alpha(\psi, R_\theta \sigma, \theta) dV(\sigma) \quad \text{for } \theta \in S_e. \end{aligned} \tag{4.27}$$

In the following, for a function  $f : S \rightarrow \mathbb{R}$ , we use the notation

$$[f; \theta_1, \theta_2] := f(\theta_1) - f(\theta_2) \quad \text{for } \theta_1, \theta_2 \in S,$$

and we note the obvious equality

$$[fg; \theta_1, \theta_2] = [f; \theta_1, \theta_2]g(\theta_1) + f(\theta_2)[g; \theta_1, \theta_2] \quad \text{for } f, g : S \rightarrow \mathbb{R}, \theta_1, \theta_2 \in S. \tag{4.28}$$

In the next results we collect helpful estimates for the functionals  $\Lambda_1$  and  $\Lambda_2$ .

**Lemma 4.5** *There exists a constant  $C > 0$  depending only on  $N$  and  $\beta$  such that for all  $\sigma, \sigma_1, \sigma_2, \theta, \theta_1, \theta_2 \in S$  and  $\psi \in C^{1,\beta}(S)$  we have*

$$|\Lambda_1(\psi, \sigma, \theta)| \leq C\|\psi\|_{C^{1,\beta}(S)}|\sigma - \theta|^{1+\beta} \tag{4.29}$$

and

$$\begin{aligned} |\Lambda_1(\psi, \sigma_1, \theta_1) - \Lambda_1(\psi, \sigma_2, \theta_2)| &\leq C\|\psi\|_{C^{1,\beta}(S)}|\theta_1 - \sigma_1|(|\theta_1 - \theta_2|^\beta + |\sigma_1 - \sigma_2|^\beta) \\ &\quad + C\|\psi\|_{C^{1,\beta}(S)}|\theta_2 - \sigma_2|^\beta(|\theta_1 - \theta_2| + |\sigma_1 - \sigma_2|). \end{aligned} \tag{4.30}$$

*Proof* To derive the estimates in the lemma, we may assume that

$$\max\{|\sigma - \theta|, |\theta_1 - \sigma_1|, |\theta_2 - \sigma_2|\} < 1 \tag{4.31}$$

(otherwise the estimates are easy to prove). Having (4.31) is essential for applying Lemma 4.2 in the sequel. We have

$$\begin{aligned} \Lambda_1(\psi, \sigma, \theta) &= \psi(\theta) - \psi(\sigma) - (\theta - \sigma) \cdot \nabla \psi(\sigma) \\ &= \int_0^1 \{ \nabla \psi(\gamma_{\theta, \sigma}(t)) - \nabla \psi(\gamma_{\theta, \sigma}(0)) \} \cdot \dot{\gamma}_{\theta, \sigma}(t) dt, \end{aligned}$$

where  $\gamma_{\theta, \sigma}$  is defined in (4.8). Therefore (4.29) follows from (4.9).

We now prove (4.30). We have

$$\begin{aligned} &\Lambda_1(\psi, \sigma_1, \theta_1) - \Lambda_1(\psi, \sigma_2, \theta_2) \\ &= \int_0^1 \{ \nabla \psi(\gamma_{\theta_1, \sigma_1}(t)) - \nabla \psi(\gamma_{\theta_1, \sigma_1}(0)) \} \cdot \dot{\gamma}_{\theta_1, \sigma_1}(t) dt \\ &\quad - \int_0^1 \{ \nabla \psi(\gamma_{\theta_2, \sigma_2}(t)) - \nabla \psi(\gamma_{\theta_2, \sigma_2}(0)) \} \cdot \dot{\gamma}_{\theta_2, \sigma_2}(t) dt \\ &= \int_0^1 \{ \nabla \psi(\gamma_{\theta_1, \sigma_1}(t)) - \nabla \psi(\gamma_{\theta_2, \sigma_2}(t)) + \nabla \psi(\sigma_2) - \nabla \psi(\sigma_1) \} \cdot \dot{\gamma}_{\theta_1, \sigma_1}(t) dt \\ &\quad + \int_0^1 \{ \nabla \psi(\gamma_{\theta_2, \sigma_2}(t)) - \nabla \psi(\gamma_{\theta_2, \sigma_2}(0)) \} \cdot (\dot{\gamma}_{\theta_1, \sigma_1}(t) - \dot{\gamma}_{\theta_2, \sigma_2}(t)) dt. \end{aligned}$$

This implies that

$$\begin{aligned} &|\Lambda_1(\psi, \sigma_1, \theta_1) - \Lambda_1(\psi, \sigma_2, \theta_2)| \\ &\leq \|\psi\|_{C^{1, \beta}(S)} \int_0^1 |\gamma_{\theta_1, \sigma_1}(t) - \gamma_{\theta_2, \sigma_2}(t)|^\beta |\dot{\gamma}_{\theta_1, \sigma_1}(t)| dt \\ &\quad + \|\psi\|_{C^{1, \beta}(S)} |\sigma_1 - \sigma_2|^\beta \int_0^1 |\dot{\gamma}_{\theta_1, \sigma_1}(t)| dt \\ &\quad + \|\psi\|_{C^{1, \beta}(S)} \int_0^1 |\gamma_{\theta_2, \sigma_2}(t) - \gamma_{\theta_2, \sigma_2}(0)|^\beta |\dot{\gamma}_{\theta_1, \sigma_1}(t) - \dot{\gamma}_{\theta_2, \sigma_2}(t)| dt \\ &\leq \|\psi\|_{C^{1, \beta}(S)} \int_0^1 |\gamma_{\theta_1, \sigma_1}(t) - \gamma_{\theta_2, \sigma_2}(t)|^\beta |\dot{\gamma}_{\theta_1, \sigma_1}(t)| dt \\ &\quad + \|\psi\|_{C^{1, \beta}(S)} |\sigma_1 - \sigma_2|^\beta \int_0^1 |\dot{\gamma}_{\theta_1, \sigma_1}(t)| dt \\ &\quad + \|\psi\|_{C^{1, \beta}(S)} \int_0^1 \left| \int_0^1 t \dot{\gamma}_{\theta_2, \sigma_2}(rt) dr \right|^\beta |\dot{\gamma}_{\theta_1, \sigma_1}(t) - \dot{\gamma}_{\theta_2, \sigma_2}(t)| dt. \end{aligned}$$

Thanks to (4.9), we get

$$\begin{aligned} & |\Lambda_1(\psi, \sigma_1, \theta_1) - \Lambda_1(\psi, \sigma_2, \theta_2)| \\ & \leq C \|\psi\|_{C^{1,\beta}(S)} |\sigma_1 - \theta_1| \left( \int_0^1 |\gamma_{\theta_1, \sigma_1}(t) - \gamma_{\theta_2, \sigma_2}(t)|^\beta dt + |\sigma_1 - \sigma_2|^\beta \right) \\ & \quad + \|\psi\|_{C^{1,\beta}(S)} \int_0^1 \left| \int_0^1 t \dot{\gamma}_{\theta_2, \sigma_2}(rt) dr \right|^\beta |\dot{\gamma}_{\theta_1, \sigma_1}(t) - \dot{\gamma}_{\theta_2, \sigma_2}(t)| dt. \end{aligned}$$

From Lemma 4.2, we have that, for every  $\theta_1, \sigma_1, \theta_2, \sigma_2 \in S$  and satisfying (4.31),

$$|\gamma_{\theta_1, \sigma_1}(t) - \gamma_{\theta_2, \sigma_2}(t)| + |\dot{\gamma}_{\theta_1, \sigma_1}(t) - \dot{\gamma}_{\theta_2, \sigma_2}(t)| \leq C(|\theta_1 - \theta_2| + |\sigma_1 - \sigma_2|)$$

for every  $t \in [0, 1]$

and

$$|\dot{\gamma}_{\theta_2, \sigma_2}(rt)| \leq C|\theta_2 - \sigma_2| \quad \text{for every } t, r \in [0, 1].$$

Therefore

$$\begin{aligned} |\Lambda_1(\psi, \sigma_1, \theta_1) - \Lambda_1(\psi, \sigma_2, \theta_2)| & \leq C \|\psi\|_{C^{1,\beta}(S)} |\theta_1 - \sigma_1| (|\theta_1 - \theta_2|^\beta + |\sigma_1 - \sigma_2|^\beta) \\ & \quad + C \|\psi\|_{C^{1,\beta}(S)} |\theta_2 - \sigma_2|^\beta (|\theta_1 - \theta_2| + |\sigma_1 - \sigma_2|). \end{aligned}$$

This ends the proof of (4.30). □

**Corollary 4.6** *There exists a constant  $C > 0$ , depending only on  $N$  and  $\beta$ , such that for all  $e \in S, \sigma \in S$ , all  $\theta, \theta_1, \theta_2 \in S_e$  and all  $\psi \in C^{1,\beta}(S)$  we have*

$$|\Lambda_1(\psi, R_\theta \sigma, \theta)| \leq C \|\psi\|_{C^{1,\beta}(S)} |e - \sigma|^{1+\beta} \tag{4.32}$$

and

$$\begin{aligned} & |\Lambda_1(\psi, R_{\theta_1} \sigma, \theta_1) - \Lambda_1(\psi, R_{\theta_2} \sigma, \theta_2)| \\ & \leq C \|\psi\|_{C^{1,\beta}(S)} (|e - \sigma| |\theta_1 - \theta_2|^\beta + |e - \sigma|^\beta |\theta_1 - \theta_2|). \end{aligned} \tag{4.33}$$

*Proof* It suffices to apply (4.29) and (4.30) with  $\sigma, \sigma_1$  and  $\sigma_2$  replaced by  $R_\theta \sigma, R_{\theta_1} \sigma$  and  $R_{\theta_2} \sigma$  respectively, to use (4.25), and the fact that

$$|R_{\theta_1} \sigma - R_{\theta_2} \sigma| \leq \|R_{\theta_1} - R_{\theta_2}\| \leq C|\theta_1 - \theta_2|.$$

□

Next, we derive estimates for  $\Lambda_2$ .

**Lemma 4.7** *There exists a constant  $C > 0$ , depending only on  $N$  and  $\beta$ , such that for all  $e \in S, \sigma \in S$ , all  $\theta, \theta_1, \theta_2 \in S_e$  and all  $\psi_1, \psi_2 \in C^{1,\beta}(S)$  we have*

$$|\Lambda_2(\psi_1, \psi_2, R_\theta\sigma, \theta)| \leq C \|\psi_1\|_{C^{1,\beta}(S)} \|\psi_2\|_{C^{1,\beta}(S)} |e - \sigma|^2 \tag{4.34}$$

and

$$\begin{aligned} & |\Lambda_2(\psi_1, \psi_2, R_{\theta_1}\sigma, \theta_1) - \Lambda_2(\psi_1, \psi_2, R_{\theta_2}\sigma, \theta_2)| \\ & \leq C \|\psi_1\|_{C^{1,\beta}(S)} \|\psi_2\|_{C^{1,\beta}(S)} |e - \sigma|^2 |\theta_1 - \theta_2|^\beta. \end{aligned} \tag{4.35}$$

*Proof* To prove the lemma, by (4.25) and (4.26) it is easy to see that we may assume  $|e - \sigma| < 1$ . This implies that

$$|R_\theta\sigma - \theta| = |R_{\theta_1}\sigma - \theta_1| = |R_{\theta_2}\sigma - \theta_2| = |\sigma - e| < 1,$$

and therefore allows us to apply Lemma 4.2 in the following. By (4.9) we have

$$\begin{aligned} |\psi(\theta) - \psi(R_\theta\sigma)| &= \left| \int_0^1 \nabla\psi(\gamma_{\theta, R_\theta\sigma}(t)) \cdot \dot{\gamma}_{\theta, R_\theta\sigma}(t) dt \right| \\ &\leq C \|\psi\|_{C^{1,\beta}(S)} |\theta - R_\theta\sigma| = C \|\psi\|_{C^{1,\beta}(S)} |e - \sigma| \quad \text{for } \psi \in C^{1,\beta}(S) \end{aligned} \tag{4.36}$$

and

$$\begin{aligned} |\Lambda_2(\psi_1, \psi_2, R_\theta\sigma, \theta)| &= |(\psi_1(\theta) - \psi_1(R_\theta\sigma))| |(\psi_2(\theta) - \psi_2(R_\theta\sigma))| \\ &\leq C \|\psi_1\|_{C^{1,\beta}(S)} \|\psi_2\|_{C^{1,\beta}(S)} |e - \sigma|^2, \end{aligned}$$

as claimed in (4.34).

Next, we note that, by (4.14) we have

$$\begin{aligned} \dot{\gamma}_{\theta, R_\theta\sigma}(t) &= \frac{\theta - R_\theta\sigma}{\Upsilon(t, \theta, R_\theta\sigma)} + \frac{(1 - 2t)|\theta - R_\theta\sigma|^2}{2} \frac{R_\theta\sigma + t(\theta - R_\theta\sigma)}{\Upsilon(t, \theta, R_\theta\sigma)^3} \\ &= R_\theta \left\{ \frac{(e - \sigma)}{\Upsilon(t, e, \sigma)} + \frac{(1 - 2t)|e - \sigma|^2}{2} \frac{\sigma + t(e - \sigma)}{\Upsilon(t, e, \sigma)^3} \right\} = R_\theta \dot{\gamma}_{e, \sigma}(t), \end{aligned}$$

since  $|\theta - R_\theta\sigma| = |R_\theta(e - \sigma)| = |e - \sigma|$  and, by (4.12),

$$\begin{aligned} \Upsilon(t, \theta, R_\theta\sigma) &= |R_\theta\sigma + t(\theta - R_\theta\sigma)| \\ &= |R_\theta(\sigma + t(e - \sigma))| = |\sigma + t(e - \sigma)| = \Upsilon(t, e, \sigma). \end{aligned}$$

Consequently,

$$\begin{aligned} & \left| \psi(\theta_1) - \psi(R_{\theta_1}\sigma) - (\psi(\theta_2) - \psi(R_{\theta_2}\sigma)) \right| \\ &= \left| \int_0^1 \left\{ \nabla\psi(\gamma_{\theta_1, R_{\theta_1}\sigma}(t)) \cdot \dot{\gamma}_{\theta_1, R_{\theta_1}\sigma}(t) - \nabla\psi(\gamma_{\theta_2, R_{\theta_2}\sigma}(t)) \cdot \dot{\gamma}_{\theta_2, R_{\theta_2}\sigma}(t) \right\} dt \right| \\ &= \left| \int_0^1 \left\{ \nabla\psi(\gamma_{\theta_1, R_{\theta_1}\sigma}(t)) \cdot R_{\theta_1} \dot{\gamma}_{e, \sigma}(t) - \nabla\psi(\gamma_{\theta_2, R_{\theta_2}\sigma}(t)) \cdot R_{\theta_2} \dot{\gamma}_{e, \sigma}(t) \right\} dt \right| \end{aligned}$$

$$\begin{aligned}
 &= \left| \int_0^1 \left( \nabla \psi(\gamma_{\theta_1, R_{\theta_1} \sigma}(t)) - \nabla \psi(\gamma_{\theta_2, R_{\theta_2} \sigma}(t)) \right) \cdot R_{\theta_1} \dot{\gamma}_{e, \sigma}(t) \right. \\
 &\quad \left. + \nabla \psi(\gamma_{\theta_2, R_{\theta_2} \sigma}(t)) \cdot \left( R_{\theta_1} \dot{\gamma}_{e, \sigma}(t) - R_{\theta_2} \dot{\gamma}_{e, \sigma}(t) \right) dt \right| \\
 &\leq \| \psi \|_{C^{1, \beta}(S)} \int_0^1 \left( |\gamma_{\theta_1, R_{\theta_1} \sigma}(t) - \gamma_{\theta_2, R_{\theta_2} \sigma}(t)|^\beta |\dot{\gamma}_{e, \sigma}(t)| + |(R_{\theta_1} - R_{\theta_2}) \dot{\gamma}_{e, \sigma}(t)| \right) dt.
 \end{aligned}$$

Hence Lemma 4.2 and the Lipschitz continuity of the map  $\theta \mapsto R_\theta$  give rise to the estimate

$$\begin{aligned}
 &| \psi(\theta_1) - \psi(R_{\theta_1} \sigma) - (\psi(\theta_2) - \psi(R_{\theta_2} \sigma)) | \\
 &\leq C \| \psi \|_{C^{1, \beta}(S)} |e - \sigma| \left( \int_0^1 |\gamma_{\theta_1, R_{\theta_1} \sigma}(t) - \gamma_{\theta_2, R_{\theta_2} \sigma}(t)|^\beta dt + \| R_{\theta_1} - R_{\theta_2} \| \right) \\
 &\leq C \| \psi \|_{C^{1, \beta}(S)} |e - \sigma| \left( \int_0^1 (|\theta_1 - \theta_2| + |R_{\theta_1} \sigma - R_{\theta_2} \sigma|)^\beta dt + |\theta_1 - \theta_2| \right) \\
 &\leq C \| \psi \|_{C^{1, \beta}(S)} |e - \sigma| |\theta_1 - \theta_2|^\beta. \tag{4.37}
 \end{aligned}$$

Using (4.36) and (4.37), applied with  $\psi$  replaced by  $\psi_1$  and  $\psi_2$ , we then find that

$$\begin{aligned}
 &| \Lambda_2(\psi_1, \psi_2, R_{\theta_1} \sigma, \theta_1) - \Lambda_2(\psi_1, \psi_2, R_{\theta_2} \sigma, \theta_2) | \\
 &= | (\psi_1(\theta_1) - \psi_1(R_{\theta_1} \sigma)) (\psi_2(\theta_1) - \psi_2(R_{\theta_1} \sigma)) \\
 &\quad - (\psi_1(\theta_2) - \psi_1(R_{\theta_2} \sigma)) (\psi_2(\theta_2) - \psi_2(R_{\theta_2} \sigma)) | \\
 &\leq \left| (\psi_1(\theta_1) - \psi_1(R_{\theta_1} \sigma) - (\psi_1(\theta_2) - \psi_1(R_{\theta_2} \sigma))) (\psi_2(\theta_1) - \psi_2(R_{\theta_1} \sigma)) \right| \\
 &\quad + \left| (\psi_1(\theta_2) - \psi_1(R_{\theta_2} \sigma)) (\psi_2(\theta_1) - \psi_2(R_{\theta_1} \sigma) - (\psi_2(\theta_2) - \psi_2(R_{\theta_2} \sigma))) \right| \\
 &\leq \| \psi_1 \|_{C^{1, \beta}(S)} \| \psi_2 \|_{C^{1, \beta}(S)} |e - \sigma|^2 |\theta_1 - \theta_2|^\beta,
 \end{aligned}$$

as claimed in (4.35). □

The following result provides some estimates related to the kernel  $\overline{\mathcal{K}}_\alpha$  and its derivatives.

**Lemma 4.8** *Let  $r > 0, k \in \mathbb{N} \cup \{0\}$ . Then there exists a constant  $c = c(N, \alpha, \beta, r, k) > 1$  such that for all  $e \in S, \sigma \in S$ , all  $\theta, \theta_1, \theta_2 \in S_e$  and  $\psi \in \mathcal{B}(r)$ , we have*

$$\| D_\psi^k \overline{\mathcal{K}}_\alpha(\psi, R_\theta \sigma, \theta) \| \leq c (1 + \| \psi \|_{C^{1, \beta}(S)})^c \tag{4.38}$$

and

$$\| D_\psi^k \overline{\mathcal{K}}_\alpha(\psi, R_{\theta_1} \sigma, \theta_1) - D_\psi^k \overline{\mathcal{K}}_\alpha(\psi, R_{\theta_2} \sigma, \theta_2) \| \leq c (1 + \| \psi \|_{C^{1, \beta}(S)})^c |\theta_1 - \theta_2|^\beta. \tag{4.39}$$

*Proof* Throughout this proof, the letter  $c$  stands for different constants greater than one and depending only on  $N, \alpha, \beta, k$  and  $r$ . We assume  $k \geq 1$  for notation coherence, but the case  $k = 0$  is simpler and can be proved similarly. We define

$$\begin{aligned} Q : C^{1,\beta}(S) \times S \times S &\rightarrow \mathbb{R}, \quad Q(\psi, \sigma, \theta) = \frac{|\psi(\theta) - \psi(\sigma)|^2}{|\theta - \sigma|^2} + \psi(\theta)\psi(\sigma) \\ &= \frac{\Lambda_2(\psi, \psi, \sigma, \theta)}{|\theta - \sigma|^2} + \psi(\theta)\psi(\sigma) \end{aligned}$$

and, for  $\alpha > 0$ ,

$$g_\alpha \in C^\infty(\mathbb{R}_+, \mathbb{R}), \quad g_\alpha(x) = x^{-(N+\alpha)/2},$$

so that

$$\bar{K}_\alpha(\psi, R_\theta\sigma, \theta) = g_\alpha(Q(\psi, R_\theta\sigma, \theta)). \tag{4.40}$$

Note that for  $\theta \in S_e$  we have

$$\begin{aligned} Q(\psi, R_\theta\sigma, \theta) &= \frac{\Lambda_2(\psi, \psi, R_\theta\sigma, \theta)}{|e - \sigma|^2} + \psi(R_\theta\sigma)\psi(\theta) \\ D_\psi Q(\psi, R_\theta\sigma, \theta)\psi_1 &= 2 \frac{\Lambda_2(\psi, \psi_1, R_\theta\sigma, \theta)}{|e - \sigma|^2} + \psi(\theta)\psi_1(R_\theta\sigma) + \psi_1(\theta)\psi(R_\theta\sigma) \\ D_\psi^2 Q(\psi, R_\theta\sigma, \theta)[\psi_1, \psi_2] &= 2 \frac{\Lambda_2(\psi_1, \psi_2, R_\theta\sigma, \theta)}{|e - \sigma|^2} + \psi_2(\theta)\psi_1(R_\theta\sigma) \\ &\quad + \psi_1(\theta)\psi_2(R_\theta\sigma) \end{aligned}$$

for  $\psi, \psi_1, \psi_2 \in C^{1,\beta}(S)$ . For a subset  $P \subset \{1, 2\}$  and  $\psi_1, \psi_2 \in C^{1,\beta}(S)$ , we thus have, by (4.28) and (4.35),

$$| [D_\psi^{|P|} Q(\psi, R_\theta\sigma, \cdot)] [\psi_j]_{j \in P}; \theta_1, \theta_2 | \leq c(1 + \|\psi\|_{C^{1,\beta}(S)}^2) |\theta_1 - \theta_2|^\beta \prod_{j \in P} \|\psi_j\|_{C^{1,\beta}(S)}. \tag{4.41}$$

Moreover, by (4.34),

$$| D_\psi^{|P|} Q(\psi, R_\theta\sigma, \cdot) [\psi_j]_{j \in P} | \leq c(1 + \|\psi\|_{C^{1,\beta}(S)}^2) \prod_{j \in P} \|\psi_j\|_{C^{1,\beta}(S)} \quad \text{on } S_e. \tag{4.42}$$



By (4.7) and recalling that  $Q$  is quadratic in  $\psi$ , we have

$$D_{\psi}^k \bar{\mathcal{K}}_{\alpha}(\psi, R_{\theta}\sigma, \theta)[\psi_1, \dots, \psi_k] = \sum_{\Pi \in \mathcal{P}_k^2} g_{\alpha}^{(|\Pi|)}(Q(\psi, R_{\theta}\sigma, \theta)) \prod_{P \in \Pi} D_{\psi}^{|P|} Q(\psi, R_{\theta}\sigma, \theta)[\psi_j]_{j \in P},$$

where  $\mathcal{P}_k^2$  denotes the set of partitions  $\Pi$  of  $\{1, \dots, k\}$  such that  $|P| \leq 2$  for every  $P \in \Pi$ . By (4.28), we now have

$$\begin{aligned} & \left[ D_{\psi}^k \bar{\mathcal{K}}_{\alpha}(\psi, R_{\theta}\sigma, \cdot)[\psi_1, \dots, \psi_k]; \theta_1, \theta_2 \right] \\ &= \sum_{\Pi \in \mathcal{P}_k^2} \left[ g_{\alpha}^{(|\Pi|)}(Q(\psi, R_{\theta}\sigma, \cdot)); \theta_1, \theta_2 \right] \prod_{P \in \Pi, |P| \leq 2} D_{\psi}^{|P|} Q(\psi, R_{\theta}\sigma, \theta_1)[\psi_j]_{j \in P} \\ &+ \sum_{\Pi \in \mathcal{P}_k^2} g_{\alpha}^{(|\Pi|)}(Q(\psi, R_{\theta}\sigma, \theta_2)) \left[ \prod_{P \in \Pi, |P| \leq 2} D_{\psi}^{|P|} Q(\psi, R_{\theta}\sigma, \cdot)[\psi_j]_{j \in P}; \theta_1, \theta_2 \right], \end{aligned} \tag{4.43}$$

whereas

$$g_{\alpha}^{(k)}(t) = (-1)^k 2^{-k} \prod_{i=0}^{k-1} (N + \alpha + 2i) t^{-\frac{N+\alpha+2k}{2}} \quad \text{for } k \in \mathbb{N} \text{ and } t > 0.$$

Consequently, by (4.41) and since  $\psi \in \mathcal{B}(r)$ , we have the estimates

$$\begin{aligned} & \left| \left[ g_{\alpha}^{(\ell)}(Q(\psi, R_{\theta}\sigma, \cdot)); \theta_1, \theta_2 \right] \right| \\ & \leq \left| \left[ Q(\psi, R_{\theta}\sigma, \cdot); \theta_1, \theta_2 \right] \int_0^1 g_{\alpha}^{(\ell+1)}(\tau Q(\psi, R_{\theta}\sigma, \theta_1) + (1-\tau)Q(\psi, R_{\theta}\sigma, \theta_2)) d\tau \right| \\ & \leq c \left( 1 + \|\psi\|_{C^{1,\beta}(S)}^2 \right) |\theta_1 - \theta_2|^{\beta} \end{aligned} \tag{4.44}$$

and

$$|g_{\alpha}^{(\ell)}(Q(\psi, R_{\theta}\sigma, \cdot))| \leq c \left( 1 + \|\psi\|_{C^{1,\beta}(S)} \right)^c \tag{4.45}$$

for  $\ell = 0, \dots, k$ . Combining (4.41), (4.42), (4.43), (4.44) and (4.45), we obtain

$$\begin{aligned} & \left| \left[ D_{\psi}^k \bar{\mathcal{K}}_{\alpha}(\psi, R_{\theta}\sigma, \cdot)[\psi_1, \dots, \psi_k]; \theta_1, \theta_2 \right] \right| \\ & \leq c \left( 1 + \|\psi\|_{C^{1,\beta}(S)} \right)^c |\theta_1 - \theta_2|^{\beta} \sum_{\Pi \in \mathcal{P}_k^2} \prod_{P \in \Pi} \prod_{j \in P} \|\psi_j\|_{C^{1,\beta}(S)} \\ & \leq c \left( 1 + \|\psi\|_{C^{1,\beta}(S)} \right)^c |\theta_1 - \theta_2|^{\beta} \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)}. \end{aligned}$$

This yields (4.39). Furthermore we easily deduce from (4.42) and (4.45) that

$$\left| D_{\psi}^k \bar{\mathcal{K}}_{\alpha}(\psi, R_{\theta}\sigma, \theta)[\psi_1, \dots, \psi_k] \right| \leq c (1 + \|\psi\|_{C^{1,\beta}(S)})^c \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)},$$

completing the proof. □

We now derive estimates for functions of a specific form which will appear in formulas for the derivatives of the transformed NMC operator  $\tilde{h}$  in (4.27).

**Lemma 4.9** *Let  $\psi \in \mathcal{B}(r)$ , with  $r > 0$ . Let  $k \in \mathbb{N}$ ,  $e \in S$ ,  $\Psi \in C^{1,\beta}(S)$  and  $\omega, \omega_1, \psi_1, \dots, \psi_k \in C^{1,\beta}(S)$ . Define the functions  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 : S_e \rightarrow \mathbb{R}$  by*

$$\begin{aligned} \mathcal{F}_1(\theta) &= \int_S \frac{\Lambda_1(\omega, R_{\theta}\sigma, \theta)}{|e - \sigma|^{N+\alpha}} \Psi(R_{\theta}\sigma) D_{\psi}^k \bar{\mathcal{K}}_{\alpha}(\psi, R_{\theta}\sigma, \theta)[\psi_i]_{i=1,\dots,k} dV(\sigma), \\ \mathcal{F}_2(\theta) &= \int_S \frac{\Lambda_2(\omega, \omega_1, R_{\theta}\sigma, \theta)}{|e - \sigma|^{N+\alpha}} \Psi(R_{\theta}\sigma) D_{\psi}^k \bar{\mathcal{K}}_{\alpha}(\psi, R_{\theta}\sigma, \theta)[\psi_i]_{i=1,\dots,k} dV(\sigma) \end{aligned}$$

and

$$\mathcal{F}_3(\theta) = \int_S \frac{\Psi(R_{\theta}\sigma)}{|e - \sigma|^{N+\alpha-2}} D_{\psi}^k \bar{\mathcal{K}}_{\alpha}(\psi, R_{\theta}\sigma, \theta)[\psi_i]_{i=1,\dots,k} dV(\sigma). \tag{4.46}$$

Then, there exists a constant  $c = c(N, \alpha, \beta, k, r) > 1$  such that

$$\|\mathcal{F}_1\|_{C^{\beta-\alpha}(S_e)} \leq c (1 + \|\psi\|_{C^{1,\beta}(S)})^c \|\omega\|_{C^{1,\beta}(S)} \|\Psi\|_{C^{1,\beta}(S)} \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)}, \tag{4.47}$$

$$\|\mathcal{F}_2\|_{C^{\beta}(S_e)} \leq c (1 + \|\psi\|_{C^{1,\beta}(S)})^c \|\omega\|_{C^{1,\beta}(S)} \|\omega_1\|_{C^{1,\beta}(S)} \|\Psi\|_{C^{1,\beta}(S)} \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)} \tag{4.48}$$

and

$$\|\mathcal{F}_3\|_{C^{\beta}(S_e)} \leq c (1 + \|\psi\|_{C^{1,\beta}(S)})^c \|\Psi\|_{C^{1,\beta}(S)} \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)}. \tag{4.49}$$

*Proof* Let  $\theta_1, \theta_2 \in S_e$ . We first note that, for  $\sigma \in S$ ,

$$|\Psi(R_{\theta_1}\sigma) - \Psi(R_{\theta_2}\sigma)| \leq C \|\Psi\|_{C^{1,\beta}(S)} |R_{\theta_1}\sigma - R_{\theta_2}\sigma| \leq C \|\Psi\|_{C^{1,\beta}(S)} |\theta_1 - \theta_2|. \tag{4.50}$$

To prove estimate (4.47) for  $\mathcal{F}_1$ , we recall that by (4.33) we have

$$|\Lambda_1(\omega, R_{\theta_1}\sigma, \theta_1) - \Lambda_1(\omega, R_{\theta_2}\sigma, \theta_2)| \leq C \|\omega\|_{C^{1,\beta}(S)} \mu(\sigma, \theta_1, \theta_2),$$

where

$$\mu(\sigma, \theta_1, \theta_2) := |e - \sigma||\theta_1 - \theta_2|^\beta + |e - \sigma|^\beta|\theta_1 - \theta_2|.$$

Combining this with the fact that

$$\begin{aligned} |\Lambda_1(\omega, R_\theta\sigma, \theta)| &\leq C\|\omega\|_{C^{1,\beta}(S)} |R_\theta\sigma - \theta|^{1+\beta} \\ &= C\|\omega\|_{C^{1,\beta}(S)} |e - \sigma|^{1+\beta} \quad \text{for } \sigma \in S, \theta \in S_e \end{aligned}$$

by (4.32), we find that

$$|\Lambda_1(\omega, R_{\theta_1}\sigma, \theta_1) - \Lambda_1(\omega, R_{\theta_2}\sigma, \theta_2)| \leq C\|\omega\|_{C^{1,\beta}(S)} \min(|e - \sigma|^{1+\beta}, \mu(\sigma, \theta_1, \theta_2)). \tag{4.51}$$

Using inductively (4.28) together with Lemma 4.8, (4.50) and (4.51), we get the estimate

$$\begin{aligned} |[\mathcal{F}_1; \theta_1, \theta_2]| &\leq c(1 + \|\psi\|_{C^{1,\beta}(S)})^c \|\omega\|_{C^{1,\beta}(S)} \|\Psi\|_{C^{1,\beta}(S)} \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)} \\ &\quad \times \left( |\theta_1 - \theta_2|^\beta \int_S |e - \sigma|^{1+\beta-N-\alpha} dV(\sigma) + \int_S \frac{\min(|e - \sigma|^{1+\beta}, \mu(\sigma, \theta_1, \theta_2))}{|e - \sigma|^{N+\alpha}} dV(\sigma) \right) \end{aligned} \tag{4.52}$$

for all  $\theta_1, \theta_2 \in S$ . Since

$$\begin{aligned} &\int_S \frac{\min(|e - \sigma|^{1+\beta}, \mu(\sigma, \theta_1, \theta_2))}{|e - \sigma|^{N+\alpha}} dV(\sigma) \\ &\leq \int_{|e-\sigma|\leq|\theta_1-\theta_2|} |e - \sigma|^{1+\beta-N-\alpha} dV(\sigma) + \int_{|\theta_1-\theta_2|\leq|e-\sigma|} \frac{\mu(\sigma, \theta_1, \theta_2)}{|e - \sigma|^{N+\alpha}} dV(\sigma) \\ &\leq \int_{|e-\sigma|\leq|\theta_1-\theta_2|} |e - \sigma|^{1+\beta-N-\alpha} dV(\sigma) \\ &\quad + \int_{|\theta_1-\theta_2|\leq|e-\sigma|} \{|\theta_1 - \theta_2|^\beta |e - \sigma|^{1-N-\alpha} + |\theta_1 - \theta_2| |e - \sigma|^{\beta-N-\alpha}\} dV(\sigma) \\ &\leq C|\theta_1 - \theta_2|^{\beta-\alpha}, \end{aligned}$$

we thus deduce from (4.52) that

$$|[\mathcal{F}_1; \theta_1, \theta_2]| \leq c|\theta_1 - \theta_2|^{\beta-\alpha} (1 + \|\psi\|_{C^{1,\beta}(S)})^c \|\omega\|_{C^{1,\beta}(S)} \|\Psi\|_{C^{1,\beta}(S)} \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)}.$$

Since, by a similar but easier argument, we have

$$|\mathcal{F}_1| \leq c \left(1 + \|\psi\|_{C^{1,\beta}(S)}\right)^c \|\omega\|_{C^{1,\beta}(S)} \|\Psi\|_{C^{1,\beta}(S)} \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)} \quad \text{on } S_e,$$

(4.47) follows.

Next we consider  $\mathcal{F}_2$ . For this we recall that

$$|\Lambda_2(\omega, \omega_1, R_\theta \sigma, \theta)| \leq C \|\omega\|_{C^{1,\beta}(S)} \|\omega_1\|_{C^{1,\beta}(S)} |e - \sigma|^2$$

by (4.34), and that

$$\begin{aligned} &|\Lambda_2(\omega, \omega_1, R_{\theta_1} \sigma, \theta_1) - \Lambda_2(\omega, \omega_1, R_{\theta_2} \sigma, \theta_2)| \\ &\leq C \|\omega_1\|_{C^{1,\beta}(S)} \|\omega\|_{C^{1,\beta}(S)} |e - \sigma|^2 |\theta_1 - \theta_2|^\beta \end{aligned}$$

by (4.35). Consequently, we find that

$$\begin{aligned} |[\mathcal{F}_2; \theta_1, \theta_2]| &\leq c \left(1 + \|\psi\|_{C^{1,\beta}(S)}\right)^c \|\omega\|_{C^{1,\beta}(S)} \|\omega_1\|_{C^{1,\beta}(S)} \\ &\quad \times \|\Psi\|_{C^{1,\beta}(S)} \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)} \\ &\quad \times |\theta_1 - \theta_2|^\beta \int_S |e - \sigma|^{2-N-\alpha} dV(\sigma) \leq c \left(1 + \|\psi\|_{C^{1,\beta}(S)}\right)^c \|\omega\|_{C^{1,\beta}(S)} \|\omega_1\|_{C^{1,\beta}(S)} \|\Psi\|_{C^{1,\beta}(S)} |\theta_1 - \theta_2|^\beta \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)}. \end{aligned} \quad (4.53)$$

Moreover, by a similar but easier argument,

$$|\mathcal{F}_2| \leq c \left(1 + \|\psi\|_{C^{1,\beta}(S)}\right)^c \|\omega\|_{C^{1,\beta}(S)} \|\omega_1\|_{C^{1,\beta}(S)} \|\Psi\|_{C^{1,\beta}(S)} \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)} \quad \text{on } S_e. \quad (4.54)$$

Combining (4.53) and (4.54), we get (4.48), as claimed. We skip the proof of (4.49), which is similar but easier.  $\square$

The following result contains all what is needed to prove the regularity of  $\tilde{h}$  and hence of  $h$ .

**Proposition 4.10** *Let  $k \in \mathbb{N} \cup \{0\}$ ,  $r > 0$ ,  $\psi_1, \dots, \psi_k \in C^{1,\beta}(S)$  and let  $\mathcal{M}_i : S \rightarrow \mathbb{R}$  with  $i = 1, 2, 3$ , be defined by*

$$\mathcal{M}_i(\theta) = \int_S D_\psi^k M_i(\psi, \sigma, \theta)[\psi_1, \dots, \psi_k] dV(\sigma),$$

where  $M_1, M_2, M_3 : \mathcal{B}(r) \times S \times S \rightarrow \mathbb{R}$  are given by

$$\begin{aligned} M_1(\psi, \sigma, \theta) &= \frac{\Lambda_1(\psi, \sigma, \theta)}{|\theta - \sigma|^{N+\alpha}} \overline{\mathcal{K}}_\alpha(\psi, \sigma, \theta) \psi^{N-2}(\sigma), \\ M_2(\psi, \sigma, \theta) &= \frac{\Lambda_2(\psi, \psi, \sigma, \theta)}{|\theta - \sigma|^{N+\alpha}} \overline{\mathcal{K}}_\alpha(\psi, \sigma, \theta) \psi^{N-2}(\sigma), \\ M_3(\psi, \sigma, \theta) &= \frac{1}{|\theta - \sigma|^{N+\alpha-2}} \overline{\mathcal{K}}_\alpha(\psi, \sigma, \theta) \psi^{N-1}(\sigma). \end{aligned}$$

Then, for  $i = 1, 2, 3$ ,  $\mathcal{M}_i \in C^{\beta-\alpha}(S)$ , and there exists a constant  $c = c(N, \alpha, \beta, k, r) > 1$  such that

$$\|\mathcal{M}_i\|_{C^{\beta-\alpha}(S)} \leq c(1 + \|\psi\|_{C^{1,\beta}(S)})^c \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)}, \tag{4.55}$$

understanding that the last product equals 1 if  $k = 0$ .

*Proof* To prove (4.55), it suffices to fix  $e \in S$  and show that

$$|\mathcal{M}_i(\theta)| \leq c(1 + \|\psi\|_{C^{1,\beta}(S)})^c \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)} \tag{4.56}$$

and

$$|\mathcal{M}_i(\theta_1) - \mathcal{M}_i(\theta_2)| \leq c|\theta_1 - \theta_2|^{\beta-\alpha} (1 + \|\psi\|_{C^{1,\beta}(S)})^c \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)} \tag{4.57}$$

for  $\theta, \theta_1, \theta_2 \in S_e$ , where  $S_e$  is defined in (4.24) and  $c > 1$  does not depend on  $e$ . For this, we define a Lipschitz continuous map  $\theta \mapsto R_\theta$  of rotations as in Remark 4.4 corresponding to  $e$ , so that the inclusion in (4.24) holds. By a change of variable, we then have

$$\mathcal{M}_i(\theta) = \int_S D_\psi^k M_i(\psi, R_\theta \sigma, \theta) [\psi_1, \dots, \psi_k] dV(\sigma) \quad \text{for } \theta \in S_e.$$

We first consider the case  $i = 1$ , and we note that

$$M_1(\psi, R_\theta \sigma, \theta) = \frac{T_1(\psi, \sigma, \theta)}{|e - \sigma|^{N+\alpha}} \psi^{N-2}(R_\theta \sigma) \quad \text{for } \theta \in S_e,$$

where

$T_1 : \mathcal{B}(r) \times S \times S_e \rightarrow \mathbb{R}$  defined by  $T_1(\psi, \sigma, \theta) = \Lambda_1(\psi, R_\theta \sigma, \theta) \overline{\mathcal{K}}_\alpha(\psi, R_\theta \sigma, \theta)$ .

By (4.4), we thus have

$$D_\psi^k M_1(\psi, R_\theta \sigma, \theta)[\psi_1, \dots, \psi_k] = \frac{1}{|e - \sigma|^{N+\alpha}} \sum_{\mathcal{N} \in \mathcal{S}_k} \Psi^{\mathcal{N}}(R_\theta \sigma) D_\psi^{|\mathcal{N}|} T_1(\psi, \sigma, \theta)[\psi_i]_{i \in \mathcal{N}},$$

where  $\Psi^{\mathcal{N}} = \psi^{N-2}$  when  $k = |\mathcal{N}|$ , and

$$\Psi^{\mathcal{N}} := \prod_{\ell=0}^{k-|\mathcal{N}|-1} (N - 2 - \ell) \psi^{N-2-(k-|\mathcal{N}|)} \prod_{j \in \mathcal{N}^c} \psi_j \quad \text{when } k > |\mathcal{N}|$$

(noting that  $|\mathcal{N}^c| = k - |\mathcal{N}|$ ). By (4.5) we have, if  $|\mathcal{N}| \geq 1$ ,

$$D_\psi^{|\mathcal{N}|} T_1(\psi, \sigma, \theta)[\psi_i]_{i \in \mathcal{N}} = \Lambda_1(\psi, R_\theta \sigma, \theta) D_\psi^{|\mathcal{N}|} \bar{\mathcal{K}}_\alpha(\psi, R_\theta \sigma, \theta)[\psi_i]_{i \in \mathcal{N}} + \sum_{j \in \mathcal{N}} \Lambda_1(\psi_j, R_\theta \sigma, \theta) D_\psi^{|\mathcal{N}|-1} \bar{\mathcal{K}}_\alpha(\psi, R_\theta \sigma, \theta)[\psi_i]_{i \in \mathcal{N} \setminus \{j\}}.$$

Consequently,

$$D_\psi^k M_1(\psi, R_\theta \sigma, \theta)[\psi_1, \dots, \psi_k] = \sum_{\mathcal{N} \in \mathcal{S}_k} M_1^{\mathcal{N}}(\sigma, \theta), \tag{4.58}$$

where

$$M_1^{\mathcal{N}}(\sigma, \theta) = \frac{\Psi^{\mathcal{N}}(R_\theta \sigma)}{|e - \sigma|^{N+\alpha}} \left( \Lambda_1(\psi, R_\theta \sigma, \theta) D_\psi^{|\mathcal{N}|} \bar{\mathcal{K}}_\alpha(\psi, R_\theta \sigma, \theta)[\psi_i]_{i \in \mathcal{N}} + \sum_{j \in \mathcal{N}} \Lambda_1(\psi_j, R_\theta \sigma, \theta) D_\psi^{|\mathcal{N}|-1} \bar{\mathcal{K}}_\alpha(\psi, R_\theta \sigma, \theta)[\psi_i]_{i \in \mathcal{N} \setminus \{j\}} \right),$$

where the second summand does not appear if  $|\mathcal{N}| = 0$ . Clearly we also have that

$$\|\Psi^{\mathcal{N}}\|_{C^{1,\beta}(S)} \leq c(1 + \|\psi\|_{C^{1,\beta}(S)})^c \prod_{i \in \mathcal{N}^c} \|\psi_i\|_{C^{1,\beta}(S)}. \tag{4.59}$$

Denoting

$$\mathcal{M}_1^{\mathcal{N}} : S_e \rightarrow \mathbb{R}, \quad \mathcal{M}_1^{\mathcal{N}}(\theta) = \int_S M_1^{\mathcal{N}}(\sigma, \theta) dV(\sigma),$$

by Lemma 4.9 it follows that  $\mathcal{M}_1^{\mathcal{N}} \in C^{\beta-\alpha}(S_e)$  and that

$$\|\mathcal{M}_1^{\mathcal{N}}\|_{C^{\beta-\alpha}(S_e)} \leq c(1 + \|\psi\|_{C^{1,\beta}(S)})^c \|\Psi^{\mathcal{N}}\|_{C^{\beta}(S)} \prod_{i \in \mathcal{N}} \|\psi_i\|_{C^{1,\beta}(S)}$$

with a constant  $c > 1$  depending only on  $N, \alpha, \beta, r$ , and  $|\mathcal{N}|$  (in particular, independent of  $e \in S$ ). Since

$$\mathcal{M}_1(\theta) = \sum_{\mathcal{N} \in \mathcal{S}_k} \mathcal{M}_1^{\mathcal{N}}(\theta) \quad \text{for } \theta \in S_e$$

by (4.58), we thus obtain the estimates (4.56) and (4.57) for  $i = 1$ , as desired.

The estimate for  $\mathcal{M}_2$  follows the same arguments as above but using (4.6) in the place of (4.5) while the one for  $\mathcal{M}_3$  is similar but easier. In these two cases, we get a  $C^\beta(S)$  estimate for  $\mathcal{M}_2$  and  $\mathcal{M}_3$ , and in particular a  $C^{\beta-\alpha}(S)$  estimate.  $\square$

We are now in position to prove the regularity of the NMC operator of perturbed spheres, thereby completing the proof of Theorem 2.1.

**Theorem 4.11** *With  $\mathcal{O}$  defined by (2.3), the map  $h : \mathcal{O} \rightarrow C^{\beta-\alpha}(S)$  defined by (2.7) is smooth.*

*Proof* We fix  $r > 0$ . By (4.17), we have

$$h(\varphi) = -(1 + \varphi)\tilde{h}_1(1 + \varphi) + \tilde{h}_2(1 + \varphi) + \frac{1 + \varphi}{2}\tilde{h}_3(1 + \varphi) \quad (4.60)$$

for  $\varphi \in \mathcal{O}$  with  $\varphi > r - 1$  on  $S$ , where the maps  $\tilde{h}_j : \mathcal{B}(r) \rightarrow C^{\beta-\alpha}(S)$  are given by

$$\tilde{h}_j(\psi)(\theta) = \int_S M_j(\psi, \sigma, \theta) dV(\sigma)$$

for  $j = 1, 2, 3$  and the function  $M_j$  is defined in Proposition 4.10—which guarantees that  $\tilde{h}_j$  takes values in  $C^{\beta-\alpha}(S)$ . Thus, it suffices to establish that  $\tilde{h}_j$ , for  $j = 1, 2, 3$ , are smooth on  $\mathcal{B}(r)$  for every  $r > 0$ .

For this, we only need to prove that, for  $k \in \mathbb{N}$ ,

$$D^k \tilde{h}_j(\psi) = \int_S D_\psi^k M_j(\psi, \sigma, \cdot) dV(\sigma) \quad \text{in Fréchet sense} \quad (4.61)$$

for  $j = 1, 2, 3$ . Then the continuity of  $D^k \tilde{h}_j$  is a well known consequence of the existence of  $D^{k+1} \tilde{h}_j$  in Fréchet sense. To prove (4.61), we proceed by induction. For  $k = 0$ , the statement is true by definition. Let us now assume that the statement holds true for some  $k \geq 0$ . Then  $D^k \tilde{h}_j(\psi)$  is given by

$$D^k \tilde{h}_j(\psi)[\psi_1, \dots, \psi_k](\theta) = \int_S D_\psi^k M_j(\psi, \sigma, \theta)[\psi_1, \dots, \psi_k] dV(\sigma). \quad (4.62)$$

We fix  $\psi_1, \dots, \psi_k \in C^{1,\beta}(S)$ . Moreover, for  $\psi \in \mathcal{B}(r)$  and  $v \in C^{1,\beta}(S)$ , we put

$$\Gamma(\psi, v, \theta) = \int_S D_\psi^{k+1} M_j(\psi, \sigma, \theta)[\psi_1, \dots, \psi_k, v] dV(\sigma).$$

Let  $\psi \in \mathcal{B}(r)$  and  $v \in C^{1,\beta}(S)$  with  $\|v\|_{C^{1,\beta}(S)} < r/2$ . We have

$$\begin{aligned} & D^k \tilde{h}_j(\psi + v)[\psi_1, \dots, \psi_k](\theta) - D^k \tilde{h}_j(\psi)[\psi_1, \dots, \psi_k](\theta) - \Gamma(\psi, v, \theta) \\ &= \int_S \int_0^1 \left\{ D_\psi^{k+1} M_j(\psi + \rho v, \sigma, \theta) - D_\psi^{k+1} M_j(\psi, \sigma, \theta) \right\} [\psi_1, \dots, \psi_k, v] d\rho dV(\sigma) \\ &= \int_0^1 \rho \int_0^1 \mathcal{H}^{\rho,\tau}(\theta) d\tau d\rho, \end{aligned}$$

where

$$\mathcal{H}^{\rho,\tau}(\theta) := \int_S D_\psi^{k+2} M_j(\psi + \tau\rho v, \sigma, \theta)[\psi_1, \dots, \psi_k, v, v] dV(\sigma).$$

Note that  $\psi + \tau\rho v \in \mathcal{B}(r/2)$  for every  $\tau, \rho \in [0, 1]$ . By Proposition 4.10, we have

$$\begin{aligned} \|\mathcal{H}^{\rho,\tau}\|_{C^{\beta-\alpha}(S)} &\leq c(1 + \|\psi + \tau\rho v\|_{C^{1,\beta}(S)})^c \|v\|_{C^{1,\beta}(\mathbb{R})}^2 \prod_{i=1}^k \|u_i\|_{C^{1,\beta}(S)} \\ &\leq c(1 + \|\psi\|_{C^{1,\beta}(S)} + \|v\|_{C^{1,\beta}(S)})^c \|v\|_{C^{1,\beta}(S)}^2 \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)} \end{aligned}$$

with a constant  $c > 1$  independent of  $\rho, \tau, \psi, \psi_1, \dots, \psi_k$  and  $v$ . Consequently,

$$\begin{aligned} & \|D^k \tilde{h}_j(\psi + v)[\psi_1, \dots, \psi_k] - D^k \tilde{h}_j(\psi)[\psi_1, \dots, \psi_k] - \Gamma(\psi, v, \cdot)\|_{C^{\beta-\alpha}(S)} \\ &\leq c(1 + \|\psi\|_{C^{1,\beta}(S)} + \|v\|_{C^{1,\beta}(S)})^c \|v\|_{C^{1,\beta}(S)}^2 \prod_{i=1}^k \|\psi_i\|_{C^{1,\beta}(S)}. \end{aligned}$$

This shows that  $D^{k+1} \tilde{h}_j(\psi)$  exists in Fréchet sense, and that

$$D^{k+1} \tilde{h}_j(\psi)[\psi_1, \dots, \psi_k, v] = \Gamma(\psi, v, \cdot) \in C^{\beta-\alpha}(S).$$

We conclude that (4.61) holds for  $k + 1$  in place of  $k$ , and thus the proof is finished.  $\square$

We close this section with an outline of the proof of Proposition 2.2, which is concerned with the maps  $G$  and  $G_p$ . Since the definition of these functions in (2.9) and (2.10) does not involve singular integrals, the proof is much easier than the proof of Theorem 2.1 but still somewhat lengthy if all details are carried out. In the following, we point out the main steps.

*Sketch of the proof of Proposition 2.2* We fix  $c_1 \in (0, c_0)$  arbitrarily, and we note that it clearly suffices to prove the statement with  $c_0$  replaced by  $c_1$ . For  $p \in \mathcal{L}_*$ , we then use polar coordinates to write  $G_p$  as follows:



$$\begin{aligned}
 -\frac{1}{\alpha} G_p(\tau, \varphi)(\theta) &= \int_{B_\varphi} \frac{1}{|\tau(y - F_\varphi(\theta)) + p|^{N+\alpha}} dy \\
 &= \int_S \int_0^{1+\varphi(\sigma)} \frac{r^{N-1}}{|\tau(r\sigma - F_\varphi(\theta)) + p|^{N+\alpha}} dr dV(\sigma) \\
 &= \int_S \int_0^1 \frac{(1 + \varphi(\sigma))^N \rho^{N-1}}{|\tau\{\rho(1 + \varphi(\sigma))\sigma - (1 + \varphi(\theta))\theta\} + p|^{N+\alpha}} d\rho dV(\sigma) \\
 &= \int_S \int_0^1 \frac{(1 + \varphi(\sigma))^N \rho^{N-1}}{|\mathcal{D}_p(\tau, \varphi)(\rho, \sigma, \theta)|^{N+\alpha}} d\rho dV(\sigma), \tag{4.63}
 \end{aligned}$$

with

$$\mathcal{D}_p(\tau, \varphi)(\rho, \sigma, \theta) := \tau\{\rho(1 + \varphi(\sigma))\sigma - (1 + \varphi(\theta))\theta\} + p.$$

We point out that, for  $\varphi \in \mathcal{O}$ ,  $\tau \in (-\frac{c_1}{4}, \frac{c_1}{4})$ ,  $p \in \mathcal{L}_*$ ,  $\rho \in [0, 1]$  and  $\sigma, \theta \in S$  we have

$$\begin{aligned}
 |\mathcal{D}_p(\tau, \varphi)(\rho, \sigma, \theta)| &\geq |p| - |\tau| \left| \rho(1 + \varphi(\sigma))\sigma - (1 + \varphi(\theta))\theta \right| \\
 &\geq |p| - \frac{c_1}{4} (2 + 2\|\varphi\|_{L^\infty(S)}) \\
 &\geq |p| - c_1 \geq c_0 - c_1 > 0 \tag{4.64}
 \end{aligned}$$

by the definition of  $c_0$  in (2.1).

We now claim that  $G_p$  is of class  $C^k$  for all  $k \in \mathbb{N} \cup \{0\}$ , and that every partial derivative  $\partial^\gamma G_p$  of order  $|\gamma| = k$  with respect to  $\tau$  and  $\varphi$  can be written as

$$\begin{aligned}
 \partial^\gamma G_p(\tau, \varphi)[\psi_1, \dots, \psi_\ell](\theta) \\
 = \int_S \int_0^1 \sum_{\mathcal{N} \subset \mathcal{S}_\ell} \prod_{i \in \mathcal{N}} \psi_i(\theta) \prod_{j \in \mathcal{N}^c} \psi_j(\sigma) \frac{P^{\gamma, \mathcal{N}}(\tau, \rho, \sigma, \theta, \varphi(\sigma), \varphi(\theta), p)}{|\mathcal{D}_p(\tau, \varphi)(\rho, \sigma, \theta)|^{N+\alpha+2k}} d\rho dV(\sigma)
 \end{aligned}$$

for  $\psi_1, \dots, \psi_\ell \in C^{1,\beta}(S)$ ,  $\theta \in S$ . Here,  $\ell \leq k$  is the number of derivatives with respect to  $\varphi$ ,  $\mathcal{S}_\ell$  is the set of subsets of  $\{1, \dots, \ell\}$  and  $\mathcal{N}^c = \{1, \dots, \ell\} \setminus \mathcal{N}$  for  $\mathcal{N} \in \mathcal{S}_\ell$ . Moreover, the functions  $P^{\gamma, \mathcal{N}}$  are polynomials in all variables which are of degree at most  $2k$  in the variable  $p = (p_1, \dots, p_N)$ . This representation follows easily from (4.63). We use it, together with a similar induction argument as in the proof of Theorem 4.11, to show that  $G_p$  is a smooth map. In this step we also use the embeddings  $C^{1,\beta}(S) \hookrightarrow C^1(S) \hookrightarrow C^{\beta-\alpha}(S)$  and the estimate

$$\begin{aligned}
 &\left\| \sum_{\mathcal{N} \subset \mathcal{S}_\ell} \prod_{i \in \mathcal{N}} \psi_i(\cdot) \prod_{j \in \mathcal{N}^c} \psi_j(\sigma) \frac{P^{\gamma, \ell}(\tau, \rho, \sigma, \cdot, \varphi(\sigma), \varphi(\cdot), p)}{|\mathcal{D}_p(\tau, \varphi)((\rho, \sigma, \cdot +))|^{N+\alpha+2k}} \right\|_{C^1(S)} \\
 &\leq c_\gamma (1 + \|\varphi\|_{C^1(S)})^{c_\gamma} |p|^{-N-\alpha} \prod_{i=1}^\ell \|\psi_i\|_{C^1(S)}, \tag{4.65}
 \end{aligned}$$

which can be deduced from (4.64) since  $|p| - c_1 = |p| - (c_1/c_0)c_0 \geq |p|(1 - c_1/c_0) > 0$ . Here  $c_\gamma > 1$  is a constant which depends on  $\gamma$  but not on  $\tau, \rho, \sigma$  and  $p$ .

It thus follows that  $G_p : (-\frac{c_1}{4}, \frac{c_1}{4}) \times \mathcal{O} \rightarrow C^{\beta-\alpha}(S)$  is of class  $C^\infty$ , and that

$$\left\| \partial^\gamma G_p(\tau, \varphi)[\psi_1, \dots, \psi_\ell] \right\|_{C^{\beta-\alpha}(S)} \leq d_\gamma (1 + \|\varphi\|_{C^{1,\beta}(S)})^{d_\gamma} |p|^{-N-\alpha} \prod_{i=1}^\ell \|\psi_i\|_{C^{1,\beta}(S)} \tag{4.66}$$

for every partial derivative  $\partial^\gamma G_p$  of the form above, and with a constant  $d_\gamma > 0$  independent of  $\tau$  and  $p$ . Consequently, the series  $\sum_{p \in \mathcal{L}_*} \partial^\gamma G_p(\tau, \varphi)$  is convergent in the space  $C^{\beta-\alpha}(S)$ , and the convergence is uniform in  $\tau \in (-\frac{c_1}{4}, \frac{c_1}{4})$  and  $\varphi \in \mathcal{O}$ . From this we deduce that the map

$$G = \sum_{p \in \mathcal{L}_*} G_p : \left(-\frac{c_1}{4}, \frac{c_1}{4}\right) \times \mathcal{O} \rightarrow C^{\beta-\alpha}(S)$$

is of class  $C^\infty$ , as claimed. □

### 5 The linearized NMC operator

In this section, we compute a simple expression for the linearization at zero of the nonlocal mean curvature operator  $h$  defined in (2.8), and we study its invertibility properties between suitably chosen function spaces. As we mentioned in the introduction, once the Fréchet differentiability of  $h$  is proved—as we have done—the expression can also be derived from the results of [9, Appendix, Proposition B.2] and [12, Section 6] applied to the special case of the sphere  $S$ . For completeness, we give a direct proof in our setting based on formula (4.60).

**Lemma 5.1** *Let  $\alpha \in (0, 1)$ ,  $\beta \in (\alpha, 1)$ , and let  $h : \mathcal{O} \subset C^{1,\beta}(S) \rightarrow C^{\beta-\alpha}(S)$  be defined by (2.8). Then, we have*

$$\frac{1}{\alpha} Dh(0)\varphi = L_\alpha \varphi - \lambda_1 \varphi \text{ in } C^{\beta-\alpha}(S) \text{ for } \varphi \in C^{1,\beta}(S), \tag{5.1}$$

with

$$L_\alpha \varphi(\theta) = PV \int_S \frac{\varphi(\theta) - \varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) \text{ for } \theta \in S \tag{5.2}$$

and  $\lambda_1$  given in (1.9) for  $k = 1$ . In addition,  $L_\alpha$  defines a continuous linear operator  $C^{1,\beta}(S) \rightarrow C^{\beta-\alpha}(S)$ .

Before proving the lemma, we first discuss the spectral representation of the operator  $L_\alpha$  and the special role of  $\lambda_1$ . For  $k \in \mathbb{N}$ , we let  $\mathcal{E}_k$  be the space of spherical harmonics of degree  $k$ , and we denote by  $n_k$  its dimension. We recall that  $n_0 = 1$  and that  $\mathcal{E}_0$  consists of constant functions, whereas  $n_1 = N$  and  $\mathcal{E}_1$  is spanned by the coordinate

functions  $\theta \mapsto \theta_i$  for  $i = 1, \dots, N$ . As already mentioned in the introduction, we have

$$L_\alpha \psi = \lambda_k \psi \quad \text{for every } k \in \mathbb{N} \text{ and } \psi \in \mathcal{E}_k, \tag{5.3}$$

where

$$\lambda_k = \frac{\pi^{(N-1)/2} \Gamma((1-\alpha)/2)}{(1+\alpha) 2^\alpha \Gamma((N+\alpha)/2)} \left( \frac{\Gamma\left(\frac{2k+N+\alpha}{2}\right)}{\Gamma\left(\frac{2k+N-\alpha-2}{2}\right)} - \frac{\Gamma\left(\frac{N+\alpha}{2}\right)}{\Gamma\left(\frac{N-\alpha-2}{2}\right)} \right), \tag{5.4}$$

see e.g. [17, Lemma 6.26]. Here  $\Gamma$  denotes the usual Gamma function, see e.g. [14, Section 8.3] for its generalization to negative non-integer real numbers. For the proof of Lemma 5.1, it will be useful to represent the eigenvalue  $\lambda_1$  in a different form. For this we note that, if  $\theta \in S$  is fixed, then the function  $\sigma \mapsto Y_\theta(\sigma) := \sigma \cdot \theta$  is a spherical harmonic of degree one, so that

$$\lambda_1 = \lambda_1 Y_\theta(\theta) = L_\alpha Y_\theta(\theta) = \int_S \frac{1 - \sigma \cdot \theta}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) = \int_S \frac{(\sigma - \theta) \cdot \sigma}{|\theta - \sigma|^{N+\alpha}} dV(\sigma). \tag{5.5}$$

Comparing this with (1.1), we see that  $\frac{2}{\alpha} d_{N,\alpha} \lambda_1$  equals the NMC of the sphere  $S$ , as stated in (1.10). Moreover, since  $|\theta - \sigma|^2 = 2(1 - \sigma \cdot \theta)$ , we may rewrite the first integral in (5.5) to obtain the equality

$$\lambda_1 = \frac{1}{2} \int_S \frac{1}{|\theta - \sigma|^{N+\alpha-2}} dV(\sigma), \tag{5.6}$$

which will be used in the proof of Lemma 5.1.

*Proof of Lemma 5.1* By (4.60), we have

$$h(\varphi) = -(1 + \varphi) \tilde{h}_1(1 + \varphi) + \tilde{h}_2(1 + \varphi) + \frac{1 + \varphi}{2} \tilde{h}_3(1 + \varphi) \tag{5.7}$$

with

$$\tilde{h}_j(\psi)(\theta) = \int_S M_j(\psi, \sigma, \theta) dV(\sigma),$$

for  $j = 1, 2, 3$ , where the functions  $M_j$  are defined in Proposition 4.10. Let  $\varphi \in C^{1,\beta}(S)$ . By (4.61), the functions  $D\tilde{h}_j(1)\varphi \in C^{0,\beta-\alpha}(S)$  are given by

$$(D\tilde{h}_j(1)\varphi)(\theta) = \int_S \partial_\psi M_j(1, \sigma, \theta) \varphi dV(\sigma). \tag{5.8}$$

For  $\sigma, \theta \in S, \sigma \neq \theta$  we have

$$M_1(1, \sigma, \theta) = M_2(1, \sigma, \theta) = 0, \quad M_3(1, \sigma, \theta) = \frac{1}{|\theta - \sigma|^{N+\alpha-2}}$$

and

$$\partial_\psi M_1(1, \sigma, \theta)\varphi = \frac{\Lambda_1(\varphi, \sigma, \theta)}{|\theta - \sigma|^{N+\alpha}} = \frac{\varphi(\theta) - \varphi(\sigma) - (\theta - \sigma) \cdot \nabla\varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}},$$

$$\partial_\psi M_2(1, \sigma, \theta)\varphi = 0,$$

$$\begin{aligned} \partial_\psi M_3(1, \sigma, \theta)\varphi &= \frac{\partial_\psi \bar{\mathcal{K}}_\alpha(1, \sigma, \theta)\varphi + (N - 1)\varphi(\sigma)}{|\theta - \sigma|^{N+\alpha-2}} \\ &= \frac{-\frac{N+\alpha}{2}(\varphi(\theta) + \varphi(\sigma)) + (N - 1)\varphi(\sigma)}{|\theta - \sigma|^{N+\alpha-2}} \\ &= -\frac{(N + \alpha)\varphi(\theta) + (2 + \alpha - N)\varphi(\sigma)}{2|\theta - \sigma|^{N+\alpha-2}}, \end{aligned}$$

since  $\Lambda_1(1, \cdot, \cdot) \equiv 0$  and  $\Lambda_2(1, \cdot, \cdot) \equiv \partial_\psi \Lambda_2(1, \cdot, \cdot)\varphi \equiv 0$  on  $S \times S$ . Combining this with (5.7) and (5.8), and also using (5.5) or (5.6), we find that

$$\begin{aligned} (Dh(0)\varphi)(\theta) &= \int_S \left\{ -\partial_\psi M_1(1, \sigma, \theta)\varphi + \partial_\psi M_2(1, \sigma, \theta)\varphi \right. \\ &\quad \left. + \frac{\varphi(\theta)M_3(1, \sigma, \theta) + \partial_\psi M_3(1, \sigma, \theta)\varphi}{2} \right\} dV(\sigma) \\ &= - \int_S \frac{\varphi(\theta) - \varphi(\sigma) - (\theta - \sigma) \cdot \nabla\varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) \\ &\quad - \int_S \frac{(N + \alpha - 2)\varphi(\theta) + (2 + \alpha - N)\varphi(\sigma)}{4|\theta - \sigma|^{N+\alpha-2}} dV(\sigma) \\ &= - \int_S \frac{\varphi(\theta) - \varphi(\sigma) - (\theta - \sigma) \cdot \nabla\varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) - \alpha\lambda_1\varphi(\theta) \\ &\quad - \frac{N - \alpha - 2}{2} \int_S \frac{(1 - \sigma \cdot \theta)(\varphi(\theta) - \varphi(\sigma))}{|\theta - \sigma|^{N+\alpha}} dV(\sigma). \end{aligned} \tag{5.9}$$

Next, for  $\theta \in S$ , we let  $B_\varepsilon(\theta)$  be a ball on  $S$  centered at  $\theta \in S$  with radius  $\varepsilon \in (0, 1)$ . We have

$$\begin{aligned} &\int_S \frac{\varphi(\theta) - \varphi(\sigma) - (\theta - \sigma) \cdot \nabla\varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{S \setminus B_\varepsilon(\theta)} \frac{\varphi(\theta) - \varphi(\sigma) + (\sigma - \theta) \cdot \nabla\varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} dV(\sigma), \end{aligned} \tag{5.10}$$

and, for  $\varepsilon > 0$  small, integrating by parts,

$$\int_{S \setminus B_\varepsilon(\theta)} \frac{(\sigma - \theta) \cdot \nabla \varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) = \int_{\partial B_\varepsilon(\theta)} \frac{(\sigma - \theta) \cdot \tilde{\nu}(\sigma)(\varphi(\sigma) - \varphi(\theta))}{|\theta - \sigma|^{N+\alpha}} d\tilde{V}(\sigma) + \int_{S \setminus B_\varepsilon(\theta)} (\varphi(\theta) - \varphi(\sigma)) \operatorname{div}_\sigma \frac{P_\sigma(\sigma - \theta)}{|\theta - \sigma|^{N+\alpha}} dV(\sigma). \tag{5.11}$$

Here and in the following,  $\partial B_\varepsilon(\theta)$  denotes the relative boundary of  $B_\varepsilon(\theta)$  in  $S$ ,  $d\tilde{V}$  denotes the  $(N - 2)$ -dimensional Hausdorff measure on  $\partial B_\varepsilon(\theta)$  and  $\tilde{\nu}$  the unit outer normal vector field of  $\partial B_\varepsilon(\theta)$  on  $S$ . Moreover, the differential operators  $\nabla = \nabla_\sigma$ ,  $\operatorname{div}_\sigma$ , and  $\Delta_\sigma$  on the sphere  $S$  are all defined with respect to the standard metric on  $S$ , and

$$P_\sigma(\sigma - \theta) = \sigma - \theta - ((\sigma - \theta) \cdot \sigma) \sigma = \sigma - \theta - (1 - \theta \cdot \sigma) \sigma = (\theta \cdot \sigma) \sigma - \theta \tag{5.12}$$

is the orthogonal projection of  $\sigma - \theta$  onto the tangent space  $T_\sigma S$ . Since

$$\varphi(\sigma) - \varphi(\theta) = \nabla \varphi(\theta) \cdot (\sigma - \theta) + O(|\sigma - \theta|^{1+\beta}) \quad \text{as } |\sigma - \theta| \rightarrow 0,$$

and, by antisymmetry with respect to reflection at the axis  $\mathbb{R}\theta$ ,

$$\int_{\partial B_\varepsilon(\theta)} \frac{((\sigma - \theta) \cdot \tilde{\nu}(\sigma))(\nabla \varphi(\theta) \cdot (\sigma - \theta))}{|\theta - \sigma|^{N+\alpha}} d\tilde{V}(\sigma) = 0,$$

we find that

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial B_\varepsilon(\theta)} \frac{(\sigma - \theta) \cdot \tilde{\nu}(\sigma)(\varphi(\sigma) - \varphi(\theta))}{|\theta - \sigma|^{N+\alpha}} d\tilde{V}(\sigma) = 0. \tag{5.13}$$

Now we note that, by (5.12), we have

$$-\nabla_\sigma(\sigma \cdot \theta) = -\{\theta - (\theta \cdot \sigma) \sigma\} = P_\sigma(\sigma - \theta) \quad \text{on } S,$$

and therefore

$$\operatorname{div}_\sigma P_\sigma(\sigma - \theta) = -\Delta_\sigma(\sigma \cdot \theta) = (N - 1)(\sigma \cdot \theta).$$

Consequently,

$$\begin{aligned} \operatorname{div}_\sigma \frac{P_\sigma(\sigma - \theta)}{|\theta - \sigma|^{N+\alpha}} &= (N - 1) \frac{\sigma \cdot \theta}{|\theta - \sigma|^{N+\alpha}} + P_\sigma(\sigma - \theta) \cdot \nabla |\theta - \sigma|^{-N-\alpha} \\ &= (N - 1) \frac{\sigma \cdot \theta}{|\theta - \sigma|^{N+\alpha}} - (N + \alpha)((\theta \cdot \sigma)\sigma - \theta) \cdot \frac{\sigma - \theta}{|\theta - \sigma|^{N+\alpha+2}} \\ &= (N - 1) \frac{\sigma \cdot \theta}{|\theta - \sigma|^{N+\alpha}} - \frac{N + \alpha}{2} \frac{1 + \sigma \cdot \theta}{|\theta - \sigma|^{N+\alpha}} \\ &= - \frac{(2 + \alpha - N)\sigma \cdot \theta + (N + \alpha)}{2|\theta - \sigma|^{N+\alpha}}. \end{aligned} \tag{5.14}$$

Combining (5.10), (5.11), (5.13) and (5.14), we conclude that

$$\begin{aligned} &\int_S \frac{\varphi(\theta) - \varphi(\sigma) - (\theta - \sigma) \cdot \nabla \varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) \tag{5.15} \\ &= \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \int_{S \setminus B_\varepsilon(\theta)} \frac{\{(2 - N - \alpha) + (N - 2 - \alpha)\sigma \cdot \theta\}(\varphi(\theta) - \varphi(\sigma))}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) \tag{5.16} \end{aligned}$$

and thus, by (5.9),

$$\begin{aligned} (Dh(0)\varphi)(\theta) &= \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \int_{S \setminus B_\varepsilon(\theta)} \frac{\{(\alpha + N - 2) + (2 + \alpha - N)\sigma \cdot \theta\}(\varphi(\theta) - \varphi(\sigma))}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) \\ &\quad - \frac{N - \alpha - 2}{2} \lim_{\varepsilon \rightarrow 0} \int_{S \setminus B_\varepsilon(\theta)} \frac{(1 - \sigma \cdot \theta)(\varphi(\theta) - \varphi(\sigma))}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) - \alpha \lambda_1 \varphi(\theta) \\ &= \alpha \lim_{\varepsilon \rightarrow 0} \int_{S \setminus B_\varepsilon(\theta)} \frac{\varphi(\theta) - \varphi(\sigma)}{|\theta - \sigma|^{N+\alpha}} dV(\sigma) - \alpha \lambda_1 \varphi(\theta), \end{aligned}$$

as claimed.

The last statement—that  $L_\alpha$  is continuous between  $C^{1,\beta}(S) \rightarrow C^{\beta-\alpha}(S)$ —is a direct consequence of our nonlinear result of Theorem 2.2.  $\square$

Next, we wish to study invertibility properties of the linearized operator  $Dh(0)$  between suitably chosen function spaces. The following theorem is the main result of this section.

**Theorem 5.2** *Let  $\alpha \in (0, 1)$ ,  $\beta \in (\alpha, 1)$ , and let the subspaces  $X \subset C^{1,\beta}(S)$ ,  $Y \subset C^{\beta-\alpha}(S)$  be defined by (1.16) and (1.17). Then, the restriction to  $X$  of the linearized NMC operator  $Dh(0) : X \rightarrow Y$  is an isomorphism onto  $Y$ .*

The remainder of this section is devoted to the proof of this theorem.

In the following, for  $k \in \mathbb{N} \cup \{0\}$ , we let  $P_k : L^2(S) \rightarrow L^2(S)$  denote the  $\langle \cdot, \cdot \rangle_{L^2}$ -orthogonal projections on  $\mathcal{E}_k$ —the space of spherical harmonics of degree  $k$ . For  $\rho \geq 0$ , we then define the Sobolev space

$$H^\rho(S) := \left\{ u \in L^2(S) : \sum_{k=0}^\infty (1 + k^2)^\rho \|P_k u\|_{L^2(S)}^2 < \infty \right\}, \tag{5.17}$$

which is a Hilbert space with the scalar product

$$(u, v) \mapsto \sum_{k=0}^{\infty} (1 + k^2)^\rho \langle P_k u, P_k v \rangle_{L^2} \quad \text{for } u, v \in H^\rho(S). \tag{5.18}$$

We need the following result on the mapping properties of the operator  $L_\alpha$  with regard to the scale of Sobolev spaces  $H^\rho(S)$ .

**Lemma 5.3** *Let  $\alpha \in (0, 1)$  and  $\beta \in (\alpha, 1)$ .*

(i) *For given  $\rho \geq 0$ , the map*

$$v \mapsto \tilde{L}_\alpha v := \sum_{k=0}^{\infty} \lambda_k P_k v = \sum_{k=0}^{\infty} L_\alpha P_k v \tag{5.19}$$

*defines a continuous linear operator  $\tilde{L}_\alpha : H^{\rho+1+\alpha}(S) \rightarrow H^\rho(S)$ .*

*Moreover,  $\tilde{L}_\alpha + \text{id} : H^{\rho+1+\alpha}(S) \rightarrow H^\rho(S)$  is an isomorphism.*

(ii) *We have  $C^{1,\beta}(S) \subset H^{1+\alpha}(S)$  and*

$$L_\alpha \psi = \tilde{L}_\alpha \psi \quad \text{in } L^2(S) \quad \text{for } \psi \in C^{1,\beta}(S) \tag{5.20}$$

*with  $\tilde{L}_\alpha : H^{1+\alpha}(S) \rightarrow L^2(S)$  given in (5.19).*

(iii) *The operator  $L_\alpha + \text{id}$  restricts to a bijective map  $C^\infty(S) \rightarrow C^\infty(S)$ .*

*Proof* (i) Since

$$\lim_{\tau \rightarrow +\infty} \frac{\Gamma(\tau + \varrho)}{\Gamma(\tau)\tau^\varrho} = 1 \quad \text{for all } \varrho \in \mathbb{R}$$

(see e.g. [16, Page 15, Problem 7]), we deduce from (5.4) that

$$\lim_{k \rightarrow +\infty} \frac{\lambda_k}{k^{1+\alpha}} = \frac{\pi^{(N-1)/2} \Gamma((1-\alpha)/2)}{(1+\alpha)2^\alpha \Gamma((N+\alpha)/2)} \in (0, \infty). \tag{5.21}$$

Using this and the fact that  $\lambda_k \geq 0$  for all  $k \in \mathbb{N} \cup \{0\}$ , we infer that  $\tilde{L}_\alpha$ , as defined in (5.19), is a well defined continuous linear operator  $H^{\rho+1+\alpha}(S) \rightarrow H^\rho(S)$ , and that  $\tilde{L}_\alpha + \text{id}$  is an isomorphism.

(ii) In the following, we let  $C_1, C_2, \dots$  denote positive constants depending only on  $N, \alpha$  and  $\beta$ . For  $\psi \in C^{1,\beta}(S)$ , by Lemma 5.1, we have

$$\|L_\alpha \psi\|_{L^2(S)} \leq C_1 \|L_\alpha \psi\|_{C^{\beta-\alpha}(S)} \leq C_2 \|\psi\|_{C^{1,\beta}(S)}$$

and thus

$$\|L_\alpha \psi\|_{L^2(S)} + \|\psi\|_{L^2(S)} \leq C_3 \|\psi\|_{C^{1,\beta}(S)}. \tag{5.22}$$

Next we remark that, as a consequence of the spectral representation of the Laplace–Beltrami operator on  $S$ , we have

$$C^\infty(S) = \bigcap_{\rho \in \mathbb{N}} H^\rho(S) = \bigcap_{\rho \geq 0} H^\rho(S). \tag{5.23}$$

Moreover, for  $\psi \in C^\infty(S)$  the series  $\sum_{k=0}^\infty \lambda_k P_k \psi$  converges in  $L^2(S)$  and the series  $\sum_{k=0}^\infty P_k \psi$  converges in  $C^m(S)$  for every  $m \in \mathbb{N}$ . From this and (5.22) we deduce that

$$L_\alpha \psi = \lim_{\ell \rightarrow \infty} L_\alpha \sum_{k=0}^\ell P_k \psi = \lim_{\ell \rightarrow \infty} \sum_{k=0}^\ell L_\alpha P_k \psi = \tilde{L}_\alpha \psi \quad \text{for } \psi \in C^\infty(S).$$

Combining this with (i) and (5.22), we find that

$$\|\psi\|_{H^{1+\alpha}(S)} \leq C_4 \left( \|L_\alpha \psi\|_{L^2(S)} + \|\psi\|_{L^2(S)} \right) \leq C_5 \|\psi\|_{C^{1,\beta}(S)} \quad \text{for } \psi \in C^\infty(S). \tag{5.24}$$

Next, let  $\psi \in C^{1,\beta}(S)$ , and let  $\psi_n \in C^\infty(S)$ ,  $n \in \mathbb{N}$  satisfy  $\psi_n \rightarrow \psi$  in  $C^{1,\beta}(S)$ . Then (5.24) implies that  $(\psi_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $H^{1+\alpha}(S)$ , and by completeness this forces  $\psi \in H^{1+\alpha}(S)$ . Moreover, by passing to the limit, we deduce that  $\psi_n \rightarrow \psi$  in  $H^{1+\alpha}(S)$ , which implies that

$$\tilde{L}_\alpha \psi = \lim_{n \rightarrow \infty} \tilde{L}_\alpha \psi_n = \lim_{n \rightarrow \infty} L_\alpha \psi_n \quad \text{in } L^2(S).$$

Since moreover  $L_\alpha \psi = \lim_{n \rightarrow \infty} L_\alpha \psi_n$  in  $C^{\beta-\alpha}(S)$  by Lemma 5.1, we obtain (5.20).

(iii) This follows immediately from (i), (ii) and (5.23). □

The following lemma provides an analogue for  $L_\alpha + id$  of the classical interior Hölder regularity estimate for the classical fractional Laplacian. In the proof, we will apply a series of changes of variables to reduce our problem to one where regularity for the classical fractional Laplacian can be applied.

**Lemma 5.4** *Let  $\alpha \in (0, 1)$ ,  $\beta \in (\alpha, 1)$ . Then there exists a constant  $C = C(N, \alpha, \beta) > 0$  such that*

$$\|\psi\|_{C^{1,\beta}(S)} \leq C \|L_\alpha \psi + \psi\|_{C^{\beta-\alpha}(S)} \quad \text{for all } \psi \in C^\infty(S). \tag{5.25}$$

To establish this lemma we need a standard interpolation estimate. We include a simple proof for the convenience of the reader.

**Lemma 5.5** *Let  $\beta \in (0, 1)$ . Then for every  $\varepsilon > 0$  there exists  $K = K(\varepsilon, N, \beta) > 0$  such that*

$$\|\psi\|_{C^\beta(S)} \leq \varepsilon \|\psi\|_{C^1(S)} + K \|\psi\|_{L^2(S)} \quad \text{for every } \psi \in C^1(S). \tag{5.26}$$



*Proof* In the following, we let  $C_1, C_2, \dots$ , denote positive constants which only depend on  $N$ . As a consequence of (4.9), we have

$$|\psi(\theta) - \psi(\sigma)| \leq C_1 \|\psi\|_{C^1(S)} |\theta - \sigma| \quad \text{for } \theta, \sigma \in S, \tag{5.27}$$

and thus

$$\frac{|\psi(\theta) - \psi(\sigma)|}{|\theta - \sigma|^\beta} \leq C_1 |\theta - \sigma|^{1-\beta} \|\psi\|_{C^1(S)} \leq C_1 \delta^{1-\beta} \|\psi\|_{C^1(S)}$$

for  $\theta, \sigma \in S$  with  $|\theta - \sigma| \leq \delta$ .

Moreover,

$$\frac{|\psi(\theta) - \psi(\sigma)|}{|\theta - \sigma|^\beta} \leq \frac{2}{\delta^\beta} \|\psi\|_{L^\infty(S)} \quad \text{for } \theta, \sigma \in S \text{ with } |\theta - \sigma| \geq \delta.$$

Combining these inequalities, we find that

$$\frac{|\psi(\theta) - \psi(\sigma)|}{|\theta - \sigma|^\beta} \leq C_1 \delta^{1-\beta} \|\psi\|_{C^1(S)} + \frac{2}{\delta^\beta} \|\psi\|_{L^\infty(S)} \quad \text{for } \theta, \sigma \in S. \tag{5.28}$$

Next, for  $0 < r < 1$  and  $\theta \in S$ , we let  $d_r$  denote the  $(N - 1)$ -dimensional volume of the ball  $B_r(\theta)$  on  $S$ , which clearly does not depend on  $\theta$ . By (5.27) we then have

$$\begin{aligned} \left| \psi(\theta) - \frac{1}{d_r} \int_{B_r(\theta)} \psi(\sigma) dV(\sigma) \right| &\leq \frac{1}{d_r} \int_{B_r(\theta)} |\psi(\theta) - \psi(\sigma)| dV(\sigma) \\ &\leq \frac{C_1}{d_r} \|\psi\|_{C^1(S)} \int_{B_r(\theta)} |\theta - \sigma| dV(\sigma) \leq r C_2 \|\psi\|_{C^1(S)} \quad \text{for all } \theta \in S, \end{aligned}$$

and thus

$$\begin{aligned} \|\psi\|_{L^\infty(S)} &\leq r C_2 \|\psi\|_{C^1(S)} + \max_{\theta \in S} \left| \frac{1}{d_r} \int_{B_r(\theta)} \psi(\sigma) dV(\sigma) \right| \\ &\leq r C_2 \|\psi\|_{C^1(S)} + \frac{\|\psi\|_{L^1(S)}}{d_r} \leq r C_2 \|\psi\|_{C^1(S)} + \frac{|S|^{1/2}}{d_r} \|\psi\|_{L^2(S)}. \end{aligned} \tag{5.29}$$

Combining (5.28) and (5.29), we find that

$$\begin{aligned} \|\psi\|_{C^\beta(S)} &= \sup_{\substack{\theta, \sigma \in S \\ \theta \neq \sigma}} \frac{|\psi(\theta) - \psi(\sigma)|}{|\theta - \sigma|^\beta} + \|\psi\|_{L^\infty(S)} \\ &\leq C_1 \delta^{1-\beta} \|\psi\|_{C^1(S)} + \left( \frac{2}{\delta^\beta} + 1 \right) \|\psi\|_{L^\infty(S)} \\ &\leq \left\{ C_1 \delta^{1-\beta} + r C_2 \left( \frac{2}{\delta^\beta} + 1 \right) \right\} \|\psi\|_{C^1(S)} + \frac{|S|^{1/2}}{d_r} \left( \frac{2}{\delta^\beta} + 1 \right) \|\psi\|_{L^2(S)}. \end{aligned}$$

For a given  $\varepsilon > 0$ , we may now choose  $\delta > 0$  such that  $C_1\delta^{1-\beta} \leq \frac{\varepsilon}{2}$  and then  $r > 0$  such that  $rC_2\left(\frac{2}{\delta^\beta} + 1\right) \leq \frac{\varepsilon}{2}$ . Then (5.26) follows with  $K = \frac{|S|^{1/2}}{dr} \left(\frac{2}{\delta^\beta} + 1\right)$ .  $\square$

We can now give the

*Proof of Lemma 5.4* In the following, the letter  $C$  stands for positive constants which may change from line to line but only depend on  $N, \alpha$  and  $\beta$ . Let  $\psi \in C^\infty(S)$  and  $g := L_\alpha\psi + \psi \in C^\infty(S)$ . We define  $u \in C^\infty(\mathbb{R}^N \setminus 0) \cap L^\infty(\mathbb{R}^N)$  by  $u(x) = \psi(x/|x|)$  for  $x \neq 0$ . For  $x \in \mathbb{R}^N \setminus \{0\}$ , a change of variable in polar coordinates gives, with  $r = |x|$  and  $\theta = \frac{x}{|x|}$ ,

$$\begin{aligned} (-\Delta)^{(1+\alpha)/2}u(x) &= C \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+1+\alpha}} dy \\ &= C \int_0^\infty \int_S \frac{\psi(\theta) - \psi(\sigma)}{((\rho - r)^2 + r\rho|\theta - \sigma|^2)^{(N+1+\alpha)/2}} \rho^{N-1} dV(\sigma) d\rho. \end{aligned}$$

We make the change of variable  $t = \frac{\rho-r}{|\theta-\sigma|}$  to get

$$\begin{aligned} &(-\Delta)^{(1+\alpha)/2}u(x) \\ &= C \int_S \frac{\psi(\theta) - \psi(\sigma)}{|\theta - \sigma|^{N+\alpha}} \int_0^\infty \frac{(t|\theta - \sigma| + r)^{N-1}}{(t^2 + r(t|\theta - \sigma| + r))^{(N+1+\alpha)/2}} dt dV(\sigma). \end{aligned} \tag{5.30}$$

To further simplify this expression, we define the function

$$Q : [0, \infty) \times (0, \infty) \rightarrow \mathbb{R}, \quad Q(a, b) := C \int_0^\infty \frac{(ta + b)^{N-1}}{(t^2 + b(ta + b))^{(N+1+\alpha)/2}} dt.$$

Using the change of variable  $s = \frac{t}{b}$ , we see that

$$Q(a, b) = b^{-1-\alpha} C \int_0^\infty \frac{(sa + 1)^{N-1}}{(s^2 + sa + 1)^{(N+1+\alpha)/2}} ds.$$

From this we see that  $Q \in C^\infty([0, \infty) \times (0, \infty))$ . Moreover, from (5.30) we get that

$$\begin{aligned} (-\Delta)^{(1+\alpha)/2}u(x) &= \int_S \frac{\psi(\theta) - \psi(\sigma)}{|\theta - \sigma|^{N+\alpha}} Q(|\theta - \sigma|, r) dV(\sigma) \\ &= Q(0, r) L_\alpha\psi(\theta) \\ &\quad + \int_S \frac{\psi(\theta) - \psi(\sigma)}{|\theta - \sigma|^{N+\alpha}} \left( Q(|\theta - \sigma|, r) - Q(0, r) \right) dV(\sigma) \\ &= Q(0, r) (g(\theta) - \psi(\theta)) \\ &\quad + \int_S \frac{\psi(\theta) - \psi(\sigma)}{|\theta - \sigma|^{N+\alpha-1}} \int_0^1 \partial_a Q(\tau|\theta - \sigma|, r) d\tau dV(\sigma). \end{aligned}$$

Next, we define

$$Q_{g,\psi}(x) := Q(0, |x|)(g(x/|x|) - \psi(x/|x|))$$

and

$$G_\psi(x) = \int_S \frac{\psi(\theta) - \psi(\sigma)}{|\theta - \sigma|^{N+\alpha-1}} \int_0^1 \partial_a Q(\tau|\theta - \sigma|, r) d\tau dV(\sigma)$$

for  $x \in \mathbb{R}^N \setminus \{0\}$ ,  $r = |x|$ ,  $\theta = \frac{x}{|x|}$ ,

so that

$$(-\Delta)^{(1+\alpha)/2} u(x) = Q_{g,\psi}(x) + G_\psi(x) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}. \tag{5.31}$$

We also put  $A := \{x \in \mathbb{R}^N : \frac{1}{2} \leq |x| \leq 2\}$ . We have  $Q_{g,\psi} \in C^{\beta-\alpha}(A)$  and

$$\|Q_{g,\psi}\|_{C^{\beta-\alpha}(A)} \leq C (\|g\|_{C^{\beta-\alpha}(S)} + \|\psi\|_{C^{\beta-\alpha}(S)}). \tag{5.32}$$

Next we show that

$$G_\psi \in C^{\beta-\alpha}(A) \quad \text{with} \quad \|G_\psi\|_{C^{\beta-\alpha}(A)} \leq C \|\psi\|_{C^\beta(S)}. \tag{5.33}$$

To this end, we write

$$G_\psi(x) = \int_S \frac{\psi(\theta) - \psi(\sigma)}{|\theta - \sigma|^{N+\alpha-1}} \int_0^1 \partial_a Q(\tau|\theta - \sigma|, r) d\tau dV(\sigma)$$

$$= \partial_a Q(0, r) L_{\alpha-1} \psi(\theta) + \tilde{G}_\psi(x)$$

with

$$L_{\alpha-1} \psi(\theta) := \int_S \frac{\psi(\theta) - \psi(\sigma)}{|\theta - \sigma|^{N+\alpha-1}} dV(\sigma)$$

and

$$\tilde{G}_\psi(x) := \int_S \frac{\psi(\theta) - \psi(\sigma)}{|\theta - \sigma|^{N+\alpha-2}} \int_0^1 \int_0^1 \tau \partial_a^2 Q(\lambda\tau|\theta - \sigma|, r) d\tau d\lambda dV(\sigma)$$

for  $\psi \in S$ . We will show that

$$\|L_{\alpha-1} \psi\|_{C^{\beta-\alpha}(S)} \leq C \|\psi\|_{C^\beta(S)} \tag{5.34}$$

and that

$$\|\tilde{G}_\psi\|_{C^{\beta-\alpha}(A)} \leq C\|\psi\|_{C^\beta(S)}. \tag{5.35}$$

From this (5.33) follows, since  $\partial_q Q(0, r)$  is equal to a constant times  $r^{-1-\alpha}$ .

To show (5.34) and (5.35), it suffices to fix  $e \in S$  arbitrarily, and prove that

$$\|L_{\alpha-1}\psi\|_{C^{\beta-\alpha}(S_e)} \leq C\|\psi\|_{C^\beta(S)} \quad \text{with } S_e := \{\theta \in S : \theta \cdot e \geq 0\} \tag{5.36}$$

and that

$$\|\tilde{G}_\psi\|_{C^{\beta-\alpha}(A_e)} \leq C\|\psi\|_{C^\beta(S)} \quad \text{with } A_e := \{x \in A : x \cdot e \geq 0\}. \tag{5.37}$$

To show these estimates, we consider again a Lipschitz continuous map of rotations  $S \mapsto SO(N)$ ,  $\theta \mapsto R_\theta$  with the property that (4.24) (4.25) holds, so that by a change of variable we have

$$L_{\alpha-1}\psi(\theta) = \int_S \frac{\psi(\theta) - \psi(R_\theta\sigma)}{|e - \sigma|^{N+\alpha-1}} dV(\sigma) \quad \text{for } \theta \in S_e. \tag{5.38}$$

Since

$$\begin{aligned} & \left| \{\psi(\theta_1) - \psi(R_{\theta_1}\sigma)\} - \{\psi(\theta_2) - \psi(R_{\theta_2}\sigma)\} \right| \\ & \leq C\|\psi\|_{C^\beta(S)} \min\{|\theta_1 - \theta_2|^\beta, |\theta_1 - R_{\theta_1}\sigma|^\beta + |\theta_2 - R_{\theta_2}\sigma|^\beta\} \\ & \leq C\|\psi\|_{C^\beta(S)} \min\{|\theta_1 - \theta_2|^\beta, |e - \sigma|^\beta\} \quad \text{for } \theta_1, \theta_2 \in S_e \text{ and } \sigma \in S, \end{aligned}$$

we may deduce by a similar integration as in the proof of Lemma 4.9 that (5.36) holds.

To prove (5.37), we write, again by a change of variable,

$$\tilde{G}_\psi(x) = \int_S \frac{\psi(\theta) - \psi(R_\theta\sigma)}{|e - \sigma|^{N+\alpha-2}} q(r, \sigma) dV(\sigma) \quad \text{for } x = r\theta \in A_e, \tag{5.39}$$

with

$$q \in C^\infty([1/2, 2] \times S), \quad q(r, \sigma) := \int_0^1 \int_0^1 \tau \partial_a^2 Q(\lambda\tau|e - \sigma|, r) d\tau d\lambda.$$

Since the function  $\sigma \mapsto \frac{1}{|e - \sigma|^{N+\alpha-2}}$  is integrable over  $S$ , it is then easy to deduce that

$$\|\tilde{G}_\psi\|_{C^{\beta-\alpha}(A_e)} \leq C\|\tilde{G}_\psi\|_{C^\beta(A_e)} \leq C\|\psi\|_{C^\beta(S)}.$$

Hence (5.37) holds as well.

In view of (5.31), (5.32) and (5.35), we can thus apply local Hölder regularity estimates for the fractional Laplacian. A version suited for our situation is that of

Theorem 1.3 of [10], which we apply rescaled and with  $k = 0$  and  $\gamma = \beta - \alpha$ . We conclude that  $u \in C_{\text{loc}}^{1,\beta}(A)$ , and that

$$\begin{aligned} \|\psi\|_{C^{1,\beta}(S)} &\leq C (\|g\|_{C^{\beta-\alpha}(S)} + \|\psi\|_{C^\beta(S)} + \|u\|_{L^\infty(\mathbb{R}^N)}) \\ &\leq C (\|g\|_{C^{\beta-\alpha}(S)} + \|\psi\|_{C^\beta(S)}). \end{aligned}$$

We finally combine this with Lemma 5.5, applied with  $\varepsilon = \frac{1}{2C}$ . We also apply the isomorphism statement in Lemma 5.3(i) to get that  $\|\psi\|_{L^2(S)} \leq \|\psi\|_{H^{1+\alpha}(S)} \leq C\|g\|_{L^2(S)}$ . We conclude the estimate

$$\begin{aligned} \|\psi\|_{C^{1,\beta}(S)} &\leq C (\|g\|_{C^{\beta-\alpha}(S)} + \|\psi\|_{L^2(S)}) \\ &\leq C (\|g\|_{C^{\beta-\alpha}(S)} + \|g\|_{L^2(S)}) \leq C\|g\|_{C^{\beta-\alpha}(S)}. \end{aligned}$$

Thus (5.25) holds. □

By a density argument, we may now deduce the following proposition from Lemma 5.3(iii) and Lemma 5.4.

**Proposition 5.6** *Let  $\alpha \in (0, 1)$ ,  $\beta \in (\alpha, 1)$ , and let the operator  $L_\alpha$  be given by (5.2). Then the operator*

$$L_\alpha + \text{id} : C^{1,\beta}(S) \rightarrow C^{\beta-\alpha}(S)$$

*is an isomorphism.*

*Proof* We first show that  $\text{Ker}(L_\alpha + \text{id}) = \{0\}$ . Let  $\psi \in C^{1,\beta}(S)$  with  $L_\alpha\psi + \psi = 0$  in  $C^{\beta-\alpha}(S)$ . By Lemma 5.3(ii) we then have  $\tilde{L}_\alpha\psi + \psi = L_\alpha\psi + \psi = 0$  in  $L^2(S)$ , and thus  $\psi = 0$  by Lemma 5.3(i).

Next we show that  $L_\alpha + \text{id}$  is onto. For this, we let  $g \in C^{\beta-\alpha}(S)$  and let  $g_n \in C^\infty(S)$  be a sequence such that  $g_n \rightarrow g$  in  $C^{\beta-\alpha}(S)$ . By Lemma 5.3(iii), there exists  $\psi_n \in C^\infty(S)$ ,  $n \in \mathbb{N}$ , with  $L_\alpha\psi_n + \psi_n = g_n$ . Moreover, by Lemma 5.4 we have

$$\|\psi_n - \psi_m\|_{C^{1,\beta}(S)} \leq C\|g_n - g_m\|_{C^{\beta-\alpha}(S)} \quad \text{for } n, m \in \mathbb{N}.$$

Consequently, the sequence  $(\psi_n)_n$  is a Cauchy sequence in  $C^{1,\beta}(S)$ , so that  $\psi_n \rightarrow \psi$  in  $C^{1,\beta}(S)$ . Moreover, by continuity we have

$$L_\alpha\psi + \psi = \lim_{n \rightarrow \infty} (L_\alpha\psi_n + \psi_n) = \lim_{n \rightarrow \infty} g_n = g \quad \text{in } C^{\beta-\alpha}(S).$$

It follows that the continuous linear map  $L_\alpha + \text{id} : C^{1,\beta}(S) \rightarrow C^{\beta-\alpha}(S)$  is bijective, and thus it is an isomorphism by the open mapping theorem. □

*Proof of Theorem 5.2 (completed)* By Proposition 5.6, we have that

$$L_\alpha + \text{id} : X \rightarrow Y \quad \text{is an isomorphism.} \tag{5.40}$$

Since the inclusion  $\text{id} : X \rightarrow Y$  is a compact operator, it follows that

$$\mathcal{L} := L_\alpha - \lambda_1 \text{id} : X \rightarrow Y \text{ is a Fredholm operator of index zero.}$$

Moreover, by Lemma 5.3(ii) and (5.3), we have that  $\text{Ker } \mathcal{L} = X \cap \mathcal{E}_1 = \{0\}$  since  $X$  is made of even functions and  $\mathcal{E}_1$  contains only odd ones. Consequently,  $\mathcal{L}$  is an isomorphism, and thus  $Dh(0) = \alpha \mathcal{L} : X \rightarrow Y$  is an isomorphism as well.  $\square$

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