On the enumeration of bipartite simple games

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Abstract

This paper provides a classification of all monotonic bipartite simple games. The problem we deal with is very versatile since simple games are inequivalent monotonic Boolean functions, functions that are used in many fields such as game theory, neural networks, artificial intelligence, reliability or multiple-criteria decision-making. The obtained classification can be implemented in an algorithm able to enumerate bipartite simple games. These numbers provide some light on enumerations of several subclasses of bipartite simple games, for which we find formulas.

Complete simple games, a subclass of all simple games for which the desirability relation is a complete preordering, were already classified by means of two parameters: a vector and a matrix fulfilling some conditions. Complete simple games are inequivalent monotonic regular Boolean functions. In this paper, we deduce a procedure for bipartite non-complete games, which allows enumerating the number of bipartite simple games. Several formulas are obtained, in particular polynomial expressions for the number of bicameral meet games and the number of bicameral join games, two types of voting systems widely used in practice.

Key words: Dedekind numbers and simple games; Inequivalent monotonic Boolean functions; Classification of bipartite simple games and bipartite Boolean functions; Enumeration of bipartite simple games and bipartite Boolean functions; Enumeration of the bicameral meet and bicameral join voting systems.
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1 Introduction

In this paper we consider monotonic simple games with two types of equivalent players, bipartite simple games, by the well-known desirability relation [26, 33]. If this relation is a complete preordering then the simple game is called complete or linear [5, 7, 43]. A classification theorem for complete simple games was obtained in [7], which made it possible to enumerate some subclasses of these simple games and to study other game theory problems. For instance, the characterization of weighted games by means of the properties of trade robustness and invariant trade robustness [19, 15] or the study of weighted games with a unique representation in integers, [20, 18, 28].

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Another consequence was the enumeration of bipartite complete games, in [21]. Let *BCG*(n) be the number of bipartite complete games with n players. The following closed formula (sequence A163250 in OEIS, [37]) was deduced:

$$BCG(n) = F(n+6) - (n^2+4n+8) \in \Theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right), \tag{1}$$

where $F(n)$ are the Fibonacci numbers which constitute a well-known sequence of integer numbers defined by the following recurrence relation: $F(n) = F(n-1) + F(n-2)$ for all $n > 1$ and $F(0) = 0$, $F(1) = 1$. See also [30] for an alternative shorter proof based on generating functions. Close games to bipartite complete games are also enumerated in [16].

The purpose of the paper is to classify all bipartite simple games, up to isomorphism. As complete games were already classified, we are concerned here with bipartite non-complete simple games. For these games, we propose a classification that allows to generate and to enumerate all of them for a moderate number of players. This enumeration together with the one in Equation (1) for bipartite complete games allows to generate and enumerate bipartite simple games for a moderate number of players. Criteria to algebraically characterize weighted games within bipartite simple games can be found in [24, 17, 15].

Many real-world examples are bipartite simple games. Bipartite simple games with a House and a Senate are common in almost all the countries in the world. A proposal passes *if and only if* it passes in both chambers (the meet of the games in the two chambers) or in either chamber (the join of the games in the two chambers). The first situation is known as the bicameral meet and the second as the bicameral join, see [43, 12] for more details on these games. The bicameral meet and bicameral join games are bipartite simple games, which are classified and enumerated in this paper with respect to the number of players.

It is worth noting that the problem we deal with in this paper is of interest in many different fields. In mathematics, the Dedekind numbers (sequence A000372 in the OEIS) form a rapidly growing sequence of integers. This sequence counts the number of monotonic Boolean functions of n variables. If two monotonic Boolean functions just differ in the labels of some variables, they are said to be equivalent. A variant of the Dedekind numbers is the sequence of the number of inequivalent monotonic Boolean functions (sequence A003182 in the OEIS). Note that monotonic simple games up to isomorphism are the same as inequivalent monotonic Boolean functions, and, complete simple games are the same as regular Boolean functions. Bipartite simple games are the same as Boolean functions with two types of equivalent variables. Thus, the results in this paper are of interest in various scientific disciplines such as Game Theory [9, 10, 42, 6, 3, 4, 30, 29], Boolean Algebra [22, 23], Reliability [2, 40, 31, 32], Neural Networks [41, 38], Threshold Logic and Coherent Structures [11, 35, 36, 39], Cryptography and Secret Sharing [34, 25, 24], Multiple-Criteria Decision Analysis (MCDA) [8] and even in Risk Analysis [1, 27].

The rest of the paper is organized as follows. Section 2 is devoted to the necessary preliminaries to follow up the paper. Section 3 contains parametrizations for bipartite complete games and for bipartite non-complete games, which allow to generate all of these games, up to isomorphism. Enumerations of bipartite simple games for small numbers of players is provided in Section 4, which follows from the parameterizations in Section 3. A formula for the number of bipartite simple games with a unique minimal winning model representative is obtained in Section 5 and another formula for the number of bipartite simple games with a maximal number of minimal winning model representatives is obtained in Section 6. Some widely-used voting systems, the bicameral meet and the bicameral join, are presented in Section 7 and we show that their enumerations are directly connected to the enumerations found in Section 5. The Conclusion ends the paper in Section 8.

2 Preliminaries

Let $N = \{1, 2, \dots, n\}$ be a set of *players*. Any subset $S \subseteq N$ is a *coalition* and s denotes its cardinality $|S|$. Let W be a set of coalitions such that: (i) $\emptyset \notin W$, (ii) if $S \subset T$ and $S \in W$ then $T \in W$, and (iii) $N \in W$. The pair (N, W) defines a (monotonic) *simple game*. The coalitions in N that are in W are called *winning* coalitions and the coalitions in N that are not in W are called *losing* coalitions. The intuition here is that a coalition S is a winning coalition *if and only if* the bill or amendment passes when the players in S are the ones who voted for it. A *minimal winning* coalition is a winning coalition all of whose proper subsets are losing. Because of monotonicity, any simple game is determined by its set of minimal winning coalitions, here denoted by W^m.

Two simple games (N, W) and (N', W') are *isomorphic* if there exists a one-to-one correspondence $f : N \to N'$ such that $S \in W$ *if and only if* $f(S) \in W'$; f is called and *isomorphism* of simple games.

Let (N, W) be a simple game. Let $W_a = \{S \in W \ : \ a \in S\}$, $\tau_{ab} : N \to N$ denotes the transposition of players $a, b \in N$. The *desirability relation* is the binary relation $\succsim$ on N: $a \succsim b$ *if and only if* $\tau_{ab}(W_b) \subseteq W_a$, and say that a is *at least as desirable as* b. The relation $\succsim$ is a preorder. The *equi-desirability relation*, is the equivalence relation $\approx$ on N: $a \approx b$ *if and only if* $a \succsim b$ and $b \succsim a$. The preorder $\succsim$ induces an ordering $\geqslant$ in the quotient set $N/\approx$ of equi-desirable classes, $N_1, N_2, \dots, N_t$. Hence, $N_p \geqslant N_q$ *if and only if* $a \succsim b$ for any $a \in N_p$ and any $b \in N_q$.

A game (N, W) is *complete* if the desirability relation is a total preordering. If the number, t, of equi-desirable classes of a simple game is $t = 1$ then the game is complete and it is called a *symmetric* game, or more specifically a *k-out-of-n game* with $k = 1, \dots, n$, which indicates that a minimum of k votes over n are required to defeat the status quo. The number of non-isomorphic symmetric games of n players is n since k can be any integer between 1 and n. In particular, the n-out-of-n game is called the *unanimity* game.

A *bipartite* simple game is a simple game with $t = 2$ equi-desirable classes, N_1 and N_2. If the bipartite simple game is complete, we assume, w.l.o.g., that $N_1 > N_2$. A bipartite simple game is not necessarily complete. With as few as $n = 4$ players one may find bipartite simple games not being complete.

Example 2.1 *Let $N = \{1, 2, 3, 4\}$ and $W^m = \{\{1, 2\}, \{3, 4\}\}$. Then, (N, W) is a bipartite game with $N_1 = \{1, 2\}$ and $N_2 = \{3, 4\}$ which is not complete since, for example, $1 \not\succsim 3$ and $3 \not\succsim 1$.*

3 Two parametrizations for bipartite simple games

The purpose of this section is to present bipartite simple games in a more compact way that allows their enumeration. We distinguish between complete and non-complete games.

Assume (N, W) is a bipartite game with classes N_1 and N_2. We can associate to each coalition S its *pair* $\overline{s} = (|S \cap N_1|, |S \cap N_2|)$. Let R and S be two coalitions, notice that if $\overline{r} = \overline{s}$, then R is winning *if and only if* S is winning, and, R is minimal winning *if and only if* S is minimal winning. In 2^N we define the equivalence relation $R \sim S$ *if and only if* $\overline{r} = \overline{s}$. Thus, the elements in the quotient set $\overline{2^N} = 2^N / \sim$ are partitioned into winning and losing pairs, denoted here $\overline{W}$ and $\overline{L}$ respectively. The minimal winning pairs $\overline{s} \in \overline{W^m}$ verify: $\overline{s} \in \overline{W}$ and $\overline{r} \in \overline{L}$ if $\overline{r} \leq \overline{s}$ and $\overline{r} \neq \overline{s}$. Thus, a bipartite game admits the more compact representation given by $\overline{n}$ and the list of minimal winning pairs $\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_r$. If $\overline{s}$ and $\overline{s}'$ are distinct minimal winning pairs, they cannot have the same first coordinate or the same second coordinate because of monotonicity, thus we can take $\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_r$ indexed so that the first coordinates of the pairs form a strictly decreasing sequence (and the second coordinates form a strictly increasing sequence).

Example 3.1 *(Example 2.1 revisited) As shown, this bipartite game is not complete. Clearly,* $1 \approx 2$ *and* $3 \approx 4$*. Thus, we can arbitrarily choose* $N_1 = \{1, 2\}$ *and* $N_2 = \{3, 4\}$ *so that* $(n_1, n_2) = (2, 2)$*. The set of winning pairs is* $\overline{W} = \{(2, 0), (0, 2), (2, 1), (1, 2), (2, 2)\}$*, the set of losing pairs is* $\overline{L} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ *and the set of minimal winning pairs is* $\overline{W^m} = \{(2, 0), (0, 2)\}$*, thus* $r = 2$*. The pair* $(2, 0)$ *represents the coalition* $\{1, 2\}$ *and the pair* $(0, 2)$ *represents the coalition* $\{3, 4\}$*. Note that the losing pair* $(1, 1)$ *represents the coalitions* $\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}$ *and the winning pair* $(2, 1)$ *represents the coalitions* $\{1, 2, 3\}, \{1, 2, 4\}$*.*

Let $\mathcal{M}(N, W)$ be the $2 \times r$ matrix whose i-th row is the pair $\overline{s}_i$, we claim that $\overline{n}$ together with the matrix $\mathcal{M}$ of minimal winning coalition pairs defines the simple game. Of course, the conditions that $\mathcal{M}$ must verify differ depending on whether the game is complete or not. These respective conditions are stated in the next two subsections.

3.1 Parameterization for bipartite non-complete games

Players a and b are incomparable *if and only if* there are minimal winning coalitions S and S' such that:

a. S contains a but not b.

b. If a is replaced by b in S, the coalition becomes losing.

c. S' contains b but not a.

d. If b is replaced by a in S, the coalition becomes losing.

We now recall a standard notation. Let $x \in (\mathbf{N} \cup \{0\})^2$ and $y \in (\mathbf{N} \cup \{0\})^2$, we write that $x \geq y$ if either $x = y$ or $x_i \geq y_i$ for $i = 1, 2$ and write $x > y$ if $x \geq y$ and $x \neq y$.

From the above comments about players incomparability and monotonicity, it follows that there is a row (s_1, s_2) in $\mathcal{M}$ with $s_1 > 0$ and $s_2 < n_2$ such that $(s_1 - 1, s_2 + 1)$ is not greater or equal than a row of $\mathcal{M}$ (i.e., the pair (s_1-1, s_2+1) represents losing coalitions) and (s_1, s_2) in $\mathcal{M}$ with $s_1 < n_1$ and $s_2 > 0$ such that (s_1+1, s_2-1) is not greater or equal than a row of $\mathcal{M}$ (i.e., the pair (s_1+1, s_2-1) represents losing coalitions).

Thus, every bipartite non-complete game can be represented by $\overline{n}$ and $\mathcal{M}$ with these two properties, and choose $n_1 \geq n_2$ to avoid duplicities in the presentation of the game. We distinguish the case $n_1 \neq n_2$ to the case $n_1 = n_2$.

If $n_1 > n_2$, and the vector $\overline{n} = (n_1, n_2)$ together with $\mathcal{M}$ define a bipartite non-complete game, then $\overline{n}' = (n_2, n_1)$ together with $\mathcal{M}'$, where $\mathcal{M}'$ is obtained from $\mathcal{M}$ by swaping the two columns and inverting their orderings, is isomorphic to the bipartite non-complete game given by $\overline{n}$ and $\mathcal{M}$. Hence, if $n_1 \neq n_2$ it is sufficient to consider only $n_1 > n_2$ for generating all feasible (non-isomorphic) bipartite non-complete games.

If $n_1 = n_2$ the previous process leads to the same bipartite non-complete game since $\overline{n} = \overline{n}'$ and $\mathcal{M} = \mathcal{M}'$. Nevertheless, another type of duplicity may arise, to avoid it consider the following relation for vectors. Let $x \in (\mathbf{N} \cup \{0\})^r$ and $y \in (\mathbf{N} \cup \{0\})^r$, we write that $x \ L \ y$ if either $x = y$ or there is some i $(1 \leq i \leq r)$ such that $x_i > y_i$ and $x_j = y_j$ for all j with $j < i$. Then the duplicities for $n_1 = n_2$ are avoided by demanding to the matrix $\mathcal{M}$ the condition:

$$(s_{1,1}, s_{2,1}, \ldots, s_{r,1}) \ \ L \ \ (s_{r,2}, s_{r-1,2}, \ldots, s_{1,2}).$$

The next example illustrates these two situations.

Example 3.2 *a. The bipartite non-complete game given by* $\overline{n} = (4,3)$ *and* $\mathcal{M} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$ *is isomorphic to the bipartite non-complete game given by* $\overline{n}' = (3,4)$ *and* $\mathcal{M}' = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$*. By convention, we will only consider the former game since* $n_1 > n_2$*. This non-complete game of* 7 *players is defined by* 16 *minimal winning coalitions:* $(3,0)$ *represents* 4 *minimal winning coalitions and* $(1,2)$ *represents the other* 12*.*

b. The bipartite non-complete game given by $\overline{n} = (4,4)$ *and* $\mathcal{M} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}$ *is isomorphic to the bipartite non-complete game* $\overline{n} = (4,4)$ *and* $\mathcal{M}' = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$*. By convention, we will only consider the former game since* $(3,1) \, L \, (2,0)$*.*

From the explanations in this section and the similarity with the classification of complete games provided in [7], we state an equivalent, but more compact, way to present bipartite non-complete games.

Definition 3.3 *The pair ($\overline{n}$, $\mathcal{M}$) associated with a bipartite non-complete simple game* (N, W) *satisfy the following properties:*

(1) $n_1 \geq n_2 > 0$,

(2) $\overline{0} < \overline{s}_i < \overline{n}$ *for* $i = 1, 2, \ldots, r$,

(3) $s_{i,1} > s_{i+1,1}$ *and* $s_{i,2} < s_{i+1,2}$ *for* $i = 1, 2, \dots, r-1$,

(4) *it exists a row* $\overline{s}_p = (s_{p,1}, s_{p,2})$ *of* $\mathcal{M}$ *such that* $\overline{0} < \overline{s}_p + (-1, 1) < \overline{n}$ *and* $\overline{s}_p + (-1, 1)$ *is not greater or equal than a row of* $\mathcal{M}$,

(5) *it exists a row* $\overline{s}_p = (s_{p,1}, s_{p,2})$ *of* $\mathcal{M}$ *such that* $\overline{0} < \overline{s}_p + (1, -1) < \overline{n}$ *and* $\overline{s}_p + (1, -1)$ *is not greater or equal than a row of* $\mathcal{M}$,

(6) $(s_{1,1}, s_{2,1}, \dots, s_{r,1})\ L\ (s_{r,2}, s_{r-1,2}, \dots, s_{1,2})$ *if* $n_1 = n_2$.

Moreover, each bipartite non-complete game (N, W) *can be obtained from a pair* $(\overline{n}, \mathcal{M})$ *satisfying the conditions (1)-(6).*

Observe that neither $\overline{0}$ nor $\overline{n}$ can be a row of $\mathcal{M}$ because $\overline{0}$ corresponds to the empty coalition, which is always losing, and, $\overline{n}$ corresponds to the grand coalition N, which cannot be minimal in a bipartite game. Condition (4) guarantees that $N_2 \not\succcurlyeq N_1$ and condition (5) that $N_1 \not\succcurlyeq N_2$. From the conditions of Definition 3.3 we can exhaustively list all bipartite non-complete games for moderate values of n. The next example shows all of these games for $n < 7$.

Example 3.4 *a. For* $n = 1$ *there are no bipartite simple games and for* $n = 2$ *and* $n = 3$ *all bipartite simple games are complete.*

b. For $n = 4$ *there are two bipartite non-complete games, which are obtained from the vector* $\overline{n} = (2, 2)$ *and the matrices:*

$$(1\ 1); \qquad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Note that the second game is a compact presentation of the game introduced in Example 2.1.

c. For $n = 5$ *there are six bipartite non-complete games, which are obtained from the vector* $\overline{n} = (3, 2)$ *and the matrices:*

$$(2\ 1); \quad (1\ 1); \quad \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}; \quad \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}; \quad \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}; \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

d. For $n = 6$ *there are* 27 *bipartite non-complete games, which are obtained from the vectors* $(3, 3)$ *and* $(4, 2)$. *The matrices for* $(3, 3)$ *are:*

$$(2\ 2); \ (2\ 1); \ (1\ 1); \ \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}; \ \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}; \ \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}; \ \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}; \ \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}; \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}; \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}; \ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}; \ \begin{pmatrix} 3 & 0 \\ 2 & 2 \\ 0 & 3 \end{pmatrix}; \ \begin{pmatrix} 3 & 0 \\ 2 & 1 \\ 0 & 3 \end{pmatrix}; \ \begin{pmatrix} 3 & 0 \\ 1 & 1 \\ 0 & 3 \end{pmatrix}.$$

The matrices for $(4, 2)$ *are:*

$$(3\ 1); \ (2\ 1); \ (1\ 1); \ \begin{pmatrix} 4 & 0 \\ 2 & 2 \end{pmatrix}; \ \begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix}; \ \begin{pmatrix} 4 & 0 \\ 1 & 2 \end{pmatrix}; \ \begin{pmatrix} 4 & 0 \\ 1 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}; \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}; \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}; \ \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}; \ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

3.2 A parameterization for bipartite complete games

A parameterization for complete simple games was obtained in [7] by using models of shift-minimal winning coalitions (a subclass of minimal winning coalitions). Here we adapt it for the bipartite case and for pairs of minimal winning coalitions. As the game is complete we can assume w.l.o.g. that $N_1 > N_2$ so that if $a \in N_1$, $b \in N_2$ and S is a minimal winning coalition containing b but not a, then $(S \setminus \{b\}) \cup \{a\}$ is winning (see condition (3) in Definition 3.5). Moreover, it exists a minimal winning coalition T that contains a but not b such that $(T \setminus \{a\}) \cup \{b\}$ is not winning (see condition (4) in Definition 3.5).

Definition 3.5 *The pair ($\overline{n}$, $\mathcal{M}$) associated with a bipartite complete simple game (N, W) satisfies the following properties:*

(1) $\overline{0} < \overline{s}_i < \overline{n}$ *for* $i = 1, 2, \dots, r$,

(2) *either* $s_{1,1} = n_1$ *or* $s_{1,2} = 0$,

(3) $s_{i+1,1} = s_{i,1} - 1$ *and* $s_{i,2} < s_{i+1,2}$ *for* $i = 1, 2, \dots, r-1$,

(4) *it exists a row* $\overline{s}_p = (s_{p,1}, s_{p,2})$ *of* $\mathcal{M}$ *such that* $\overline{0} < \overline{s}_p + (-1, 1) < \overline{n}$ *and* $\overline{s}_p + (-1, 1)$ *is not greater or equal than a row of* $\mathcal{M}$.

Moreover, each bipartite complete game (N, W) can be obtained from a pair $(\overline{n}, \mathcal{M})$ satisfying the conditions (1)-(4).

Condition (1) expresses that the pairs of minimal winning coalitions are well-defined for the bipartite game. Condition (2) guarantees that $N_1 \geqslant N_2$ if $r = 1$, and, conditions (2) and (3) guarantee that $N_1 \geqslant N_2$ if $r > 1$, because for each row $\overline{s}_p$ of $\mathcal{M}$ such that $\overline{0} < \overline{s}_p + (1, -1) < \overline{n}$, it holds that $\overline{s}_p + (1, -1)$ is equal or greater than a row of $\mathcal{M}$. Finally, condition (4) witnesses the existence of a minimal winning coalition T such that $(T \setminus \{a\}) \cup \{b\}$ is not winning with $a \in N_1$ and $b \in N_2$. Hence, $N_2 \not\geqslant N_1$ and therefore $N_1 > N_2$.

Observe that if $s_{r,1} > 0$ and $s_{i+1,2} = s_{i,2} + 1$ for all $i = 1, 2, \dots, r-1$, then $s_{r,2} < n_2$, otherwise (4) would fail.

4 Enumeration of bipartite simple games for a small number of players

Some enumerations can be deduced from the classifications given in Definitions 3.3 and 3.5. For this purpose we respectively denote by *BCG*(n), *BNCG*(n) and *BSG*(n) the number of bipartite complete games, bipartite non-complete games and bipartite simple games of n players. Let r be the number of minimal winning pairs of a bipartite simple game, we respectively denote by *BCG*(n, r), *BNCG*(n, r) and *BSG*(n, r) the number of bipartite complete games, bipartite non-complete games and bipartite simple games of n players and r minimal winning pairs.

The aim of the next result is to identify the infeasible combinations for n and r.

Lemma 4.1 BNCG$(n) = 0$ if $n \leq 3$ *and* BNCG$(n, r) = 0$ if $n > 3$ *and* $r > \left\lfloor \frac{n}{2} \right\rfloor$.

Proof: If $n \leq 3$, $\overline{n}$ has at least a component equal to 1 so that either (4) or (5) fail in Definition 3.3. From Definition 3.3 (conditions (1) and (3)) we have $n_1 \geq n_2$ and $s_{i,1} > s_{i+1,1}$ and $s_{i,2} < s_{i+1,2}$ for $i = 1, \ldots, r-1$. These conditions together with (2) imply that the maximal number of minimal winning pairs r is $r \leq \min\{n_1 + 1, n_2 + 1\}$, but the conditions (4) and (5) in Definition 3.3 imply that none of the two columns in the matrix of minimal winning pairs is formed by consecutive numbers. Thus, the maximal number of rows is $r = \min\{n_1, n_2\}$. As $n_2 = n - n_1$ and $n_1 = 1, \ldots, n-1$, it holds:

$$\max_{n_1} \min\{n_1, n - n_1\} = \left\lfloor \frac{n}{2} \right\rfloor$$

and, therefore, $\lfloor \frac{n}{2} \rfloor$ is an upper bound for r if the bipartite game is not complete and $n \geq 4$. □

In Section 6 we will see that *BNCG*$(n, r) \neq 0$ *if* $n > 3$ and $r \leq \left\lfloor \frac{n}{2} \right\rfloor$. The following result is immediately deduced from Lemma 4.1.

Corollary 4.2

$$\text{BNCG}(n) = \begin{cases} 0 & \textit{if } n \leq 3 \\ \sum_{r=1}^{\lfloor n/2 \rfloor} \text{BNCG}(n, r) & \textit{if } n > 3 \end{cases}$$

From the conditions of Definition 3.3 we have obtained, by the implementation of a routine, the number of bipartite non-complete games for some small combinations of n and r, see Table 1.

n ↓ / r →	1	2	3	4	5	6
4	1	1				
5	2	4				
6	6	18	3			
7	10	45	16			
8	19	107	72	6		
9	28	206	210	39		
10	44	381	543	190	10	
11	60	634	1190	633	76	
12	85	1025	2425	1817	406	15
13	110	1556	4528	4480	1522	130

Table 1: The positive numbers of bipartite non-complete games *BNCG*(n, r), up to isomorphisms, for $n < 14$.

Table 2 shows *BNCG*(n), *BCG*(n) and *BSG*(n) for small values of n. The numbers *BNCG*(n) are deduced from Corollary 4.2 and Table 1, which are both consequences of Definition 3.3. The numbers *BCG*(n) are derived from Equation (1) and the numbers *BSG*(n) are simply the sum of *BCG*(n) and *BNCG*(n).

n	*BCG*(n)	*BNCG*(n)	*BSG*(n)
1	0	0	0
2	1	0	1
3	5	0	5
4	15	2	17
5	36	6	42
6	76	27	103
7	148	71	219
8	273	204	477
9	485	483	968
10	839	1168	2007
11	1424	2593	4017
12	2384	5773	8157
13	3952	12326	16278

Table 2: The numbers of bipartite non-complete *BNCG*(n), bipartite complete *BCG*(n), and bipartite simple games *BSG*(n), up to isomorphisms, for $n < 14$.

5 A formula for the number of bipartite simple games with a unique minimal winning pair

The aim of this section is to determine the number of bipartite simple games with a unique pair of minimal winning coalitions, i.e., to determine *BSG*$(n, r = 1)$ for all n. We start by finding *BNCG*$(n, r = 1)$ for all n.

Lemma 5.1 *Let $r = 1$ and $n = n_1 + n_2$. The conditions in Definition 3.3 imply that:*

a. there are $(n_1 - 1)(n_2 - 1)$ bipartite non-complete games with $r = 1$ for each vector decomposition (n_1, n_2) such that $n_1 > n_2$,

b. there are $n(n-2)/8$ bipartite non-complete games with $r = 1$ for each vector decomposition (n_1, n_2) such that $n_1 = n_2$.

Proof:

a. Assume $n_1 > n_2$. For the row (s_1, s_2) of $\mathcal{M}$ we can choose any s_1 such that $0 < s_1 < n_1$ and any s_2 such that $0 < s_2 < n_2$. Any of these choices verify the conditions in Definition 3.3. Thus, there are $(n_1 - 1)(n_2 - 1)$ bipartite non-complete games.

b. Assume $n_1 = n_2 = n/2$. For the row (s_1, s_2) of $\mathcal{M}$ we can choose any s_1 such that $0 < s_1 < n/2$ and any s_2 such that $0 < s_2 \leq s_1$. Any of these choices verify the conditions in Definition 3.3. Thus, there are $n(n-2)/8$ bipartite non-complete games.

□

Theorem 5.2 *The number of bipartite non-complete games with one pair of minimal winning coalitions is:*

$$\text{BNCG}(n, r=1) = \begin{cases} \dfrac{(n-2)(n^2-4n+6)}{12}, & \textit{if } n \textit{ is even} \\[2ex] \dfrac{(n-1)(n-2)(n-3)}{12}, & \textit{if } n \textit{ is odd} \end{cases}$$

Proof: Assume n is even. We apply the result obtained in Lemma 5.1 to each vector (n_1, n_2) with integer components satisfying $n_1 \geq n_2$, $n_2 > 0$ and $n_1 + n_2 = n$. The vectors $\overline{n}$ we need to consider are:

$$(a+1, a+1), (a+2, a), \ldots, (a+(n-a-3), 3), (a+(n-a-2), 2)$$

where $a = n/2 - 1$. According to Lemma 5.1 the respective number of games is: $(a+k)(a-k)$ for $k = 1, \ldots, n/2 - 2$. Thus,

$$BNCG(n, r=1) = \frac{n(n-2)}{8} + \sum_{k=1}^{\frac{n}{2}-2} (a+k)(a-k),$$

where

$$\sum_{k=1}^{\frac{n}{2}-2} (a+k)(a-k) = \sum_{k=1}^{\frac{n}{2}-2} a^2 - \sum_{k=1}^{\frac{n}{2}-2} k^2 = \left(\frac{n}{2} - 2\right) a^2 - \sum_{k=1}^{\frac{n}{2}-2} k^2. \tag{2}$$

We now use the identity

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \tag{3}$$

and replace a by $n/2 - 1$ in equation (2). Then, for n even we obtain:

$$BNCG(n, r=1) = \frac{(n-4)(n-2)^2}{8} - \frac{(n-2)(n-3)(n-4)}{24} + \frac{n(n-2)}{8} = \frac{(n-2)(n^2-4n+6)}{12}$$

Assume n is odd. The procedure followed is, *mutatis mutandis*, the same. The vectors $\overline{n}$ we need to consider are:

$$(a+2, a+1), (a+3, a), \ldots, (a+(n-a-3), 3), (a+(n-a-2), 2)$$

where $a = (n-3)/2$. According to Lemma 5.1 the respective number of games is: $(a+1+k)(a-k)$ for $k = 0, 1, \ldots, (n-5)/2$. By using again the identity (3) and $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ we deduce the final expression for n odd.

$$BNCG(n, r=1) = \sum_{k=0}^{\frac{n-5}{2}} (a+1+k)(a-k) = \sum_{k=0}^{\frac{n-5}{2}} a^2 + a - \sum_{k=1}^{\frac{n-5}{2}} k + k^2 = \frac{(n-1)(n-2)(n-3)}{12}.$$

□

Once determined a closed formula for *BNCG*$(n, r = 1)$ we proceed to compute *BCG*$(n, r = 1)$.

We recall here that a player $a \in N$ is *null* in a simple game (N, W) if $a \notin S$ for every $S \in W^m$. The null players of a game (if any) form an equi-desirable class, which is the minimum for $\geqslant$. A player $a \in N$ has *veto* if $a \in S$ for every $S \in W$. The veto players of a game (if any) form an equi-desirable class, which is the maximum for $\geqslant$.

Proposition 5.3 *The number of bipartite complete games with one pair of minimal winning coalitions is:*

$$\mathrm{BCG}(n, r = 1) = (n-1)^2.$$

Proof: As the game is bipartite and complete it follows $t = 2$ and $n \geq 2$ and we assumed, w.l.o.g., that $N_1 > N_2$ with respective cardinalities n_1 and n_2 verifying: $n_1 > 0$, $n_2 > 0$ and $n_1 + n_2 = n$.

As $r = 1$, let (s_1, s_2) be the unique pair of minimal winning coalitions. As stated in condition (2) of Definition 3.5 it is either $s_1 = n_1$ or $s_2 = 0$, otherwise $(s_1 + 1, s_2 - 1)$ is not greater or equal than (s_1, s_2) and therefore $N_1 \geqslant N_2$ would be false.

For an arbitrary number of players $n \geq 2$, let us consider all the vector decompositions (n_1, n_2) verifying $n_1 > 0$, $n_2 > 0$ and $n_1 + n_2 = n$. For each of these $n - 1$ decompositions there are $n - 1$ pairs (s_1, s_2) verifying either $s_1 = n_1$ or $s_2 = 0$ because (n_1, n_2) and $(0, 0)$ are not possible for condition (1) in Definition 3.5. Thus, *BCG*$(n, r = 1) = (n - 1)^2$. $\square$

In words, the $(n - 1)^2$ bipartite complete games with a unique pair of minimal winning coalitions have either veto players or nulls.

From Theorem 5.2 and Proposition 5.3 we obtain the formula for the number of bipartite simple games with one pair of minimal winning coalitions.

Corollary 5.4 *The number of bipartite simple games with one pair of minimal winning coalitions is:*

$$\mathrm{BSG}(n, r = 1) = \begin{cases} \dfrac{(n-2)(n^2 - 4n + 6)}{12} + (n-1)^2, & \textit{if } n \textit{ is even} \\[2ex] \dfrac{(n-1)(n-2)(n-3)}{12} + (n-1)^2, & \textit{if } n \textit{ is odd} \end{cases}$$

The sequence *BNCG*$(n, r = 1)$ obtained in Theorem 5.2 appears (sequence A005993 in OEIS) for an alternative enumeration and its generating function is: $\frac{1+x^2}{(1-x)^2(1-x^2)^2}$.

6 A formula for the number of bipartite simple games with a maximal number of minimal winning pairs

The aim of this section is to determine the number of bipartite simple games, as a function of the number of players n, with a maximal number of minimal winning pairs. The next result is the analogous to Lemma 4.1 for bipartite complete games.

Lemma 6.1 *For* $n > 1$, $\mathrm{BCG}(n, r) = 0$ if $r > \left\lceil \frac{n}{2} \right\rceil$.

Proof: As the game is complete and bipartite we can assume $N_1 > N_2$, where $|N_i| = n_i > 0$ for $i = 1, 2$ and $n_1 + n_2 = n$. From conditions (1), (2) and (3) in Definition 3.5 it follows that $r \leq \min\{n_1, n_2\} + 1$. As $n_2 = n - n_1$ and $n_1 = 1, \ldots, n-1$, it holds:

$$\max_{n_1} \min\{n_1, n - n_1\} + 1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$$

and, therefore, $\lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor + 1$ is an upper bound for r, if the bipartite game is complete, n is odd and $n \neq 1$. If n is even, the upper bound for r is $\frac{n}{2} + 1$, but the only matrix that verifies conditions (1), (2) and (3) fails to verify condition (4). Thus, there are not complete games for n players and $r > \left\lceil \frac{n}{2} \right\rceil$. □

The next result provides *BCG*(n, r) if $r = \left\lceil \frac{n}{2} \right\rceil$.

Proposition 6.2 *The number of bipartite complete games with a maximal number of minimal winning pairs* $\left\lceil \frac{n}{2} \right\rceil$*, is:*

$$\mathrm{BCG}(n, r = \left\lceil \frac{n}{2} \right\rceil) = \begin{cases} \dfrac{1}{8}(n^2 + 14n - 24), & \textit{if } n \textit{ is even} \\ \\ \dfrac{1}{2}(n-1), & \textit{if } n \textit{ is odd} \end{cases}$$

Proof: Assume n even. Only the vectors $(n/2+1, n/2-1)$, $(n/2, n/2)$, and $(n/2-1, n/2+1)$ allow for matrices with $n/2$ rows. For the matrices of each of these vectors $\overline{n}$, we have:

a. $\overline{n} = (n/2+1, n/2-1)$. The second column is then formed by all the consecutive numbers from 0 to $n/2 - 1$. Thus, condition (4) in Definition 3.5 is not verified.

b. $\overline{n} = (n/2, n/2)$. As the first column is formed by consecutive numbers there are only two choices for it. Only one number from 0 to $n/2$ does not appear in the second column. Thus, we have $2(n/2+1) = n+2$ possible matrices. If the missing number in the second column is any number between 1 and $n/2-1$ then the conditions (1)-(4) in Definition 3.5 are verified. The remaining 4 matrices have the second column formed by consecutive numbers, from these only the matrix verifying $s_{1,1} = n/2$ and $s_{1,2} = 0$ satisfies the conditions (1)-(4) in Definition 3.5. Hence for the vector $\overline{n} = (n/2, n/2)$ we have $(n+2) - 3 = n - 1$ matrices.

c. $\overline{n} = (n/2-1, n/2+1)$. The first column is formed by all the consecutive numbers from 0 to $n/2 - 1$ in decreasing order. The second column is formed by $n/2$ numbers from 0 to $n/2 + 1$ in increasing order. Thus, there are $\binom{\frac{n}{2}+2}{\frac{n}{2}}$ choices, which all of them verify the conditions (1)-(4) in Definition 3.5, with the exception of condition (4) when the $n/2$ numbers of the second column are consecutive. Thus, there are $\binom{\frac{n}{2}+2}{\frac{n}{2}} - 3 = \frac{1}{8}(n^2+6n-16)$ bipartite complete games for the vector $(n/2-1, n/2+1)$.

Then, the result for n even is obtained by adding $(n-1)+\frac{1}{8}(n^2+6n-16)$ which equals $\frac{1}{8}(n^2+14n-24)$.

Assume n odd. Only the vectors $((n-1)/2,(n+1)/2)$ and $((n+1)/2,(n-1)/2)$ allow for matrices with $(n+1)/2$ rows. For the matrices of each of these vectors $\overline{n}$, we have:

a. $((n-1)/2,(n+1)/2)$. All matrices have the first column formed by all the consecutive numbers from 0 to $(n-1)/2$ in decreasing order and only one number from 0 to $(n+1)/2$ is missing in the second column. Thus, we have $(n+3)/2$ possible matrices. Two of these matrices have the second column formed by consecutive numbers, which do not verify condition (4) in Definition 3.5. The remaining matrices verify all the conditions in Definition 3.5. Thus, we have $(n-1)/2$ matrices.

b. $((n+1)/2,(n-1)/2)$. All matrices have the second column formed by all the consecutive numbers from 0 to $(n-1)/2$ in increasing order and only one number from 0 to $(n+1)/2$ is missing in the first column. If the missing number in the first column is not 0 nor $(n+1)/2$ then condition (3) in Definition 3.5 is not verified. If the missing number in the first column is either 0 or $(n+1)/2$ then condition (4) in Definition 3.5 is not verified. Thus, there are not bipartite complete games for the vector $((n+1)/2,(n-1)/2)$ with $(n+1)/2$ pairs of minimal winning coalitions.

Then, for n odd we have $(n-1)/2$ bipartite complete games. □

The next more elaborated result determines *BNCG*$(n,\lfloor\frac{n}{2}\rfloor)$ for $n\geq 4$, which together with Proposition 6.2 allows to find the number of bipartite simple games with a maximal number of minimal winning pairs.

Theorem 6.3 *The number of bipartite non-complete simple games, for $n\geq 4$, with a maximal number of minimal winning pairs $\lfloor\frac{n}{2}\rfloor$, is:*

$$\text{BNCG}(n,r=\left\lfloor\frac{n}{2}\right\rfloor)=\begin{cases}\frac{1}{8}n(n-2), & \textit{if } n \textit{ is even}\\[2ex] \frac{1}{16}(n^3+n^2-25n+39), & \textit{if } n \textit{ is odd}\end{cases}$$

Proof:

a. Let n be an even number. As $r=\lfloor\frac{n}{2}\rfloor=\frac{n}{2}$, the only decompositions for (n_1,n_2) with $n_1\geq n_2$ that achieve $r=\frac{n}{2}$ are: $(\frac{n}{2},\frac{n}{2})$ and $(\frac{n}{2}+1,\frac{n}{2}-1)$ because the conditions (1), (2) and (3) in Definition 3.3 establish that the numbers in the first column should appear in decreasing ordering and those in the second column in increasing ordering. Nevertheless, conditions (4) and (5) tell that none of the two columns can be formed by consecutive numbers. Thus, no matrix with the decomposition $(\frac{n}{2}+1,\frac{n}{2}-1)$ can achieve $r=\frac{n}{2}$ because the second column of the matrix has only room for $n/2$ numbers, which necessarily would need to be consecutive, contradicting the requirements in Definition 3.3.

Assume from now on that the bipartite non-complete game has the decomposition $(\frac{n}{2}, \frac{n}{2})$, by the same stated reason only one number p from 0 to $\frac{n}{2}$ is missed in the first column and only one number q from 0 to $\frac{n}{2}$ is missed in the second column, but from the conditions (4) and (5) in Definition 3.3 no column can be formed by consecutive numbers. Thus, the missing number p of the first column of the matrix verifies $1 \le p \le \frac{n}{2} - 1$ and the same applies for the missing number q of the second column. Hence, potentially there are $(\frac{n}{2} - 1)^2$ bipartite non-complete games with $r = \frac{n}{2}$, but some of them are isomorphic. Indeed, condition (6) in Definition 3.3 avoids these duplicities. Verification of condition (6) means that only those the (p, q) such that $p + q \ge r$ need to be counted. Hence, the cardinality of the set:

$$\{(p, q) \in \mathbb{N} \times \mathbb{N} \, : \, p + q \ge r, 1 \le p \le r \text{ and } 1 \le q \le r\}$$

provides the number of bipartite non-complete games with a maximal number of minimal winning pairs for an even number of players n. This number is

$$\sum_{j=1}^{r-1} j = \frac{r(r-1)}{2} = \frac{1}{8}n(n-2).$$

b. Let n be an odd number. As $r = \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$, the only decompositions for (n_1, n_2) with $n_1 > n_2$, as assumed in Definition 3.3, are: $(\frac{n+1}{2}, \frac{n-1}{2})$ and $(\frac{n+3}{2}, \frac{n-3}{2})$, but no matrix verifies the conditions in Definition 3.3 for $(\frac{n+3}{2}, \frac{n-3}{2})$ because all the numbers from 0 to $\frac{n-3}{2}$ appear in the second column. Thus, the only possibility is the decomposition $(\frac{n+1}{2}, \frac{n-1}{2})$.

Given any matrix, for this vector we have $\frac{n+3}{2}$ feasible digits for the first column and $\frac{n+1}{2}$ feasible digits for the second column. According to the conditions (1), (2) and (3) in Definition 3.3 the numbers in the first column should appear in decreasing ordering and those in the second column in increasing ordering. Thus, potentially we have:

$$\binom{\frac{n+3}{2}}{2} \cdot \binom{\frac{n+1}{2}}{1} = \frac{(n+3)(n+1)^2}{16} \tag{4}$$

feasible choices with these orderings. But, there are three types of these matrices that do not verify some condition in Definition 3.3. These three types are those matrices which have at least a column formed by consecutive numbers. The remaining matrices fulfill conditions (4) and (5) in Definition 3.3. We distinguish the three different cases formed by consecutive numbers in some column:

- Matrices with $s_{1,2} \neq 0$.
  For all these matrices we have $s_{i,2} = i$ for all row i. But then condition (4) in Definition 3.3 fails, which means that $N_2 \geqslant N_1$ and the game is complete, a contradiction with the assumed non-completeness of the game. The number of matrices with $s_{1,2} \neq 0$ is:
  $$\binom{\frac{n+3}{2}}{2} = \frac{(n+3)(n+1)}{8} \tag{5}$$
- Matrices with $s_{1,2} = 0$ and all numbers in the first column being consecutive.
  This implies that Condition (5) in Definition 3.3 fails, which implies $N_1 \geqslant N_2$ and

the game is complete, a contradiction. The number of matrices with $s_{1,2} = 0$ and all numbers in the first column being consecutive is:

$$3\frac{n-1}{2} \tag{6}$$

- Matrices with $s_{r,1} = 0$ and $s_{i,2} = i - 1$ for all $i = 1, \ldots, r$.
  Then condition (4) in Definition 3.3 fails, which implies $N_2 \geqslant N_1$ and the game is complete. The number of matrices with $s_{r,1} = 0$ and $s_{i,2} = i-1$ for all $i = 1, \ldots, r-1$ is:

$$\binom{\frac{n+1}{2}}{2} = \frac{(n+1)(n-1)}{8} \tag{7}$$

  but one of these matrices has been already counted in the previous item.

Thus, *BNCG*$(n, r = \frac{n-1}{2})$ for an odd integer n is obtained by subtracting the sum of the expressions (5), (6) and (7) to the expression in equation (4) plus 1. Hence,

$$\textit{BNCG}\left(n, r = \frac{n-1}{2}\right) = \frac{(n+3)(n+1)^2}{16} + 1 - \frac{(n+3)(n+1)}{8} - 3\frac{n-1}{2} - \frac{(n+1)(n-1)}{8},$$

which after simplification becomes

$$\textit{BNCG}\left(n, r = \frac{n-1}{2}\right) = \frac{1}{16}(n^3 + n^2 - 25n + 39).$$

□

Note that the last number in each row in Table 1 is given by the formula in Theorem 6.3. As a consequence of Proposition 6.2 and Theorem 6.3 we can deduce the number of bipartite simple games with a maximal number of minimal winning pairs.

Corollary 6.4

$$\mathrm{BSG}\left(n, r = \left\lceil \frac{n}{2} \right\rceil\right) = \begin{cases} \frac{1}{4}(n^2 + 6n - 12), & \textit{if } n \textit{ even} \\ \frac{1}{2}(n-1), & \textit{if } n \textit{ odd} \end{cases}$$

7 Bicameral meet and bicameral join: classification and enumeration

Two of the most widely used voting systems in practice are the bicameral meet and the bicameral join. The purpose of this section is to enumerate up to isomorphism the number of these games for all n. In addition, a strong link between the results of this section and those of the previous one is proved.

Let N_1 and N_2 be two independent chambers (for example a House and a Senate), i.e. $N_1 \cap N_2 = \emptyset$, and (N_i, W_i) for $i = 1, 2$ is a k_i-out-of-n_i game.

a. The *bicameral meet* of the two chambers is the simple game (N, W) such that $N = N_1 \cup N_2$ and $S \in W$ *if and only if* $S = S_1 \cup S_2$ with $S_1 \in W_1$ and $S_2 \in W_2$.

b. The *bicameral join* of the two chambers is the simple game (N, W) such that $N = N_1 \cup N_2$ and $S \in W$ *if and only if* either $S \supseteq S_1$ or $S \supseteq S_2$ with $S_1 \in W_1$ and $S_2 \in W_2$.

Whenever N_1 and N_2 are known (and thus n_1 and n_2 are also known), we refer to the first game as the (k_1, k_2)-*bicameral meet* and to the second as the (k_1, k_2)-*bicameral join*. Note that neither the (k_1, k_2)-*bicameral meet* game nor the (k_1, k_2)-*bicameral join* have null voters because the integer numbers k_i are positive.

These two types of games are linked by duality as shown in the next result. Recall that the *dual game* (N, W^*) of a simple game (N, W) is defined as $W^* = \{S \subseteq N : N \setminus S \notin W\}$. It is easy to verify that $\succsim^* = \succsim$ and $\approx^* = \approx$ and a game is complete *if and only if* the dual is.

Proposition 7.1 *The dual game of the* (k_1, k_2)*-bicameral meet game is the* $(n_1 - k_1 + 1, n_2 - k_2 + 1)$*-bicameral join game.*

Proof: Consider the (k_1, k_2)-bicameral meet game. Any coalition S formed by either $k_1 - 1$ members of the first type and n_2 members of the second type or by n_1 members of the first type and $k_2 - 1$ members of the second type is not winning, moreover any superset of S is a winning coalition. Thus, $N \setminus S$ is a minimal winning coalition in the dual game and it has either $n_1 - k_1 + 1$ members of the first type or $n_2 - k_2 + 1$ members of the second type. □

Proposition 7.2 *The* (k_1, k_2)*-bicameral meet game of two chambers with respective cardinalities* n_1 *and* n_2 *is:*

a. a bipartite game if and only if $k_1 + k_2 < n_1 + n_2$.

b. a bipartite complete game if $k_i = n_i$ *for one of the two chambers* $i = 1, 2$.

c. a bipartite non-complete game if $k_i < n_i$ *for every* $i = 1, 2$.

Proof:

a. The conditions $k_1 = n_1$ and $k_2 = n_2$ imply unanimity in both chambers and then unanimity in the bicameral meet game. Thus, the game has only one minimal winning coalition, N, and all players are equi-desirable. Hence, the game is not bipartite but symmetric. The next two parts prove the other implication.

b. As $k_i = n_i$ for one of the two chambers $i = 1, 2$, the players of such chamber have veto in the (k_1, k_2)-bicameral meet game of the two chambers and the players in the other chamber have no veto in the (k_1, k_2)-bicameral meet game. Thus, the players in the former chamber, that with $k_i = n_i$, strictly dominate players in the other chamber in the (k_1, k_2)-bicameral meet game. Hence, the game is bipartite and complete.

c. As $0 < k_i < n_i$ for every $i = 1, 2$. Both (k_1+1, k_2-1) and (k_1-1, k_2+1) are well-defined pairs of losing coalitions, which imply that neither $N_1 \geq N_2$ nor $N_2 \geq N_1$ are true. Thus, the game is bipartite and non-complete.

□

The next result is a consequence of the duality result, Proposition 7.1, and Proposition 7.2.

Proposition 7.3 *The (k_1, k_2)-bicameral join game of two chambers with respective cardinalities n_1 and n_2 is:*

a. a bipartite game if and only if $k_1 + k_2 > 2$.

b. a bipartite complete game if $k_i = 1$ for one of the two chambers $i = 1, 2$.

c. a bipartite non-complete game if $k_i > 1$ for every $i = 1, 2$.

Let *BMCG*(n), *BMNCG*(n) and *BMSG*(n) be the respective number of bicameral meet: complete games, non-complete games, and simple games of n players. Let *BJCG*(n), *BJNCG*(n) and *BJSG*(n) be the respective number of bicameral join: complete games, non-complete games, and simple games of n players. From Proposition 7.2-(c) and Proposition 7.3-(c) it follows:

Corollary 7.4

$$\text{BMNCG}(n) = \text{BJNCG}(n) = \text{BNCG}(n, r = 1).$$

Moreover, *BMCG*(n) = *BJCG*(n) but they do not coincide with *BCG*$(n, r = 1)$ because null players are admitted when enumerating bipartite complete games with $r = 1$, but this is not the case when enumerating bipartite complete games coming from a bicameral meet or a bicameral join. The enumeration of *BMCG*(n) or *BJCG*(n) is obtained by subtracting the number of bipartite complete games with $r = 1$ and having null players to *BCG*$(n, r = 1)$. From this observation, we obtain:

$$\mathit{BMCG}(n) = \mathit{BJCG}(n) = \frac{1}{2}(n-1)(n-2).$$

From the last equalities, Corollary 7.4 and Theorem 5.2 it follows:

$$\mathit{BMSG}(n) = \mathit{BJSG}(n) = \begin{cases} \dfrac{(n-2)(n^2-4n+6)}{12} + \dfrac{(n-1)(n-2)}{2}, & \text{if } n \text{ is even} \\[2ex] \dfrac{(n-1)(n-2)(n-3)}{12} + \dfrac{(n-1)(n-2)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

which after simplification becomes:

$$\mathit{BMSG}(n) = \mathit{BJSG}(n) = \begin{cases} \dfrac{n(n-2)(n+2)}{12}, & \text{if } n \text{ is even} \\[2ex] \dfrac{(n-1)(n-2)(n+3)}{12}, & \text{if } n \text{ is odd} \end{cases}$$

8 Conclusion

In this paper, we have provided two parametrizations useful to generate all bipartite non-complete games up to isomorphism and all bipartite complete games up to isomorphism; putting them together we could generate all bipartite simple games. The problem we have dealt with in this paper is very versatile since monotonic simple games up to isomorphisms are inequivalent monotonic Boolean functions. Thus, the problem of enumerating bipartite simple games is equivalent to the problem of enumerating inequivalent monotonic Boolean functions with two types of equivalent variables.

We have obtained polynomial expressions for the number of bipartite games with a unique pair of minimal winning coalitions and with a maximal number of minimal winning coalition pairs. These enumerations concern the number of bipartite: complete games, non-complete games and simple games. These closed formulas have also equivalences in terms of inequivalent monotonic Boolean functions.

The bicameral meet and the bicameral join are two real-world voting systems widely used in practice. These commonly used voting systems are bipartite games. We have studied for them their relation and their common enumeration.

As far as we know, the enumerations obtained in this paper do not appear in “The On-Line Encyclopedia of Integer Sequences” (www.oeis.org). The only exception is the sequence A005993, which appears there in a different context.

We suggest some lines of future research related to our paper. First, the generalization of our parametrizations from bipartite games to tripartite games or to games with more than two equivalence classes, i.e., the extension to $t = 3$ or any $t > 2$ of Definitions 3.3 and 3.5. The classification we obtain in this paper for non-complete bipartite games gives some hints on a general classification for more than two equivalence classes. Second, finding closed formulas for the number of bipartite games with two or more pairs of minimal winning coalitions, i.e., the extensions to $r \geq 2$ of Theorem 5.2, Proposition 5.3 and Corollary 5.4. Third, proving that

$$\mathit{BNCG}(n) \in \Theta\left(2^n\right).$$

which would imply that $\mathit{BSG}(n) \in \Theta\left(2^n\right)$ since $\mathit{BSG}(n) = \mathit{BCG}(n) + \mathit{BNCG}(n)$ and $\frac{1+\sqrt{5}}{2} < 2$. The analogous enumerations obtained in this paper for tripartite games with either vetoes or nulls and for quadripartite games with vetoes and nulls have been obtained, in [13], quite recently as an application of the results found in this paper.

A complementary study could be to determine the dimension (and codimension), the minimum number of weighted games required to express the game as intersection (union) of them, of bipartite simple games or finding an upper bound for the dimension of bipartite games depending on the number of players.

Another interesting problem is to determine whether the characterization of weighted games within the class of simple games in terms of pseudoweightings obtained in Theorem 1 in [14], can be relaxed to inferior levels for bipartite simple games.

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References

[1] T. Aven. *Reliability and risk analysis*. Elsevier Applied Science, London, Great Britain, 1992.

[2] T. Aven and U. Jensen. *Stochastic Models in Reliability*. Springer, New York, 1999.

[3] M. Axenovich and S. Roy. On the structure of minimal winning coalitions in simple voting games. *Social Choice and Welfare*, 34(3):429–440, 2010.

[4] S. Bolus. Power indices of simple games and vector-weighted majority games by means of binary decision diagrams. *European Journal of Operational Research*, 210(2):258–272, 2011.

[5] F. Carreras. A characterization of the Shapley–Shubik index of power via automorphisms. *Stochastica*, 8:171–179, 1984.

[6] F. Carreras. A decisiveness index for simple games. *European Journal of Operational Research*, 163(2):370–387, 2005.

[7] F. Carreras and J. Freixas. Complete simple games. *Mathematical Social Sciences*, 32(2):139–155, 1996.

[8] W.D. Cook. Distance–based and ad hoc consensus models in ordinal preference ranking. *European Journal of Operational Research*, 172:369–385, 2006.

[9] E. Einy. The desirability relation of simple games. *Mathematical Social Sciences*, 10(2):155–168, 1985.

[10] E. Einy and E. Lehrer. Regular simple games. *International Journal of Game Theory*, 18(2):195–207, 1989.

[11] C.C. Elgot. Truth functions realizable by single threshold organs. In *AIEE Conference Paper 60-1311 (October), revised November 1960; paper presented at IEEE Symposium on Switching Circuit Theory and Logical Design*, September 1961.

[12] D.S. Felsenthal, M. Machover, and W.S. Zwicker. The bicameral postulates and indices of a priori voting power. *Theory and Decision*, 44(1):83–116, 1998.

[13] J. Freixas. On the enumeration of Boolean functions with distinguished variables. *Soft Computing*, 2020. https://doi.org/10.1007/s00500-020-05422-5

[14] J. Freixas. A characterization of weighted simple games based on pseudoweightings. *Optimization Letters*, 2020. https://doi.org/10.1007/s11590-020-01647-3

[15] J. Freixas, M. Freixas, and S. Kurz. On the characterization of weighted simple games. *Theory and Decision*, 83(4):469–498, 2017.

[16] J. Freixas and S. Kurz. The golden number and Fibonacci sequences in the design of voting systems. *European Journal of Operational Research*, 226(2):246–257, 2013.

[17] J. Freixas and S. Kurz. Enumerations of weighted games with minimum and an analysis of voting power for bipartite complete games with minimum. *Annals of Operations Research*, 222:317–339, 2014.

[18] J. Freixas and X. Molinero. On the existence of a minimum integer representation for weighted voting systems. *Annals of Operations Research*, 166(1):243–260, 2009.

[19] J. Freixas and X. Molinero. Simple games and weighted games: a theoretical and computational viewpoint. *Discrete Applied Mathematics*, 157(7):1496–1508, 2009.

[20] J. Freixas and X. Molinero. Weighted games without a unique minimal representation in integers. *Optimization Methods and Software*, 25(2):203–215, 2010.

[21] J. Freixas, X. Molinero, and S. Roura. Complete voting systems with two types of voters: weightedness and counting. *Annals of Operations Research*, 193(1):273–289, 2012.

[22] P.L. Hammer and R. Holzman. Approximations of pseudoboolean functions; applications to game theory. *ZOR Methods and Models of Operations Research*, 36:3–21, 1992.

[23] P.L. Hammer, T. Ibaraki, and U.N. Peled. Threshold numbers and threshold completions. In P. Hansen, editor, *Studies on Graphs and Discrete Programming*, pages 125–145. NorthHolland, Amsterdam, The Netherlands, 1981.

[24] J. Herranz. Any 2-asummable bipartite function is weighted threshold. *Discrete Applied Mathematics*, 159:1079–1084, 2011.

[25] J. Herranz and G. Sáez. New results on multipartite access structures. *IEEE Proceedings-Information Security*, 153(4):153–162, 2006.

[26] J.R. Isbell. A class of majority games. *Quart. J. Math. Oxford Ser.*, 7(2):183–187, 1956.

[27] W. Kuo and X. Zhu. *Importance measures in reliability, risk, and optimization.* Wiley, Malaysia, 2012.

[28] S. Kurz. On minimum sum representation for weighted voting games. *Annals of Operations Research*, 196(1):361–369, 2012.

[29] S. Kurz, A. Mayer, and S. Napel. Weighted committee games. *European Journal of Operational Research*, 282(3):972–979, 2020.

[30] S. Kurz and N. Tautenhahn. On Dedekind's problem for complete simple games. *International Journal of Game Theory*, 42(2):411–437, 2013.

[31] G. Levitin. Optimal allocation of multi-state elements in linear consecutively connected systems with vulnerable nodes. *European Journal of Operational Research*, 150(2):406–419, 2003.

[32] G. Levitin, L. Podofillini, and E. Zio. Generalised importance measures for multi–state elements based on performance level restrictions. *Reliability Engineering and System Safety*, 82(3):287–298, 2003.

[33] M. Maschler and B. Peleg. A characterization, existence proof, and dimension bounds for the kernel of a game. *Pacific Journal of Mathematics*, 18:289–328, 1966.

[34] C. Mitchell. Enumerating boolean functions of cryptographic significance. *Journal of Cryptology*, 2(3):155–170, 1990.

[35] S. Muroga, I. Toda, and S. Takasu. Theory of majority decision elements. *Journal Franklin Institute*, 271(5):376–418, 1961.

[36] S. Muroga, T. Tsuboi, and R. Baugh. Enumeration of threshold functions of eight variables. *IEEE Transactions on Computers*, C-19(9):818–825, 1970.

[37] N.J.A. Sloane. `https://oeis.org/`, 1964.

[38] P.D. Picton. *Neural Networks*. The Macmillan Press LTD (2nd Edition), London, Great Britain, 2000.

[39] K.G. Ramamurthy. *Coherent structures and simple games*. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1990.

[40] M. Rausand and A. Høyland. *System reliability theory: Models, statistical methods and applications. 2nd ed. Hoboken*. John Wiley & Sons, New Jersey, USA, 2004.

[41] V.P. Roychowdhury, K.Y. Siu, and A. Orlitsky (Eds.). *Theoretical Advances in Neural Computation and Learning*. Kluwer Academic Publishers, Stanford, USA, 1994.

[42] A.D. Taylor and W.S. Zwicker. Weighted voting, multicameral representation, and power. *Games and Economic Behavior*, 5:170–181, 1993.

[43] A.D. Taylor and W.S. Zwicker. *Simple games: desirability relations, trading, and pseudoweightings*. Princeton University Press, New Jersey, USA, 1999.