# Identifying codes in line digraphs *

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**Abstract**

Given an integer $\ell \geq 1$, a $(1, \leq \ell)$-identifying code in a digraph is a dominating subset $C$ of vertices such that all distinct subsets of vertices of cardinality at most $\ell$ have distinct closed in-neighborhoods within $C$. In this paper, we prove that every line digraph of minimum in-degree one does not admit a $(1, \leq \ell)$-identifying code for $\ell \geq 3$. Then we give a characterization so that a line digraph of a digraph different from a directed cycle of length 4 and minimum in-degree one admits a $(1, \leq 2)$-identifying code. The identifying number of a digraph $D$, $\overrightarrow{\gamma}^{ID}(D)$, is the minimum size of all the identifying codes of $D$. We establish for digraphs without digons with both vertices of in-degree one that $\overrightarrow{\gamma}^{ID}(LD)$ is lower bounded by the number of arcs of $D$ minus the number of vertices with out-degree at least one. Then we show that $\overrightarrow{\gamma}^{ID}(LD)$ attains the equality for a digraph having a 1-factor with minimum in-degree two and without digons with both vertices of in-degree two. We finish by giving an algorithm to construct identifying codes in oriented digraphs with minimum in-degree at least two and minimum out-degree at least one.

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## 1 Introduction

In this paper, we consider simple digraphs without loops or multiple edges. Unless otherwise stated, we follow the book by Bang-Jensen and Gutin [3] for terminology and definitions.

Let $D$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. A vertex $u$ is *adjacent to* a vertex $v$ if $(u,v) \in A(D)$. If both arcs $(u,v),(v,u) \in A(D)$, then we say that they form a *digon.* A digraph is *symmetric* if $(u,v) \in A(D)$ implies $(v,u) \in A(D)$, so it can be studied as a graph. A digon is often referred to as a *symmetric arc* of $D$. An *oriented graph* is a digraph without digons. The *out-neighborhood* of a vertex $u$ is $N^+(u) = \{v \in V : (u,v) \in A(D)\}$ and the *in-neighborhood* of $u$ is $N^-(u) = \{v \in V(D) : (v,u) \in A(D)\}$. The *closed in-neighborhood* of $u$ is $N^-[u] = \{u\} \cup N^-(u)$. Given a vertex subset $U \subset V(D)$, let $N^-[U] = \bigcup_{u\in U} N^-[u]$ and $N^+[U] = \bigcup_{u\in U} N^+[u]$. *A dominating set* is a subset of vertices $S \subseteq V$ such that $N^+[S] = V$. The *out-degree* of $u$ is $d^+(u) = |N^+(u)|$, and its *in-degree* $d^-(u) = |N^-(u)|$. We denote by $\delta^+ = \delta^+(D)$ the minimum out-degree of the vertices in $D$, and by $\delta^- = \delta^-(D)$ the minimum in-degree. The minimum degree is $\delta = \delta(D) = \min\{\delta^+, \delta^-\}$. A digraph $D$ is said to be *d-in-regular* if $|N^-(v)| = d$ for all $v \in V(D)$, and *d-regular* if $|N^+(v)| = |N^-(v)| = d$ for all $v \in V(D)$. For each vertex $v \in V(D)$, we denote $\Omega^-(v) = \{(u,v) \in A(D)\}$ and $\Omega^+(v) = \{(v,u) \in A(D)\}$.

In this paper, we study the concept of $(1, \le \ell)$-identifying codes, where $\ell \ge 1$ is an integer. For a given integer $\ell \ge 1$, a vertex subset $C \subset V(D)$ is a *$(1, \le \ell)$-identifying code* in $D$ if it is a dominating set and for all distinct subsets $X, Y \subset V(D)$, with $1 \le |X|, |Y| \le \ell$, we have

$$N^-[X] \cap C \ne N^-[Y] \cap C. \tag{1}$$

The definition of a $(1, \le \ell)$-identifying code for graphs was introduced by Karpovsky, Chakrabarty and Levitin [12], and their definition can be obtained from (1) by omitting the superscript signs minus. Thus, the definition for digraphs is a natural extension of the concept of $(1, \le \ell)$-identifying codes in graphs. A $(1, \le 1)$-identifying code is known as an *identifying code.* Therefore, an identifying code of a graph is a dominating set such that any two vertices of the graph have distinct closed neighborhoods within this set. Identifying codes model fault-diagnosis in multiprocessor systems, and these are used in other applications, such as the design of emergency sensor networks. For more information on these applications, see Karpovsky, Chakrabarty, and Levitin [12] and Laifenfeld, Trachtenberg, Cohen, and Starobinski [13].

Note that if $C$ is a $(1, \le \ell)$-identifying code in a digraph $D$, then the whole set of vertices $V(D)$ also is. Thus, a digraph $D$ admits a *$(1, \le \ell)$-identifying code* if and only if for all distinct subsets $X, Y \subset V(D)$ with $|X|, |Y| \le \ell$, we have

$$N^-[X] \ne N^-[Y]. \tag{2}$$

In [2], the authors studied the $(1, \le \ell)$-identifying codes in digraphs and proved the following results.

**Proposition 1.1** ([2])**.** *Let $D$ be a digraph admitting a $(1, \leq \ell)$-identifying code. Let $u$ be a vertex such that $d^+(u) \geq 1$. Then $\ell \leq d^-(u) + 1$. Furthermore, if $u$ belongs to a digon, then $\ell \leq d^-(u)$.*

**Corollary 1.2** ([2])**.** *Let $D$ be a digraph admitting a $(1, \leq \ell + 1)$-identifying code. Then any vertex $u$ with $d^-(u) = \ell$ does not lie on a digon.*

According to Corollary 1.2 if $D$ is a digraph admitting a $(1, \leq 2)$-identifying code, then there is no vertex of in-degree 1 lying on a digon.

Regarding line graphs, Foucaud, Gravier, Naserasr, Parreau, and Valicov [7] studied $(1, \leq 1)$-identifying codes and Junnila and Laihonen [11] studied $(1, \leq \ell)$-identifying codes for $\ell \geq 2$. Regarding identifying codes in digraphs, Foucaud, Naserasr, and Parreau [8] characterized extremal digraphs for identifying codes; Charon, Gravier, Hudry, et. al., [5] gave a linear algorithm to find a minimum $(1, \leq 1)$-identifying code in oriented trees; and, Charon, Hudry, and Lobstein [6] gave some results about complexity. Other results were given by Skaggs in his Ph.D. thesis [15].

The main objective of this paper is to study $(1, \leq \ell)$-identifying codes in line digraphs. In the line digraph $LD$ of a digraph $D$, each vertex represents an arc of $D$. Thus, $V(LD) = \{uv : (u, v) \in A(D)\}$; and a vertex $uv$ is adjacent to a vertex $wz$ if and only if $v = w$, that is, when the arc $(u, v)$ is adjacent to the arc $(w, z)$ in $D$. For any integer $k \geq 1$, the *$k$-iterated line digraph* $L^kD$ is defined recursively by $L^kD = LL^{k-1}D$, where $L^0D = D$. From the definition, it is evident that the order of $LD$ equals the size of $D$, that is, $|V(LD)| = |A(D)|$. Due to the bijection between the set of arcs in the digraph D and the set of vertices in the digraph $LD$, when it is clear from the context, we use $uv$ to denote both the arc in $A(D)$ and the vertex in $V(LD)$. Hence, for each vertex $v \in V(D)$, the set of arcs $\Omega^+(v), \Omega^-(v)$ in $D$ corresponds to a set of vertices in $LD$. Moreover, $d^+(v) = |\Omega^+(v)|$ and $d^-(v) = |\Omega^-(v)|$, so if $D$ has minimum degree $\delta$, then the iterated line digraph $L^kD$ has minimum degree $\delta$. Other properties of line digraph can be seen in Aigner [1], Fiol, Yebra, and Alegre [9], and Reddy, Kuhl, Hosseini, and Lee [14].

This paper is organized as follows. In Section 2, we prove that a line digraph of minimum in-degree one does not admit a $(1, \leq \ell)$-identifying code for $\ell \geq 3$. Then we give a characterization so that a line digraph of a digraph different from a directed cycle of length 4 and minimum in-degree one admits a $(1, \leq 2)$-identifying code. As a direct consequence, we obtain that a Kautz digraph $K(d, k)$ with $d \geq 3$ admits a $(1, \leq 2)$-identifying code. The identifying number of a digraph $D$, $\overrightarrow{\gamma}^{ID}(D)$, is the minimum size of all the identifying codes of $D$. In Section 3, we establish for digraphs without digons with both vertices of in-degree one that $\overrightarrow{\gamma}^{ID}(LD) \geq |A(D)| - |V^+_{\geq 1}(D)|$, where $V^+_{\geq 1}(D)$ is the set of vertices of $D$ with out-degree at least one. Using this lower bound we get that a digraph having a 1-factor with minimum in-degree two and without digons with both vertices of in-degree two satisfies that $\overrightarrow{\gamma}^{ID}(LD) = |A(D)| - |V(D)|$. We finish by giving an algorithm to construct identifying codes in oriented digraphs with minimum in-degree at least two and minimum out-degree at least one. This algorithm allows us to prove that an oriented graph with minimum in-degree and out-degree at least two satisfies that $\overrightarrow{\gamma}^{ID}(LD) = |A(D)| - |V(D)|$.

## 2 Identifying codes in line digraphs

Line digraphs were characterized by Heuchenne [10] with the following property.

**Lemma 2.1** ([10]). *A digraph $D$ is a line digraph if and only if it has no multiple arcs, and for any pair of vertices $u$ and $v$, either $N^-(u) \cap N^-(v) = \emptyset$ or $N^-(u) = N^-(v)$. (The similar result is obtained replacing $N^-$ by $N^+$.)*

Recall that a *transitive tournament* on 3 vertices is denoted by $TT_3$, as it is shown in Figure 1. Another useful characterization of line digraphs was given by Beineke and Zamfirescu [4].

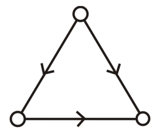


Figure 1: A transitive tournament on 3 vertices.

**Theorem 2.2** ([4]). *A (simple) digraph $D$ is a line digraph if and only if $D$ is $TT_3$-free, the paths of length two are unique, there are no two digons incident to the same vertex, and if there are two vertices $u, v$ such that $N^+(u) \cap N^+(v) \neq \emptyset$, then $N^+(u) = N^+(v)$.*

The following remark is very useful for our next results.

**Remark 2.3.** *Let $D$ be a $TT_3$-free digraph. Then for every arc $(x, y)$ of $D$, we have $N^-(x) \cap N^-(y) = \emptyset$ and $N^+(x) \cap N^+(y) = \emptyset$.*

**Remark 2.4.** *Since there is a bijection between the digons of the original digraph and the digons of its line digraph, it follows that a digraph contains a digon with both vertices of in-degree 1 if and only if its line digraph also contains a digon with both vertices of in-degree 1.*

**Proposition 2.5.** *Let $D$ be a digraph. Then, its line digraph admits a $(1, \leq 1)$-identifying code if and only if there is no digon with both vertices of in-degree 1 in $D$.*

*Proof.* We know that a digraph $G$ admits a $(1, \leq 1)$-identifying code if and only if for any two different vertices $x, y \in V(G)$ we have $N_G^-[x] \neq N_G^-[y]$. Let $LD$ be the line digraph of a digraph, and suppose that $LD$ does not admit a $(1, \leq 1)$-identifying code. This is equivalent to have two different vertices $x, y \in V(LD)$ such that $N^-[x] = N^-[y]$ which implies that $x$ and $y$ form a digon. By Theorem 2.2, $LD$ is $TT_3$-free, and by Remark 2.3, $N^-(x) \cap N^-(y) = \emptyset$, yielding that $d^-(x) = d^-(y) = 1$. Hence, $LD$ contains a digon with both vertices of in-degree 1, and by Remark 2.4, $D$ contains a digon with both vertices of in-degree 1, which is a contradiction with the hypothesis. Conversely, suppose that there is a digon $uv, vu \in A(D)$ with $d^-(u) = d^-(v) = 1$, then $uv, vu \in V(LD)$ form a digon in $LD$, and these vertices are twins since $N^-_{LD}[uv] = \{uv, vu\} = N^-_{LD}[vu]$, implying that $LD$ does not admit a $(1, \leq 1)$-identifying code, a contradiction. □

**Corollary 2.6.** *Let $D$ be a digraph with minimum in-degree $\delta^- \geq 2$. Then its line digraph admits a $(1, \leq 1)$-identifying code.*

Next, we establish that if a line digraph admits a $(1, \leq \ell)$-identifying code, then $\ell \leq 2$. To this end, we need to prove some preliminary results.

**Lemma 2.7.** *Let $D$ be a digraph admitting a $(1, \leq \ell)$-identifying code. If there are two different vertices $x, y \in V(D)$ such that $d^+(y) \geq 1$, then $\ell < d^-(y) - |N^-(x) \cap N^-(y)| + 3$. Moreover, if $x \in N^+(y)$, then $\ell < d^-(y) - |N^-(x) \cap N^-(y)| + 2$.*

*Proof.* Let $x, y$ be two distinct vertices satisfying the hypothesis of the lemma, and let $w \in N^+(y)$. First, assume that $w \neq x$. Consider the set $X = (N^-(y) \setminus N^-(x)) \cup \{w, x, y\}$. Since $y \in N^-(w)$ and $w \in X - y$, we can check that $N^-[y] \subset N^-[X - y]$, which implies that $N^-[X] = N^-[X - y]$. Then, $\ell < |X| \leq d^-(y) - |N^-(x) \cap N^-(y)| + 3$. Finally, if $w = x$ repeating the same reasoning, we obtain that $\ell < |X| \leq d^-(y) - |N^-(x) \cap N^-(y)| + 2$. This completes the proof. □

**Corollary 2.8.** *Let $D$ be a digraph admitting a $(1, \leq \ell)$-identifying code. If there are two different vertices $x, y \in V(D)$ such that $d^+(y) \geq 1$ and $N^-(y) \subseteq N^-(x)$, then $\ell \leq 2$.*

**Lemma 2.9.** *Let $D$ be a digraph with minimum in-degree $\delta^- \geq 2$. Then, there exists a vertex $u \in V(D)$ with $d^+(u) \geq 2$ and such that there is at least two out-neighbours $x, y \in N^+(u)$ such that $d^+(x), d^+(y) \geq 1$.*

*Proof.* Let $D$ be a digraph with minimum in-degree $\delta^- \geq 2$, and consider the subdigraph $D' = D - \{w \in V(D) \mid d^+(w) = 0\}$. Then, $\delta^-(D') \geq 2$. If $d^+_{D'}(u) < 2$ for all $u \in V(D')$, then we would reach the contradiction:

$$2|V(D')| \leq \sum_{v \in V(D')} d^-_{D'}(v) = \sum_{v \in V(D')} d^+_{D'}(v) \leq |V(D')|.$$

Hence, there is $u \in V(D')$ such that $d^+_{D'}(u) \geq 2$ and therefore, $d^+(u) \geq 2$. Since for any $v \in N^+_{D'}(u) \subseteq N^+(u)$ we have $d^+(v) \geq 1$, the proof is completed. □

**Proposition 2.10.** *Let $LD$ be a line digraph of a digraph $D$ with minimum in-degree $\delta^- \geq 1$, then $LD$ does not admit a $(1, \leq \ell)$-identifying code for $\ell \geq 3$.*

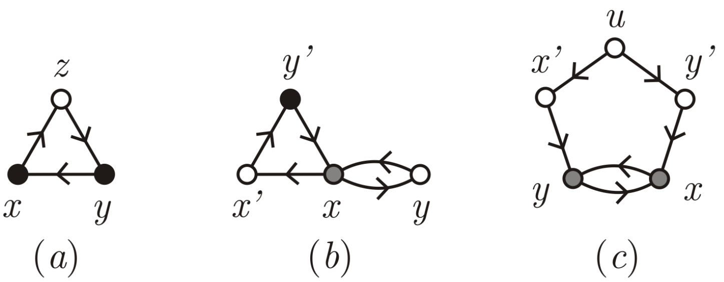


Figure 2: The forbidden subdigraphs of Theorem 2.12 and Corollary 2.13, where the vertices of in-degree one are indicated in black color and the vertices of in-degree two in gray color.

*Proof.* Note that $\delta^-(LD) = \delta^-(D) = \delta^-$. If $\delta^- \geq 2$, by Lemma 2.9, there exists a vertex $v$ in $LD$ with $d^+(v) \geq 2$ and two vertices $x, y \in N^+(v)$ such that $d^+(x), d^+(y) \geq 1$. By Lemma 2.1, we have $N^-(x) = N^-(y)$. Hence, by Corollary 2.8, if $LD$ admits a $(1, \leq \ell)$-identifying code, then $\ell \leq 2$, and the result is valid. Suppose that $\delta^- = 1$. Take a vertex $u$ with $d^-(u) = 1$. If $d^+(u) \geq 1$, then by Proposition 1.1, we get that $\ell \leq 2$ and we obtain the result. Therefore we assume that every vertex with in-degree one has out-degree zero. Let $F$ be the digraph obtained from $LD$ by removing all the vertices of in-degree one. Observe that $\delta^-(F) \geq 2$, then reasoning as in the first part of the proof, $F$ does not admit a $(1, \leq 3)$-identifying code. This means that there are two different sets $X, Y \subseteq F \subset V(LD)$ such that $1 \leq |X| \leq |Y| \leq 3$ and $N^-_F[X] = N^-_F[Y]$. Since for any vertex $u \in V(F)$, $N^-_F[u] = N^-_{LD}[u]$, it follows that $N^-_{LD}[X] = N^-_{LD}[Y]$. Hence, $LD$ does not admit a $(1, \leq 3)$-identifying code. □

Remember that according to Corollary 1.2, if $D$ is a digraph admitting a $(1, \leq 2)$-identifying code, then there is no vertex of in-degree 1 belonging to a digon. In the following result, we give sufficient and necessary conditions for a line digraph to admit a $(1, \leq 2)$-identifying code. To do that, we use the following result which follows from the fact that in a line digraph the paths of length two are unique by Theorem 2.2.

**Corollary 2.11.** *Let $LD$ be a line digraph. If $u, v \in V(LD)$ are two different vertices such that $N^+(u) \cap N^+(v) \neq \emptyset$, then $N^-(u) \cap N^-(v) = \emptyset$.*

**Theorem 2.12.** *Let $LD$ be a line digraph with minimum in-degree $\delta^- \geq 1$, different from a directed 4-cycle and such that the vertices of in-degree 1 (if any) do not lie on a digon. Then, $LD$ admits a $(1, \leq 2)$-identifying code if and only if $LD$ satisfies the following conditions:*

*(i) There are no directed 3-cycles with at least 2 vertices of in-degree 1 (see Figure 2 (a)).*

*(ii) There do not exist four vertices $x, x', y$ and $y'$ such that $N^-(x) = \{y, y'\}$, $N^-(y') = \{x'\}$ and $x \in N^-(x') \cap N^-(y)$ (see Figure 2 (b)).*

(*iii*) *There do not exist four vertices* $x, x', y$ *and* $y'$ *in* $V(LD)$ *such that* $N^-(x) = \{y, y'\}$, $N^-(y) = \{x, x'\}$ *and* $N^-(x') \cap N^-(y') \neq \emptyset$ *(see Figure 2 (c)).*

(*iv*) *There is no directed 4-cycle with the four vertices of in-degree 1.*

*Proof.* First, suppose that $LD$ admits a $(1, \leq 2)$-identifying code and let us show that $LD$ satisfies all the conditions $(i)$-$(iv)$.

$(i)$ Suppose that $LD$ does not satisfy $(i)$. Hence, let $(z, y, x, z)$ be a directed 3-cycle in $LD$ such that $d^-(x) = 1 = d^-(y)$ (see Figure 2 (a)). Then, $N^-[\{x, z\}] = \{x, z\} \cup \{y\} \cup N^-(z) = \{y\} \cup N^-[z]$, and $N^-[\{y, z\}] = \{y, z\} \cup N^-(z) = \{y\} \cup N^-[z]$, implying that $LD$ does not admit a $(1, \leq 2)$-identifying code which is a contradiction.

$(ii)$ Suppose that $LD$ does not satisfy $(ii)$. Let $X = \{x, x'\}$ and $Y = \{y, y'\}$, where $x, x', y, y'$ are four different vertices of $LD$ such that $N^-(x) = \{y, y'\}$, $N^-(y') = \{x'\}$, and $x \in N^-(x') \cap N^-(y)$ (see Figure 2 (b)). Hence, by Lemma 2.1, we get $N^-(x') = N^-(y)$, and it follows that

$$\begin{aligned} N^-[X] &= N^-(x) \cup N^-(x') \cup \{x, x'\} \\ &= \{y, y'\} \cup N^-(y) \cup \{x, x'\} \\ &= \{y, y'\} \cup N^-(y) \cup \{x'\} \\ &= \{y, y'\} \cup N^-(y) \cup N^-(y') \\ &= N^-[Y]. \end{aligned}$$

Therefore, $LD$ does not admit a $(1, \leq 2)$-identifying code which is a contradiction.

$(iii)$ Suppose that $LD$ does not satisfy $(iii)$. Let $X = \{x, x'\}$ and $Y = \{y, y'\}$, where $N^-(x) = \{y, y'\}$, $N^-(y) = \{x, x'\}$, and $N^-(x') \cap N^-(y') \neq \emptyset$ (see Figure 2 (c)). Since, by Lemma 2.1, $N^-(x') = N^-(y')$, it follows that

$$\begin{aligned} N^-[X] &= N^-(x) \cup N^-(x') \cup \{x, x'\} \\ &= \{y, y'\} \cup N^-(y') \cup N^-(y) \\ &= N^-[Y]. \end{aligned}$$

Therefore, $LD$ does not admit a $(1, \leq 2)$-identifying code which is a contradiction.

$(iv)$ Suppose that $LD$ does not satisfy $(iv)$. Let $(u_1, u_2, u_3, u_4, u_1)$ be a 4-cycle of $LD$ such that $d^-(u_i) = 1$ for all $i \in \{1, 2, 3, 4\}$. Then, $N^-[\{u_1, u_3\}] = N^-[\{u_2, u_4\}]$, implying that $LD$ does not admit a $(1, \leq 2)$-identifying code which is a contradiction.

For the converse, suppose that $LD$ satisfies all the conditions $(i)$-$(iv)$ and that does not admit a $(1, \leq 2)$-identifying code. Let $X, Y \subseteq V(LD)$ be two different subsets such that $1 \leq |X| \leq |Y| \leq 2$ and $N^-[X] = N^-[Y]$. Since the vertices of in-degree one do not lie on a digon, by Proposition 2.5, $|Y| = 2$. If $|X| = 1$, say $X = \{x\}$, then $N^-[Y] = N^-[X] = N^-[x]$. It follows that $N^-[y] \subseteq N^-(x)$ for all $y \in Y \setminus X$, hence

$y \in N^-(x)$. By Theorem 2.2, $LD$ is $TT_3$-free which allows us to apply Remark 2.3, so $N^-(y) \cap N^-(x) = \emptyset$, then $N^-(y) = \emptyset$ which contradicts that $\delta^- \geq 1$.

Suppose $|X| = 2$ and consider two cases according to $X \cap Y \neq \emptyset$ or if $X \cap Y = \emptyset$.

$(a)$ Suppose that $X \cap Y \neq \emptyset$. Let $X = \{x, z\}$ and $Y = \{y, z\}$. We will consider two cases, when there is at least one arc between $x$ and $y$, and the case when there is no arc between $x$ and $y$.

$(a.1)$ If there is an arc between $x$ and $y$, say $yx \in A(LD)$, then by Remark 2.3, $N^-(x) \cap N^-(y) = \emptyset$. Then, $N^-(y) \subseteq N^-[z] \cup \{x\}$ and $N^-(x) \subseteq N^-[z] \cup \{y\}$. First, suppose that $d^-(x) \geq 2$ and let $u \in N^-(x) \setminus \{y\}$. Hence, $u \in N^-[z]$. If $u = z$, then $N^-(x) \cap N^-(z) = \emptyset$ by Remark 2.3, and $N^-(y) \cap N^-(z) = \emptyset$. Hence, $N^-(x) = \{y, z\}$ and $N^-(y) = \{x\}$ (since $\delta^-(LD) \geq 1$). Then $y$ is a vertex of in-degree 1 lying on a digon, a contradiction to the hypothesis. Therefore, $u \neq z$, that is, $u \in N^-(z) \cap N^-(x)$ implying, by Lemma 2.1, that $N^-(z) = N^-(x)$, hence $y \in N^-(z)$, implying that $N^-(y) \cap N^-(z) = \emptyset$ by Remark 2.3. Then $N^-(y) \subseteq \{x, z\}$. Since $\delta^-(LD) \geq 1$, it follows that $N^-(y) = \{x\}$, $N^-(y) = \{z\}$ or $N^-(y) = \{x, z\}$. The first two cases are not possible because vertices of degree one do not lie on digons, and the third case is not possible because by Theorem 2.2, $LD$ does not contain two digons incident to the same vertex. Second, suppose that $d^-(x) = 1$, then $N^-(x) = \{y\}$. Since $x \in N^-[Y]$ and $x$ does not lie on a digon, $x \in N^-(z)$. Since, $x \notin N^-(y)$, $N^-(y) \cap N^-(z) = \emptyset$ by Lemma 2.1, implying that $N^-(y) = \{z\}$ because $N^-(y) \subseteq N^-[z]$. Therefore, $(x, z, y, x)$ is a directed 3-cycle of $LD$ with two vertices of in-degree 1, implying that $LD$ does not satisfy $(i)$.

$(a.2)$ Now, suppose that there is no arc between $x$ and $y$. Since $x \in N^-[Y]$ and $y \in N^-[X]$, it follows that $x, y \in N^-(z)$. Since $LD$ is $TT_3$-free, $y \notin N^-(x)$, by Remark 2.3, $N^-(x) \cap N^-(z) = \emptyset$, and by Corollary 2.11, $N^-(x) \cap N^-(y) = \emptyset$ implying that $N^-(x) = \{z\}$ and $x, z$ form a digon, a contradiction since there are no vertices of in-degree 1 lying on a digon.

$(b)$ Suppose $X \cap Y = \emptyset$, with $X = \{x, x'\}$ and $Y = \{y, y'\}$. Notice that we can assume $y \in N^-(x)$, that is, $yx \in A(LD)$. Then by Remark 2.3, $N^-(x) \cap N^-(y) = \emptyset$ implying that $N^-(y) \subseteq N^-(x') \cup \{x, x'\}$. Since $x \in N^-[Y]$, there are two cases to be considered.

$(b.1)$ Suppose that $x \in N^-(y)$. Then $d^-(x), d^-(y) \geq 2$, since both vertices lie on a digon. If there is $u \in N^-(y) \setminus (X \cup Y)$, then $u \in N^-(x')$, and by Lemma 2.1, $N^-(y) = N^-(x')$ implying that $x \in N^-(x')$. Hence, since $x' \in N^-[Y]$ and $N^-(x') = N^-(y)$, it follows that $x' \in N^-(y')$. Furthermore, $y' \in N^-(x') \cup N^-(x)$. If $y' \in N^-(x')$, then by Remark 2.3, $N^-(x') \cap N^-(y') = \emptyset$, and by Corollary 2.11, $N^-(x) \cap N^-(y') = \emptyset$, because $x' \in N^+(x) \cap N^+(y')$. Moreover, by Theorem 2.2, $x \notin N^-(y')$ because $LD$ is $TT_3$-free, and $y \notin N^-(y')$ because, otherwise $LD$ would have two digons incident to the vertex $y$. This implies that $N^-(y') = \{x'\}$, that is, $d^-(y') = 1$, a contradiction because $y'$ lies on a digon. Then $y' \in N^-(x)$ and by Remark 2.3, $N^-(y') \cap N^-(x) = \emptyset$. Moreover, $x \notin N^-(y')$ because otherwise $LD$ would have two digons incident to vertex $x$. If $N^-(y') \cap N^-(x') \neq \emptyset$ by Lemma 2.1, $N^-(y') = N^-(x')$ implying that $x \in N^-(y')$ which is a contradiction.

Therefore, $N^-(y') \cap N^-(x') = \emptyset$ and so $N^-(y') = \{x'\}$ and $N^-(x) = \{y, y'\}$. Therefore, $LD$ does not satisfy $(ii)$. Thus, we have proved that $N^-(y) \subseteq X \cup Y$. Reasoning similarly for $x$ as we did for $y$, this time considering the arc $xy$, we get that $N^-(x) \subseteq X \cup Y$.

If $x' \in N^-(x)$, then $x' \in N^-(y')$. Since $LD$ is $TT_3$-free, $x' \notin N^-(y)$ and by Corollary 2.11, $N^+(x) \cap N^+(y') = \emptyset$ implying $y \notin N^+(y')$, therefore $|N^-(y) \cap \{x', y'\}| = 0$ implying that $d^-(y) = 1$, a contradiction. Therefore, $x' \in N^-(y)$ and $y' \in N^-(x)$. Since $x \in N^+(y) \cap N^+(y')$, by Corollary 2.11, $N^-(y) \cap N^-(y') = \emptyset$, that is, $x' \notin N^-(y')$. Moreover, since $LD$ is $TT_3$-free, $y \notin N^-(y')$. And since there are no two digons incident to the same vertex $x \notin N^-(y')$. Therefore there is a vertex $u \in N^-(y') \setminus (X \cup Y)$ and since $N^-(y') \cap N^-(x) = \emptyset$, $u \in N^-(x')$. Hence, $LD$ does not satisfy $(iii)$.

$(b.2)$ Suppose that $x \in N^-(y') \setminus N^-(y)$. Then $N^-(x) \cap (N^-(y) \cup N^-(y')) = \emptyset$ by Remark 2.3, implying that $N^-(x) \subseteq \{y, y'\}$.

$(b.2.1)$ If $N^-(x) = \{y\}$, then $y' \in N^-(x')$, implying that $N^-(y') \cap (N^-(x') \cup N^-(x)) = \emptyset$, and consequently $N^-(y') \subseteq \{x, x'\}$. If $x' \in N^-(y)$, then $N^-(x') \cap (N^-(y) \cup N^-(y')) = \emptyset$, implying that $N^-(x') \subseteq \{y, y'\}$. Observe that if $y \in N^-(x')$, then by Lemma 2.1, $N^-(x) = N^-(x') = \{y, y'\}$, a contradiction with the assumption that $N^-(x) = \{y\}$. Hence, $N^-(x') = \{y'\}$. Moreover, $N^-(y') \subseteq \{x, x'\}$ and $N^-(y') = \{x\}$ because otherwise the vertices $x', y'$ form a digon with vertex $x'$ of degree one contradicting the hypothesis. Also, $N^-(y) \subseteq \{x', y'\}$ and since $N^-(x') \cap N^-(y) = \emptyset$, $y' \notin N^-(y)$, we have $N^-(y) = \{x'\}$. Therefore, $(x, y', x', y, x)$ is a directed 4-cycle in $LD$ with four vertices of in-degree one, and $LD$ does not satisfy $(iv)$.

$(b.2.2)$ If $N^-(x) = \{y, y'\}$, we have a digon formed by vertices $x$ and $y'$, also $N^-(x) \cap (N^-(y) \cup N^-(y')) = \emptyset$, and consequently $N^-(y) \subseteq \{x'\} \cup N^-(x')$ (recall that we are assuming that $x \in N^-(y') \setminus N^-(y)$). First, suppose that $x' \in N^-(y)$. Then $N^-(y) \cap N^-(x') = \emptyset$ and so $N^-(y) = \{x'\}$, and therefore $y \notin N^-(x')$ because $LD$ has no digons consisting of vertices of in-degree one. Hence, by Lemma 2.1, we have $N^-(x) \cap N^-(x') = \emptyset$. Also $x \notin N^-(x')$ because otherwise $LD$ does not satisfy $(ii)$, a contradiction, and then $N^-(x') \cap N^-(y') = \emptyset$, concluding that $N^-(x') = \emptyset$ which is a contradiction. Therefore, suppose that $x' \in N^-(y') \setminus N^-(y)$. By Theorem 2.2, $x, y' \notin N^-(x')$, implying by Lemma 2.1 that $N^-(x') \cap N^-(x) = \emptyset$, $N^-(x') \cap N^-(y') = \emptyset$. Hence, $N^-(y') = \{x, x'\}$ and $N^-(x') \subseteq N^-(y)$ and, since $\delta^-(LD) \geq 1$, there is $u \in N^-(x') \setminus (X \cup Y)$. Therefore, $LD$ does not satisfy $(iii)$, a contradiction. This completes the proof. □

Notice that, according to the above theorem, if a line digraph with minimum in-degree $\delta^-_{LD} \geq 2$ does not admit a $(1, \leq 2)$-identifying code, then $\delta^-_{LD} = 2$. In the following corollary we give some sufficient conditions for a line digraph with minimum in-degree at least two, to admit a $(1, \leq 2)$-identifying code.

**Corollary 2.13.** *Let $D$ be a digraph with minimum in-degree $\delta^-(D) \geq 2$. Then the following assertions hold.*

*(i) The line digraph $LD$ admits a $(1, \leq 2)$-identifying code if and only if $LD$ does not*

*contain the subdigraph of Figure 2 (c).*

*(ii) If $\delta^- \geq 3$, then $LD$ admits a $(1, \leq 2)$-identifying code.*

*(iii) If $k \geq 2$, then $L^k D$ admits a $(1, \leq 2)$-identifying code.*

*Proof.* Let $D$ be a digraph with minimum in-degree $\delta^-(D) \geq 2$. Items $(i)$ and $(ii)$ follow directly from Theorem 2.12. To prove $(iii)$ observe that if $k \geq 2$, then $L^k D$ does not contain the subdigraph of Figure 2 (c), otherwise $L^{k-1}D$ would contain a $TT_3$, a contradiction to Theorem 2.2. More precisely, suppose that $u, x, x', y, y' \in V(L^k D)$ are five different vertices such that $L^k D[\{u, x, x', y, y'\}]$ is isomorphic to Figure 2 (c). Let $u = (u_1, u_2)$ with $u_1, u_2 \in V(L^{k-1}D)$. Then $x' = (u_2, x'_2)$ and $y' = (u_2, y'_2)$ for some two different vertices $x'_2, y'_2 \in V(L^{k-1}D)$, $x = (y'_2, x'_2)$ and $y = (x'_2, y'_2)$. Therefore, $L^{k-1}D[\{u_2, x'_2, y'_2\}] \cong TT_3$. □

A large known family of digraphs obtained with the line digraph technique is the family of Kautz digraphs. The *Kautz digraph* of degree $d$ and diameter $k$ is defined as the $(k-1)$-iterated line digraph of the symmetric complete digraph of $d+1$ vertices $K_{d+1}$, that is, $K(d, k) \cong L^{k-1}K_{d+1}$. For instance, the Kautz digraph $K(2, 2)$ shown in Figure 3, is the line digraph of the symmetric complete digraph on three vertices.

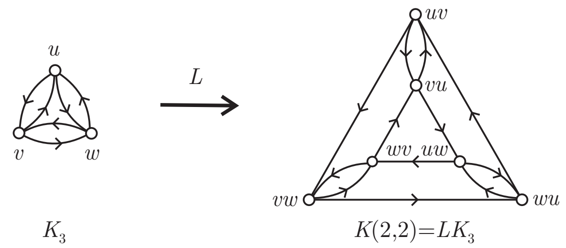


Figure 3: The Kautz digraph $K(2, 2)$ as the line digraph of the symmetric complete digraph $K_3$.

**Corollary 2.14.** *For each $d \geq 3$, the Kautz digraph $K(d, 2) \cong LK_{d+1}$ admits a $(1, \leq 2)$-identifying code.*

By Corollary 2.13 $(iii)$, the Kautz digraph $K(2, 2) = LK_3$ (see Figure 3) does not admit a $(1, \leq 2)$-identifying code. Then the condition $k \geq 2$ in Corollary 2.13 $(iii)$ is necessary.

## 3 Arc-identifying codes

As mentioned in the Introduction a $(1, \leq 1)$-identifying code is known as an identifying code. The identifying number of a digraph $D$, $\overrightarrow{\gamma}^{ID}(D)$, is the minimum size of all the identifying codes of $D$. Foucaud, Naserasr, et al. [7] characterized the digraphs that only admit as identifying code the whole set of vertices. Let us introduce the terminology used for this characterization.

Given two digraphs $D_1$ and $D_2$ on disjoint sets of vertices, we denote $D_1 \oplus D_2$ the disjoint union of $D_1$ and $D_2$, that is, the digraph whose vertex set is $V(D_1) \cup V(D_2)$ and whose arc set is $A(D_1) \cup A(D_2)$. Given a digraph $D$ and a vertex $x \notin V(D)$, $x\overrightarrow{\lhd}(D)$ is the digraph with vertex set $V(D) \cup \{x\}$, and whose arcs are the arcs of $D$ together with each arc $(x, v)$ for every $v \in V(D)$.

**Definition 3.1.** *We define $(K_1, \oplus, \overrightarrow{\lhd})$ as the* closure *of the one-vertex graph $K_1$ with respect to the operations $\oplus$ and $\overrightarrow{\lhd}$. That is, the class of all digraphs that can be built from $K_1$ by repeated applications of $\oplus$ and $\overrightarrow{\lhd}$.*

Foucaud, Naserasr, et al. [8] proved that for any digraph $D$, $\overrightarrow{\gamma}^{ID}(D) = |V(D)|$ if and only if $D \in (K_1, \oplus, \overrightarrow{\lhd})$.

Since, as they pointed out, every $D \in (K_1, \oplus, \overrightarrow{\lhd})$ is the transitive closure of a rooted oriented forest, if $LD$ is a line digraph with minimum in-degree $\delta^- \geq 2$, then $LD \notin (K_1, \oplus, \overrightarrow{\lhd})$. Hence, $\overrightarrow{\gamma}^{ID}(LD) \leq |V(LD)| - 1$. Next, we give a lower bound on $\overrightarrow{\gamma}^{ID}(LD)$.

With this goal, we define the relation $\sim$ over the set of vertices $V(LD)$ as follows. For all $u, v \in V(LD)$, $u \sim v$ if and only if $N^-(u) = N^-(v)$. Clearly, $\sim$ is an equivalence relation. For any $u \in V(LD)$, let $[u]_\sim = \{v \in V(LD) : v \sim u\}$.

**Lemma 3.2.** *Let $C$ be an identifying code of a line digraph $LD$. Then, for any vertex $w \in V(LD)$,*

$$|[w]_\sim \setminus C| \leq 1.$$

*Proof.* Let $w \in V(LD)$ and $u, v \in [w]_\sim \setminus C$. Then, $N^-(u) = N^-(v)$ and, since $u, v \notin C$, it follows that $N^-[u] \cap C = N^-(u) \cap C = N^-(v) \cap C = N^-[v] \cap C$, which is a contradiction if $u \neq v$. □

**Definition 3.3.** *Given a digraph $D$, a subset $\widetilde{C}$ of $A(D)$ is an* arc-identifying code *of $D$ if $\widetilde{C}$ is both:*

- *an arc-dominating set of $D$, that is, for each arc $uv \in A(D)$, $(\{uv\} \cup \Omega^-(u)) \cap \widetilde{C} \neq \emptyset$, and*
- *an arc-separating set of $D$, that is, for each pair $uv, wz \in A(D)$ (with $uv \neq wz$), $(\{uv\} \cup \Omega^-(u)) \cap \widetilde{C} \neq (\{wz\} \cup \Omega^-(w)) \cap \widetilde{C}$.*

Hence, an arc-identifying code of $D$ is an identifying code of its line digraph $LD$. As a consequence, given a digraph $D$, the minimum size of an identifying code of its line digraph, $\overrightarrow{\gamma}^{ID}(LD)$, is equivalent to the minimum size of an arc-identifying code of $D$.

With the following result, we characterize the arc-identifying codes.

**Theorem 3.4.** *Let $D$ be a digraph and $\widetilde{C} \subseteq A(D)$. Then, $\widetilde{C}$ is an arc-identifying code of $D$ if and only if $\widetilde{C}$ satisfies the following two conditions:*

*(i) For all $v \in V(D)$, $|\Omega^+(v) \setminus \widetilde{C}| \leq 1$, and if $|\Omega^+(v) \setminus \widetilde{C}| = 1$, then $\Omega^-(v) \cap \widetilde{C} \neq \emptyset$;*

*(ii) for all $uv \in \widetilde{C}$, if $vu \in \widetilde{C}$ or $|\Omega^+(v) \setminus \widetilde{C}| = 1$, then $((\Omega^-(u) \cup \Omega^-(v)) \setminus \{uv, vu\}) \cap \widetilde{C} \neq \emptyset$.*

*Proof.* Suppose that $\widetilde{C}$ is an arc-identifying code of $D$. Hence, $\widetilde{C}$ is an identifying code of $LD$, and by Lemma 3.2, we have for all $vw \in V(LD)$, $|[vw]_\sim \setminus \widetilde{C}| \leq 1$. Observe that $rs \in [vw]_\sim$ if and only if $N^-_{LD}(rs) = N^-_{LD}(vw)$, which only can occur if and only if $r = v$. Therefore, we get that for all $v \in V(D)$, $|\Omega^+(v) \setminus \widetilde{C}| \leq 1$ holds. Moreover, let $v \in V(D)$ be such that $|\Omega^+(v) \setminus \widetilde{C}| = 1$ and let $vx \in \Omega^+(v) \setminus \widetilde{C}$. Hence, $(\{vx\} \cup \Omega^-(v)) \cap \widetilde{C} = \Omega^-(v) \cap \widetilde{C}$. Since $\widetilde{C}$ is an arc-identifying code, $(\{vx\} \cup \Omega^-(v)) \cap \widetilde{C} \neq \emptyset$, hence $\widetilde{C}$ satisfies $(i)$. To prove $(ii)$, let $uv \in \widetilde{C}$ be such that $((\Omega^-(u) \cup \Omega^-(v)) \setminus \{vu, uv\}) \cap \widetilde{C} = \emptyset$. If $vu \in \widetilde{C}$, then $(\{uv\} \cup \Omega^-(u)) \cap \widetilde{C} = \{uv, vu\} = (\{vu\} \cup \Omega^-(v)) \cap \widetilde{C}$, contradicting that $\widetilde{C}$ is an arc-identifying code. Hence, $vu \notin \widetilde{C}$. If $|\Omega^+(v) \setminus \widetilde{C}| = 1$, say $\Omega^+(v) \setminus \widetilde{C} = \{vx\}$, then $(\{uv\} \cup \Omega^-(u)) \cap \widetilde{C} = \{uv\} = (\{vx\} \cup \Omega^-(v)) \cap \widetilde{C}$, a contradiction. Therefore, $\widetilde{C}$ satisfies $(ii)$.

Now, suppose that $\widetilde{C}$ is a set of arcs of $D$ satisfying $(i)$ and $(ii)$, and let us show that $\widetilde{C}$ is an arc-identifying code. To see that $\widetilde{C}$ is an arc-dominating set of $D$, let $ab \in A(D)$. By $(i)$, $\Omega^+(a) \subseteq \widetilde{C}$ or $\Omega^-(a) \cap \widetilde{C} \neq \emptyset$, implying that $(\{ab\} \cup \Omega^-(a)) \cap \widetilde{C} \neq \emptyset$. Therefore, $\widetilde{C}$ is an arc-dominating set of $D$. Next, let us prove that $\widetilde{C}$ is an arc-separating set of $D$. On the contrary, suppose that there are two different arcs $ab$ and $cd$, such that $(\{ab\} \cup \Omega^-(a)) \cap \widetilde{C} = (\{cd\} \cup \Omega^-(c)) \cap \widetilde{C}$. First, let us assume that $ab, cd \notin \widetilde{C}$ and take an arc $uv \in (\{ab\} \cup \Omega^-(a)) \cap \widetilde{C} = (\{cd\} \cup \Omega^-(c)) \cap \widetilde{C}$. Then $v = a = c$, implying that $ab, cd \in \Omega^+(v) \setminus \widetilde{C}$, contradicting $(i)$. Second, assume that $ab \in \widetilde{C}$, hence, $ab \in \Omega^-(c)$, implying that $c = b$. If $bd \notin \widetilde{C}$, then $|\Omega^+(b) \setminus \widetilde{C}| = 1$ and by $(ii)$, $((\Omega^-(a) \cup \Omega^-(b)) \setminus \{ba, ab\}) \cap \widetilde{C} \neq \emptyset$. Then $(\{ab\} \cup \Omega^-(a)) \cap \widetilde{C} \neq (\{bd\} \cup \Omega^-(b)) \cap \widetilde{C}$, a contradiction with our assumption. Therefore, $bd \in \widetilde{C}$ implying that $bd \in \Omega^-(a)$ and $d = a$. Again by $(ii)$, $((\Omega^-(a) \cup \Omega^-(b)) \setminus \{ab, ba\}) \cap \widetilde{C} \neq \emptyset$, yielding that $(\{ab\} \cup \Omega^-(a)) \cap \widetilde{C} \neq (\{ba\} \cup \Omega^-(b)) \cap \widetilde{C}$, a contradiction. Therefore, $\widetilde{C}$ is an arc-separating set. This completes the proof. □

Let $D$ be a digraph and $i \geq 1$ be an integer. Then, denote $V^+_{\geq i}(D) = \{v \in V(D) : d^+(v) \geq i\}$, and $V^+_i(D) = \{v \in V(D) : d^+(v) = i\}$.

Recall that by Proposition 2.5, a digraph $D$ admits an arc-identifying code if and only if there is no digon with both vertices of in-degree one.

**Theorem 3.5.** *Let $D$ be a digraph without digons with both vertices of in-degree 1. Then,*

$$\overrightarrow{\gamma}^{ID}(LD) \geq |A(D)| - |V^+_{\geq 1}(D)|.$$

*Proof.* By Proposition 2.5, $LD$ admits a $(1, \leq 1)$-identifying code. Let $\widetilde{C}$ be an arc-identifying code of $D$. Then, by Theorem 3.4,

$$\begin{aligned}|\widetilde{C}| &\geq \sum_{u\in V^+_{\geq 1}(D)} (d^+_D(u)-1)\\ &= \sum_{u\in V^+_{\geq 1}(D)} d^+_D(u) - |V^+_{\geq 1}(D)|\\ &= |A(D)| - |V^+_{\geq 1}(D)|.\end{aligned}$$

□

**Remark 3.6.** *Notice that by the proof of Theorem 3.5, we have $\gamma^{ID}(LD) = |A(D)| - |V^+_{\geq 1}(D)|$ if and only if $|\Omega^+(v) \setminus \widetilde{C}| = 1$ for each vertex $v \in V^+_{\geq 1}(D)$. In particular, if $d^+(v) = 1$ and the lower bound is reached, then $\Omega^+(v) \cap \widetilde{C} = \emptyset$.*

A 1-factor in a digraph is a 1-regular spanning subdigraph. Next, we show that some digraphs with a 1-factor have an identifying number that attains the equality in Theorem 3.5.

**Theorem 3.7.** *Let $D$ be a digraph having a 1-factor with minimum in-degree $\delta^- \geq 2$ and without digons with both vertices of in-degree two. Then, $\gamma^{ID}(LD) = |A(D)| - |V(D)|$.*

*Proof.* Let $F$ denote a 1-factor in $D$. Let $\widetilde{C} = A(D) \setminus A(F)$. Let us show that $\widetilde{C}$ satisfies the requirements of Theorem 3.4. By definition of $\widetilde{C}$, $|\Omega^+(v) \setminus \widetilde{C}| = 1$ and $|\Omega^-(v) \setminus \widetilde{C}| = 1$ for each vertex $v \in V(D)$. Hence, Theorem 3.4 (*i*) holds because $|\Omega^+(v) \setminus \widetilde{C}| = 1$ and $\Omega^-(v) \cap \widetilde{C} \neq \emptyset$, since $\delta^- \geq 2$. Moreover, for any arc $uv \in \widetilde{C}$ not in a digon we have $((\Omega^-(v) \cup \Omega^-(u)) \setminus \{uv\}) \cap \widetilde{C} \neq \emptyset$ because $|\Omega^-(v) \setminus \widetilde{C}| = 1$ which implies that $|\Omega^-(v) \cap \widetilde{C}| = d^-(v) - 1 \geq 1$. And if $uv \in \widetilde{C}$ belongs to a digon, since one of the two vertices, say $v$, must have $d^-(v) \geq 3$ we have $((\Omega^-(v) \cup \Omega^-(u)) \setminus \{uv, vu\}) \cap \widetilde{C} \neq \emptyset$ because $|\Omega^-(v) \cap \widetilde{C}| = d^-(v) - 1 \geq 2$. In either case Theorem 3.4 (*ii*) holds. Therefore, $\widetilde{C}$ is an arc-identifying code of $D$ and $\gamma^{ID}(LD) \leq |A(D)| - |V(D)|$. Furthermore, by Theorem 3.5, $\gamma^{ID}(LD) = |A(D)| - |V(D)|$. □

Clearly, a Hamiltonian digraph has a 1-factor consisting of a directed cycle $W$ such that $V(W) = V(D)$. The following result is an immediate consequence of Theorem 3.7.

**Corollary 3.8.** *Let $D$ be a Hamiltonian digraph with minimum in-degree $\delta^- \geq 2$ and without digons with both vertices of in-degree two. Then, $\gamma^{ID}(LD) = |A(D)| - |V(D)|$.*

**Corollary 3.9.** *The identifying number of a Kautz digraph $K(d,k)$ is $\gamma^{ID}(K(d,k)) = d^k - d^{k-2}$ for $d \geq 3$ and $k \geq 2$.*

*Proof.* Note that $K(d,2) = LK_{d+1}$. Since $K_{d+1}$ is Hamiltonian and $d \geq 3$, by Corollary 3.8, $\gamma^{ID}(K(d,2)) = \gamma^{ID}(LK_{d+1}) = |A(K_{d+1})| - |V(K_{d+1})| = d(d+1) - (d+1) = d^2 - 1$, and the result holds for $k = 2$. For any $k \geq 3$, the Kautz digraph $K(d,k) = L^{k-1}K_{d+1} = LL^{k-2}K_{d+1} = LK(d,k-1)$. Since $K(d,k-1)$ is a Hamiltonian digraph and $d \geq 3$, by Corollary 3.8, $\gamma^{ID}(K(d,k)) = \gamma^{ID}(LK(d,k-1)) = d^k + d^{k-1} - (d^{k-1} + d^{k-2}) = d^k - d^{k-2}$, and the result holds. □

To extend Corollary 3.9 to $K(2,k)$, we use the 1-factorization of Kautz digraphs obtained by Tvrdík [16]. This 1-factorization uses the following operation.

**Definition 3.10.** *[16] If $x = x_1 \dots x_k \in V(K(d,k))$, then*

- $\sigma_1(x) = x_2 \dots x_{k-1}x_k x_1$ *if* $x_1 \neq x_k$.
- $\sigma_1(x) = x_2 \dots x_{k-1}x_k x_2$ *if* $x_1 = x_k$.

*Let* $Inc : V(K(d,k)) \times \mathbb{Z}_d \to V(K(d,k))$ *denote a binary operation such that*

$$Inc(x_1 \dots x_{k-1}x_k, i) = x_1 \dots x_{k-1}x'_k,$$

*where*

$$x'_k = \begin{cases} x_k + i \mod (d+1) & \text{if } x_{k-1} > x_k \text{ and } x_{k-1} > x_k + i, \\ & \text{or } x_{k-1} < x_k \text{ and } x_{k-1} + d + 1 > x_k + i; \\ x_k + i + 1 \mod (d+1) & \text{otherwise.} \end{cases}$$

*Then, the generalized K-shift operation is defined as follows:*

$$\begin{aligned} \sigma_1^{+i}(x) &= Inc(\sigma_1(x), i), \\ \sigma_k^{+i}(x) &= \sigma_1^{+i}(\sigma_{k-1}^{+i}(x)). \end{aligned}$$

**Theorem 3.11.** *[16] The arc set of $K(d,k)$ can be partitioned into $d$ 1-factors $\mathcal{F}_0, \dots, \mathcal{F}_{d-1}$ such that the cycles of $\mathcal{F}_i$ are closed under the operation $\sigma_1^{+i}$.*

**Theorem 3.12.** *The identifying number of a Kautz digraph $K(2,k)$ is $\gamma^{ID}(K(2,k)) = 2^k - 2^{k-2}$ for $k \geq 2$.*

*Proof.* We can check in Figure 3 that $\widetilde{C} = \{uv, vw, wu\}$ is an identifying code of $K(2,2)$, then $\gamma^{ID}(K(2,2)) = 3$, and the theorem holds for $k = 2$. Suppose that $k \geq 3$ and let us consider the Kautz digraph $K(2,k-1)$. By Theorem 3.11, we can take a partition of the arcs of $K(2,k-1)$ into two 1-factors $\mathcal{F}_0$ and $\mathcal{F}_1$, such that the cycles of $\mathcal{F}_i$ are closed under the operation called $\sigma_1^{+i}$, given in Definition 3.10. In particular the relation $\sigma_1^{+0}$ preserves digons, implying that all the digons of $K(2,k-1)$ belong to the family $\mathcal{F}_0$. Hence, since $\mathcal{F}_1$ is a 1-factor of $K(2,k-1)$, the set of arcs in $\mathcal{F}_1$, say $A_1$, satisfies the conditions of Theorem 3.4. Therefore, $A_1$ is an arc-identifying code of $K(2,k-1)$, that is, an identifying code of $K(2,k)$ and, $\gamma^{ID}(K(2,k)) = |A_1| = |V(K(2,k-1)| = 3 \cdot 2^{k-2} = 2^k - 2^{k-2}$. □

### 3.1 Arc-identifying codes in oriented graphs

Now we present an algorithm for constructing an arc-identifying code $\widetilde{C}$ of an oriented graph $D$ with minimum in-degree $\delta^- \geq 2$ and minimum out-degree $\delta^+ \geq 1$. The idea of this algorithm is to add to $\widetilde{C}$ all the arcs but one from $\Omega^+(v)$, for each vertex $v \in V(D)$ trying to reach an arc-identifying code of order $|A(D)|-|V(D)|$. Notice that in particular, for each vertex $v \in V_1^+(D)$ we have $\Omega^+(v) \cap \widetilde{C} = \emptyset$ or $\Omega^+(v) \subset \widetilde{C}$, and in the latter case the obtained arc-identifying code has order strictly greater than $|A(D)| - |V(D)|$.

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**Algorithm 1** Let $D$ be an oriented graph $D$ with minimum out-degree $\delta^+ \geq 1$ and minimum in-degree $\delta^- \geq 2$.

---

1: let $X := \emptyset$, $Y := \emptyset$ and $\widetilde{C} := \emptyset$
2: **while** $Y \neq V(D)$ **do**
3:     **take** $xy \in A(D)$ such that $y \in V(D) \setminus Y$
4:     $X := X \cup \{x\}$, $Y := Y \cup (N^+(x) \setminus \{y\})$ and $\widetilde{C} := \widetilde{C} \cup (\Omega^+(x) \setminus \{xy\})$
5:     **if** $N^-(y) \setminus X \neq \emptyset$ **then**
6:         **if** there is $t \in N^-(y) \setminus X$ such that $t \in V^+_{\geq 2}(D)$
7:             **take** $tz \in A(D)$ such that $z \neq y$
8:             **let** $x := t$ and $y := z$
9:             **return** to 4
10:         **else**
11:             **take** $t \in N^-(y) \setminus X$
12:             $X := X \cup \{t\}$, $Y := Y \cup \{y\}$ and $\widetilde{C} := \widetilde{C} \cup \{ty\}$
13:             **return** to 3
14:         **end if then**
15:     **else**
16:         **return** to 3
17:     **end if**
18: **end while**
19: **if** $Y = V(D)$ **then**
20:     **while** $X \neq V(D)$ **do**
21:         **take** $uv \in A(D)$ such that $u \notin X$
22:         $X := X \cup \{u\}$ and $\widetilde{C} := \widetilde{C} \cup (\Omega^+(u) \setminus \{uv\})$
23:     **end while**
24:     **if** $X = V(D)$ **then**
25:         **return** $\widetilde{C}$
26:     **end if**
27: **end if**

---

To illustrate Algorithm 3.1, as an example we run the algorithm for the oriented graph of Figure 4 with 7 vertices and 14 arcs.

**Input:** Oriented graph depicted in Figure 4.

Steps 1-4: Start with the arc $(1,5)$, then $X := \{1\}$, $Y := \{2\}$ and $\widetilde{C} := \{(1,2)\}$.

Step 5: $N^-(5) \setminus X = \{6\} \subseteq V_1^+(D)$, then we go to step 11.

Steps 11-13: $X = \{1,6\}$, $Y := \{2,5\}$ and $\widetilde{C} := \{(1,2),(6,5)\}$, go to step 3.

Steps 3-4: Take the arc $(4,3)$, then $X = \{1,6,4\}$, $Y := \{2,5,1,6,7\}$ and $\widetilde{C} := \{(1,2),(6,5),(4,1),(4,7),(4,6)\}$.

Step 5: $N^-(3) \setminus X = \{5\} \subseteq V_{\geq 2}^+(D)$.

Steps 6-9: Take the arc $(5,7)$, that is $t = 5$ and $z = 7$.

Step 4: $X = \{1,4,6,5\}$, $Y := \{1,2,5,6,7,3,4\}$ and $\widetilde{C} := \{(1,2),(6,5),(4,1),(4,7),(4,6),(5,4),(5,3)\}$.

Step 5: $N^-(7) \setminus X = \emptyset$, then go to step 3 but since $Y = V(D)$, go to step 19.

Step 20: Since $X \neq V(D)$, start steps 21 to 22 until $X = V(D)$.

Steps 21-22: Take the arc $(7,6)$, then $X = \{1,4,6,5,7\}$, and $\widetilde{C} := \{(1,2),(4,1),(4,7),(4,6),(5,3),(5,4),(6,5),(7,1)\}$.

Steps 20-22: Take the arc $(3,2)$, then $X = \{1,4,6,5,7,3\}$, and $\widetilde{C} := \widetilde{C}$.

Steps 20-22: Take the arc $(2,4)$, then $X = \{1,2,3,4,5,6,7\}$, and $\widetilde{C} := \widetilde{C}$

Step 25: **Output** $\widetilde{C} := \{(1,2),(4,1),(4,7),(4,6),(5,3),(5,4),(6,5),(7,1)\}$.

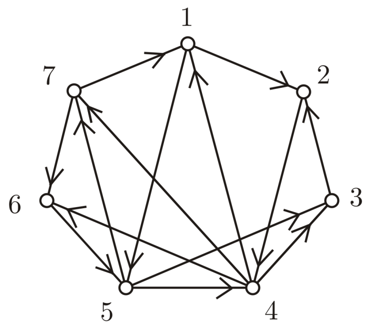


Figure 4: A digraph to illustrate Algorithm 3.1.

**Theorem 3.13.** *Let $D$ be an oriented graph with minimum degree $\delta = \min\{\delta^+, \delta^-\} \geq 2$. Then, Algorithm 3.1 produces a subset $\widetilde{C} \subset A(D)$ of size*

$$|\widetilde{C}| = |A(D)| - |V(D)|,$$

*satisfying the requirements of Theorem 3.4.*

*Proof.* By construction, the Algorithm 3.1 produces a set $Y$ such that every $v \in V(D)$ satisfies that $v \in Y$ at a certain step of the algorithm. Then, $v \in N^+(x) \setminus \{y\}$ for certain $x$ and $y$ in the algorithm such that $xv \in \Omega^+(x) \setminus \{xy\} \subset \widetilde{C}$ because $\delta^+ \geq 2$. Then, $\Omega^-(v) \cap \widetilde{C} \neq \emptyset$ and Theorem 3.4 $(i)$ holds. Finally, since $D$ is oriented, for all $uv \in \widetilde{C}$, clearly $vu \notin A(D)$, and we have $|(\Omega^-(u) \cup (\Omega^-(v) \setminus \{uv\})) \cap \widetilde{C}| \geq 1$ because $\Omega^-(u) \cap \widetilde{C} \neq \emptyset$. Hence, Theorem 3.4 $(ii)$ also holds. Therefore, $\widetilde{C}$ is an arc-identifying code of $D$ and $|\widetilde{C}| = |A(D)| - |V(D)|$. This completes the proof. □

As a consequence of Theorems 3.5 and 3.13, we can conclude the following.

**Corollary 3.14.** *Let $D$ be an oriented graph $D$ with minimum degree $\delta \geq 2$. Then,*

$$\overrightarrow{\gamma}^{ID}(LD) = |A(D)| - |V(D)|.$$

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