Triangulations and a discrete Brunn-Minkowski inequality in the plane

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Abstract

For a set A of points in the plane, not all collinear, we denote by $tr(A)$ the number of triangles in a triangulation of A, that is, $tr(A) = 2i + b - 2$, where b and i are the numbers of boundary and interior points of the convex hull $[A]$ of A respectively. We conjecture the following discrete analog of the Brunn–Minkowski inequality: for any two finite point sets $A, B \subset \mathbb{R}^2$ one has

$$tr(A + B) \geq tr(A)^{1/2} + tr(B)^{1/2}.$$

We prove this conjecture in the cases where $[A] = [B]$, $B = A \cup \{b\}$, $|B| = 3$ and if A and B have no interior points. A generalization to larger dimensions is also discussed.

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1 Introduction

In this paper we write A, B to denote finite subsets of $\mathbb{R}^d$, and $|\cdot|$ stands for their cardinality. We say that $A \subset \mathbb{R}^d$ is *d–dimensional* if it is not contained in any affine hyperplane of $\mathbb{R}^d$. Equivalently, the real affine span of A is $\mathbb{R}^d$. For subsets $X_1, \dots, X_k$ of $\mathbb{R}^d$, $[X_1, \dots, X_k]$ denotes their convex hull. Here and in what follows we denote $A + B := \{a + b : a \in A, b \in B\}$ and $A - B := A + (-B)$. The *lattice generated by* A is the additive subgroup $\Lambda = \Lambda(A) \subset \mathbb{R}^d$ generated by $A - A = \{x - y : x, y \in A\}$, and A is called *saturated* if it satisfies $A = [A] \cap \Lambda(A)$.

Our starting point are two classical results. The first one is from the 1950's, due to Kemperman [10], and popularized by Freiman [4]: if A and B are finite nonempty subsets of $\mathbb{R}$, then

$$|A + B| \geq |A| + |B| - 1, \tag{1}$$

with equality if and only if A and B are arithmetic progressions of the same difference. The other result, the Brunn-Minkowski inequality, dates back to the 19th century. It says that if $X, Y \subset \mathbb{R}^d$ are compact nonempty sets then

$$\lambda(X + Y)^{\frac{1}{d}} \geq \lambda(X)^{\frac{1}{d}} + \lambda(Y)^{\frac{1}{d}}$$

where λ stands for the Lebesgue measure. Moreover, provided that $\lambda(X)\lambda(Y) > 0$, equality holds if and only if X and Y are convex homothetic sets.

Various discrete analogues of the Brunn-Minkowski inequality have been established in Bollobás, Leader [1], Gardner, Gronchi [5], Green, Tao [6], González-Merino, Henze [11], Hernández, Iglesias and Yepes [8], Huicochea [9] in any dimension, and Grynkiewicz, Serra [7] in the planar case. Most of these papers use the method of compression, which changes a finite set into a set better suited for sumset estimates, but does not control the convex hull.

Unfortunately the known analogues are not as simple in their form as the original Brunn–Minkowski inequality. For instance, a formula due to Gardner and Gronchi [5] says that, if A is d–dimensional, then

$$|A + B| \geq (d!)^{-\frac{1}{d}}(|A| - d)^{\frac{1}{d}} + |B|^{\frac{1}{d}}. \tag{2}$$

Concerning the case $A = B$, Freiman [4] proved that, if the dimension of A is d, then

$$|A + A| \geq (d + 1)|A| - \binom{d+1}{2}. \tag{3}$$

Both estimates are optimal. In particular, we can not expect a true discrete analogue of the Brunn–Minkowski inequality if the notion of volume is replaced by cardinality.

We here conjecture and discuss a more direct version of the Brunn–Minkowski inequality where the notion of volume is replaced by the number of full dimensional simplices in a triangulation of the convex hull of the finite set.

For any finite d–dimensional set $A \subset \mathbb{R}^d$ we write T_A to denote some triangulation of A, by which we mean a triangulation of $[A]$ with set of vertices equal to A. We denote $|T_A|$ the number of d-dimensional simplices in T_A.

In dimension two the number $|T_A|$ is the same for all triangulations of A, so we denote it $\mathrm{tr}(A)$. More precisely, if Δ_A and Ω_A denote the number of points of A in the boundary $\partial[A]$ and in the interior $\mathrm{int}[A]$, respectively, then it is easy (see, e.g., [3, Lemma 3.1.3]) to show that

$$\mathrm{tr}(A) = \Delta_A + 2\Omega_A - 2 = 2|A| - \Delta_A - 2. \tag{4}$$

Therefore around 2005, Matolcsi and Ruzsa conjectured in dimension two the following discrete analogue of the Brunn–Minkowski inequality (see Böröczky, Hoffman [2]).

Conjecture 1 *If finite $A, B \subset \mathbb{R}^2$ in the plane are not collinear, then*

$$\mathrm{tr}(A+B)^{\frac{1}{2}} \geq \mathrm{tr}(A)^{\frac{1}{2}} + \mathrm{tr}(B)^{\frac{1}{2}}.$$

One case where Conjecture 1 holds with equality is when A and B are homothetic saturated sets with respect to the same lattice; that is, $A = \Lambda \cap k \cdot P$ and $B = \Lambda \cap m \cdot P$ for a lattice Λ, polygon P and integers $k, m \geq 1$. This follows from the original Brunn-Minkowski equality as follows: for saturated sets $tr(A) = 2\,\mathrm{area}([A])/\det\Lambda$, because every triangle in a triangulation is a fundamental lattice triangle, of area $\frac{1}{2}\det\Lambda$. On the other hand, $A + B = \Lambda \cap (k+m) \cdot P$ and $tr(S) \leq 2\,\mathrm{area}([S])/\det\Lambda$ for every subset $S \subset \Lambda$, such as $S = A + B$.

Concerning Δ_A and Δ_B in (4), we observe that any side of $[A+B]$ is of the form $e+f$ where e and f is a side or a vertex of $[A]$ and $[B]$, respectively, with the same exterior unit normal, and $|(e+f)\cap(A+B)| \geq |e \cap A| + |f \cap B| - 1$ by (1). This implies that

$$\Delta_{A+B} \geq \Delta_A + \Delta_B. \tag{5}$$

We also note that Conjecture 1, together with the equality (4) and (5), would imply the following inequality of Gardner and Gronchi [5, Theorem 7.2] for sets A and B saturated with respect to the same lattice:

$$|A+B| \geq |A| + |B| + (2|A| - \Delta_A - 2)^{1/2}(2|B| - \Delta_B - 2)^{1/2} - 1.$$

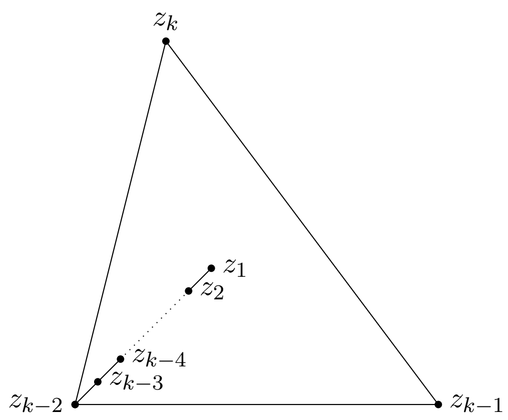

Figure 1: An illustration of case (b) in Theorem 2.

Unfortunately we have not been able to prove Conjecture 1 in full generality. Our main results are the following four cases of it: if $[A] = [B]$ (Theorem 2), in which case we also determine the conditions for equality in Conjecture 1; if A and B differ by one element (Theorem 4); if either $|A| = 3$ or $|B| = 3$ (Theorem 7); and if none of A and B have interior points (Theorem 8). Actually, the last two theorems satisfy a stronger conjecture (Conjecture 5) discussed below.

We start with the case $[A] = [B]$, which naturally include the case $A = B$.

Theorem 2 *Let $A, B \subset \mathbb{R}^2$ be finite two dimensional sets. If $[A] = [B]$ then Conjecture 1 holds. Moreover equality holds if and only if $A = B$, and*

(a) either A is a saturated set, or

(b) $A = \{z_1, \dots, z_k\}$ for $k \geq 4$, where $z_1, \dots, z_{k-3} \in \text{int}[z_{k-2}, z_{k-1}, z_k]$, and $z_1, \dots, z_{k-2}$ are collinear and equally spaced in this order (see Figure 1).

Let us mention that Theorem 2 (in fact, its particular case $A = B$) gives a simple proof of the following structure theorem of Freiman [4] for a planar set with small doubling. We recall that according to (3), if finite $A \subset \mathbb{R}^d$ is two dimensional, then $|A + A| \geq 3|A| - 3$ and, if the dimension of A is at least 3, then $|A + A| \geq 4|A| - 6$.

Corollary 3 (Freiman) *Let $A \subset \mathbb{R}^2$ be a finite two dimensional set and $\varepsilon \in (0, 1)$. If $|A| \geq 48/\varepsilon^2$ and*

$$|A + A| \leq (4 - \varepsilon)|A|,$$

then there exists a line l such that A is covered by at most

$$\frac{2}{\varepsilon} \cdot \left(1 + \frac{32}{|A|\varepsilon^2}\right)$$

lines parallel to l.

We note that, for A the grid $\{1, \ldots, k\} \times \{1, \ldots, k^2\}$ and large k,

$$|A + A| \leq (4 - \varepsilon)\,|A|, \tag{6}$$

with $\varepsilon = \varepsilon_k = \frac{2}{k}$ and A can not be covered by less than k parallel lines. Therefore the constant 2 in the numerator of $\frac{2}{\varepsilon}$ is asymptotically optimal in Corollary 3.

The next case we address is when A and B differ by one element.

Theorem 4 *Let $A \subset \mathbb{R}^2$ be a finite two dimensional set. If $B = A \cup \{b\}$ for some $b \notin A$ then Conjecture 1 holds.*

For our next results we need the notion of *mixed subdivision* (see De Loera, Rambau, Santos [3] for details). For finite d–dimensional sets $A, B \subset \mathbb{R}^d$ and triangulations T_A and T_B corresponding to A and B, we call a polytopal subdivision M of $[A + B]$ a *mixed subdivision* corresponding to T_A and T_B if

(i) every k-cell of M is of the form $F + G$ where F is an i-simplex of T_A and G is a j-simplex of T_B with $i + j = k$; in particular, all vertices of M are in $A + B$;

(ii) for any d-simplices F of T_A and G of T_B, there is a unique $b \in B$ and a unique $a \in A$ such that $F + b \in M$ and $a + G \in M$.

In dimension two, every mixed subdivision consists of $|T_A| + |T_B|$ triangles, translated from those of T_A and T_B, together with a certain number of parallelograms that we denote M_{11}. Since we can triangulate each parallelogram into two triangles, the following is stronger than Conjecture 1, and offers a geometric and algorithmic approach to prove Conjecture 1.

Conjecture 5 *For every finite two dimensional sets $A, B \subset \mathbb{R}^2$ there exist triangulations T_A and T_B of $[A]$ and $[B]$ using A and B, respectively, as the set of vertices, and a corresponding mixed subdivision M of $[A+B]$ such that*

$$|M_{11}| \geq \sqrt{|T_A| \cdot |T_B|}. \tag{7}$$

The following example shows that one cannot a priori fix any of the triangulations T_A and T_B in Conjecture 5:

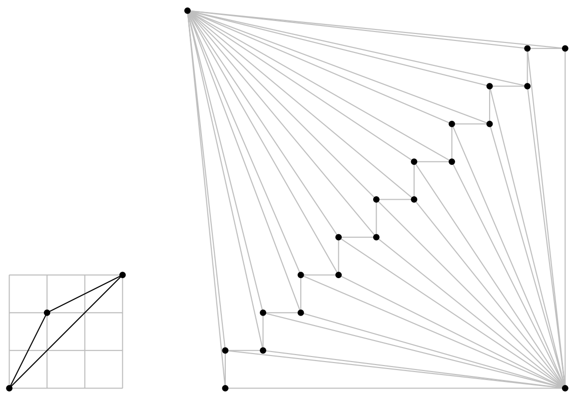

Figure 2: An illustration of the example described in Proposition 6.

Proposition 6 *Let*

$$A = \{(0,0), (-1,-2), (2,1)\}.$$

For $k \geq 145$*, let*

$$B = \{p, q, l_0, \dots, l_k, r_0, \dots, r_{k-1}\},$$

where $p = (-1, k+1)$*,* $q = (k+1, -1)$*,* $l_i = (i,i)$ *for* $i = 0, \dots, k$ *and* $r_i = (i, i+1)$ *for* $i = 0, \dots, k-1$*.*

Let T_B *be the triangulation of* B *consisting of the triangles*

$$[p, l_i, r_i], [q, l_i, r_i],\ i = 0, \dots, k-1 \text{ and } [p, l_i, r_{i-1}], [q, l_i, r_{i-1}],\ i = 1, \dots, k.$$

Then, no mixed subdivision of $A + B$ *corresponding to* T_B *and any triangulation* T_A *of* A *satisfies* (7) *for* $d = 2$*.*

Now Conjecture 5 is verified if either A or B has only three elements.

Theorem 7 *If* $|B| = 3$*, then Conjecture 5 holds for any finite two dimensional set* $A \subset \mathbb{R}^2$*.*

Remark It follows that if B is the sum of sets of cardinality three, then Conjecture 1 holds for any finite two dimensional set $A \subset \mathbb{R}^2$. For example, if $m \geq 1$ is an integer, and $B = \{(t,s) \in \mathbb{Z}^2 : t, s \geq 0 \text{ and } t + s \leq m\}$, or $B = \{(t,s) \in \mathbb{Z}^2 : |t|, |s| \leq m \text{ and } |t+s| \leq m\}$.

Conjecture 1 was verified by Böröczky, Hoffman [2] if A and B are in convex position; that is, if $A \subset \partial[A]$ and $B \subset \partial[B]$. Here we even verify Conjecture 5 under these conditions.

Theorem 8 *Let $A, B \subset \mathbb{R}^2$ be finite two dimensional sets. If $A \subset \partial[A]$ and $B \subset \partial[B]$ then Conjecture 5 holds.*

Part of the reason why we could not verify Conjecture 1 in general is that, except for Theorem 7, our arguments actually prove the inequality $\mathrm{tr}(A+B) \geq 2(\mathrm{tr}(A)+\mathrm{tr}(B))$, which is stronger than Conjecture 1, but which does not hold for all pairs with $A \subset B$. For example, if A are the lattice points with nonnegative coordinates and with the sum of coordinates at most k, and B is the same with sum of coordinates at most l, we have $\mathrm{tr}(A+B) = (k+l)^2$, $\mathrm{tr}(A) = k^2$ and $\mathrm{tr}(B) = l^2$. So we have $\mathrm{tr}(A+B) < 2(\mathrm{tr}(A)+\mathrm{tr}(B))$ if $k \neq l$.

We now turn to higher dimensions. The first difference is that we can no longer define $\mathrm{tr}(A)$ for a point configuration, since different triangulations of A have different numbers of d-simplices (see Example 11 below). Still, there is the following analogue of Conjecture 5. For a mixed subdivision M corresponding to triangulations T_A and T_B of A and B, let us denote by $\|M\|$ the weighted number of d-polytopes in M, where $F+G$ has weight $\binom{i+j}{i}$ if F is an i-simplex of T_A, and G is a j-simplex of T_B with $i+j=d$. The reason for these weights is that every triangulation (without additional vertices) of such an $F+G$ has exactly $\binom{i+j}{i}$ d-simplices (see e.g. [3, Proposition 6.2.11]). Thus, $||M||$ is the number of d-simplices of any triangulation of $A+B$ that refines M without additional vertices.

Hence, we may ask for which triangulations T_A and T_B there exists a corresponding mixed subdivision M for $[A+B]$ such that

$$\|M\|^{\frac{1}{d}} \geq |T_A|^{\frac{1}{d}} + |T_B|^{\frac{1}{d}}. \tag{8}$$

Question 9 *Is it true that for every finite sets $A, B \subset \mathbb{R}^d$ there are triangulations T_A and T_B and a corresponding mixed subdivision M of $[A+B]$ satisfying* (8)*?*

It is easy to show that the answer is positive if $A = B$:

Theorem 10 *For a finite d–dimensional set $A \subset \mathbb{R}^d$ and for any triangulation T_A of $[A]$ using A as the set of vertices there exists a corresponding mixed subdivision M of $[A+A]$ such that*

$$\|M\| = 2^d |T_A|.$$

Therefore in certain cases, mixed subdivisions point to a higher dimensional generalization of Conjecture 1. This is specially welcome knowing that, if $d \geq 3$, then the order of the number of d-simplices in a triangulation of the convex hull of a finite $A \subset \mathbb{R}^d$ spanning $\mathbb{R}^d$ might be as low as $|A|d$ and as high as $\Theta(|A|^{\lceil d/2 \rceil})$ for the same A, as the following example shows. In particular, one can not assign the number of d-simplices as a natural notion of discrete volume if $d \geq 3$.

Example 11 Let A be any set of n points in general position in $\mathbb{R}^d$ (that is, no $d+1$ in any affine hyperplane) and such that $[A]$ is a simplex. Any such A has triangulations of size $1+d(n-d-1)$ via the following construction: in a first step, consider $[A]$ as the single d-simplex in your triangulation. Then, one by one add the $n-d-1$ interior points to the triangulation as follows: at each step you stellarly subdivide the simplex containing the new point into $d+1$ simplices, all having the new point as a common vertex. At the end, as claimed, we have a triangulation of A of size $1+d(n-d-1)$.

If, moreover, the $n-d-1$ interior points of A are the vertices of a cyclic polytope, then you can also triangulate A with size $\Theta(n^{\lceil d/2 \rceil})$ (and this is optimal by [3, Corollary 6.1.20]): triangulate first the cyclic polytope with size $\Theta(n^{\lceil d/2 \rceil})$ and then add one by one the $d+1$ outer points, at each step conning the new point to the part of the boundary of the previous triangulation that is visible from that point.

2 Proof of Theorem 2

We will actually prove that

$$\mathrm{tr}(A+B) \geq 2\mathrm{tr}(A) + 2\mathrm{tr}(B), \tag{9}$$

a stronger inequality than Conjecture 1.

For a finite two dimensional set $X \subset \mathbb{R}^2$, we define

$$f_X(z) = \begin{cases} 1 & \text{if } z \in \partial[X] \\ 2 & \text{if } z \in \operatorname{int}[X] \end{cases},$$

thus (4) yields that

$$\mathrm{tr}(X) = \left(\sum_{z \in X} f_X(z) \right) - 2. \tag{10}$$

Lemma 12 *Let $A, B \subset \mathbb{R}^2$ satisfy $[A] = [B]$. Then inequality (9) holds. Moreover, equality in (9) yields $A = B$.*

Proof: Let T be a triangulation of $[A] = [B]$ such that the set of vertices is $A \cap B$. One nice thing about inequality (9) is that, since it is linear, it is additive over the triangles of T. Therefore, it suffices to show that, for each triangle t of T, if $A_t = A \cap t$ and $B_t = B \cap t$, then

$$\mathrm{tr}(A_t + B_t) \geq 2\mathrm{tr}(A_t) + 2\mathrm{tr}(B_t), \tag{11}$$

and that equality in (11) implies that $A_t = B_t$ consists of the three vertices of t alone. According to (10), inequality (11) is equivalent to

$$\sum_{p \in A_t + B_t} f_{A_t+B_t}(p) \geq 2\left(\sum_{p \in A_t} f_{A_t}(p)\right) + 2\left(\sum_{p \in B_t} f_{B_t}(p)\right) - 6. \tag{12}$$

Let $A_t \cap B_t = \{v_1, v_2, v_3\}$ be the three vertices of the triangle $t = [A_t] = [B_t]$. We claim that if $i, j \in \{1, 2, 3\}$, $p \in (A_t \cup B_t) \backslash \{v_1, v_2, v_3\}$ and $q \in A_t \cup B_t$, then

$$v_i + p = v_j + q \ \text{ yields } \ v_i = v_j \ \text{ and } \ p = q. \tag{13}$$

We may assume that v_i is the origin and, to get a contradiction, $v_i \neq v_j$. Then the line l passing through v_j and parallel to the side of t opposite to v_j separates t and $v_j + t$, and intersects t only in $v_j \neq p$. Since $v_j + q \in v_j + t$, we get the desired contradiction.

It follows from (13) that the six points $v_i + v_j$, $1 \leq i \leq j \leq 3$, and the points of the form $v_i + p$, $i = 1, 2, 3$ and $p \in (A_t \cup B_t) \backslash \{v_1, v_2, v_3\}$ are all different. Since the six points $v_i + v_j$, $1 \leq i \leq j \leq 3$, belong to $\partial(A_t + B_t)$, we have

$$\sum_{i,j=1,2,3} f_{A_t+B_t}(v_i + v_j) = \left(\sum_{i=1}^{3} f_{A_t}(v_i)\right) + \left(\sum_{j=1}^{3} f_{B_t}(v_j)\right) = 6. \tag{14}$$

On the other hand, we claim that, if $p \in A_t \backslash \{v_1, v_2, v_3\}$ and $q \in B_t \backslash \{v_1, v_2, v_3\}$, then

$$\begin{array}{lcl} \sum_{j=1}^{3} f_{A_t+B_t}(p + v_j) & > & 2f_{A_t}(p) \\ \sum_{i=1}^{3} f_{A_t+B_t}(v_i + q) & > & 2f_{B_t}(q). \end{array} \tag{15}$$

Indeed, if $p \in \partial[A_t]$, then the inequality readily holds, and if $p \in \mathrm{int}\,[A_t]$, then $p + v_j \in \mathrm{int}\,[A_t + B_t]$ for $j = 1, 2, 3$, as well, yielding (15).

By combining (14) and (15) we get (12) and in turn (9). Moreover, (15) shows that if equality holds in (11) for a triangle t of T, then $A_t = B_t$, and, therefore, if equality holds in (9), then $A = B$. $\square$

For a finite two dimensional set $A \subset \mathbb{R}^2$ and a triangulation T of A we denote by A_T the union of A and the set of midpoints of the edges of T (see Figure 3).

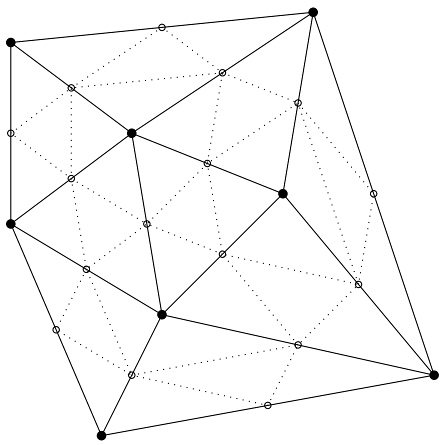

Figure 3: A triangulation and its midpoints.

Lemma 13 *Let $A \subset \mathbb{R}^2$ be a finite two-dimensional set. Then the equality*

$$\operatorname{tr}(A+A) = 4 \cdot \operatorname{tr}(A)$$

holds if, and only if, for every triangulation T of $[A]$, we have $A_T = \frac{1}{2}(A+A)$.

Proof: Divide each triangle t of T into four triangles using the vertices of t and the midpoints of the sides of t. This way we have obtained a triangulation of $[A] = [A_T]$ using A_T as the vertex set. Therefore

$$\operatorname{tr}(A+A) = \operatorname{tr}(\tfrac{1}{2}(A+A)) \geq \operatorname{tr}(A_T) = 4 \cdot \operatorname{tr}(A).$$

Moreover, there is equality if and only if $A_T = \frac{1}{2}(A+A)$. □

We observe that the equation in Lemma 13 is equivalent to Conjecture 1 for the case $A = B$. Therefore all we have left to prove is that $\operatorname{tr}(A+A) = 4 \cdot \operatorname{tr}(A)$ if and only if A is of the form either (a) or (b) in Theorem 2. The if part is simple.

Lemma 14 *Suppose that either (a) or (b) in Theorem 2 hold for the finite set A. Then*

$$A_T = \frac{1}{2}(A+A).$$

Proof: Suppose first that we have property (b). Then there is a unique triangulation T of $[A]$ using A as vertex set. For $1 \leq i < j \leq k$, $[z_i, z_j]$ is an edge of T, unless $j \leq k-2$, an hence we have $A_T = \frac{1}{2}(A+A)$.

So, for the rest of the proof we assume (a): $A = [A] \cap \Lambda$ for a lattice Λ. For a triangulation T corresponding to A, readily the midpoints of sides of triangles of T are in $\frac{1}{2}(A + A)$. On the other hand, let $m \in \frac{1}{2}(A + A)$, and let t be a triangle of T containing m. We may assume that the origin o is a vertex of t, and hence the other two vertices p and q form a basis of Λ. Since $m \in \frac{1}{2}(\Lambda + \Lambda)$, both of its coordinates in the basis p and q are integers or half of integers, thus m is either a vertex of t, or the midpoint of a side of t. Therefore $m \in A_T$. $\square$

The next Lemma shows the reverse direction and concludes the proof of Theorem 2.

Lemma 15 *Let $A \subset \mathbb{R}^2$ be a finite two dimensional set. If every triangulation T of A satisfies*

$$A_T = \frac{1}{2}(A + A),$$

then either (a) or (b) from Theorem 2 hold.

Proof: We prove the Lemma by induction on $|A| \geq 3$. If $|A| = 3$, then A is readily a saturated set.

If $|A| \geq 4$, then we claim that

$$\begin{array}{l}\text{there exists a vertex } v \text{ of } [A] \text{ such that } A\backslash\{v\} \\ \text{is two dimensonal and does not satify (b).}\end{array} \tag{16}$$

Let v' be any vertex of $[A]$. If $A\backslash\{v'\}$ is collinear, then we can choose v to be any other vertex of the triangle $[A]$. If $\widetilde{A} = A\backslash\{v'\}$ is two-dimensional and satifies (b), then there exists a line ℓ such that $\widetilde{A} = \{v_1, v_2\} \cup (\ell \cap \widetilde{A})$ where v_1 and v_2 are strictly separated by ℓ. We may assume that the closed half plane bounded by ℓ and containing v_1 also contains v'. Then we may choose $v = v_2$, as $A' = A\backslash\{v_2\}$ satisfies that ℓ is a supporting line of $[A']$ and $|\ell \cap A'| \geq 3$, proving (16). This finishes the proof of claim (16).

Now, let $v \in A$ be as in (16), and let $A' = A \setminus \{v\}$. We fix a triangulation T' of A', and extend it to a triangulation T of A. We observe that the triangles in $T\backslash T'$ are of the form $[v, u, w]$ where there exists side e of $[A']$ whose line strictly separates v and $\mathrm{int}\,[A']$ and $u, v \in e \cap A'$ are consecutive points. Applying the induction hypothesis to $A'_{T'}$, we deduce from (16) that A' satisfies (a); it is a saturated set with respect to some lattice Λ.

For any side e of $[A']$, let ℓ_e be the line parallel to e and intersecting $[A'] \cap \Lambda$, which is closest to e among the lines with these properties and not containing e. We claim that

$$\ell_e \cap A' \neq \emptyset. \tag{17}$$

To prove (17), we may assume that $\Lambda = \mathbb{Z}^2$, $(0,0),(1,0) \in e$ and $(x,y) \in A'$ for $y \geq 1$. It follows from the convexity of $[A']$ that $(\frac{x}{y},1),(\frac{x+y-1}{y},1) \in [A'] \cap \ell_e$. Since there exists a multiple $z \cdot y$, $z \in \mathbb{Z}$, of y among $x,\ldots,x+y-1$, we have $(z,1) \in \ell_e \cap A'$ by the saturatedness of A'.

We distiguish two cases depending on whether A would eventually satify (a) or (b).

Case 1. For any side e of $[A']$ whose line strictly separates v and $\mathrm{int}\,[A']$, there exists a $p \in \ell_e \cap A'$ such that $[p,v] \cap [A'] \neq \{p\}$.

In this case, we prove that A is also saturated with respect to Λ; namely,

if e is a side of $[A']$ whose line strictly separates v and $\mathrm{int}\,[A']$, then $[e,v] \cap \Lambda = \{v\} \cup (e \cap \Lambda)$. (18)

To prove (18) for e, let $p \in \ell_e \cap A'$ such that $[p,v] \cap [A'] \neq \{p\}$. It follows from $[p,v] \cap [A'] \neq \{p\}$ that $\frac{1}{2}(p+v)$ can't lie in $A_T \backslash A'_{T'}$, therefore it lies in $A'_{T'}$ by $A_T = \frac{1}{2}(A+A)$. Since $p \in \ell_e$, we have $\frac{1}{2}(p+v) \in e$, and actually $\frac{1}{2}(p+v) = \frac{1}{2}(u+w)$ for $u,w \in e \cap \Lambda$. In turn, we conclude (18), and hence A is a saturated set.

Case 2. There exists a side e of $[A']$ whose line strictly separates v and $\mathrm{int}\,[A']$, and $[p,v] \cap [A'] = \{p\}$ for any $p \in \ell_e \cap A'$.

In this case, we prove that A satifies (b). Let $p \in \ell_e \cap A'$. Since $p \in \ell_e$ and $[p,v] \cap [A'] = \{p\}$, there exists a side f of $[A']$ such that f meets e in a vertex of $[A']$ and $p \in f$. Since $[p,v] \cap [A'] = \{p\}$ and the line of e strictly separates v and $\mathrm{int}\,[A']$, we may also assume that the line of f strictly separates v and $\mathrm{int}\,[A']$. In particular, we may asssume that $\Lambda = \mathbb{Z}^2$, $e \cap f = (0,0)$, $w = (1,0) \in e$ and $p = (0,1)$, and then $v = (s,t)$ where $s,t < 0$. For $q = (1,1)$, we have $[q,v] \cap \mathrm{int}[A'] \neq \emptyset$, and hence $q \notin A'$ in Case 2. Therefore either $A' = \{p\} \cup (e \cap \mathbb{Z}^2)$ or $A' = \{w\} \cup (f \cap \mathbb{Z}^2)$, thus A satisfies (b) in Case 2, verifying Lemma 15. □

3 Proof of Theorem 4

The inequality between the quadratic and arithmetic means gives that, if $a,k > 0$, then

$$(4a+2k)^{\frac{1}{2}} > a^{\frac{1}{2}} + (a+k)^{\frac{1}{2}}.$$

Therefore to prove Theorem 4, it is sufficient to verify the following: Let $B = A \cup \{b\}$ for $b \notin A$.

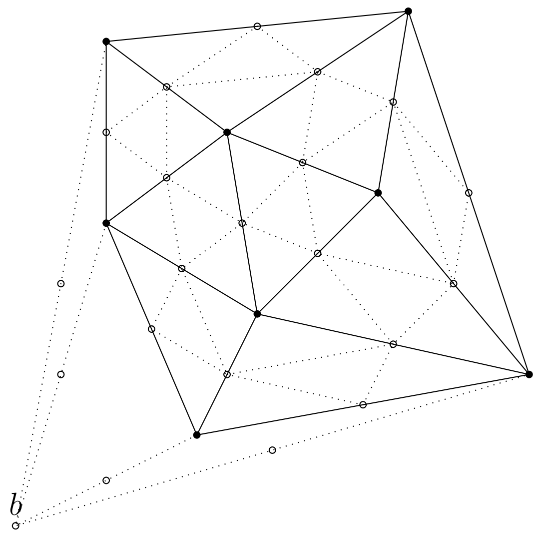

Figure 4: An illustration of Case 1.

(∗) If $\mathrm{tr}(A) = a$ and $\mathrm{tr}(B) = a + k$, then $\mathrm{tr}(A + B) \geq 4a + 2k$.

We fix a triangulation T of A, and let A_T be the union of A and the set of midpoints of the edges of T. It follows by (4) that

$$\Delta_{A_T} + 2\Omega_{A_T} - 2 = \mathrm{tr}(A_T) = 4a.$$

To estimate $\mathrm{tr}(A + B) = \mathrm{tr}(\frac{1}{2}(A + B))$, we isolate certain subset V of A in a way such that

$$A_T \cap (\tfrac{1}{2}(V + \{b\})) = \emptyset. \tag{19}$$

Therefore, equation (4) and (19) give,

$$\begin{aligned} \mathrm{tr}(A + B) \;\; \geq \;\; & 4a + 2|\tfrac{1}{2}(V + \{b\}) \cap \mathrm{int}[B]| + \\ & |\tfrac{1}{2}(V + \{b\}) \cap \partial[B]| + |A_T \cap \partial[A] \cap \mathrm{int}[B]|. \end{aligned} \tag{20}$$

We distinguish two cases depending on how to define V.

Case 1 $b \notin [A]$

We say that $x \in [A]$ is visible if $[b, x] \cap [A] = \{x\}$. In this case $x \in \partial[A]$. We note that there are exactly two visible points on $\partial[B]$, which are on the two supporting lines to $[A]$ passing through b (see Figure 4). Let $k+1$ be the number of visible points of A, and hence $k \geq 1$. Now $k - 1$ visible points of A lie in $\mathrm{int}[B]$, thus (4) yields that $\mathrm{tr}(B) = a + k$. Let V be the set of visible points of A. The condition (19) is satisfied because $[A] \cap (\frac{1}{2}(V + \{b\})) = \emptyset$.

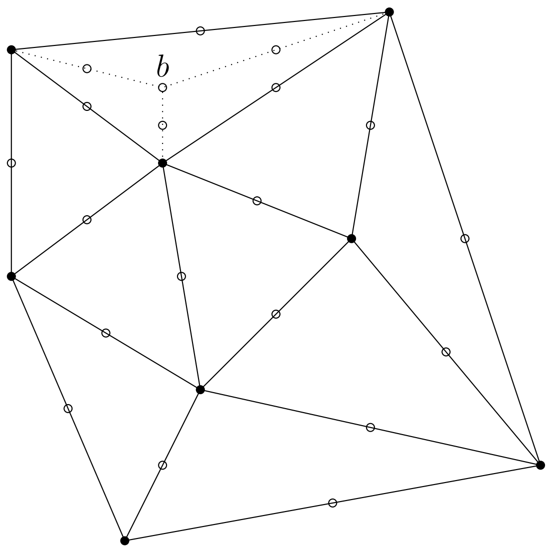

Figure 5: An illustration of Case 2.

We have $|\frac{1}{2}(V+\{b\})| = k+1$, and $2k-1$ visible points of A_T lie in $\mathrm{int}[B]$. In particular, $(*)$ follows as (20) yields

$$\mathrm{tr}(A+B) \geq 4a + 2k - 1 + k + 1 = 4a + 3k > 4a + 2k.$$

Case 2 $b \in [A]$

In this case $\mathrm{tr}(B) = a+k$ for $k \leq 2$ by (4), and b is contained in a triangle $T = [p,q,r]$ of T (see Figure 5). We may assume that b is not contained in the sides $[r,p]$ and $[r,q]$ of T. We take $V = \{p,q,r\}$, which satisfies (19). Since $\frac{1}{2}(b+q) \in \mathrm{int}T \subset \mathrm{int}[A]$, (20) yields $\mathrm{tr}(A+B) \geq 4a+4$. In turn, we conclude Theorem 4.

Remark: The argument does not work if we only assume that $A \subset B$, because we may have equality in Conjecture 1 in this case.

4 Proof of Theorem 7

Let $A \subset \mathbb{R}^2$ be finite and not contained in any line. By a *path* σ on A we mean a concatenation of segments $[a_0,a_1]$,..., $[a_{\ell-1},a_\ell]$ where $a_0,\dots,a_\ell \in A$ are distinct points and the segments do not intersect A or one another except at their endpoints. We call the number ℓ of segments the *length* of σ, and denote it $|\sigma|$. We allow the case that σ is a point, and in this case we set $|\sigma| = 0$. We say that σ is *transversal* to a non-zero vector u if every line parallel to u intersects σ in at most one point; equivalently, if $u \cdot (a_{i+1} - a_i)$ is non-zero and of the same sign for all i. In this case, the segments in σ induce a subdivision of $\sigma + [o, u]$ into $|\sigma|$ parallelograms if $|\sigma| \geq 1$. For the proof of Theorem 7 the idea is to find an appropriate set of paths on A with total length at least $\sqrt{\mathrm{tr}(A)}$.

First, we explore the possibilities using only one or two paths. We will see in Remark 16 that one path is not enough, but Proposition 17 shows that using two paths σ_1, σ_2 almost does the job.

Observe that for any given non-zero vector w, the length of the longest path on A transversal to w equals the number of lines parallel to w intersecting A, minus one. The next remark indicates that we may need a least two paths to get the total length close to $\sqrt{T_A}$.

Remark 16 *Given pairwise independent vectors $w_1, \ldots, w_n$ let $f(w_1, \ldots, w_n, s)$ be the minimal number such that, for every finite set $A \subset \mathbb{R}^2$ with $\mathrm{tr}(A) = s$, there is a w_i and a path on A transversal to w_i of length $f(w_1, \ldots, w_n, s)$.*

For $n = 2$, $f(w_1, w_2, s) \geq \sqrt{s/2}$, with equality provided that $k := \sqrt{s/2}$ is an integer. An extremal configuration consists of the points $\{iw_1 + jw_2 : i, j \in \{0, \ldots, k\}\}$.

For $n = 3$, $f(w_1, w_2, w_3, s) \geq \sqrt{2s/3}$ and equality holds provided that $s = 6k^2$. Assuming without loss of generality that $w_1 + w_2 + w_3 = 0$, an extremal configuration is given by the points of the lattice generated by w_1, w_2 in the affine regular hexagon $[\pm kw_1, \pm kw_2, \pm kw_3]$.

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$, and let σ_1, σ_2 be paths on A. We say that the ordered pair (σ_1, σ_2) is a *horizontal–vertical* path if

(i') σ_i is transversal with respect e_{3-i} (possibly a point), $i = 1, 2$;

(ii') the right endpoint a of σ_1 equals the upper endpoint of σ_2

(iii') writing $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$, if $|\sigma_1|, |\sigma_2| > 0$, then

$$((\sigma_1 \backslash \{a\}) + \mathbb{R}_+ e_2) \cap ((\sigma_2 \backslash \{a\}) + \mathbb{R}_+ e_1) = \emptyset.$$

We call σ_1 the horizontal branch, and σ_2 the vertical branch, and a the center.

We observe that if σ_i' is the image of σ_i by reflection through the line $\mathbb{R}(e_1 + e_2)$, then the ordered pair (σ_2', σ_1') is also a horizontal–vertical path.

For any polygon P and non-zero vector u, we write $F(P, u)$ to denote the face of P with exterior normal u. In particular, $F(P, u)$ is either an edge or a vertex.

Proposition 17 *For every finite $A \subset \mathbb{R}^2$ not contained in a line, and for every triangulation T of $[A]$ having A as the set of vertices, there exists a horizontal–vertical path (σ_1, σ_2) whose vertices belong to A, and satisfies*

$$|\sigma_1| + |\sigma_2| \geq \sqrt{|T| + 1} - \tfrac{1}{2}.$$

Proof: Let us write

$$\begin{aligned} \xi &= |F([A], -e_1) \cap F([A], -e_2)| \leq 1, \\ \Delta'_A &= |(A \cap \partial[A]) \backslash (F([A], -e_1) \cup F([A], -e_2))| . \end{aligned}$$

By the invariance with respect to reflection through the line $\mathbb{R}(e_1 + e_2)$, we may assume that

$$|F([A], -e_2) \cap A| \geq |F([A], -e_1) \cap A|. \tag{21}$$

We set $\{\langle e_1, p\rangle \,:\, p \in A\} = \{\alpha_0, \ldots, \alpha_k\}$ with $\alpha_0 < \ldots < \alpha_k$, $k \geq 1$. For $i = 0, \ldots, k$, let $A_i = \{p \in A : \langle e_1, p\rangle = \alpha_i\}$, let $x_i = |A_i|$, and let a_i be the top-most point of A_i; that is, $\langle e_2, a_i\rangle$ is maximal. In particular, $x_0 = |F([A], -e_1) \cap A|$. For each $i = 1, \ldots, k$, we consider the horizontal–vertical path $(\sigma_{1i}, \sigma_{2i})$ where

$$\sigma_{1i} = \{[a_0, a_1], \ldots, [a_{i-1}, a_i]\},$$

and the vertex set of σ_{2i} is A_i. In particular, the total length of the horizontal–vertical path is $(\sigma_{1i}, \sigma_{2i})$ is

$$|\sigma_{1i}| + |\sigma_{2i}| = i + x_i - 1.$$

The average length of these paths for $i = 1, \ldots, k$ is

$$\frac{\sum_{i=1}^k (|\sigma_{1i}| + |\sigma_{2i}|)}{k} = \frac{\sum_{i=1}^k (i + x_i - 1)}{k} = \frac{|A| - x_0}{k} + \frac{k}{2} - \frac{1}{2}.$$

We observe that $2|A| = |T| + \Delta_A + 2$, according to (4), and (21) yields

$$2 + \Delta_A - 2x_0 = 2 + \Delta'_A + |F([A], -e_2) \cap A| - \xi - x_0 \geq \Delta'_A + 1.$$

Therefore we deduce from the inequality between the arithmetic and geometric mean that

$$\begin{aligned} \frac{\sum_{i=1}^k (|\sigma_{1i}| + |\sigma_{2i}|)}{k} &= \frac{2|A| - 2x_0}{2k} + \frac{k}{2} - \frac{1}{2} \\ &\geq \frac{1}{2}\left(\frac{|T| + \Delta'_A + 1}{k} + k\right) - \frac{1}{2} \qquad (22) \\ &\geq \sqrt{|T| + \Delta'_A + 1} - \frac{1}{2}. \qquad (23) \end{aligned}$$

Therefore there exists some horizontal–vertical path $(\sigma_{1i}, \sigma_{2i})$ satisfying (23).
□

The estimate of Proposition 17 is close to be optimal according to the following example.

Example 18 *Let* $k \geq 2$ *and* $t > 0$*. Let* A' *be the saturated set with* $[A']$ *having vertices* $(0,0),(0,k),(k-1,0)$ *and* $(k-1,1)$*, and let* $A = A' \cup \{(k+t,0)\}$*. A triangulation* T *of* A *has* k^2+k-1 *triangles and every horizontal–vertical path* (σ_1, σ_2) *on* A *has total length*

$$|\sigma_1| + |\sigma_2| \leq k < \sqrt{|T|+2} - \tfrac{1}{2}.$$

□

We next proceed to the proof of Theorem 7 by a similar strategy using three paths. Let $B = \{v_1, v_2, v_3\}$ and, for $\{i,j,k\} = \{1,2,3\}$ denote by u_i the exterior unit normal to the side $[v_j, v_k]$ of B. A set of three paths $(\sigma_1, \sigma_2, \sigma_3)$ on A with a common endpoint a is called a *proper star* (with respect to $B = \{v_1, v_2, v_3\}$) if the following conditions hold:

(i) σ_i is transversal with respect $v_j - v_k$ (possibly $\sigma_i = \{a\}$);

(ii) writing $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$, if $|\sigma_j|, |\sigma_k| > 0$, then

$$((\sigma_j \setminus \{a\}) + \mathbb{R}_+(v_k - v_i)) \cap ((\sigma_k \setminus \{a\}) + \mathbb{R}_+(v_j - v_i)) = \emptyset;$$

(iii) the other endpoint b_i of σ_i lies in $\partial[A]$ and u_i is an exterior unit normal to $[A]$ at b_i; in particular,

$$\langle b_i, u_i \rangle = \max\{\langle x, u_i \rangle : x \in A\}.$$

We note that the three paths are allowed to have common vertices and edges, but they do not cross one another by (ii).

If the paths $\sigma_i \setminus \{a\}$, $i = 1,2,3$, are all non-empty and pairwise disjoint (except for their common end-point a), then (ii) means that they come around a in the same order as the orientation of the triangle $[v_1, v_2, v_3]$ (see Figure 6 for an illustration).

The next Lemma shows how to construct an appropriate mixed subdivision of $A + B$ using a proper star.

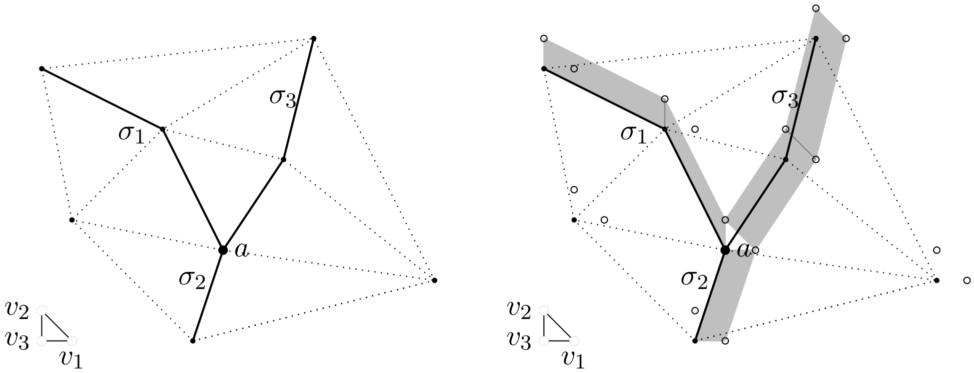

Figure 6: A proper star with respect to v_1, v_2, v_3 centered at a. On the right, paralellograms based on the proper star

Lemma 19 *Let A and B be finite non-collinear sets in $\mathbb{R}^2$ with $B = \{v_1, v_2, v_3\}$, and let us a consider a proper star on A with respect to B with rays $\sigma_1, \sigma_2, \sigma_3$ and center a such that $|\sigma_1|+|\sigma_2|+|\sigma_3| > 0$. Then there exists a triangulation T_A for A extending the paths $\sigma_1, \sigma_2, \sigma_3$, and a mixed subdivision M for $A+B$ satisfying*

$$|M_{11}| = |\sigma_1| + |\sigma_2| + |\sigma_3|.$$

Proof: We may assume that $|\sigma_1| > 0$ and $v_3 = o$. Let T_A be a triangulation using all the edges in the given proper star, and partition the triangles of T_A into three subsets $\Sigma_1, \Sigma_2, \Sigma_3$ (some of the Σ_i might be empty). The idea is that if the semi-open paths $\sigma_i \backslash \{a\}$, $i = 1, 2, 3$, are all non-empty and pairwise disjoint and $\{i, j, k\} = \{1, 2, 3\}$, then Σ_i consists of the triangles of T_A cut off by $\sigma_j \cup \sigma_k$. We also use Jordan's theorem for a simple closed polygonal path σ; namely, it encloses an open bounded set D such that $x \in D$ if and if whenever a ray ℓ emanating from x does not contain any edge of σ, then $|\ell \cap \sigma|$ is finite and odd.

A triangle τ of T_A is in Σ_1 if and only if for any $p \in (\mathrm{int}\, \tau) \backslash (a + \mathbb{R}v_1)$ such that $p - \mathbb{R}_+ v_1$ does not contain any edge of σ_2 or σ_3, we have

$$|(p - \mathbb{R}_+ v_1) \cap \sigma_2| + |(p - \mathbb{R}_+ v_1) \cap \sigma_3|$$

is finite and odd. Similarly, $\tau \in T_A$ is in Σ_2 if and only if for any $p \in (\mathrm{int}\, \tau) \backslash (a + \mathbb{R}v_2)$ such that $p - \mathbb{R}_+ v_2$ does not contain any edge of σ_1 or σ_3, we have

$$|(p - \mathbb{R}_+ v_2) \cap \sigma_1| + |(p - \mathbb{R}_+ v_2) \cap \sigma_3|,$$

is finite and odd. The rest of the triangles of T_A form Σ_3.

The mixed subdivision M is constructed as follows. Concerning triangles, $[B]+a$ is in M, and if $\tau \in \Sigma_i$, then the corresponding triangle in M is $\tau + v_i$. For the parallelograms, if $\{i,j,k\} = \{1,2,3\}$ and e is an edge of σ_i, then $e + [v_j, v_k]$ is in M. It follows from properties (i) and (ii) of the proper star that these parallelograms do not overlap, and taking also (iii) into account, we obtain a mixed triangulation of $A+B$. □

For the rest of the section, we fix finite $A \subset \mathbb{R}^2$ and $B = \{v_1, v_2, v_3\} \subset \mathbb{R}^2$ such that both of them span $\mathbb{R}^2$ affinely, and confirm Conjecture 5 in this case.

The following statement is a simple consequence of the definition of a proper star.

Lemma 20 *Assuming $B = \{v_1, v_2, v_3\}$ with $v_1 = (1,0) = -u_1$, $v_2 = (0,1) = -u_2$ and $v_3 = (0,0)$, and hence $u_3 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, if (σ_1, σ_2) is a horizontal–vertical path for A centered at $a \in A$, then*

- *(a) there exists a proper star $(\sigma_1', \sigma_2', \sigma_3')$ centered at a such that $\sigma_1 \subset \sigma_1'$, $\sigma_2 \subset \sigma_2'$,*
- *(b) if in addition $a \notin F([A], u_3)$, then $|\sigma_3'| \geq 1$.*

Proof: A triple of paths $(\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3)$ meeting at a will be called a semi-proper star extending (σ_1, σ_2) if it satisfies properties (i) and (ii) above and $\sigma_i \subset \tilde{\sigma}_i$ for $i = 1, 2$. In particular, $(\sigma_1, \sigma_2, \{a\})$ is a semi-proper star extending (σ_1, σ_2). We show that if $(\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3)$ is a semi-proper star extending (σ_1, σ_2) and

$$\max\{\langle x, u_i \rangle : x \in \tilde{\sigma}_i\} < \max\{\langle x, u_i \rangle : x \in A\} \text{ for an } i \in \{1,2,3\},$$

then there exists a semi-proper star $(\sigma_1', \sigma_2', \sigma_3')$ extending (σ_1, σ_2) such that

$$\sigma_j' = \tilde{\sigma}_j \text{ for } j \neq i,\ \tilde{\sigma}_i \subset \sigma_i' \text{ and } \tilde{\sigma}_i \neq \sigma_i'. \tag{24}$$

Let $b_i \in \sigma_i$ be the other endpoint of $\tilde{\sigma}_i$; namely,

$$\langle b_i, u_i \rangle = \max\{\langle x, u_i \rangle : x \in \tilde{\sigma}_i\}.$$

To prove (24), we consider the open half plane $H_i^+ = \{x \in \mathbb{R}^2 : \langle x, u_i \rangle > \langle b_i, u_i \rangle\}$, and distinguish two cases. First, if $H_i^+ \cap \tilde{\sigma}_j = \emptyset$ for $j \neq i$, then we choose any $z \in A \cap H_i^+$. The points of $A \cap [b_i, z]$ divide $[b_i, z]$ into a path, and adding this path to $\tilde{\sigma}_i$ we obtain the required σ_i' in (24).

The second case in proving (24) is that if there exists $j \neq i$ such that $H_i^+ \cap \tilde{\sigma}_j \neq \emptyset$. We consider the $z \in A \cap \tilde{\sigma}_j \cap H_i^+$ such that

$$\langle u_j, x \rangle \geq \langle u_j, z \rangle \quad \text{for } x \in A \cap \tilde{\sigma}_j \cap H_i^+.$$

Let $\{1,2,3\} = \{i,j,k\}$. Since

$$\tilde{\sigma}_j + \mathbb{R}_+(v_i - v_k) \subset b_i + \mathbb{R}_+(v_i - v_k) + \mathbb{R}_+(z - b_i)$$

by the choice of z and as $\tilde{\sigma}_j$ is transversal with respect to $v_i - v_k$, and in addition, $v_k - v_j \in \mathbb{R}_+(v_i - v_k) + \mathbb{R}_+(z - b_i)$, we deduce that

$$([z, b_i] + \mathbb{R}_+(v_j - v_k)) \cap (\tilde{\sigma}_j + \mathbb{R}_+(v_i - v_k)) = \emptyset. \tag{25}$$

Similarly,

$$\begin{aligned} \langle x, u_k \rangle &< \langle b_i, u_k \rangle \quad \text{for } x \in [z, b_i] + \mathbb{R}_+(v_k - v_j) \\ \langle x, u_k \rangle &> \langle b_i, u_k \rangle \quad \text{for } x \in \tilde{\sigma}_k + \mathbb{R}_+(v_i - v_j) \end{aligned}$$

imply that

$$([z, b_i] + \mathbb{R}_+(v_k - v_j)) \cap (\tilde{\sigma}_k + \mathbb{R}_+(v_i - v_j)) = \emptyset. \tag{26}$$

Again, the points of $A \cap [b_i, z]$ divide $[b_i, z]$ into a path, and adding this path to $\tilde{\sigma}_i$ we obtain the σ'_i, which, together with $\sigma'_j = \tilde{\sigma}_j$ and $\sigma'_k = \tilde{\sigma}_k$, satifies (ii) by (25) and (26). In turn, we conclude (24).

Since A is finite, repeated application of (24) leads to the required proper star satifying (iii), as well. □

Proof of Theorem 7 We apply the same idea as in the proof of Proposition 17, only applying Lemma 20 at a certain point to improve the bound.

We may assume that $B = \{v_1, v_2, v_3\}$ with $v_1 = (1,0) = -u_1$, $v_2 = (0,1) = -u_2$ and $v_3 = (0,0)$, and hence $u_3 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. In addition, we may assume that

$$|F([A], u_2) \cap A| \geq |F([A], u_1) \cap A|.$$

Using the notation of the proof of (22), we set $\{\langle -u_1, p \rangle : p \in A\} = \{\alpha_0, \ldots, \alpha_k\}$ with $\alpha_0 < \ldots < \alpha_k$, and $\Delta'_A = |(A \cap \partial[A]) \backslash (F([A], u_1) \cup F([A], u_2))|$. For $i = 0, \ldots, k$, let $A_i = \{p \in A : \langle u_1, p \rangle = \alpha_i\}$, let $x_i = |A_i|$ and let a_i be the top-most point of A_i; namely, $\langle -u_2, a_i \rangle$ is maximal. According to (22) and (23), we have

$$\frac{\sum_{i=1}^k (i + x_i - 1)}{k} \geq \frac{|T_A| + \Delta'_A + 1}{2k} + \frac{k}{2} - \frac{1}{2} \geq \sqrt{|T_A| + 1} - \frac{1}{2}. \tag{27}$$

Let I be the set of all $i \in \{1, \ldots, k\}$ such that

$$i + x_i - 1 \geq \left\lceil \frac{|T_A| + \Delta'_A + 1}{2k} + \frac{k}{2} - \frac{1}{2} \right\rceil = \xi. \tag{28}$$

Since $\xi \geq \sqrt{|T_A| + 1} - \frac{1}{2}$, if strict inequality holds for some i in (28), then using Lemma 19 for the proper star constructed in Lemma 20 (a) concludes the proof of Theorem 7. Thus we assume that

$$i + x_i - 1 = \xi \quad \text{for } i \in I.$$

If $i \in I$ and $a_i \notin F([A], u_3)$, then $\xi \geq \sqrt{|T_A| + 1} - \frac{1}{2}$ and using Lemma 19 for the proper star constructed in Lemma 20 (b) concludes the proof of Theorem 7.

Therefore we may assume that

$$a_i \in F([A], u_3) \quad \text{for } i \in I. \tag{29}$$

Let $\theta = |I|$. Since $i \geq 1$ for $i \in I$ and $|F([A], u_3) \cap F([A], u_2))| \leq 1$, we deduce that

$$\theta \leq |F([A], u_3) \backslash F([A], u_1)| \leq \min\{\Delta'_A + 1, k\}. \tag{30}$$

Since $i + x_i - 1 \leq \xi - 1$, if $i \notin I$, we have

$$\xi - \frac{\sum_{i=1}^{k}(i + x_i - 1)}{k} \geq \xi - \frac{\theta \cdot \xi + (k - \theta) \cdot (\xi - 1)}{k} = \frac{k - \theta}{k}.$$

We deduce from (27) that if $i \in I$, then

$$\begin{aligned} i + x_i - 1 &= \xi \geq \frac{\sum_{i=1}^{k}(i + x_i - 1)}{k} + \frac{k - \theta}{k} \\ &\geq \frac{|T_A| + \Delta'_A + 1}{2k} + \frac{k}{2} - \frac{1}{2} + \frac{k - \theta}{k} \\ &= \frac{|T_A| + \Delta'_A + 1}{2k} + \frac{k}{2} + \frac{1}{2} - \frac{\theta}{k}. \end{aligned}$$

Finally, if (29) holds and $i \in I$, then we apply both inequalities in (30) and later the inequality between the arithmetic and the geometric mean to obtain

$$\begin{aligned} i + x_i - 1 &\geq \frac{|T_A| + \theta}{2k} + \frac{k}{2} + \frac{1}{2} - \frac{\theta}{k} = \frac{|T_A|}{2k} + \frac{k}{2} + \frac{1}{2} - \frac{\theta}{2k} \\ &\geq \frac{|T_A|}{2k} + \frac{k}{2} \geq \sqrt{|T_A|}. \end{aligned}$$

Therefore, we conlude Theorem 7 by Lemma 19 and Lemma 20 (a). □

5 Proof of Theorem 8

We assume in this section that there are no points of A (resp. B) in the interior of $[A]$, (resp. $[B]$).

Recall that Δ_X denotes the number of points of X in the boundary of $[X]$. It is easy to check that Δ_{A+B} has at least as many points as Δ_A and Δ_B together, that is:

$$\Delta_{A+B} \geq \Delta_A + \Delta_B = \mathrm{tr}(A) + \mathrm{tr}(B) + 4.$$

As a motivation for the proof, we note that Conjecture 1 follows if the number Ω_{A+B} of points of $A+B$ in $\mathrm{int}([A+B])$ is at least

$$\frac{\mathrm{tr}(A) + \mathrm{tr}(B) - 2}{2} = \frac{\Delta_A + \Delta_B}{2} - 3.$$

Naturally we aim at the stronger Conjecture 5. Given Theorem 7, Theorem 8 follows if A and B being in convex position and $|A|, |B| \geq 4$ yield that there exists a mixed subdivision of $A+B$ satisfying

$$|M_{11}| \geq \frac{\mathrm{tr}(A) + \mathrm{tr}(B)}{2}. \tag{31}$$

Throughout the proof we assume that $[B]$ has at most as many vertices as $[A]$ and v denotes a unit vector (which we assume pointing upwards) not parallel to any side of $[A+B]$. We denote by a_0 and a_1 the leftmost and rightmost vertex of $[A]$ and by b_0 and b_1 the leftmost and rightmost vertex of $[B]$.

To prove (31), we say that A and B form a *strange pair* if $[B]$ is a triangle and the three exterior normals to $[B]$ are also exterior normals of edges of $[A]$.

We will use that, for $t, s \geq 1$,

$$ts \geq t + s - 1. \tag{32}$$

Case 1 A and B are not strange pairs.

We choose a unit vector v as above in the following way: if B is a triangle, then the upper arc of $\partial[B]$ is an side such that $[A]$ has no side with same exterior unit normal; if $[B]$ has at least four edges, then the two supporting lines of $[B]$ parallel to v touch at non-consecutive vertices of $[B]$. For the existence of the latter pair of supporting lines, we note that while continuously rotating $[B]$, the number of upper minus lower vertices changes by either zero or two units at a time when an edge of $[B]$ is parallel to v, and

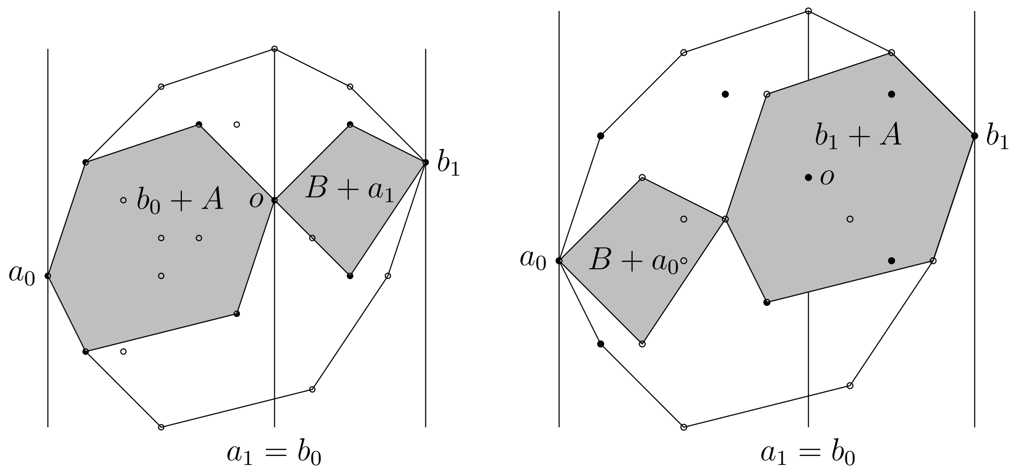

Figure 7: An illustration of the proof of Claim 21.

after rotation by π it changes to its opposite. Hence, at some position that difference is zero or one which implies, since $[B]$ has at least four vertices, that at that position there is at least one upper and one lower vertex, as required.

Claim 21 *One of the two following statements hold:*

$$\left|\Big((A+b_0)\cup(a_1+B)\Big)\cap \mathrm{int}[A+B]\right| \geq \tfrac{\Delta_A+\Delta_B}{2}-3, \text{ or}$$
$$\left|\Big((a_0+B)\cup(A+b_1)\Big)\cap \mathrm{int}[A+B]\right| \geq \tfrac{\Delta_A+\Delta_B}{2}-3. \tag{33}$$

Proof: We may assume that $b_1 = a_0 = o$ (see Fig. 7). Observe first that the only repetitions $x + b_0 = a_1 + y$ or $x + b_1 = a_0 + y$ in these configurations are the points a_1+b_0 and a_0+b_1 (which are interior to $[A+B]$ by our hypothesis). To prove (33), we verify first that

(i) for every $x \in A \setminus \{a_0, a_1\}$ except perhaps two of them, at least one of $x + b_0$ or $x + b_1$ is interior in $A + B$,

(ii) for every $y \in B \setminus \{b_0, b_1\}$ except perhaps two of them, at least one of $a_0 + y$ or $a_1 + y$ is interior in $A + B$.

For (i), we note that if both $x + b_0$ or $x + b_1$ are in $\partial[A + B]$, then they are the endpoints of a segment translated from $[b_0, b_1]$ and only two such translations have their endpoints in $\partial[A + B]$ because A and B are not a strange pair. The argument for (ii) is similar.

Now (i) and (ii) say that counting the interior points of $(A+b_0)\cup(a_1+B)$ and $(a_0+B)\cup(A+b_1)$ except a_0+b_1 and a_1+b_0 we have altogether at least $|\Delta_A|+|\Delta_B|-8$ of them. Including the latter we have at least $|\Delta_A|+|\Delta_B|-6$ of them and at least half of these in either $(A+b_0)\cup(a_1+B)$ or $(a_0+B)\cup(A+b_1)$, which yields (33). $\square$

Let us construct the suitable mixed triangulation of $[A+B]$. For every path σ on A, we assume that every point of A in σ is a vertex of σ. According to (33), we may assume that

$$|(A\cup B)\cap \operatorname{int}[A+B]| \geq \frac{\Delta_A+\Delta_B}{2}-3. \tag{34}$$

Let $a_{\rm upp}$ ($a_{\rm low}$) be the neighboring vertex of $[A]$ to o on the upper (lower) arc of $\partial[A]$, and let $b_{\rm upp}$ ($b_{\rm low}$) be the neighboring vertex of $[B]$ to o on the upper (lower) arc of $\partial[B]$. We write $\omega^A_{\rm upp}$ and $\omega^A_{\rm low}$ to denote the paths determined by $[o,a_{\rm upp}]$ and $[o,a_{\rm low}]$ and $\omega^B_{\rm upp}$ and $\omega^B_{\rm low}$ to denote the paths determined by $[o,b_{\rm upp}]$ and $[o,b_{\rm low}]$, and hence the two dimensionality of $[A]$ and $[B]$ implies

$$|\omega^A_{\rm upp}|,\ |\omega^A_{\rm low}|,\ |\omega^B_{\rm upp}|,\ |\omega^B_{\rm low}|\geq 1.$$

Next let $\sigma^A_{\rm upp}$ ($\sigma^A_{\rm low}$) be the longest path on the upper (lower) arc of $\partial[A]$ starting from o such that every segment s of $\sigma^A_{\rm upp}$ ($\sigma^A_{\rm low}$) satisfies that $s+[o,b_{\rm upp}]$ ($s+[o,b_{\rm low}]$) is a parallelogram that does not intersect $\operatorname{int}[A]$. Similarly, let $\sigma^B_{\rm upp}$ ($\sigma^B_{\rm low}$) be the longest path on the upper (lower) arc of $\partial[B]$ starting from o such that every segment s of $\sigma^B_{\rm upp}$ ($\sigma^B_{\rm low}$) satisfies that $s+[o,a_{\rm upp}]$ ($s+[o,a_{\rm low}]$) is a parallelogram that does not intersect $\operatorname{int}[B]$. Since $a_1=b_0=o$ is a common point of $\sigma^A_{\rm upp}$, $\sigma^A_{\rm low}$, $\sigma^B_{\rm upp}$, $\sigma^B_{\rm low}$, we deduce from (34) that

$$1+(|\sigma^A_{\rm upp}|-1)+(|\sigma^A_{\rm low}|-1)+(|\sigma^B_{\rm upp}|-1)+(|\sigma^B_{\rm low}|-1)\geq \frac{\Delta_A+\Delta_B}{2}-3,$$

equivalently,

$$|\sigma^A_{\rm upp}|+|\sigma^A_{\rm low}|+|\sigma^B_{\rm upp}|+|\sigma^B_{\rm low}|\geq \frac{\Delta_A+\Delta_B}{2}. \tag{35}$$

We construct the mixed subdivision by considering the subdivisions into suitable paralleograms of $\sigma^A_{\rm upp}+\omega^B_{\rm upp}$ and $\sigma^B_{\rm upp}+\omega^A_{\rm upp}$ that have $\omega^A_{\rm upp}+\omega^B_{\rm upp}$ in common, and the subdivisions into suitable parallelograms of $\sigma^A_{\rm low}+\omega^B_{\rm low}$ and $\sigma^B_{\rm low}+\omega^A_{\rm low}$ that have $\omega^A_{\rm low}+\omega^B_{\rm low}$ in common (see Figure 8).

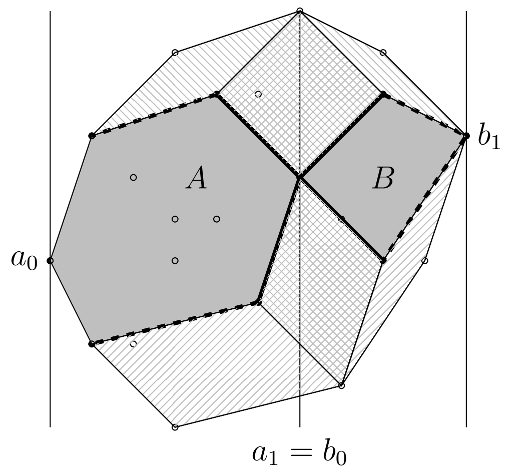

Figure 8: An illustration of the parallelograms of the mixed subdivision in Case 1.

In particular, $|\omega^A_{\text{upp}}|, |\omega^B_{\text{upp}}| \geq 1$, (32) and (35) yield that

$$\begin{aligned}
|M_{11}| &\geq (|\sigma^A_{\text{upp}}| - |\omega^A_{\text{upp}}|)|\omega^B_{\text{upp}}| + (|\sigma^B_{\text{upp}}| - |\omega^B_{\text{upp}}|)|\omega^A_{\text{upp}}| + |\omega^A_{\text{upp}}| \cdot |\omega^B_{\text{upp}}| + \\
&\quad + (|\sigma^A_{\text{low}}| - |\omega^A_{\text{low}}|)|\omega^B_{\text{low}}| + (|\sigma^B_{\text{low}}| - |\omega^B_{\text{low}}|)|\omega^A_{\text{low}}| + |\omega^A_{\text{low}}| \cdot |\omega^B_{\text{low}}| \\
&\geq (|\sigma^A_{\text{upp}}| - |\omega^A_{\text{upp}}|) + (|\sigma^B_{\text{upp}}| - |\omega^B_{\text{upp}}|) + |\omega^A_{\text{upp}}| + |\omega^B_{\text{upp}}| - 1 + \\
&\quad + (|\sigma^A_{\text{low}}| - |\omega^A_{\text{low}}|) + (|\sigma^B_{\text{low}}| - |\omega^B_{\text{low}}|) + |\omega^A_{\text{low}}| + |\omega^B_{\text{low}}| - 1 \\
&\geq \frac{\Delta_A + \Delta_B}{2} - 2 = \frac{\text{tr}(A) + \text{tr}(B)}{2},
\end{aligned}$$

proving (31) in Case 1.

Case 2 A and B form a strange pair with $|A|, |B| \geq 4$, and $[A]$ and $[B]$ are not similar triangles

We write α_{upp} (α_{low}) to denote the number of segments that the points of A divide the upper (lower) arc of $\partial[A]$. We denote by b_2 the third vertex of $[B]$ and by $[x_0, x_1]$ the side of A with $x_1 - x_0 = t(b_1 - b_0)$ for $t > 0$. For $i = 0, 1, 2$, let s_i be the number of segments that the points of B divide the side of $[B]$ opposite to b_i.

Claim 22 *There exists a v such that one of the following holds:*

$$\alpha_{\text{upp}} \geq 2 \text{ and } \alpha_{\text{upp}} + s_0 + s_1 \geq \frac{1}{2}(\Delta_A + \Delta_B), \text{ or} \tag{36}$$

$$\alpha_{\text{low}}, s_2 \geq 2 \text{ and } \alpha_{\text{low}} + s_2 \geq \frac{1}{2}(\Delta_A + \Delta_B). \tag{37}$$

Proof: Since $\alpha_{\text{upp}}+\alpha_{\text{low}}=\Delta_A$ and $s_0+s_1+s_2=\Delta_B$, the claim easily follows if there is a v such that, for each the sets A and B, both the upper arc and the lower arc contain a point of the set strictly between the two supporting lines parallel to v.

Otherwise, choose a v such that the side $[b_0,b_1]$ of $[B]$ contains at least 3 points of B (this is possible since $|B|\geq 4$). Then $[x_0,x_1]$ has no other point of A than x_0,x_1 and the other side of $[A]$ at x_i, $i=0,1$ is parallel to $[b_i,b_2]$. As $[A]$ and $[B]$ are not similar triangles , $[A]$ has some more edges, which in turn yields that $[b_i,b_2]\cap B=\{b_i,b_2\}$ for $i=0,1$. In summary, we have $\alpha_{\text{upp}}=s_0=s_1=1$ and $\alpha_{\text{low}},s_2\geq 2$. Since $\alpha_{\text{low}}+s_2>\alpha_{\text{upp}}+s_0+s_1$, we conclude (37). □

To prove (31) based on (36) and (37), we introduce some further notation. After a linear transformation, we may assume that v is an exterior normal to the edge $[b_0,b_1]$ of $[B]$. We say that $p,q\in\partial[A]$ are opposite if there exists a unit vector w such that w is an exterior normal at p and $-w$ is an exterior normal at q. If $p,q\in\partial[A]$ are not opposite, then we write $\overline{pq}$ the arc of $\partial[A]$ connecting p and q and not containing opposite pair of points.

First we assume that (36) holds and $b_2=o$. Since $[x_0,x_1]$ has exterior normal v and $\alpha_{\text{upp}}\geq 2$, there exists $a\in A\backslash\{x_0,x_1\}$ such that v is an exterior normal to $\partial[A]$ at a. We write l_{upp} and r_{upp} to denote the number of segments the points of A divide the arcs $\overline{ax_0}$ and $\overline{ax_1}$, respectively. To construct a mixed subdivision, we observe that every exterior normal u to a side of $[A]$ in $\overline{ax_0}$ satisfies $\langle u,b_0\rangle>0$, and every exterior normal w to a side of $[A]$ in $\overline{ax_1}$ satisfies $\langle w,b_1\rangle>0$. We divide $\overline{ax_0}+[o,b_0]$ into suitable s_1l_{upp} parallelograms, and $\overline{ax_1}+[o,b_1]$ into suitable s_0r_{upp} parallelograms. It follows from (32) that

$$\begin{aligned}|M_{11}| &= s_1l_{\text{upp}}+s_0r_{\text{upp}}\geq l_{\text{upp}}+r_{\text{upp}}+s_0+s_1-2=\alpha_{\text{upp}}+s_0+s_1-2\\ &\geq \tfrac{1}{2}(\Delta_A+\Delta_B)-2=\tfrac{1}{2}(\text{tr}(A)+\text{tr}(B)).\end{aligned}$$

Secondly we assume that (37) holds. Since $s_2\geq 2$, we may assume that $o\in([b_0,b_1]\backslash\{b_0,b_1\})\cap B$. For $i=0,1$, we write s_{2i} to denote the number of segments the points of B divide $[o,b_i]$. Let $\tilde{x}_0$ and $\tilde{x}_1$ be the leftmost and rightmost points of A such that $-v$ is an exterior normal to $\partial[A]$, where possibly $\tilde{x}_0=\tilde{x}_1$. Since $[A]$ has sides parallel to the sides $[b_2,b_0]$ and $[b_2,b_1]$ of $[B]$, we deduce that $\tilde{x}_0\neq x_0$ and $\tilde{x}_1\neq x_1$. To construct a mixed subdivision, we set $m_{\text{low}}=0$ if $\tilde{x}_0=\tilde{x}_1$, and m_{low} to be the number of segments the points of A divide $\overline{\tilde{x}_0,\tilde{x}_1}$ if $\tilde{x}_0\neq\tilde{x}_1$. In addition, we write $l_{\text{low}}\geq 1$ and $r_{\text{low}}\geq 1$ to denote the number of segments the points of A divide the arcs $\overline{\tilde{x}_0x_0}$ and $\overline{\tilde{x}_1x_1}$, respectively. We divide $\overline{\tilde{x}_0x_0}+[o,b_0]$ into suitable $l_{\text{low}}s_{20}$ parallelograms, and $\overline{\tilde{x}_1x_1}+[o,b_1]$ into suitable $r_{\text{upp}}s_{21}$ parallelograms. In addition, if $\tilde{x}_0\neq\tilde{x}_1$, then we divide $[\tilde{x}_0\tilde{x}_1] + [o, b_2]$ into suitable $m_{\rm low}$ parallelograms. It follows from (32) that

$$\begin{aligned} |M_{11}| &= l_{\rm low}s_{20} + r_{\rm low}s_{21} + m_{\rm low} \geq l_{\rm low} + r_{\rm low} + m_{\rm low} + s_{20} + s_{21} - 2 \\ &= \alpha_{\rm low} + s_2 - 2 \geq \tfrac{1}{2}(\Delta_A + \Delta_B) - 2 = \tfrac{1}{2}({\rm tr}(A) + {\rm tr}(B)), \end{aligned}$$

finishing the proof of (31) in Case 2.

Case 3 $[A]$ and $[B]$ are similar triangles and $|A|, |B| \geq 4$

We recall that s_1, s_2 and s_3 denote the number of segments the points of B divide the sides of $[B]$ and let s_1', s_2', s_3' be the number of segments the points of A divide the corresponding sides of $[A]$. We have ${\rm tr}(A) = s_1' + s_2' + s_3' - 2$ and ${\rm tr}(B) = s_1 + s_2 + s_3 - 2$. We may assume that s_1 is the largest among the six numbers and that $s_2' \geq s_3'$. Readily

$$|M_{11}| \geq \max\{s_1s_2', s_1's_2, s_1's_3\}. \tag{38}$$

If $s_2' \geq 3$, then

$$|M_{11}| \geq 3s_1 \geq \tfrac{1}{2}(s_1 + s_2 + s_3 + s_1' + s_2' + s_3') > \tfrac{1}{2}({\rm tr}(A) + {\rm tr}(B)).$$

If $s_2' = 2$, then $s_3' \leq 2$ and

$$|M_{11}| \geq 2s_1 \geq \tfrac{1}{2}(s_1 + s_2 + s_3 + s_1' + s_2' + s_3' - 4) = \tfrac{1}{2}({\rm tr}(A) + {\rm tr}(B)).$$

Therefore we assume that $s_2' = s_3' = 1$. In particular, we may also assume that $s_2 \geq s_3$. Since $s_1' \geq 2$ and $s_2 \geq 1$ we have $s_1's_2 \geq s_1' + 2s_2 - 2$. Therefore,

$$\begin{aligned} |M_{11}| &\geq \max\{s_1, s_1's_2\} \\ &\geq \frac{1}{2}(s_1 + s_2 + s_3 + s_1' - 2) \\ &= \frac{1}{2}({\rm tr}(A) + {\rm tr}(B)), \end{aligned}$$

and we conclude (31) in Case 3, as well. □

6 Proof of Theorem 10

Let $A = \{a_1, \ldots, a_n\}$. Naturally, $[A + A]$ has a triangulation $\{F + F : F \in T_A\}$, which we subdivide in order to obtain M. We define M to be the collection of the sums of the form

$$[a_{i_0}, \ldots, a_{i_m}] + [a_{i_m}, \ldots, a_{i_k}],$$

where $k \geq 0$, $0 \leq m \leq k$, $i_j < i_l$ for $j < l$, and $[a_{i_0}, \ldots, a_{i_k}] \in T_A$.

To show that we obtain a cell decomposition, let

$$F = [a_{i_0}, \ldots, a_{i_k}] \in T_A$$

be a k-simplex with $k > 0$ where $i_j < i_l$ for $j < l$, and hence

$$F + F = \left\{ \sum_{i=0}^{k} \alpha_j a_{i_j} : \sum_{i=0}^{k} \alpha_j = 2 \ \& \ \forall\, \alpha_j \geq 0 \right\}.$$

We write relint C to denote the relative interior of a compact convex set C. For some $0 \leq m \leq k$, $\alpha_0, \ldots, \alpha_k \geq 0$ with $\sum_{i=0}^{k} \alpha_j = 2$, we have

$$\sum_{i=0}^{k} \alpha_j a_{i_j} \in \text{relint}\ ([a_{i_0}, \ldots, a_{i_m}] + [a_{i_m}, \ldots, a_{i_k}]) \subset F + F$$

if and only if $\sum_{j<m} \alpha_j < 1$ and $\sum_{i=0}^{m} \alpha_j > 1$ where we set $\sum_{j<0} \alpha_j = 0$. We conclude that M forms a cell decomposition of $[A + A]$.

For any d-simplex $F \in T_A$, and for any $m = 0, \ldots, d$, we have constructed one d-cell of M that is the sum of an m-simplex and a $(d - m)$-simplex. Therefore

$$\|M\| = |T_A| \sum_{m=0}^{d} \binom{d}{m} = 2^d |T_A|.$$

7 Proof of Corollary 3

In this section, let $A \subset \mathbb{R}^2$ be finite and not contained in a line. We prove four auxiliary statements about A. The first is an application of the case $A = B$ of Conjecture 1 (see Theorem 2).

Lemma 23

$$|A + A| \geq 4|A| - \Delta_A - 3.$$

Proof: We have readily $\Delta_{A+A} \geq 2\Delta_A$. Thus (4) and Theorem 2 yield

$$|A + A| = \frac{1}{2} \left(\text{tr}(A + A) + \Delta_{A+A} + 2 \right) \geq 2\text{tr}(A) + \Delta_A + 1 = 4|A| - \Delta_A - 3. \ \Box$$

We note that the estimate of Lemma 23 is optimal, the configuration of Theorem 2 (b) being an extremal set.

Next we provide the well–known elementary estimate for $|A + A|$ only in terms of boundary points.

Lemma 24 *Let m_A denote the maximal number of points of A contained in a side of $[A]$. We have,*

$$|A+A| \geq \frac{\Delta_A^2}{4} - \frac{\Delta_A(m_A-1)}{2}.$$

Proof: We choose a line l not parallel to any side of $[A]$, that we may assume to be a vertical line, and denote by $s_1, \dots, s_k$ the sides of $[A]$ on the upper chain of $[A]$ in left to right order. Let A_i be the set obtained from $A \cap s_i$ by removing its rightmost point. We may assume that

$$|A_1| + \cdots + |A_k| \geq \frac{\Delta_A}{2}.$$

We observe that, for $1 \leq i < j \leq k$, we have

$$|A_i + A_j| = |A_i| \cdot |A_j| \text{ and } (A_i + A_j) \cap (A_{i'} + A_{j'}) = \emptyset \text{ if } \{i, j\} \neq \{i', j'\}.$$

It follows that

$$\begin{aligned} |A+A| &\geq \sum_{1 \leq i < j \leq k} |A_i + A_j| = \sum_{1 \leq i < j \leq k} |A_i| \cdot |A_j| = (\sum_{i=1}^{k} |A_i|)^2 - \sum_{i=1}^{k} |A_i|^2 \\ &\geq \left(\frac{\Delta_A}{2}\right)^2 - (m_A - 1)\frac{\Delta_A}{2}. \ \square \end{aligned}$$

The following Lemma can be found in Freiman [4].

Lemma 25 *Let ℓ be a line intersecting $[A]$ in m points of A. If A is covered by exactly s lines parallel to ℓ, then*

$$|A+A| \geq 2|A| + (s-1)m - s. \tag{39}$$

Moreover,

$$|A+A| \geq (4 - \frac{2}{s})|A| - (2s-1). \tag{40}$$

Proof: We may assume that ℓ is the vertical line through the origin, that $a_1, \dots, a_s$ are s points of A ordered left to right such that $A = \cup_{i=1}^{s}(A \cap (\ell + a_i))$ and that $|A \cap (\ell + a_1)| = m$. Let $A_i = A \cap (a_i + \ell)$. Then,

$$\begin{aligned} |A+A| &= |A_1 + A| + |(A \setminus A_1) + A_s| \\ &\geq \sum_{i=1}^{s}(|A_1| + |A_i| - 1) + \sum_{i=2}^{s}(|A_i| + |A_s| - 1) \\ &= 2|A| + (s-1)(|A_1| + |A_s|) - (2s-1), \end{aligned}$$

from which (39) follows. On the other hand,

$$\begin{aligned}|A+A| &= \sum_{i=1}^{s}|2A_i| + \sum_{i=1}^{s-1}|A_i+A_{i+1}| \\ &\geq \sum_{i=1}^{s}(2|A_i|-1) + \sum_{=1}^{s-1}(|A_i|+|A_{i+1}|-1) \\ &= 4|A| - (|A_1|+|A_s|) - (2s-1).\end{aligned}$$

If the latter estimate is larger than the former one we obtain (40), otherwise we get the stronger inequality $|A+A| \geq (4-2/s^2)|A| - (2s-1)$. □

Proof of Corollary 3 Let $|A+A| \leq (4-\varepsilon)|A|$ where $\varepsilon \in (0,1)$ and $\varepsilon^2|A| \geq 48$. To simply formulae, we set $\Delta = \Delta_A$ and $m = m_A$.

We deduce from Lemma 23 that $\Delta \geq \varepsilon|A| - 3$. Substituting this into Lemma 24 yields

$$\begin{aligned}(4-\varepsilon)|A| &\geq \frac{\Delta^2}{4} - \frac{\Delta(m-1)}{2} \geq \frac{\Delta(\varepsilon|A|-3)}{4} - \frac{\Delta(m-1)}{2} \\ &= \frac{\Delta}{2}\cdot(\tfrac{1}{2}\varepsilon|A| - m - \tfrac{1}{2}) \geq \frac{\varepsilon|A|-3}{2}\cdot(\tfrac{1}{2}\varepsilon|A| - m - \tfrac{1}{2}).\end{aligned}$$

Therefore

$$\tfrac{1}{2}\varepsilon|A| - (m-1) \leq \frac{8}{\varepsilon}\left(1-\frac{\varepsilon}{4}\right)\left(1+\frac{3}{\varepsilon|A|-3}\right) + \frac{3}{2} < \frac{12}{\varepsilon},$$

as $\varepsilon|A| - 3 \geq \frac{48}{\varepsilon} - 3 > \frac{12}{\varepsilon}$. In particular, $m - 1 > \frac{1}{2}\varepsilon|A| - \frac{12}{\varepsilon}$.

Next let l be the line determined by a side of $[A]$ containing $m = m_A$ points of A, and let s be the number of lines parallel to l intersecting A. According to (39),

$$(4-\varepsilon)|A| \geq 2|A| + (s-1)(m-1) - 1 > 2|A| + (s-1)(\tfrac{1}{2}\varepsilon|A| - \tfrac{12}{\varepsilon}) - 1,$$

thus first rearranging, and then applying $\varepsilon^2|A| \geq 48$ yield

$$2|A| > s\cdot(\tfrac{1}{2}\varepsilon|A| - \tfrac{12}{\varepsilon}) \geq s\cdot\tfrac{1}{4}\varepsilon|A|.$$

Therefore $s < \frac{8}{\varepsilon}$.

We deduce from (40) and $s < \frac{8}{\varepsilon}$ that

$$(4-\varepsilon)|A| > (4-\tfrac{2}{s})|A| - 2s > (4-\tfrac{2}{s})|A| - \tfrac{16}{\varepsilon}.$$

Rearranging, and then applying $\varepsilon^2|A| \geq 48$ imply

$$s < \frac{2}{\varepsilon}\left(1-\frac{16}{\varepsilon^2|A|}\right)^{-1} < \frac{2}{\varepsilon}\left(1+\frac{32}{\varepsilon^2|A|}\right). \ \square$$

8 Proof of Proposition 6

We call the points of A,

$$a_0 = (0,0), \quad a_1 = (-1,-2), \quad a_2 = (2,1).$$

If $k \geq 2$, then we show that every mixed subdivision M corresponding to T_A and T_B satisfies

$$|M_{11}| \leq 24. \tag{41}$$

We prove (41) in several steps. First we verify

$$[a_1, a_2] + l_i \quad \text{is not an edge of } M \quad \text{for } i = 0, \ldots, k, \tag{42}$$

$$[a_1, a_2] + r_i \quad \text{is not an edge of } M \quad \text{for } i = 0, \ldots, k-1. \tag{43}$$

For (42), we observe that $a_1 + l_{i+1}$ if $i \leq k-1$ or $a_1 + l_{i-1}$ if $i \geq 1$ is a point of $A + B$ in $[a_1, a_2] + l_i$ different from the endpoints. Similarly, for (43), we observe that $a_1 + r_{i+1}$ if $i \leq k-2$ or $a_1 + r_{i-1}$ if $i \geq 1$ is a point of $A + B$ in $[a_1, a_2] + r_i$ different from the endpoints.

Next, we have

$$[a_0, a_2] + [l_i, r_i] \quad \text{is not a parallelogram of } M \quad \text{for } i = 0, \ldots, k-1, \tag{44}$$

$$[a_0, a_1] + [r_i, l_{i+1}] \quad \text{is not a parallelogram of } M \quad \text{for } i = 0, \ldots, k-1, \tag{45}$$

as $l_{i+1} \in \operatorname{int}[a_0, a_2] + [l_i, r_i]$ and $l_i \in \operatorname{int}[a_0, a_1] + [r_i, l_{i+1}]$.

Let us call the edges of T_B of the form either $[l_i, r_i]$ or $[r_i, l_{i+1}]$ for $i = 0, \ldots, k-1$ *small edges*, and the edges of T_B of the form either $[p, l_i]$, $[q, l_i]$ for $i = 0, \ldots, k$, or $[p, r_i]$, $[q, r_i]$ for $i = 0, \ldots, k-1$ *long edges*. In other words, long edges of T_B contain either p or q, while small edges of T_B contain neither.

Concerning long edges, we prove that that the number of parallelograms of M of the form

$$e_A + e_B \text{ for an edge } e_A \text{ of } T_A \text{ and a long edge } e_B \text{ of } T_B \text{ is at most 12.} \tag{46}$$

If e_A is an edge of T_A, then there exist at most two cells of M whose sides are $p + e_A$. Since T_A has three edges, there are at most six of parallelograms of M of the form $e_A + e_B$ where e_A is an edge of T_A and e_B is an edge of T_B with $p \in e_B$. Since the same estimate holds if $q \in e_B$, we conclude (46).

Finally, we prove that that the number of parallelograms of M of the form

$$e_A + e_B \text{ for an edge } e_A \text{ of } T_A \text{ and a small edge } e_B \text{ of } T_B \text{ is at most 12.} \tag{47}$$

The argument for (47) is based on the claim that if $e_A + e_B$ is a parallelogram of M for an edge e_A of T_A and a small edge e_B of T_B, then there is a long edge e'_B of T_B such that

$$e_A + e'_B \text{ is a neighboring parallelogram of } M. \tag{48}$$

We have $e_A \neq [a_1, a_2]$ according to (42) and (43). If $e_A = [a_0, a_1]$, then $e_B = [l_i, r_i]$ for some $i \in \{1, \dots, k-1\}$ according to (45). Now $r_i + e_A$ intersects the interior of $[A+B]$ as $r_i \in \text{int}\,[A]$, thus it is the edge of another cell of M, as well. This other cell is either a translate of $[A]$, which is impossible by (42), (43), and as $r_i \notin p + [A], q + [A]$, or of the form $e_A + e'_B$ for an edge $e'_B \neq e_B$ of T_B containing r_i. However, $e'_B \neq [r_i, l_{i+1}]$ by (45), therefore e'_B is a long edge.

On the other hand, if $e_A = [a_0, a_2]$, then $e_B = [r_i, l_{i+1}]$ for some $i \in \{1, \dots, k-1\}$ according to (44), and (48) follows as above.

Now if $e_A + e'_B$ is a parallelogram of M for an edge e_A of T_A and a long edge e'_B of T_B, then there is at most one neighboring paralellogram of the form $e_A + e_B$ for a small edge e_B of T_B because $e_A + e_B$ does not intersect $e_A + p$ and $e_A + q$. In turn, (47) follows from (46) and (48). Moreover, we conclude (41) from (46) and (47).

Finally, it follows from (41) that if $k \geq 145$, then

$$|M_{11}| \leq 24 < \sqrt{4k} = \sqrt{|T_A| \cdot |T_B|}. \ \square$$

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