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ON EXISTENCE AND NUMERICAL APPROXIMATION IN PHASE–LAG THERMOELASTICITY WITH TWO TEMPERATURES

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Abstract. In this work we study from both variational and numerical points of view a thermoelastic problem which appears in the dual-phase-lag theory with two temperatures. An existence and uniqueness result is proved in the general case of different Taylor approximations for the heat flux and the inductive temperature. Then, in order to provide the numerical analysis, we restrict ourselves to the case of second-order approximations of the heat flux and first-order approximations for the inductive temperature. First, variational formulation of the corresponding problem is derived and an energy decay property is proved. Then, a fully discrete scheme is introduced by using the finite element method for the approximation of the spatial variable and the implicit Euler scheme for the discretization of the time derivatives. A discrete stability property is shown and a priori error estimates are provided, from which the linear convergence of the algorithm is derived under suitable additional regularity on the continuous solution. Finally, some numerical simulations are presented in two-dimensional numerical examples.

1. **Introduction.** It is widely accepted the Fourier formulation to describe the heat conduction. However, when we adjoin this relation with the usual energy equation:

$$c\dot{\theta} + \text{ div } \mathbf{q} = 0, \quad c > 0, \tag{1}$$

we arrive to the instantaneous propagation of heat. This is a drawback of the model because this fact is incompatible with basic axioms of the physics. In the above equation $\mathbf{q} = (q_i)$ is the heat flux vector and θ is the temperature. In order to overcome this difficulty, alternative proposals have been stated. In fact, in the late years different authors have developed new theories for the heat conduction. The most known is the hyperbolic damped equation [5] studied by Cattaneo and Maxwell which eliminates the drawback. Moreover, Green and Nagdhi [14, 15] proposed three thermoelastic theories where, in each case, the heat conduction is described in alternative forms.

In 1995, Tzou [33] introduced a theory in such a way that the heat flux and the gradient of the temperature have a delay in the constitutive equations. When delay parameters are taken into account, it is usual to speak about phase-lag theories. The constitutive equations proposed by Tzou are given by

$$q_i(\boldsymbol{x}, t+\tau_1) = -K\theta_{,i}(\boldsymbol{x}, t+\tau_2), \qquad K > 0, \tag{2}$$

where τ_1 and τ_2 are the delay parameters which are assumed to be positive. As usual, the notation $\theta_{,i}$ means the derivative of θ with respect to the variable x_i, and repeated subscripts means summation. The derivative with respect to the time is denoted using a dot over the function. This equation suggests that the temperature gradient established across a material volume, at position $\boldsymbol{x}$ and time $t+\tau_2$, results in a heat flux to flow at a different time $t+\tau_1$. These time delays can be understood in terms of the microstructure of the material.

Some time later, in 2007 Choudhuri [9] extended Tzou's theory to propose that the heat flux is described using the following constitutive equation:

$$q_i(\boldsymbol{x}, t+\tau_1) = -K_1\alpha_{,i}(\boldsymbol{x}, t+\tau_3) - K_2\theta_{,i}(\boldsymbol{x}, t+\tau_2), \tag{3}$$

where $\dot{\alpha} = \theta$. The variable α is called the *thermal displacement*, and the parameter τ_3 is another time delay parameter.

These two aforementioned theories have several derivations when the heat flux and the gradients of the temperature and the thermal displacement are replaced by Taylor approximations. In fact, one can think that Choudhuri's proposal tries to recover Green and Naghdi theories when different Taylor approximations are considered. This new approach gives rise to different equations (depending on the selected Taylor approximation) to describe heat conduction that have been analyzed by many authors (see, for example, [1, 3, 16, 20, 23, 27, 28, 29, 30, 31, 32, 35]).

Unfortunately, the proposals of Tzou and Choudhuri lead to *ill-posed* problems in the sense of Hadamard. In fact, it can be shown that combining equation (2) (or (3)) with the energy equation (1) leads to the existence of a sequence of elements in the point spectrum such that its real part tends to infinity [11]. At the same time, the Tzou's theory is not compatible with the basic axioms of the thermomechanics [13]. Therefore, we see that this theory cannot be accepted nor from the mathematical point of view neither from the thermodynamical point of view.

In order to obtain a heat conduction theory with delays but without such an explosive behavior, Quintanilla [24, 25] combined the delay parameters of Tzou and Choudhuri with the two-temperatures theory proposed by Chen and Gurtin [6, 7, 8, 34]. The basic constitutive equation reads

$$q_i(\boldsymbol{x}, t+\tau_1) = -K_1\beta_{,i}(\boldsymbol{x}, t+\tau_3) - K_2 T_{,i}(\boldsymbol{x}, t+\tau_2), \tag{4}$$

where $\alpha = \beta - m\Delta\beta$, $\theta = T - m\Delta T$ and m is a positive constant. In this paper, we are going to consider the general case when the material is not isotropic, but we restrict our attention to the dual-phase-lag theory. We have the constitutive equation:

$$q_i(\boldsymbol{x}, t+\tau_1) = -K_{ij} T_{,j}(\boldsymbol{x}, t+\tau_2). \tag{5}$$

In fact, we are going to consider the Taylor approximation to the heat flux vector and the inductive temperature and we assume that

$$\begin{aligned} \boldsymbol{q}(\boldsymbol{x}, t+\tau_1) &\approx a_0\boldsymbol{q}(\boldsymbol{x}, t) + a_1\dot{\boldsymbol{q}}(\boldsymbol{x}, t) + \ldots + a_n\boldsymbol{q}^{(n)}(\boldsymbol{x}, t), \\ T(\boldsymbol{x}, t+\tau_2) &\approx b_0 T(\boldsymbol{x}, t) + b_1\dot{T}(\boldsymbol{x}, t) + \ldots + b_l T^{(l)}(\boldsymbol{x}, t), \end{aligned}$$

where $l \leq n$.[1]

This theory has also been extended to the thermoelasticity context [24, 25]. To do so, one must assume the equation of motion:

$$t_{ji,j} = \rho\ddot{u}_i, \tag{6}$$

the energy equation:

$$\dot{\eta} = -q_{i,i}, \tag{7}$$

and the constitutive equations:

$$\begin{aligned} t_{ji} &= C_{ijkl}u_{k,l} + \beta_{ij}\theta, \\ \eta &= -\beta_{ij}u_{i,j} + c\theta, \end{aligned} \tag{8}$$

where t_{ji} represents the stress tensor, η is the entropy, (u_i) is the displacement vector, C_{ijkl} and β_{ij} are constitutive tensors and ρ and c are the mass density and the thermal capacity, respectively.

It is worth noting that these new thermomechanical theories have attracted much attention [2, 12, 26, 18, 19, 21, 22, 35].

Finally, we point out that, in this work, we restrict our attention to the homogeneous case to make the calculations easier, but the extension to the case of nonhomogeneous materials is direct.

The paper is outlined as follows. The thermomechanical problem with two temperatures and the general Taylor developments presented above is described in Section 2, with the assumptions on the constitutive data. Then, in Section 3 it is

[1] It is worth noting that we can recover the values of a_i and b_j in terms of τ_1 and τ_2.

written as a Cauchy problem in a suitable Hilbert space, and an existence and uniqueness result is proved in Section 4. Next, a fully discrete approximation is introduced in Section 5, based on the finite element method to approximate the spatial domain and the backward Euler scheme to discretize the time derivatives. Under some conditions on the constitutive parameters, we prove that the discrete energy decays. Moreover, a priori error estimates are obtained, from which, under suitable additional regularity conditions, the linear convergence of the algorithm is deduced. Finally, some numerical simulations are presented in Section 6.

2. Basic Equations and Assumptions.

In this section, we recall the basic equations and the assumptions under what we are going to work in this paper. The basic system of equations for the phase-lag thermoelasticity with two temperatures is given by the system:

$$\begin{aligned}
&\rho \ddot{u}_i = (C_{ijkl}u_{k,l} + \beta_{ij}\theta)_{,j},\\
&c\frac{d}{dt}(a_0\theta + a_1\dot{\theta} + ... + a_n\theta^{(n)}) = (K_{ij}(b_0T_{,j} + ... + b_lT^{(l)}_{,j})_{,i})\\
&\qquad\qquad +\beta_{ij}\frac{d}{dt}(a_0u_{i,j} + ...a_nu^{(n)}_{i,j}),
\end{aligned} \tag{9}$$

and the relation

$$\theta = T - m(K_{ij}T_{,i})_{,j}. \tag{10}$$

In this system of equations, C_{ijkl} is the elasticity tensor, β_{ij} is the coupling term, K_{ij} is the thermal conductivity, ρ is the mass density, c is the thermal capacity, a_i and b_j are constants determined by the approximation we consider and m is a constant which is typical for the two temperatures theory. As usual, (u_i) is the displacement vector and θ and T are the thermodynamical temperature and the inductive temperature, respectively.

We are going to consider this system in a multi-dimensional domain B in $\mathbb{R}^d$ $(d = 2, 3)$ such that the boundary is smooth enough to apply the divergence theorem. In order to define the problem, we need to assume the initial and boundary conditions and, to simplify the arguments, we assume homogeneous Dirichlet boundary conditions:

$$u_i(\boldsymbol{x}, t) = T(\boldsymbol{x}, t) = 0 \quad \text{for a.e.} \quad \boldsymbol{x} \in \partial B,\ t > 0. \tag{11}$$

We also impose the initial conditions:

$$\begin{aligned}
&u_i(\boldsymbol{x}, 0) = u_i^0(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x}, 0) = v_i^0(\boldsymbol{x}) \quad \text{for a.e.} \quad \boldsymbol{x} \in B,\\
&\theta(\boldsymbol{x}, 0) = \theta^0(\boldsymbol{x}), \ldots, \theta^{(n)}(\boldsymbol{x}, 0) = \theta^n(\boldsymbol{x}) \quad \text{for a.e.} \quad \boldsymbol{x} \in B.
\end{aligned} \tag{12}$$

As it is usual, we assume that the elasticity and the thermal conductivity tensors satisfy the symmetries

$$C_{ijkl} = C_{klij}, \quad K_{ij} = K_{ji}.$$

In this paper, we assume that the constitutive tensors are upper bounded and that

- The mass density and the thermal conductivity are strictly positive; that is,

$$\rho(\boldsymbol{x}) \geq \rho_0 > 0, \quad c(\boldsymbol{x}) \geq c_0 > 0.$$

- The elasticity tensor is positive definite. That is, there exists a positive constant C such that

$$C_{ijkl}\xi_{ij}\xi_{kl} \geq C\xi_{ij}\xi_{ij},$$

  for every tensor ξ_{ij}.

- The thermal conductivity tensor is also positive definite. That is, there exists a positive constant K such that
$$K_{ij}\xi_i\xi_j \geq K\xi_i\xi_i,$$
for every vector ξ_i.
- We also assume that parameter a_n is strictly positive.

We now introduce the notation
$$\hat{f} = a_0 f + a_1 \dot{f} + ...a_n f^{(n)}.$$
System (9) can be written as
$$\begin{aligned} &\rho\ddot{\hat{u}}_i = (C_{ijkl}\hat{u}_{k,l} + \beta_{ij}(a_0\theta + a_1\dot{\theta} + \ldots + a_n\theta^{(n)}))_{,j}, \\ &c\frac{d}{dt}\left(a_0\theta + a_1\dot{\theta} + ... + a_n\theta^{(n)}\right) = K_{ij}(b_0 T_{,j} + \ldots + b_l T_{,j}^{(l)})_{,i} + \beta_{ij}\frac{d}{dt}\hat{u}_{i,j}. \end{aligned} \tag{13}$$
To make the notation easier we drop the hat in our system of equations. Therefore, we can write
$$\begin{aligned} &\rho\ddot{u}_i = (C_{ijkl}u_{k,l} + \beta_{ij}(a_0\theta + a_1\dot{\theta} + \ldots + a_n\theta^{(n)}))_{,j}, \\ &c\frac{d}{dt}\left(a_0\theta + a_1\dot{\theta} + ... + a_n\theta^{(n)}\right) = K_{ij}(b_0 T_{,j} + \ldots + b_l T_{,j}^{(l)})_{,i} + \beta_{ij}\dot{u}_{i,j}. \end{aligned} \tag{14}$$
The aim of this paper is to study this system jointly with initial conditions (12) and boundary conditions (11).

3. The Cauchy Problem.

In this section, we will transform our initial-boundary-value problem into a Cauchy problem in a suitable Hilbert space. We will consider the space
$$\mathcal{X} = \mathbf{W}_0^{1,2}(B) \times \mathbf{L}^2(B) \times L^2(B) \times \ldots \times L^2(B),$$
where $W_0^{i,j}$ and L^i are the usual Sobolev spaces and the boldface means that this is the two or three times product.

We denote by
$$\Omega = (u_i, v_i, \theta^{\{0\}}, \theta^{\{1\}}, \ldots, \theta^{\{n\}})$$
the elements in our Hilbert space and we define the inner product
$$< \Omega, \Omega^* >= \frac{1}{2}\int_B W dx,$$
where
$$\begin{aligned} W = &\rho v_i v_i{}^* + C_{ijkl}u_{i,j}u_{k,l}^* + \alpha_0\theta^{\{0\}}\theta^{\{0\}*} + \ldots + \alpha_{n-1}\theta^{\{n-1\}}\theta^{\{n-1\}*} \\ &+c(a_0\theta^{\{0\}} + \ldots + a_n\theta^{\{n\}})(a_0\theta^{\{0\}*} + \ldots + a_n\theta^{\{n\}*}), \end{aligned}$$
where α_i are positive constants large enough to guarantee that W defines a positive definite bilinear form. We note that
$$\begin{aligned} ||\Omega||^2_{\mathcal{X}} = \frac{1}{2}\int_B &\rho v_i v_i + C_{ijkl}u_{i,j}u_{k,l} + \alpha_0\theta^{\{0\}}\theta^{\{0\}} + \ldots + \alpha_{n-1}\theta^{\{n-1\}}\theta^{\{n-1\}} \\ &+c(a_0\theta^{\{0\}} + \ldots + a_n\theta^{\{n\}})(a_0\theta^{\{0\}} + \ldots + a_n\theta^{\{n\}})\, d\boldsymbol{x}. \end{aligned}$$
This norm is equivalent to the usual one in the Hilbert space.

It is worth noting that $\theta(T) = T - m(K_{ij}T_{,j})_{,i}$ defines an isomorphism between $L^2(B)$ and $W_0^{1,2}(B) \cap W^{2,2}(B)$. We shall denote by $\Phi(\theta)$ the inverse of this operator. We also note that
$$||\theta||_{L^2} = \int_B (T^2 + 2mK_{ij}T_{,i}T_{,j} + m^2[(K_{ij}T_{,i})_{,j}]^2)d\boldsymbol{x}.$$
Therefore, it is clear that the L^2-norm of θ is equivalent to the $W^{2,2}$-norm of T.

We shall define the following operators [2]:

$$
\begin{aligned}
&A_i\boldsymbol{u} = \rho^{-1}(C_{ijkl}u_{k,l})_{,j},\\
&B_{i0}\theta^{\{0\}} = \rho^{-1}(\beta_{ij}a_0\theta^{\{0\}})_{,j}, \quad B_{i1}\theta^{\{1\}} = \rho^{-1}(\beta_{ij}a_1\theta^{\{1\}})_{,j}, \ldots, B_{in}\theta^{\{n\}} = \rho^{-1}(\beta_{ij}a_n\theta^{\{n\}})_{,j},\\
&L_0\theta^{\{0\}} = (ca_n)^{-1}\left[(K_{ij}b_0\Phi(\theta^{\{0\}})_{,j})_{,i}\right],\\
&L_1\theta^{\{1\}} = (ca_n)^{-1}\left[(K_{ij}b_1\Phi(\theta^{\{1\}})_{,j})_{,i} - ca_0\theta^{\{1\}}\right],\\
&\ldots\\
&L_n\theta^{\{n\}} = (ca_n)^{-1}\left[(K_{ij}b_n\Phi(\theta^{\{n\}})_{,j})_{,i} - ca_{n-1}\theta^{\{n\}}\right],\\
&D\mathbf{v} = (ca_n)^{-1}\beta_{ij}v_{i,j},\\
&\mathbf{A} = (A_i), \ \ \mathbf{B}_k = (B_{ik}),
\end{aligned}
$$

and the matrix operator

$$
\mathcal{A} = \begin{pmatrix}
0 & I & 0 & 0 & 0 & \ldots & \mathbf{0}\\
\mathbf{A} & 0 & \mathbf{B}_0 & \mathbf{B}_1 & \mathbf{B}_2 & \ldots & \mathbf{B}_n\\
0 & 0 & 0 & I & 0 & \ldots & 0\\
. & . & . & . & . & \ldots & .\\
. & . & . & . & . & \ldots & .\\
. & . & . & . & . & \ldots & .\\
0 & 0 & 0 & 0 & 0 & \ldots & I\\
0 & D & L_0 & L_1 & L_2 & \ldots & L_n
\end{pmatrix}, \tag{15}
$$

where I is the identity operator. Our initial-boundary-value problem can be written as

$$
\frac{d\omega}{dt} = \mathcal{A}\omega, \quad \omega_0 = (\boldsymbol{u}^0, \boldsymbol{v}^0, \theta^0, \theta^1, \ldots, , \theta^n).
$$

It is worth noting that the domain of the operator $\mathcal{A}$ is the set of $\omega \in \mathcal{X}$ such that $\mathcal{A}\omega \in \mathcal{X}$ is a dense subspace of space $\mathcal{X}$.

4. The existence theorem.

The aim of this section is to prove a theorem of existence and uniqueness for the solutions to the Cauchy problem proposed previously. We will need a couple of lemmata.

Lemma 4.1. *There exists a positive constant M such that*

$$
\langle \mathcal{A}\omega, \omega\rangle \leq M||\omega||^2
$$

for every ω in the domain.

Proof. If we take into account the evolution equations and the boundary conditions we see that

$$
\langle \mathcal{A}\omega, \omega\rangle = \frac{1}{2}\int_B D^* d\boldsymbol{x},
$$

where

$$
\begin{aligned}
D^* = &\Big[C_{ijkl}u_{k,l} + \beta_{ij}\Big(\sum_{k=0}^{n} a_k\theta^{\{k\}}\Big)\Big]_{,j} v_i + C_{ijkl}v_{k,l}u_{i,j} + \sum_{k=0}^{n-1}\alpha_k\theta^{\{k+1\}}\theta^{\{k\}}\\
&+\Big[\sum_{k=0}^{n-1} a_k\theta^{\{k+1\}} + \Big[\Big(\sum_{k=0}^{n} K_{ij}b_k\Phi(\theta^{\{k\}})\Big)_{,j}\Big]_{,i} - \sum_{k=0}^{n-1} a_k\theta^{\{k+1\}} + \beta_{ij}v_{i,j}\Big]\Big(\sum_{k=0}^{n} a_k\theta^{\{k\}}\Big).
\end{aligned}
$$

---

[2] We recall that eventually vector b_k could be zero in case that $l < k \leq n$.

If we apply the divergence theorem we obtain

$$\begin{aligned}D^* &= \alpha_0\theta^{\{1\}}\theta^{\{0\}} + \alpha_1\theta^{\{2\}}\theta^{\{1\}} + \ldots + \alpha_{n-1}\theta^{\{n\}}\theta^{\{n-1\}} \\ &\quad + \sum_{k=0}^{l}\Big(K_{ij}b_k\Phi(\theta^{\{k\}})_{,i}\Big)_{,j}\sum_{s=0}^{n}a_s\theta^{\{s\}}.\end{aligned}$$

As $l \leq n$ and in view of the equivalence between L^2-norm of θ and the $W^{2,2}$-norm of T, we find that there exists a positive constant M_1 such that

$$\int_B D^* d\boldsymbol{x} \leq M_1 \int_B ((\theta^{\{0\}})^2 + \ldots + (\theta^{\{n\}})^2) d\boldsymbol{x}.$$

As the inner product defines a bilinear form which is equivalent to the usual one in the Hilbert space, we conclude that the lemma is proved. □

Lemma 4.2. *For λ large enough the range of $\lambda\mathcal{I} - \mathcal{A}$ is the Hilbert space $\mathcal{X}$.*

Proof. Let $(\boldsymbol{u}^*, \boldsymbol{v}^*, \theta^{\{0\}*}, \theta^{\{1\}*}, \ldots, \theta^{\{n\}*}) \in \mathcal{X}$, we have to show that for λ large enough the system

$$\begin{aligned}&\lambda\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u}^*,\\ &\lambda\boldsymbol{v} - \mathbf{A}\boldsymbol{u} - \mathbf{B}_0\theta^{\{0\}} - \ldots - \mathbf{B}_n\theta^{\{n\}} = \boldsymbol{v}^*,\\ &\lambda\theta^{\{0\}} - \theta^{\{1\}} = \theta^{\{0\}*},\\ &\lambda\theta^{\{1\}} - \theta^{\{2\}} = \theta^{\{1\}*},\\ &\ldots\\ &\lambda\theta^{\{n-1\}} - \theta^{\{n\}} = \theta^{\{n-1\}*},\\ &\lambda\theta^{\{n\}} - D\boldsymbol{v} - L_0\theta^{\{0\}} - L_1\theta^{\{1\}} - \ldots - L_n\theta^{\{n\}} = \theta^{\{n\}*},\end{aligned}$$

has a solution.

It follows that

$$\begin{aligned}\theta^{\{n\}} &= \lambda\theta^{\{n-1\}} - \theta^{\{n-1\}*} = \lambda^2\theta^{\{n-2\}} - \lambda\theta^{\{n-2\}*} - \theta^{\{n-1\}*}\\ &= \lambda^n\theta^{\{0\}} - \lambda^{n-1}\theta^{\{0\}^*} - \lambda^{n-2}\theta^{\{1\}^*} - \ldots - \lambda\theta^{\{n-2\}^*} - \theta^{\{n-1\}^*}.\end{aligned}$$

In a similar way, we have

$$\theta^{\{n-1\}} = \lambda^{n-1}\theta^{\{0\}} - \lambda^{n-2}\theta^{\{0\}^*} - \ldots - \lambda\theta^{\{n-3\}^*} - \theta^{\{n-2\}^*},$$

and, in general, we obtain

$$\theta^{\{k\}} = \lambda^k\theta^{\{0\}} - \lambda^{k-1}\theta^{\{0\}^*} - \ldots - \lambda\theta^{\{k-2\}^*} - \theta^{\{k-1\}^*}.$$

If we substitute these expressions in our system we see

$$\begin{aligned}&\lambda^2\boldsymbol{u} - \mathbf{A}\boldsymbol{u} - (\mathbf{B}_0 + \lambda\mathbf{B}_1 + \ldots + \lambda^n\mathbf{B}_n)\theta^{\{0\}} = \mathbf{F}_1,\\ &\lambda^{n+1}\theta^{\{0\}} - (L_0 + \lambda L_1 + \ldots + \lambda^n L_n)\theta^{\{0\}} - \lambda D\boldsymbol{u} = F_2,\end{aligned}$$

where

$$\mathbf{F}_1 = \lambda\boldsymbol{u}^* + \boldsymbol{v}^* + \mathbf{P}_1(\lambda, \theta^{\{0\}^*}, \theta^{\{1\}^*}, \ldots, \theta^{\{n-1\}^*})$$

and

$$F_2 = \theta^{\{n\}^*} + D\boldsymbol{u}^* + P_2(\lambda, \theta^{\{0\}^*}, \theta^{\{1\}^*}, \ldots, \theta^{\{n-1\}^*}).$$

Here, $\mathbf{P}_1$ and P_2 are polynomials in λ, but linear in the other components. It is clear that $(\mathbf{F}_1, F_2) \in \mathbf{W}^{-1,2} \times L^2$. Therefore, to prove the lemma it will be sufficient to show that the bilinear form

$$\begin{aligned}&\mathcal{B}[(\boldsymbol{u}, \theta^{\{0\}}), (\tilde{\boldsymbol{u}}, \tilde{\theta}^{\{0\}})]\\ &\quad =< (\lambda^2\boldsymbol{u} - \mathbf{A}\mathbf{u} - \sum_{i=0}^{n}\lambda^i\mathbf{B}_i\theta^{\{0\}}, -\lambda D\boldsymbol{u} + \lambda^{n+1}\theta^{\{0\}} - \sum_{j=0}^{n}\lambda^j L_i\theta^{\{0\}})(\tilde{\boldsymbol{u}}, \tilde{\theta}^{\{0\}}) >_{L^2\times L^2},\end{aligned}$$

is a coercive and bounded bilinear form on $\mathbf{W}_0^{1,2} \times L^2$. In our case, we consider the weights given to multiply the first components by λ and the second components by $Q(\lambda) = \sum_{i=0}^{n} \lambda^i a_i$. We note that $Q(\lambda)$ is positive for λ large enough because a_n is strictly positive. It is clear that our product is bounded. On the other side, we have

$$\mathcal{B}[(\boldsymbol{u}, \theta^{\{0\}}), (\boldsymbol{u}, \theta^{\{0\}})]$$
$$= \int_B (Q(\lambda)\lambda^2 u_i u_i + Q(\lambda) C_{ijkl} u_{i,j} u_{k,l} + \lambda[\lambda^{n+1} (\theta^{\{0\}})^2 - \sum_{j=1}^{l} \lambda^j L_j \theta^{\{0\}} \theta^{\{0\}}]) d\boldsymbol{x}.$$

As L_j are bounded we can take λ large enough to guarantee that this integral is equivalent to the inner product in the corresponding Sobolev space. Therefore, it leads to the coerciveness of the bilinear form and the lemma is proved. □

Theorem 4.3. *The operator $\mathcal{A}$ defined previously is the generator of a quasi-contractive semigroup.*

As a consequence, we have the following result.

Theorem 4.4. *Assume that conditions proposed previously are satisfied and that the initial conditions belong to the domain of the operator. Then, there exists a unique solution $\omega(t)$ which satisfies our system with the aforementioned initial conditions.*

Moreover, we know now that there is continuous dependence of the solutions with respect to the initial data.

Since $\mathcal{A}$ is the generator of a quasi-contractive semigroup, we can also obtain the existence and continuous dependence result when supply terms are imposed.

5. Numerical approximation.

In order to simplify the calculations, we assume that the material is homogeneous and isotropic, and we take $n = 2$ and $l = 1$. Hence, system (14) becomes

$$\begin{aligned} &\rho \ddot{u}_i = \mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + \beta(a_0\theta + a_1\dot{\theta} + a_2\ddot{\theta})_{,i} \quad \text{in} \quad B \times (0, T_f), \\ &c\frac{d}{dt}\left\{a_0\theta + a_1\dot{\theta} + a_2\ddot{\theta}\right\} = K(b_0\Delta T + b_1\Delta\dot{T}) + \beta \text{div}\, \boldsymbol{v} \quad \text{in} \quad B \times (0, T_f), \end{aligned} \tag{16}$$

where div represents the divergence operator and $[0, T_f]$, $T_f > 0$, is the time interval of interest.

We also consider the following boundary and initial conditions:

$$u_i(\boldsymbol{x}, t) = T(\boldsymbol{x}, t) = 0 \quad \text{for a.e.} \quad (x, t) \in \partial B \times (0, T_f), \tag{17}$$
$$u_i(\boldsymbol{x}, 0) = u_i^0(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x}, 0) = v_i^0(\boldsymbol{x}) \quad \text{for a.e.} \quad \boldsymbol{x} \in \partial B, \tag{18}$$
$$\theta(\boldsymbol{x}, 0) = \theta^0(\boldsymbol{x}), \quad \dot{\theta}(\boldsymbol{x}, 0) = \theta^1(\boldsymbol{x}), \quad \ddot{\theta}(\boldsymbol{x}, 0) = \theta^2(\boldsymbol{x}) \quad \text{for a.e.} \quad \boldsymbol{x} \in \partial B. \tag{19}$$

In this section, we will assume the following conditions on the constitutive coefficients:

$$\begin{aligned} &\rho > 0, \quad \mu > 0, \quad \lambda + \mu > 0, \quad a_2 > 0, \quad m > 0, \quad K > 0, \quad c > 0, \\ &a_1 > 0, \quad a_0 > 0, \quad b_1 > 0, \quad b_2 > 0. \end{aligned} \tag{20}$$

We note that conditions (20) are slightly stronger than those required in Section 2 but they are needed in the proof of the energy decay property for the variational solution and the a priori error estimates. Although we could weaken some of the conditions, we have imposed all for the sake of simplicity in the analysis.

In order to simplify the writing and the calculations, in this section we will redefine constant mK as m, making an abuse of the notation.

If we denote by $\boldsymbol{v} = \dot{\boldsymbol{u}}$ the velocity field, $e = \dot{\theta}$ the thermal velocity, $\xi = \ddot{\theta}$ the thermal acceleration, $\phi = \dot{T}$ the inductive thermal velocity and $\psi = \ddot{T}$ the inductive thermal acceleration, integrating by parts and using boundary conditions (17) we obtain the following variational formulation of problem (16)-(19).

Problem VP. *Find the velocity field $\boldsymbol{v} : [0, T_f] \to V$, the thermal acceleration $\xi : [0, T_f] \to Y$ and the inductive thermal acceleration $\psi : [0, T_f] \to E$ such that $\boldsymbol{v}(0) = \boldsymbol{v}^0$, $\xi(0) = \theta^2$, $\psi(0) = T^2$, and, for a.e. $t \in (0, T_f)$ and for all $\boldsymbol{w} \in V$, $z, \eta \in Y$,*

$$\begin{aligned}\rho(\dot{\boldsymbol{v}}(t), \boldsymbol{w})_H + \mu(\nabla \boldsymbol{u}(t), \nabla \boldsymbol{w})_Q + (\lambda + \mu)(\operatorname{div} \boldsymbol{u}(t), \operatorname{div} \boldsymbol{w})_Y \\ = -\beta(a_0\theta(t) + a_1 e(t) + a_2\xi(t), \operatorname{div} \boldsymbol{w})_Y,\end{aligned} \tag{21}$$

$$\begin{aligned}&(a_0\theta(t) + a_1 e(t) + a_2\xi(t), z)_Y \\ &= (a_0(T(t) - m\Delta T(t)) + a_1(\phi(t) - m\Delta\phi(t)) + a_2(\psi(t) - m\Delta\psi(t)), z)_Y,\end{aligned} \tag{22}$$

$$\begin{aligned}c(a_0 e(t) + a_1\xi(t) + a_2\dot{\xi}(t), \eta)_Y - K(b_0\Delta T(t) + b_1\Delta\phi(t), \eta)_Y \\ = \beta(\operatorname{div} \boldsymbol{v}(t), \eta)_Y,\end{aligned} \tag{23}$$

where the displacement, thermal and inductive thermal velocities and temperature and inductive temperature fields are then recovered from the relations

$$\begin{aligned}&\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)\, ds + \boldsymbol{u}^0, \quad e(t) = \int_0^t \xi(s)\, ds + \theta^1, \quad \theta(t) = \int_0^t e(s)\, ds + \theta^0, \\ &\phi(t) = \int_0^t \psi(s)\, ds + T^1, \quad T(t) = \int_0^t \phi(s)\, ds + T^0,\end{aligned} \tag{24}$$

and the variational spaces $Y = L^2(B)$, $H = [L^2(B)]^d$, $Q = [L^2(B)]^{d\times d}$ and

$$\begin{aligned}V &= \{\boldsymbol{w} \in [H^1(B)]^d\, ; \, \boldsymbol{w} = \boldsymbol{0} \quad \text{on} \quad \Gamma\}, \\ E &= \{z \in H^2(B)\, ; \, z = 0 \quad \text{on} \quad \Gamma\}.\end{aligned}$$

Here, we note that the initial conditions for the inductive temperature T, given by T^0, T^1 and T^2, are obtained from θ^0, θ^1 and θ^2, respectively, taking into account that $\theta^i = T^i - m\Delta T^i$ for $i = 0, 1, 2$.

Using some of the arguments already employed in [18], we will show that the energy system decays under some conditions on the constitutive coefficients.

Lemma 5.1. *Assume that $a_1 b_1 - b_0 a_2 \geq 0$ and conditions (20). Let us define the energy functional:*

$$\begin{aligned}E(t) = &\frac{1}{2}\Big(\rho\|\boldsymbol{v}(t)\|_H^2 + \mu\|\nabla \boldsymbol{u}(t)\|_Q^2 + (\lambda + \mu)\|\operatorname{div} \boldsymbol{u}(t)\|_Y^2 + c\|a_0\theta(t) + a_1 e(t) + a_2\xi(t)\|_Y^2\Big) \\ &+ \frac{K}{2}\Big((b_0 a_1 + b_1 a_0)(\|\nabla T(t)\|_H^2 + m\|\Delta T(t)\|_Y^2) + b_1 a_2(\|\nabla\phi(t)\|_H^2 + m\|\Delta\phi(t)\|_Y^2)\Big) \\ &+ K b_0 a_2\Big((\nabla T(t), \nabla\phi(t))_H + m(\Delta T(t), \Delta\phi(t))_Y\Big),\end{aligned}$$

then we have the following energy decay property:

$$\frac{d}{dt}E(t) \leq 0.$$

Proof. Taking $\boldsymbol{w} = \boldsymbol{v}(t)$ and $\eta = a_0\theta(t) + a_1 e(t) + a_2\xi(t)$ we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\rho\|\boldsymbol{v}(t)\|_H^2 + \mu\|\nabla \boldsymbol{u}(t)\|_Q^2 + (\lambda + \mu)\|\operatorname{div} \boldsymbol{u}(t)\|_Y^2 + c\|a_0\theta(t) + a_1 e(t) + a_2\xi(t)\|_Y^2\Big) \\ &\qquad - K(b_0\Delta T(t) + b_1\Delta\phi(t), a_0\theta(t) + a_1 e(t) + a_2\xi(t))_Y = 0.\end{aligned} \tag{25}$$

Now, using (22) we have

$$\begin{aligned}
-K(b_0\Delta T(t)&+b_1\Delta\phi(t),a_0\theta(t)+a_1e(t)+a_2\xi(t))_Y\\
&=-K((b_0\Delta T(t)+b_1\Delta\phi(t),a_0T(t)+a_1\phi(t)+a_2\psi(t))_Y\\
&\quad+Km(b_0\Delta T(t)+b_1\Delta\phi(t),a_0\Delta T(t)+a_1\Delta\phi(t)+a_2\Delta\psi(t))_Y\\
&=K(b_0\nabla T(t)+b_1\nabla\phi(t),a_0\nabla T(t)+a_1\nabla\phi(t)+a_2\nabla\psi(t))_H\\
&\quad+Km(b_0\Delta T(t)+b_1\Delta\phi(t),a_0\Delta T(t)+a_1\Delta\phi(t)+a_2\Delta\psi(t))_Y\\
&=Kb_0a_0\|\nabla T(t)\|_H^2+\frac{K}{2}(b_0a_1+b_1a_0)\frac{d}{dt}\|\nabla T(t)\|_H^2+Kb_0a_2(\nabla T(t),\nabla\psi(t))_H\\
&\quad+Kb_1a_1\|\nabla\phi(t)\|_H^2+\frac{Kb_1a_2}{2}\frac{d}{dt}\|\nabla\phi(t)\|_H^2\\
&\quad+Kmb_0a_0\|\Delta T(t)\|_Y^2+\frac{Km}{2}(b_0a_1+b_1a_0)\frac{d}{dt}\|\Delta T(t)\|_Y^2+Kmb_0a_2(\Delta T(t),\Delta\psi(t))_Y\\
&\quad+Kmb_1a_1\|\Delta\dot T(t)\|_Y^2+\frac{Kmb_1a_2}{2}\frac{d}{dt}\|\Delta\phi(t)\|_Y^2.
\end{aligned}$$

Thus, equation (25) becomes

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big(\rho\|\boldsymbol{v}(t)\|_H^2&+\mu\|\nabla\boldsymbol{u}(t)\|_Q^2+(\lambda+\mu)\|\mathrm{div}\,\boldsymbol{u}(t)\|_Y^2+c\|a_0\theta(t)+a_1e(t)+a_2\xi(t)\|_Y^2\Big)\\
&+\frac{K}{2}\frac{d}{dt}\Big((b_0a_1+b_1a_0)(\|\nabla T(t)\|_H^2+m\|\Delta T(t)\|_Y^2)+b_1a_2(\|\nabla\dot T(t)\|_H^2+m\|\Delta\phi(t)\|_Y^2)\Big)\\
&+Kb_0a_2\Big((\nabla T(t),\nabla\psi(t))_H+m(\Delta T(t),\Delta\psi(t))_Y\Big)\\
&+Kb_0a_2\Big((\nabla\phi(t),\nabla\phi(t))_H+m(\Delta\phi(t),\Delta\phi(t))_Y\Big)\\
&=Kb_0a_2\Big((\nabla\phi(t),\nabla\phi(t))_H+m(\Delta\phi(t),\Delta\phi(t))_Y\Big)-Kb_0a_0(\|\nabla T(t)\|_H^2+m\|\Delta T(t)\|_Y^2)\\
&\quad-Kb_1a_1(\|\nabla\phi(t)\|_H^2+m\|\Delta\phi(t)\|_Y^2).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big(\rho\|\boldsymbol{v}(t)\|_H^2&+\mu\|\nabla\boldsymbol{u}(t)\|_Q^2+(\lambda+\mu)\|\mathrm{div}\,\boldsymbol{u}(t)\|_Y^2+c\|a_0\theta(t)+a_1e(t)+a_2\xi(t)\|_Y^2\Big)\\
&+\frac{K}{2}\frac{d}{dt}\Big((b_0a_1+b_1a_0)(\|\nabla T(t)\|_H^2+m\|\Delta T(t)\|_Y^2)+b_1a_2(\|\nabla\phi(t)\|_H^2+m\|\Delta\phi(t)\|_Y^2)\Big)\\
&+Kb_0a_2\frac{d}{dt}\Big((\nabla T(t),\nabla\phi(t))_H+m(\Delta T(t),\Delta\phi(t))_Y\Big)\\
&=-K(a_1b_1-b_0a_2)(\|\nabla\phi(t)\|_H^2+m\|\Delta\phi(t)\|_Y^2)-Kb_0a_0(\|\nabla T(t)\|_H^2+m\|\Delta T(t)\|_Y^2).
\end{aligned}$$

Taking into account $a_1b_1-b_0a_2\geq 0$ and observing that

$$\det\begin{pmatrix} b_0a_1+b_1a_0 & b_0a_2\\ b_0a_2 & b_1a_2\end{pmatrix}=a_2\Big[b_0(a_1b_1-b_0a_2)+a_0b_1^2\Big]>0,$$

we conclude the desired property. □

Remark 1. We note that condition $a_1b_1-b_0a_2=\tau_1(\tau_2-\tau_1/2)\geq 0$ is the same condition that appears in [18], with the following coefficients:

$$b_0=a_0=1,\quad a_1=\tau_1,\quad a_2=\tau_1^2/2,\quad b_1=\tau_2.$$

Now, we consider a fully discrete approximation of Problem VP. This is done in two steps. First, we assume that the domain $\overline{B}$ is polyhedral and we denote by $\mathcal{T}^h$ a regular triangulation in the sense of [10]. Thus, we construct the finite dimensional spaces $V^h \subset V$ and $E^h \subset E$ given by

$$\begin{aligned} V^h = \{ & \boldsymbol{w}^h \in [C(\overline{B})]^d \cap [H^1(B)]^d \; ; \; \boldsymbol{w}^h_{|Tr} \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h, \\ & \boldsymbol{w}^h = \boldsymbol{0} \quad \text{on} \quad \Gamma \}, \end{aligned} \tag{26}$$

$$\begin{aligned} E^h = \{ & z^h \in C^1(\overline{B}) \cap H^2(B) \; ; \; z^h_{|Tr} \in P_3(Tr) \quad \forall Tr \in \mathcal{T}^h, \\ & z^h = 0 \quad \text{on} \quad \Gamma \}, \end{aligned} \tag{27}$$

$$W^h = \{ \eta^h \in L^2(B) \; ; \; \eta^h_{|Tr} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h \}, \tag{28}$$

where $P_q(Tr)$ represents the space of polynomials of degree less or equal to q in the element Tr. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $\boldsymbol{u}^{0h}$, $\boldsymbol{v}^{0h}$, θ^{0h}, θ^{1h} and θ^{2h} are given by

$$\boldsymbol{u}^{0h} = \mathcal{P}^h_1 \boldsymbol{u}^0, \quad \boldsymbol{v}^{0h} = \mathcal{P}^h_1 \boldsymbol{v}^0, \quad \theta^{0h} = \mathcal{P}^h_2 \theta^0, \quad \theta^{1h} = \mathcal{P}^h_2 \theta^1, \quad \theta^{2h} = \mathcal{P}^h_2 \theta^2, \tag{29}$$

where $\mathcal{P}^h_1$ and $\mathcal{P}^h_2$ are the classical finite element interpolation operators over V^h and E^h, respectively (see, e.g., [10]).

Secondly, we consider a partition of the time interval $[0, T_f]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T_f$. In this case, we use a uniform partition with step size $k = T_f/N$ and nodes $t_n = n\,k$ for $n = 0, 1, \ldots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

Problem VP^{hk}. *Find the discrete velocity field $\boldsymbol{v}^{hk} = \{\boldsymbol{v}^{hk}_n\}_{n=0}^N \subset V^h$, the discrete thermal acceleration $\xi^{hk} = \{\xi^{hk}_n\}_{n=0}^N \subset W^h$ and the discrete inductive thermal acceleration $\psi^{hk} = \{\psi^{hk}_n\}_{n=0}^N \subset E^h$ such that $\boldsymbol{v}^{hk}_0 = \boldsymbol{v}^{0h}$, $\xi^{hk}_0 = \theta^{2h}$, $\psi^{hk}_0 = T^{2h}$ and, for $n = 1, \ldots, N$ and for all $\boldsymbol{w}^h \in V^h$, z^h, $\eta^h \in W^h$,*

$$\begin{aligned} & \rho(\delta \boldsymbol{v}^{hk}_n, \boldsymbol{w}^h)_H + \mu(\nabla \boldsymbol{u}^{hk}_n, \nabla \boldsymbol{w}^h)_Q + (\lambda + \mu)(\operatorname{div} \boldsymbol{u}^{hk}_n, \operatorname{div} \boldsymbol{w}^h)_Y \\ & \qquad = -\beta(a_0 \theta^{hk}_n + a_1 e^{hk}_n + a_2 \xi^{hk}_n, \operatorname{div} \boldsymbol{w}^h)_Y, \end{aligned} \tag{30}$$

$$\begin{aligned} & (a_0 \theta^{hk}_n + a_1 e^{hk}_n + a_2 \xi^{hk}_n, z^h)_Y \\ & \quad = (a_0(T^{hk}_n - m\Delta T^{hk}_n) + a_1(\phi^{hk}_n - m\Delta \phi^{hk}_n) + a_2(\psi^{hk}_n - m\Delta \psi^{hk}_n), z^h)_Y, \end{aligned} \tag{31}$$

$$\begin{aligned} & c(a_0 e^{hk}_n + a_1 \xi^{hk}_n + a_2 \delta \xi^{hk}_n, \eta^h)_Y - K(b_0 \Delta T^{hk}_n + b_1 \Delta \phi^{hk}_n, \eta^h)_Y \\ & \qquad = \beta(\operatorname{div} \boldsymbol{v}^{hk}_n, \eta^h)_Y, \end{aligned} \tag{32}$$

where the discrete displacement $\boldsymbol{u}^{hk}_n$, thermal velocity e^{hk}_n, temperature θ^{hk}_n, inductive thermal velocity ϕ^{hk}_n and inductive temperature T^{hk}_n are then recovered from the relations:

$$\begin{aligned} & \boldsymbol{u}^{hk}_n = k \sum_{j=1}^n \boldsymbol{v}^{hk}_j + \boldsymbol{u}^{0h}, \quad e^{hk}_n = k \sum_{j=1}^n \xi^{hk}_j + \theta^{1h}, \quad \theta^{hk}_n = k \sum_{j=1}^n e^{hk}_j + \theta^{0h}, \\ & \phi^{hk}_n = k \sum_{j=1}^n \psi^{hk}_j + T^{1h}, \quad T^{hk}_n = k \sum_{j=1}^n \phi^{hk}_j + T^{0h}, \end{aligned} \tag{33}$$

and $T^{0h} = P^h_2 T^0$, $T^{1h} = P^h_2 T^1$ and $T^{2h} = P^h_2 T^2$.

We have the following discrete version of the energy decay property.

Lemma 5.2. *Let*

$$b_0 = a_0 = 1, \quad a_1 = \tau_1, \quad a_2 = \tau_1^2/2, \quad b_1 = \tau_2,$$

where τ_1, τ_2 are given positive constants with $\tau_2 - \tau_1/2 \geq 0$ and conditions (20). Let us define the discrete energy functional:

$$\begin{aligned} E_n^{hk} &= \frac{1}{2}\Big(\|\boldsymbol{v}_n^{hk}\|_H^2 + \mu\|\nabla \boldsymbol{u}_n^{hk}\|_Q^2 + (\lambda+\mu)\|\operatorname{div} \boldsymbol{u}_n^{hk}\|_Y^2 + c\|a_0\theta_n^{hk} + a_1 e_n^{hk} + a_2\xi_n^{hk}\|^2\Big) \\ &\quad + \frac{1}{2}\Big[K(b_0a_1 + b_1a_0)(\|\nabla T_n^{hk}\|^2 + m\|\Delta T_n^{hk}\|_Y^2) + Kb_1a_2(\|\nabla\phi_n^{hk}\|_H^2 + m\|\Delta\phi_n^{hk}\|_Y^2)\Big] \\ &\quad + 2Kb_0a_2\Big[(\nabla T_n^{hk}, \nabla\phi_n^{hk})_Y + m(\Delta T_n^{hk}, \Delta\phi_n^{hk})_Y\Big], \end{aligned} \tag{34}$$

then it satisfies

$$\frac{E_n^{hk} - E_{n-1}^{hk}}{k} \leq 0.$$

Proof. Choosing $\boldsymbol{w}^h = v_n^{hk}$ and $\eta^h = a_0\theta_n^{hk} + a_1 e_n^{hk} + a_2\xi_n^{hk}$ we obtain

$$\begin{aligned} &\frac{\rho}{2k}\Big(\|\boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1}^{hk}\|_H^2\Big) + \frac{\mu}{2k}\Big(\|\nabla \boldsymbol{u}_n^{hk}\|_Q^2 - \|\nabla \boldsymbol{u}_{n-1}^{hk}\|_Q^2\Big) + \frac{\lambda+\mu}{2k}\Big(\|\operatorname{div} \boldsymbol{u}_n^{hk}\|_Y^2 - \|\operatorname{div} \boldsymbol{u}_n^{hk}\|_Y^2\Big) \\ &\qquad + \frac{1}{2k}\Big(c\|a_0\theta_n^{hk} + a_1e_n^{hk} + a_2\xi_n^{hk}\|^2 - c\|a_0\theta_{n-1}^{hk} + a_1e_{n-1}^{hk} + a_2\xi_{n-1}^{hk}\|_Y^2\Big) \\ &\qquad - K(b_0\Delta T_n^{hk} + b_1\Delta\phi_n^{hk}, a_0\theta_n^{hk} + a_1e_n^{hk} + a_2\xi_n^{hk})_Y = 0. \end{aligned} \tag{35}$$

Using (31) with $z^h = b_0\Delta T_n^{hk} + b_1\Delta\phi_n^{hk}$ and integration by parts we deduce that

$$\begin{aligned} &-K(b_0\Delta T_n^{hk} + b_1\Delta\phi_n^{hk}, a_0\theta_n^{hk} + a_1e_n^{hk} + a_2\xi_n^{hk})_Y \\ &\quad = K(b_0\nabla T_n^{hk} + b_1\nabla\phi_n^{hk}, a_0\nabla T_n^{hk} + a_1\nabla\phi_n^{hk} + a_2\nabla\psi_n^{hk})_H \\ &\qquad + Km(b_0\Delta T_n^{hk} + b_1\Delta\phi_n^{hk}, a_0\Delta T_n^{hk} + a_1\Delta\phi_n^{hk} + a_2\Delta\psi_n^{hk})_Y \\ &\quad = Kb_0a_0\|\nabla T_n^{hk}\|_H^2 + \frac{K}{2k}(b_0a_1 + b_1a_0)\Big(\|\nabla T_n^{hk} - \nabla T_{n-1}^{hk}\|_H^2 + \|\nabla T_n^{hk}\|_H^2 - \|\nabla T_{n-1}^{hk}\|_H^2\Big) \\ &\qquad + Kb_1a_1\|\nabla\phi_n^{hk}\|_H^2 + \frac{Kb_1a_2}{2k}\Big(\|\nabla\phi_n^{hk} - \nabla\phi_{n-1}^{hk}\|_H^2 + \|\nabla\phi_n^{hk}\|_H^2 - \|\nabla\phi_{n-1}^{hk}\|_H^2\Big) \\ &\qquad + Kmb_0a_0\|\Delta T_n^{hk}\|_Y^2 + \frac{Km}{2k}(b_0a_1 + b_1a_0)\Big(\|\Delta T_n^{hk} - \Delta T_{n-1}^{hk}\|_Y^2 + \|\Delta T_n^{hk}\|_Y^2 - \|\Delta T_{n-1}^{hk}\|_Y^2\Big) \\ &\qquad + Kmb_1a_1\|\Delta\phi_n^{hk}\|_Y^2 + \frac{Kmb_1a_2}{2k}\Big(\|\Delta\phi_n^{hk} - \Delta\phi_{n-1}^{hk}\|_Y^2 + \|\Delta\phi_n^{hk}\|_Y^2 - \|\Delta\phi_{n-1}^{hk}\|_H^2\Big) \\ &\qquad + Kb_0a_2\Big((\nabla T_n^{hk}, \nabla\psi_n^{hk})_H + m(\Delta T_n^{hk}, \Delta\psi_n^{hk})_Y\Big). \end{aligned} \tag{36}$$

Combining (35) with (36) it follows that

$$\begin{aligned}
&\frac{\rho}{2k}\Big(\|\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}^{hk}\|_H^2\Big)+\frac{\mu}{2k}\Big(\|\nabla\boldsymbol{u}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{u}_{n-1}^{hk}\|_Q^2\Big)+\frac{\lambda+\mu}{2k}\Big(\|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2-\|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2\Big)\\
&\quad+\frac{1}{2k}\Big(c\|a_0\theta_n^{hk}+a_1e_n^{hk}+a_2\xi_n^{hk}\|_Y^2-c\|a_0\theta_{n-1}^{hk}+a_1e_{n-1}^{hk}+a_2\xi_{n-1}^{hk}\|_Y^2\Big)\\
&\quad+\frac{K}{2k}(b_0a_1+b_1a_0)\Big(\|\nabla T_n^{hk}-\nabla T_{n-1}^{hk}\|_H^2+\|\nabla T_n^{hk}\|_H^2-\|\nabla T_{n-1}^{hk}\|_H^2\Big)\\
&\quad+\frac{Kb_1a_2}{2k}\Big(\|\nabla\phi_n^{hk}-\nabla\phi_{n-1}^{hk}\|_H^2+\|\nabla\phi_n^{hk}\|_H^2-\|\nabla\phi_{n-1}^{hk}\|_H^2\Big)\\
&\quad+\frac{Km}{2k}(b_0a_1+b_1a_0)\Big(\|\Delta T_n^{hk}-\Delta T_{n-1}^{hk}\|_Y^2+\|\Delta T_n^{hk}\|_Y^2-\|\Delta T_{n-1}^{hk}\|_Y^2\Big)\\
&\quad+\frac{Kmb_1a_2}{2k}\Big(\|\Delta\phi_n^{hk}-\Delta\phi_{n-1}^{hk}\|_Y^2+\|\Delta\phi_n^{hk}\|_Y^2-\|\Delta\phi_{n-1}^{hk}\|_Y^2\Big)\\
&\quad+Kb_0a_2\Big[(\nabla T_n^{hk},\nabla\psi_n^{hk})_H+m(\Delta T_n^{hk},\Delta\psi_n^{hk})_Y+(\nabla\phi_n^{hk},\nabla\phi_n^{hk})_H+m(\Delta\phi_n^{hk},\Delta\phi_n^{hk})_Y\Big]\\
&\quad=-K(b_1a_1-b_0a_2)\Big(\|\nabla\phi_n^{hk}\|_H^2+m\|\Delta\phi_n^{hk}\|_Y^2\Big)-Kb_0a_0\Big(\|\nabla T_n^{hk}\|_H^2+m\|\Delta T_n^{hk}\|_Y^2\Big).
\end{aligned}$$

Keeping in mind that

$$\begin{aligned}
&Kb_0a_2\Big[(\nabla T_n^{hk},\nabla\psi_n^{hk})_H+(\nabla\phi_n^{hk},\nabla\phi_n^{hk})_H\Big]\\
&\quad=\frac{Kb_0a_2}{k}\Big[(\nabla T_n^{hk},\nabla\phi_n^{hk}-\nabla\phi_{n-1}^{hk})_H+(\nabla T_n^{hk}-\nabla T_{n-1}^{hk},\nabla\phi_n^{hk})_H\Big]\\
&\quad=\frac{Kb_0a_2}{k}\Big[(\nabla T_n^{hk},\nabla\phi_n^{hk})_H-(\nabla T_{n-1}^{hk},\nabla\phi_{n-1}^{hk})_H+(\nabla T_n^{hk}-\nabla T_{n-1}^{hk},\nabla\phi_n^{hk}-\nabla\phi_{n-1}^{hk})_H\Big],\\
&Kmb_0a_2\Big[(\Delta T_n^{hk},\Delta\psi_n^{hk})_Y+(\Delta\phi_n^{hk},\Delta\phi_n^{hk})_Y\Big]\\
&\quad=\frac{Kmb_0a_2}{k}\Big[(\Delta T_n^{hk},\Delta\phi_n^{hk})_Y-(\Delta T_{n-1}^{hk},\Delta\phi_{n-1}^{hk})_Y+(\Delta T_n^{hk}-\Delta T_{n-1}^{hk},\Delta\phi_n^{hk}-\Delta\phi_{n-1}^{hk})_Y\Big],
\end{aligned}$$

it results that, for $\epsilon>0$,

$$\begin{aligned}
&\frac{\rho}{2k}\Big(\|\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}^{hk}\|_H^2\Big)+\frac{\mu}{2k}\Big(\|\nabla\boldsymbol{u}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{u}_{n-1}^{hk}\|_Q^2\Big)+\frac{\lambda+\mu}{2k}\Big(\|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2-\|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2\Big)\\
&\quad+\frac{1}{2k}\Big(\|a_0\theta_n^{hk}+a_1e_n^{hk}+a_2\xi_n^{hk}\|_Y^2-\|a_0\theta_{n-1}^{hk}+a_1e_{n-1}^{hk}+a_2\xi_{n-1}^{hk}\|_Y^2\Big)\\
&\quad+\frac{K}{2k}(b_0a_1+b_1a_0)\Big(\|\nabla T_n^{hk}-\nabla T_{n-1}^{hk}\|_H^2+\|\nabla T_n^{hk}\|_H^2-\|\nabla T_{n-1}^{hk}\|_H^2\Big)\\
&\quad+\frac{Kb_1a_2}{2k}\Big(\|\nabla\phi_n^{hk}-\nabla\phi_{n-1}^{hk}\|_H^2+\|\nabla\phi_n^{hk}\|_H^2-\|\nabla\phi_{n-1}^{hk}\|_H^2\Big)\\
&\quad+\frac{Km}{2k}(b_0a_1+b_1a_0)\Big(\|\Delta T_n^{hk}-\Delta T_{n-1}^{hk}\|_Y^2+\|\Delta T_n^{hk}\|_Y^2-\|\Delta T_{n-1}^{hk}\|_Y^2\Big)\\
&\quad+\frac{Kmb_1a_2}{2k}\Big(\|\Delta\phi_n^{hk}-\Delta\phi_{n-1}^{hk}\|_Y^2+\|\Delta\phi_n^{hk}\|_Y^2-\|\Delta\phi_{n-1}^{hk}\|_Y^2\Big)\\
&\quad+\frac{Kb_0a_2}{k}\Big[(\nabla T_n^{hk},\nabla\phi_n^{hk})_H-(\nabla T_{n-1}^{hk},\nabla\phi_{n-1}^{hk})_H+m(\Delta T_n^{hk},\Delta\phi_n^{hk})_Y-m(\Delta T_{n-1}^{hk},\Delta\phi_{n-1}^{hk})_Y\Big]\\
&\quad\leq-K(b_1a_1-b_0a_2)\Big(\|\nabla\phi_n^{hk}\|_H^2+m\|\Delta\phi_n^{hk}\|_Y^2\Big)-Kb_0a_0\Big(\|\nabla T_n^{hk}\|_H^2+m\|\Delta T_n^{hk}\|_Y^2\Big)\\
&\quad+\frac{Kb_0a_2\epsilon}{2k}\Big(\|\nabla T_n^{hk}-\nabla T_{n-1}^{hk}\|_H^2+m\|\Delta T_n^{hk}-\Delta T_{n-1}^{hk}\|_Y^2\Big)\\
&\quad+\frac{Kb_0a_2}{2k\epsilon}\Big(\|\nabla\phi_n^{hk}-\nabla\phi_{n-1}^{hk}\|_H^2+m\|\Delta\phi_n^{hk}-\Delta\phi_{n-1}^{hk}\|_Y^2\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\rho}{2k}\Big(\|\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}^{hk}\|_H^2\Big)+\frac{\mu}{2k}\Big(\|\nabla\boldsymbol{u}_n^{hk}\|_Q^2-\|\nabla\boldsymbol{u}_{n-1}^{hk}\|_Q^2\Big)+\frac{\lambda+\mu}{2k}\Big(\|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2-\|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2\Big)\\
&\quad+\frac{1}{2k}\Big(c\|a_0\theta_n^{hk}+a_1e_n^{hk}+a_2\xi_n^{hk}\|_Y^2-c\|a_0\theta_{n-1}^{hk}+a_1e_{n-1}^{hk}+a_2\xi_{n-1}^{hk}\|_Y^2\Big)\\
&\quad+\frac{K}{2k}(b_0a_1+b_1a_0)\Big(\|\nabla T_n^{hk}\|_H^2-\|\nabla T_{n-1}^{hk}\|_H^2\Big)+\frac{Kb_1a_2}{2k}\Big(\|\nabla\phi_n^{hk}\|_H^2-\|\nabla\phi_{n-1}^{hk}\|_H^2\Big)\\
&\quad+\frac{Km}{2k}(b_0a_1+b_1a_0)\Big(\|\Delta T_n^{hk}\|_Y^2-\|\Delta T_{n-1}^{hk}\|_Y^2\Big)\\
&\quad+\frac{Kmb_1a_2}{2k}\Big(\|\Delta\phi_n^{hk}\|_Y^2-\|\Delta\phi_{n-1}^{hk}\|_Y^2\Big)\\
&\quad+\frac{Kb_0a_2}{k}\Big[(\nabla T_n^{hk},\nabla\phi_n^{hk})_H-(\nabla T_{n-1}^{hk},\nabla\phi_{n-1}^{hk})_H+m(\Delta T_n^{hk},\Delta\phi_n^{hk})_Y-m(\Delta T_{n-1}^{hk},\Delta\phi_{n-1}^{hk})_Y\Big]\\
&\quad+\frac{K}{2k}(b_0a_1+b_1a_0-b_0a_2\epsilon)\|\nabla T_n^{hk}-\nabla T_{n-1}^{hk}\|_H^2+\frac{K}{2k}(b_1a_2-b_0a_2/\epsilon)\|\nabla\phi_n^{hk}-\nabla\phi_{n-1}^{hk}\|_H^2\\
&\quad+\frac{Km}{2k}(b_0a_1+b_1a_0-b_0a_2\epsilon)\|\Delta T_n^{hk}-\Delta T_{n-1}^{hk}\|_Y^2+\frac{K}{2k}(b_1a_2-b_0a_2/\epsilon)\|\Delta\phi_n^{hk}-\Delta\phi_{n-1}^{hk}\|_Y^2\\
&\quad=-K(b_1a_1-b_0a_2)\Big(\|\nabla\phi_n^{hk}\|_H^2+m\|\Delta\phi_n^{hk}\|_Y^2\Big)-Kb_0a_0\Big(\|\nabla T_n^{hk}\|_H^2+m\|\Delta T_n^{hk}\|_Y^2\Big).
\end{aligned}$$

Then, setting $\epsilon=1/\tau_1$, we have

$$b_0a_1+b_1a_0-b_0a_2\epsilon=\tau_1/2+\tau_2>0\quad\text{and}\quad b_1a_2-b_0a_2/\epsilon=\tau_2-\tau_1/2>0.$$

Summing over n we then obtain the desired discrete version of the energy decay property. □

As a consequence, we find the following discrete stability.

Lemma 5.3. *Under the assumptions of Lemma 5.2, we have the following discrete stability property for $n=1,\dots,N$,*

$$\begin{aligned}
&\|\boldsymbol{v}_n^{hk}\|_H^2+\|\nabla\boldsymbol{u}_n^{hk}\|_Q^2+\|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2+\|a_0\theta_n^{hk}+a_1e_n^{hk}+a_2\xi_n^{hk}\|_Y^2+\|\nabla T_n^{hk}\|_H^2+\|\Delta T_n^{hk}\|_Y^2\\
&\quad+\|\nabla\phi_n^{hk}\|_H^2+\|\Delta\phi_n^{hk}\|_Y^2\le C,
\end{aligned}$$

where C is a positive constant independent of the discretization parameters h and k.

We obtain now some a priori error estimates.

In what follows, we use the notations $R(t)=a_0\theta(t)+a_1e(t)+a_2\xi(t)$ (so $R_n=R(t_n)$ and $\dot R_n=\dot R(t_n)$) and $R_n^{hk}=a_0\theta_n^{hk}+a_1e_n^{hk}+a_2\xi_n^{hk}$.

First, we provide some estimates for the velocity field. Then, we subtract variational equation (21) at time $t=t_n$ for a test function $\boldsymbol{w}=\boldsymbol{w}^h\in V^h\subset V$ and discrete variational equation (30) to obtain, for all $\boldsymbol{w}^h\in V^h$,

$$\begin{aligned}
&\rho(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{w}^h)_H+\mu(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla\boldsymbol{w}^h)_Q+(\lambda+\mu)(\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\mathrm{div}\,\boldsymbol{w}^h)_Y\\
&\quad=-\beta(R_n-R_n^{hk},\mathrm{div}\,\boldsymbol{w}^h)_Y,
\end{aligned}$$

and so, we have, for all $\boldsymbol{w}^h\in V^h$,

$$\begin{aligned}
&\rho(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H+\mu(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Q+\beta(R_n-R_n^{hk},\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Y\\
&\quad+(\lambda+\mu)(\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Y\\
&\quad=\rho(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}^h)_H+\mu(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{v}_n-\boldsymbol{w}^h))_Q+\beta(R_n-R_n^{hk},\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{w}^h))_Y\\
&\quad+(\lambda+\mu)(\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{w}^h))_Y.
\end{aligned}$$

Taking into account that

$$
\begin{aligned}
&(\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H \geq (\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n,\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H\\
&\qquad +\frac{1}{2k}\left\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\},\\
&(\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Y \geq (\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\mathrm{div}\,(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n))_Y\\
&\qquad +\frac{1}{2k}\left\{\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2-\|\mathrm{div}\,(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\},\\
&(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Q \geq (\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n))_Q\\
&\qquad +\frac{1}{2k}\left\{\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Q^2\right\},
\end{aligned}
$$

using Cauchy-Schwarz and Young's inequalities it follows that, for all $\boldsymbol{w}^h\in V^h$,

$$
\begin{aligned}
&\frac{\rho}{2k}\left\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\}+\beta(R_n-R_n^{hk},\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Y\\
&\quad +\frac{\lambda+\mu}{2k}\left\{\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2-\|\mathrm{div}\,(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\}\\
&\quad +\frac{\mu}{2k}\left\{\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Q^2\right\}\\
&\quad \leq C\Big(\|\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_V^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+\|\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n\|_V^2+\|R_n-R_n^{hk}\|_Y^2\\
&\quad +\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+(\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}^h)_H\Big).
\end{aligned}
\tag{37}
$$

Now, we subtract variational equation (22) at time $t=t_n$ for a test function $\eta=\eta^h\in E^h\subset E$ and discrete variational equation (32) to obtain, for all $\eta^h\in E^h$,

$$
c(\dot{R}_n-\delta R_n^{hk},\eta^h)_Y-\beta(\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),\eta^h)_Y-K(b_0\Delta(T_n-T_n^{hk})+b_1\Delta(\phi_n-\phi_n^{hk}),\eta^h)_Y=0,
$$

and so, we have, for all $\eta^h\in E^h$,

$$
\begin{aligned}
&c(\dot{R}_n-\delta R_n^{hk},R_n-R_n^{hk})_Y-K(b_0\Delta(T_n-T_n^{hk})+b_1\Delta(\phi_n-\phi_n^{hk}),R_n-R_n^{hk})_Y\\
&\quad -\beta(\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),R_n-R_n^{hk})_Y\\
&\quad = c(\dot{R}_n-\delta R_n^{hk},R_n-\eta^h)_Y-K(b_0\Delta(T_n-T_n^{hk})+b_1\Delta(\phi_n-\phi_n^{hk}),R_n-\eta^h)_Y\\
&\quad -\beta(\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),R_n-\eta^h)_Y.
\end{aligned}
$$

Keeping in mind that

$$
\begin{aligned}
&(\dot{R}_n-\delta R_n^{hk},R_n-R_n^{hk})_Y\geq(\dot{R}_n-\delta R_n,R_n-R_n^{hk})_Y+\frac{1}{2k}\Big\{\|R_n-R_n^{hk}\|_Y^2-\|R_{n-1}-R_{n-1}^{hk}\|_Y^2\Big\},\\
&(\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),R_n-\eta^h)_Y=-(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\nabla(R_n-\eta^h))_H,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\frac{1}{2k}\Big\{\|R_n-R_n^{hk}\|_Y^2-\|R_{n-1}-R_{n-1}^{hk}\|_Y^2\Big\}-K(b_0\Delta(T_n-T_n^{hk})+b_1(\phi_n-\Delta\phi_n^{hk}),R_n-R_n^{hk})_Y\\
&\quad -\beta(R_n-R_n^{hk},\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Y\\
&\quad \leq C\Big(\|\dot{R}_n-\delta R_n\|_Y^2+\|R_n-\eta^h\|_Y^2+\|\nabla(R_n-\eta^h)\|_H^2+\|R_n-R_n^{hk}\|_Y^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2\\
&\quad +\|\Delta(T_n-T_n^{hk})\|_Y^2+\|\Delta(\phi_n-\phi_n^{hk})\|_Y^2+(\delta R_n-\delta R_n^{hk},R_n-\eta^h)_Y\Big).
\end{aligned}
\tag{38}
$$

Combining estimates (37) and (38), multiplying the resulting estimates by k and summing up to n, we find that

$$\begin{aligned}
&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+\|R_n-R_n^{hk}\|_Y^2\\
&\quad\leq Ck\sum_{j=1}^n\Big(\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2+\|\nabla(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Q^2+\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2\\
&\qquad+\|\mathrm{div}\,(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Y^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_H^2+(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H+\|\dot{R}_j-\delta R_j\|_Y^2\\
&\qquad+\|R_j-\eta_j^h\|_Y^2+\|\nabla(R_j-\eta_j^h)\|_H^2+\|\Delta(T_j-T_j^{hk})\|_Y^2+\|\Delta(\phi_j-\phi_j^{hk})\|_Y^2\\
&\qquad+\|R_j-R_j^{hk}\|_Y^2+(\delta R_j-\delta R_j^{hk},R_j-\eta_j^h)_Y\Big)+C\Big(\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_H^2+\|\mathrm{div}\,(\boldsymbol{u}^0-\boldsymbol{u}^{0h})\|_Y^2\\
&\qquad+\|\nabla(\boldsymbol{u}^0-\boldsymbol{u}^{0h})\|_Q^2+\|R^0-R^{0h}\|_Y^2\Big).
\end{aligned}\tag{39}$$

Finally, we subtract variational equation (22) at time $t=t_n$ and discrete variational equation (31) for a test function $z=z^h\in W^h\subset W$ to obtain

$$\begin{aligned}
&(R_n-R_n^{hk},z^h)_Y-(a_2(\psi_n-\psi_n^{hk}-m\Delta(\psi_n-\psi_n^{hk})),z^h)_Y\\
&\quad-(a_0(T_n-T_n^{hk}-m\Delta(T_n-T_n^{hk}))+a_1(\phi_n-\phi_n^{hk}-m\Delta(\phi_n-\phi_n^{hk})),z^h)_Y=0,
\end{aligned}$$

and therefore,

$$\begin{aligned}
&(R_n-R_n^{hk},\Delta(\psi_n-\psi_n^{hk}))_Y+a_1(\phi_n-\phi_n^{hk}-m\Delta(\phi_n-\phi_n^{hk})),\Delta(\psi_n-\psi_n^{hk}))_Y\\
&\quad-(a_0(T_n-T_n^{hk}-m\Delta(T_n-T_n^{hk}))-(a_2(\psi_n-\psi_n^{hk}-m\Delta(\psi_n-\psi_n^{hk})),\Delta(\psi_n-\psi_n^{hk}))_Y\\
&\quad=(R_n-R_n^{hk},\Delta(\psi_n-z^h))_Y+a_1(\phi_n-\phi_n^{hk}-m\Delta(\phi_n-\phi_n^{hk})),\Delta(\psi_n-z^h))_Y\\
&\quad-(a_0(T_n-T_n^{hk}-m\Delta(T_n-T_n^{hk}))-(a_2(\psi_n-\psi_n^{hk}-m\Delta(\psi_n-\psi_n^{hk})),\Delta(\psi_n-z^h))_Y.
\end{aligned}$$

Taking into account that

$$\begin{aligned}
&-(a_0(T_n-T_n^{hk}),\Delta(\psi_n-\psi_n^{hk}))_Y=a_0(\nabla(T_n-T_n^{hk}),\nabla(\psi_n-\psi_n^{hk}))_H,\\
&-(a_1(\phi_n-\phi_n^{hk}),\Delta(\psi_n-\psi_n^{hk}))_Y=a_1(\nabla(\phi_n-\phi_n^{hk}),\nabla(\psi_n-\psi_n^{hk}))_H\\
&\qquad\geq a_1(\nabla(\phi_n-\phi_n^{hk}),\nabla(\dot{\phi}_n-\delta\phi_n))_H+\frac{a_1}{2k}\Big\{\|\nabla(\phi_n-\phi_n^{hk})\|_H^2-\|\nabla(\phi_{n-1}-\phi_{n-1}^{hk})\|_H^2\Big\},\\
&-(a_2(\psi_n-\psi_n^{hk}),\Delta(\psi_n-\psi_n^{hk}))_Y=a_2(\nabla(\psi_n-\psi_n^{hk}),\nabla(\psi_n-\psi_n^{hk}))_H\geq a_2\|\nabla(\psi_n-\psi_n^{hk})\|_H^2,\\
&(\Delta(\phi_n-\phi_n^{hk}),\Delta(\psi_n-\psi_n^{hk}))_Y\geq(\Delta(\phi_n-\phi_n^{hk}),\Delta(\dot{\phi}_n-\delta\phi_n))_Y\\
&\qquad\qquad+\frac{1}{2k}\Big\{\|\Delta(\phi_n-\phi_n^{hk})\|_Y^2-\|\Delta(\phi_{n-1}-\phi_{n-1}^{hk})\|_Y^2\Big\},\\
&a_2m(\Delta(\psi_n-\psi_n^{hk}),\Delta(\psi_n-\psi_n^{hk}))_Y=a_2m\|\Delta(\psi_n-\psi_n^{hk})\|_Y^2,\\
&-(\psi_n-\psi_n^{hk},\Delta(\psi_n-z^h))_Y=(\nabla(\psi_n-\psi_n^{hk}),\nabla(\psi_n-z^h))_H,\\
&-(\phi_n-\phi_n^{hk},\Delta(\psi_n-z^h))_Y=(\nabla(\phi_n-\phi_n^{hk}),\nabla(\psi_n-z^h))_H,\\
&-(T_n-T_n^{hk},\Delta(\psi_n-z^h))_Y=(\nabla(T_n-T_n^{hk}),\nabla(\psi_n-z^h))_H,
\end{aligned}$$

we find that

$$\begin{aligned}
&\frac{1}{2k}\Big\{\|\nabla(\phi_n-\phi_n^{hk})\|_H^2-\|\nabla(\phi_{n-1}-\phi_{n-1}^{hk})\|_H^2\Big\}+\|\Delta(\psi_n-\psi_n^{hk})\|_Y^2+\|\nabla(\psi_n-\psi_n^{hk})\|_H^2\\
&\qquad+\frac{1}{2k}\Big\{\|\Delta(\phi_n-\phi_n^{hk})\|_Y^2-\|\Delta(\phi_{n-1}-\phi_{n-1}^{hk})\|_Y^2\Big\}\\
&\quad\leq C\Big(\|R_n-R_n^{hk}\|_Y^2+\|\Delta(\psi_n-z^h)\|_Y^2+\|\nabla(\dot{\phi}_n-\delta\phi_n)\|_H^2+\|\Delta(\dot{\phi}_n-\delta\phi_n)\|_Y^2\\
&\qquad+\|\nabla(\phi_n-\phi_n^{hk})\|_H^2+\|\Delta(\phi_n-\phi_n^{hk})\|_Y^2+\|\Delta(T_n-T_n^{hk})\|_Y^2+\|\nabla(\psi_n-z^h)\|_H^2\\
&\qquad+\|\nabla(T_n-T_n^{hk})\|_H^2\Big),
\end{aligned}$$

and therefore,

$$
\begin{aligned}
&\|\nabla(\phi_n-\phi_n^{hk})\|_H^2+\|\Delta(\phi_n-\phi_n^{hk})\|_Y^2+k\sum_{j=1}^n\Big\{\|\Delta(\psi_j-\psi_j^{hk})\|_Y^2+\|\nabla(\psi_j-\psi_j^{hk})\|_H^2\Big\}\\
&\quad\le Ck\sum_{j=1}^n\Big(\|R_j-R_j^{hk}\|_Y^2+\|\Delta(\psi_j-z_j^h)\|_Y^2+\|\nabla(\dot\phi_j-\delta\phi_j)\|_H^2+\|\Delta(\dot\phi_j-\delta\phi_j)\|_Y^2\\
&\qquad+\|\nabla(\phi_j-\phi_j^{hk})\|_H^2+\|\Delta(\phi_j-\phi_j^{hk})\|_Y^2+\|\Delta(T_j-T_j^{hk})\|_Y^2+\|\nabla(\psi_j-z_j^h)\|_H^2\\
&\qquad+\|\nabla(T_j-T_j^{hk})\|_H^2\Big)+C\|T^1-T^{1h}\|_{H^2(B)}^2. \qquad (40)
\end{aligned}
$$

Now, if we combine estimates (39) and (40) it follows that

$$
\begin{aligned}
&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+\|R_n-R_n^{hk}\|_Y^2\\
&\qquad+\|\nabla(\phi_n-\phi_n^{hk})\|_H^2+\|\Delta(\phi_n-\phi_n^{hk})\|_Y^2+k\sum_{j=1}^n\Big\{\|\Delta(\psi_j-\psi_j^{hk})\|_Y^2+\|\nabla(\psi_j-\psi_j^{hk})\|_H^2\Big\}\\
&\quad\le Ck\sum_{j=1}^n\Big(\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2+\|\nabla(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Q^2+\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2\\
&\qquad+\|\mathrm{div}\,(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Y^2+\|R_j-R_j^{hk}\|_Y^2+\|\nabla(R_j-\eta_j^h)\|_H^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_H^2\\
&\qquad+(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H+\|\dot R_j-\delta R_j\|_Y^2+\|R_j-\eta_j^h\|_Y^2+\|\Delta(T_j-T_j^{hk})\|_Y^2\\
&\qquad+\|\Delta(\phi_j-\phi_j^{hk})\|_Y^2+(\delta R_j-\delta R_j^{hk},R_j-\eta_j^h)_Y+\|\Delta(\psi_j-z_j^h)\|_Y^2+\|\nabla(\dot\phi_j-\delta\phi_j)\|_H^2\\
&\qquad+\|\Delta(\dot\phi_j-\delta\phi_j)\|_Y^2+\|\nabla(T_j-T_j^{hk})\|_H^2+\|\nabla(\phi_j-\phi_j^{hk})\|_H^2\Big)+C\Big(\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_H^2\\
&\qquad+\|\mathrm{div}\,(\boldsymbol{u}^0-\boldsymbol{u}^{0h})\|_Y^2+\|\nabla(\boldsymbol{u}^0-\boldsymbol{u}^{0h})\|_Q^2+\|R^0-R^{0h}\|_Y^2+\|T^1-T^{1h}\|_{H^2(B)}^2\Big).
\end{aligned}
$$

Now, we observe that

$$
\begin{aligned}
k\sum_{j=1}^n(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H&=\sum_{j=1}^n(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}-(\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk}),\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H\\
&=(\rho(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H+(\rho(\boldsymbol{v}^{0h}-\boldsymbol{v}^0),\boldsymbol{v}_1-\boldsymbol{w}_1^h)_H\\
&\quad+\sum_{j=1}^{n-1}(\rho(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}),\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h))_H,\\
k\sum_{j=1}^n(\delta R_j-\delta R_j^{hk},R_j-z_j^h)_Y&=\sum_{j=1}^n(R_j-R_j^{hk}-(R_{j-1}-R_{j-1}^{hk}),R_j-z_j^h)_Y\\
&=(R_n-R_n^{hk},R_n-z_n^h)_Y+(R^{0h}-R^0,R_1-z_1^h)_Y\\
&\quad+\sum_{j=1}^{n-1}(R_j-R_j^{hk},R_j-z_j^h-(R_{j+1}-z_{j+1}^h))_Y,\\
\|\nabla(T_n-T_n^{hk})\|_H^2&\le C\Big(\|\nabla(T^0-T^{0h})\|_H^2+I_n+k\sum_{j=1}^n\|\nabla(\phi_n-\phi_n^{hk})\|_H^2\Big),\\
\|\Delta(T_n-T_n^{hk})\|_Y^2&\le C\Big(\|\Delta(T^0-T^{0h})\|_Y^2+J_n+k\sum_{j=1}^n\|\Delta(\phi_n-\phi_n^{hk})\|_Y^2\Big),
\end{aligned}
$$

where I_n and J_n are the integration errors given by

$$I_n = \Big\| \int_0^{t_n} \nabla\phi(s)\,ds - k\sum_{j=1}^n \nabla\phi_j \Big\|_H^2, \quad J_n = \Big\| \int_0^{t_n} \Delta\phi(s)\,ds - k\sum_{j=1}^n \Delta\phi_j \Big\|_Y^2. \quad (41)$$

Using Poincaré inequality for the inductive temperature and a discrete version of Gronwall's inequality ([4]) we have the following.

Theorem 5.4. *Let the assumptions (20) hold. If we denote by* $(\boldsymbol{u}, \boldsymbol{v}, \theta, e, \xi, T, \phi, \psi)$ *and* $(\boldsymbol{u}^{hk}, \boldsymbol{v}^{hk}, \theta^{hk}, e^{hk}, \xi^{hk}, T^{hk}, \phi^{hk}, \psi^{hk})$ *the respective solutions to problems* VP *and* VP^{hk}*, then we have the following a priori error estimates, for all* $\boldsymbol{w}^h = \{\boldsymbol{w}_j^h\}_{j=0}^N \subset V^h$ *and* $\eta^h = \{\eta_j^h\}_{j=0}^N$, $z^h = \{z_j^h\}_{j=0}^N \subset W^h$,

$$\begin{aligned}
&\max_{0\le n\le N} \Big\{ \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 + \|R_n - R_n^{hk}\|_Y^2 \\
&\quad + \|\nabla(\phi_n - \phi_n^{hk})\|_H^2 + \|\Delta(\phi_n - \phi_n^{hk})\|_Y^2 + \|\nabla(T_n - T_n^{hk})\|_H^2 + \|\Delta(T_n - T_n^{hk})\|_Y^2 \Big\} \\
&\quad + k\sum_{j=1}^N \Big\{ \|\Delta(\psi_j - \psi_j^{hk})\|_Y^2 + \|\nabla(\psi_j - \psi_j^{hk})\|_H^2 \Big\} \\
&\le Ck\sum_{j=1}^N \Big(\|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_H^2 + \|\boldsymbol{v}_j - \boldsymbol{w}_j^h\|_V^2 + \|\dot{\boldsymbol{u}}_j - \delta\boldsymbol{u}_j\|_V^2 + \|\nabla(R_j - \eta_j^h)\|_H^2 + \|\Delta(\dot{\phi}_j - \delta\phi_j)\|_Y^2 \\
&\quad + \|\dot{R}_j - \delta R_j\|_Y^2 + \|R_j - \eta_j^h\|_Y^2 + \|\Delta(\psi_j - z_j^h)\|_Y^2 + \|\nabla(\dot{\phi}_j - \delta\phi_j)\|_H^2 + I_j + J_j \Big) \\
&\quad + C\max_{0\le n\le N} \Big\{ \|\boldsymbol{v}_n - \boldsymbol{w}_n^h\|_H^2 + \|R_n - z_n^h\|_Y^2 \Big\} + \frac{C}{k}\sum_{j=1}^{N-1} \|R_j - z_j^h - (R_{j+1} - z_{j+1}^h)\|_Y^2 \\
&\quad + \frac{C}{k}\sum_{j=1}^{N-1} \|\boldsymbol{v}_j - \boldsymbol{w}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{w}_{j+1}^h)\|_H^2 + C\Big(\|\boldsymbol{v}^0 - \boldsymbol{v}^{0h}\|_H^2 + \|\boldsymbol{u}^0 - \boldsymbol{u}^{0h}\|_V^2 \\
&\quad + \|R^0 - R^{0h}\|_Y^2 + \|T^1 - T^{1h}\|_{H^2(B)}^2 + \|T^0 - T^{0h}\|_{H^2(B)}^2 \Big),
\end{aligned}$$

where $C > 0$ *is a positive constant assumed to be independent of the discretization parameters* h *and* k *but depending on the continuous solution, and the integration errors* I_j *and* J_j *are given by (41). We also recall the notations:*

$$R(t) = a_0\theta(t) + a_1 e(t) + a_2\xi(t), \quad R_n^{hk} = a_0\theta_n^{hk} + a_1 e_n^{hk} + a_2\xi_n^{hk}.$$

We note that we can study the convergence order from the previous estimates. Therefore, as an example, we have the following result which states the linear convergence of the approximation under suitable additional regularity conditions (see [4] for details regarding the estimates of the non-usual finite element terms).

Corollary 1. *If we assume that the continuous solution to Problem* VP *has the regularity:*

$$\begin{aligned}
&\boldsymbol{u} \in H^3(0,T;H) \cap C^1([0,T];[H^2(B)]^d) \cap H^2(0,T;V), \\
&\theta \in W^{2,\infty}(0,T;H^2(B)) \cap H^3(0,T;H^1(B)), \quad T \in W^{2,\infty}([0,T];H^3(B)) \cap H^3(0,T;H^2(B)),
\end{aligned}$$

then the approximations provided by Problem VP^{hk} are linearly convergent; i.e., there exists a positive constant $C > 0$ such that

$$\max_{0\le n\le N}\Big\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q+\|R_n-R_n^{hk}\|_Y$$
$$+\|\nabla(\phi_n-\phi_n^{hk})\|_H+\|\Delta(\phi_n-\phi_n^{hk})\|_Y+\|\nabla(T_n-T_n^{hk})\|_H+\|\Delta(T_n-T_n^{hk})\|_Y\Big\}\le C(h+k).$$

Remark 2. If we assume the material to be viscoelastic, that is, if we replace equation (16) by

$$\rho\ddot{u}_i=\mu u_{i,jj}+(\lambda+\mu)u_{j,ji}+\mu^*\dot{u}_{i,jj}+(\lambda^*+\mu^*)\dot{u}_{j,ji}+\beta(a_0\theta+a_1\dot{\theta}+a_2\ddot{\theta})_{,i},$$

where λ^* and μ^* are viscosity parameters, then the above error estimates can be improved.

6. Numerical results.

In order to verify the behavior of the numerical method described in the previous section, some numerical experiments have been performed in two-dimensional problems.

6.1. Numerical scheme.

First, we describe the numerical algorithm used to solve Problem VP^{hk}. So, given $\boldsymbol{v}_{n-1}^{hk}$, ξ_{n-1}^{hk} and ψ_{n-1}^{hk}, the discrete velocity, the discrete thermal acceleration and the discrete inductive thermal acceleration, $\boldsymbol{v}_n^{hk}$, ξ_n^{hk} and ψ_n^{hk}, respectively, are the solution to the following coupled linear system:

$$\begin{aligned}&\rho(\boldsymbol{v}_n^{hk},\boldsymbol{w}^h)_H+\mu k^2(\nabla\boldsymbol{v}_n^{hk},\nabla\boldsymbol{w}^h)_Q+k^2(\lambda+\mu)(\mathrm{div}\,\boldsymbol{v}_n^{hk},\mathrm{div}\,\boldsymbol{w}^h)_Y\\&\qquad+k\beta((a_0k^2+a_1k+a_2)\xi_n^{hk},\mathrm{div}\,\boldsymbol{w}^h)_Y\\&\qquad=\rho(\boldsymbol{v}_{n-1}^{hk},\boldsymbol{w}^h)_H-\mu k(\nabla\boldsymbol{u}_{n-1}^{hk},\nabla\boldsymbol{w}^h)_Q-k(\lambda+\mu)(\mathrm{div}\,\boldsymbol{u}_{n-1}^{hk},\mathrm{div}\,\boldsymbol{w}^h)_Y\\&\qquad\quad-\beta k(a_0\theta_{n-1}^{hk}+(a_0k+a_1)e_{n-1}^{hk},\mathrm{div}\,\boldsymbol{w}^h)_Y,\end{aligned}$$

$$\begin{aligned}&((a_0k^2+a_1k+a_2)\xi_n^{hk}-(a_0k^2+a_1k+a_2)\psi_n^{hk}+mK(a_0k^2+a_1k+a_2)\Delta\psi_n^{hk},z^h)_Y\\&\qquad=(-a_0k\theta_{n-1}^{hk}-(a_0k+a_1)e_{n-1}^{hk}+a_0kT_{n-1}^{hk}+(a_0k+a_1)\phi_{n-1}^{hk}\\&\qquad\quad-mK(a_0k\Delta T_{n-1}^{hk}+a_1\Delta\phi_{n-1}^{hk}),z^h)_Y,\end{aligned}$$

$$\begin{aligned}&c((a_0k^2+a_1k+a_2)\xi_n^{hk},\eta^h)_Y-K((b_0k^3+b_1k^2)\Delta\psi_n^{hk},\eta^h)_Y-\beta(\mathrm{div}\,\boldsymbol{v}_n^{hk},\eta^h)_Y\\&\qquad=c(a_2\xi_{n-1}^{hk}-a_0ke_{n-1}^{hk},\eta^h)_Y+kK(b_0\Delta T_{n-1}^{hk}+(b_0k+b_1)\Delta\phi_{n-1}^{hk},\eta^h)_Y,\end{aligned}$$

where the discrete displacement $\boldsymbol{u}_n^{hk}$, thermal velocity e_n^{hk}, temperature θ_n^{hk}, inductive thermal velocity ϕ_n^{hk} and inductive temperature T_n^{hk} are then recovered from the relations:

$$\begin{aligned}&\boldsymbol{u}_n^{hk}=k\boldsymbol{v}_n^{hk}+\boldsymbol{u}_{n-1}^{hk},\quad e_n^{hk}=k\xi_n^{hk}+e_{n-1}^{hk},\quad\theta_n^{hk}=ke_n^{hk}+\theta_{n-1}^{hk},\\&\phi_n^{hk}=k\psi_n^{hk}+\phi_{n-1}^{hk},\quad T_n^{hk}=k\phi_n^{hk}+T_{n-1}^{hk},\end{aligned}$$

We note that this discrete problem consists of three coupled symmetric linear equations, and so Cholesky's method is used for the matrix factorization written in terms of a product variable.

The numerical scheme was implemented using FreeFEM++ (see [17] for details) on a Intel Core i5-3337U @ 1.80GHz and a typical run (100 step times and 1000 nodes) took about 200 seconds of CPU time.

6.2. **Numerical convergence.** We consider the following academic problem:

Problem $\mathbf{P}^{ex}$. *Find the displacements* $\boldsymbol{u} : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}^2$, *the temperature* $\theta : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}$ *and the inductive temperature* $T : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}$ *such that*

$$
\begin{aligned}
&\ddot{u}_i - 10u_{i,jj} - 20u_{j,ji} - (\theta + \dot{\theta} + \frac{1}{2}\ddot{\theta})_{,i} = H_i \quad \textit{in } (0,1)\times(0,1)\times(0,1),\\
&\dot{\theta} + \ddot{\theta} + \frac{1}{2}\dddot{\theta} - \Delta T - \Delta \dot{T} = \operatorname{div} \dot{\boldsymbol{u}} + P, \quad \textit{in } (0,1)\times(0,1)\times(0,1),\\
&u_i(x,y,t) = T(x,y,t) = 0 \quad \textit{for} \quad i = 1,2 \textit{ and } (x,y,t) \in \partial([0,1]\times[0,1])\times(0,1),\\
&u_i(x,y,0) = x^2y^2(1-x)^2(1-y)^2 \quad \textit{for} \quad i=1,2 \textit{ and } (x,y) \in [0,1]\times[0,1]\\
&\dot{u}_i(x,y,0) = x^2y^2(1-x)^2(1-y)^2 \quad \textit{for} \quad i=1,2 \textit{ and } (x,y) \in [0,1]\times[0,1],\\
&T(x,y,0) = x^2y^2(1-x)^2(1-y)^2 \quad \textit{for} \quad (x,y) \in [0,1]\times[0,1],\\
&\dot{T}(x,y,0) = x^2y^2(1-x)^2(1-y)^2 \quad \textit{for} \quad (x,y) \in [0,1]\times[0,1],\\
&\ddot{T}(x,y,0) = x^2y^2(1-x)^2(1-y)^2 \quad \textit{for} \quad (x,y) \in [0,1]\times[0,1],
\end{aligned}
$$

where the (artificial) body forces $\mathbf{H} = (H_1, H_2)$ and the heat supply P are given by

$$
\begin{aligned}
H_1(x,y,t) = e^t \Big(& x^4y^4 - 2x^4y^3 - 119x^4y^2 + 120x^4y - 20x^4 - 14x^3y^4 - 292x^3y^3 + 850x^3y^2\\
&-544x^3y + 64x^3 - 341x^2y^4 + 1162x^2y^3 - 1397x^2y^2 + 576x^2y - 56x^2 + 426xy^4\\
&-1012xy^3 + 738xy^2 - 152xy + 12x - 96y^4 + 192y^3 - 96y^2\Big),\\
H_2(x,y,t) = e^t \Big(& x^4y^4 - 14x^4y^3 - 341x^4y^2 + 426x^4y - 96x^4 - 2x^3y^4 - 292x^3y^3 + 1162x^3y^2\\
&-1012x^3y + 192x^3 - 119x^2y^4 + 850x^2y^3 - 1397x^2y^2 + 738x^2y - 96x^2 + 120xy^4\\
&-544xy^3 + 576xy^2 - 152xy - 20y^4 + 64y^3 - 56y^2 + 12y\Big),\\
P(x,y,t) = e^t \Big(& 5x^2y^2(x-1)^2(y-1)^2/2 - 9x^2y^2(y-1)^2 - 9x^2(x-1)^2(y-1)^2\\
&-9y^2(x-1)^2(y-1)^2 - x^2y^2(2x-2)(y-1)^2 - x^2y^2(2y-2)(x-1)^2 - 9x^2y^2(x-1)^2\\
&-18xy^2(2x-2)(y-1)^2 - 18x^2y(2y-2)(x-1)^2 - 2xy^2(x-1)^2(y-1)^2\\
&-2x^2y(x-1)^2(y-1)^2\Big).
\end{aligned}
$$

We note that Problem P^{ex} corresponds to Problem (16)-(19) with the following data:

$$
\begin{aligned}
&B = (0,1)\times(0,1), \quad T_f = 1, \quad \rho = 1, \quad \lambda = \mu = 10, \quad a_0 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{2},\\
&\beta = 1, \quad m = 1, \quad b_0 = 1, \quad b_1 = 1, \quad K = 1, \quad c = 1,\\
&u_i^0(x,y) = v_i^0(x,y) = x^2y^2(x-1)^2(y-1)^2 \quad \text{for all } (x,y) \in [0,1]\times[0,1],\\
&\theta^0(x,y) = \theta^1(x,y) = \theta^2(x,y) = x^2y^2(x-1)^2(y-1)^2 \quad \text{for all } (x,y) \in [0,1]\times[0,1].
\end{aligned}
$$

The exact solution to Problem P^{ex} has the following form:

$$
\begin{aligned}
&T(x,y,t) = e^t x^2y^2(x-1)^2(y-1)^2 \quad \text{for} \quad (x,y,t) \in [0,1]\times[0,1]\times[0,1],\\
&u_i(x,y,t) = e^t x^2y^2(x-1)^2(y-1)^2 \quad \text{for i=1,2} \quad \text{and} \quad (x,y,t) \in [0,1]\times[0,1]\times[0,1].
\end{aligned}
$$

The numerical errors given by

$$
\begin{aligned}
\max_{0\le n\le N} \Big\{ & \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H + \|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q + \|R_n - R_n^{hk}\|_Y\\
&+\|\nabla(\phi_n - \phi_n^{hk})\|_H + \|\Delta(\phi_n - \phi_n^{hk})\|_Y + \|\nabla(T_n - T_n^{hk})\|_H + \|\Delta(T_n - T_n^{hk})\|_Y \Big\},
\end{aligned}
$$

$n_{el} \downarrow k \rightarrow$	0.1	0.05	0.02	0.01	0.005
8	0.9251309	0.6235920	0.4613584	0.4168623	0.3989068
16	0.6973870	0.3659938	0.1719483	0.1109517	0.0833035
32	0.6726871	0.3381078	0.1391413	0.0734105	0.0411872
64	0.6725218	0.3349879	0.1340351	0.0677103	0.0347045
128	0.6732862	0.3355613	0.1338216	0.0668053	0.0334754

TABLE 1. Example 1: Numerical errors ($\times 10^3$) for some discretization parameters.

and obtained for different discretization parameters nd and k, are depicted in Table 1 (being nd the number of subdivisions on each outer side of the square). Moreover, their evolution depending on the parameter $h+k$ is plotted in FIGURE 1. We observe that the convergence of the numerical scheme is clearly obtained but the linear convergence, shown in Corollary 1, is not achieved.

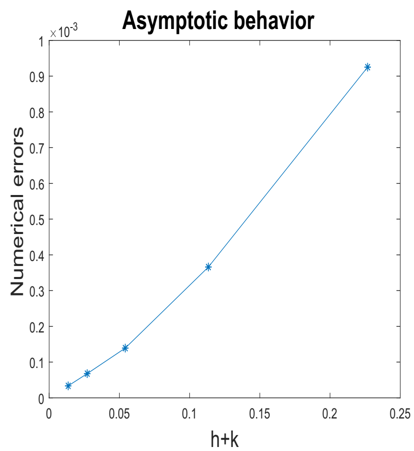

FIGURE 1. Example 1: Asymptotic behavior of the numerical scheme.

If we assume now that there are not volume forces nor heat supply, and we use as final time $T_f = 2\ s$, (with the same data and mechanical initial conditions than in the previous example), taking the discretization parameters $nd = 32$ and $k = 0.01$, the evolution in time of the discrete energy E_n^{hk}, defined by (34), is plotted in FIGURE 2 in both usual and semilog scales. The energy converges to zero and an exponential decay seems to be achieved; however, we note that such behavior is not found in the continuous case (see, for instance, [19] in the analysis of the dual-phase-lag case).

6.3. **Dependence on the thermal coefficient m.** In this second example, we study the dependence of the solution with respect to parameter m. In particular, we consider the domain $B = (0,8) \times (0,1)$ and the final time $T_f = 0.5$. In these simulations we use the following data:

$$\rho = 1, \quad \lambda = \mu = 10, \quad a_0 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{2}, \quad \beta = 1, \quad b_0 = 1,$$
$$K = 1, \quad c = 1, \quad b_1 = 1,$$

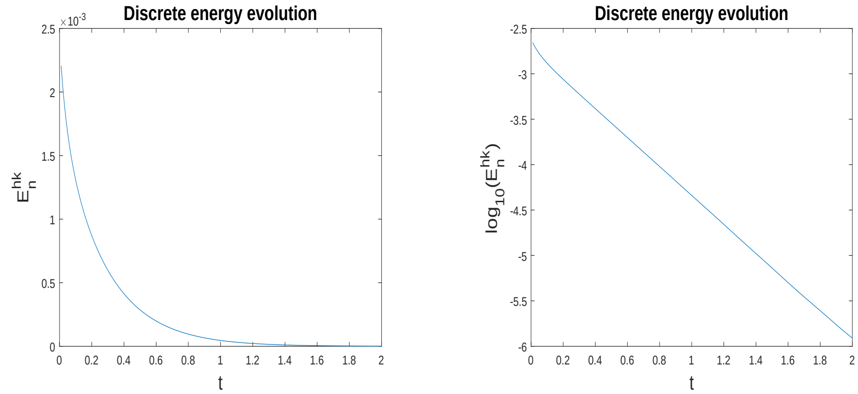

FIGURE 2. Example 1: Energy evolution in absolute and semilogarithmic scales.

and the initial conditions, for all $(x, y) \in [0, 8] \times [0, 1]$,

$$u_i^0(x, y) = v_i^0(x, y) = 0,$$
$$T^0(x, y) = \max\{(x - 3)(5 - x)y(1 - y), 0\}, \quad T^1(x, y) = T^2(x, y) = 0.$$

We solve Problem VP^{hk} with the time discretization parameter $k = 0.001$ and a fixed spatial finite element mesh. Thus, in FIGURE 3 we plot the evolution in time of the temperature and inductive temperature at point $\boldsymbol{x} = (4, 0.5)$ for different values of parameter m (varying between 0.5 and 0.005). As we can see, the inductive temperatures almost coincide and important differences appear for the temperature.

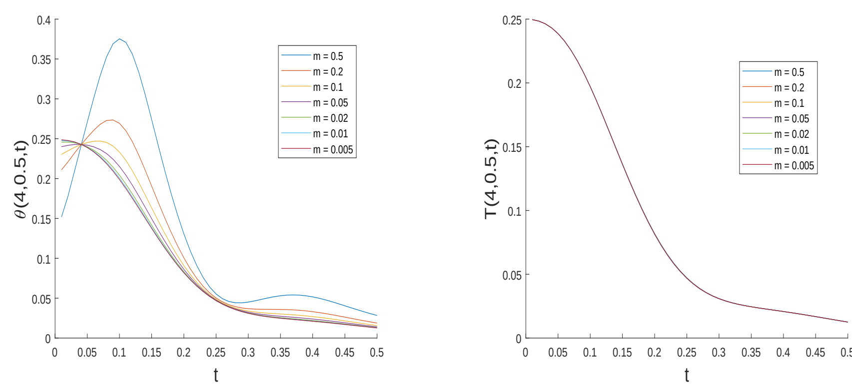

FIGURE 3. Example 2: Evolution in time of the temperature and inductive temperature at point $\boldsymbol{x} = (4, 0.5)$ for different values of parameter m.

Now, in FIGURE 4 we plot the evolution in time of the temperature and inductive temperature at point $\boldsymbol{x} = (1, 0.5)$ for the same values of parameter m. Again, the inductive temperatures are rather similar, although important differences are found for the temperatures.

Finally, in FIGURE 5 the evolution in time of the horizontal and vertical displacements at point $\boldsymbol{x} = (1, 0.5)$ for the above values of parameter m. We can also observe the differences among the corresponding solutions.

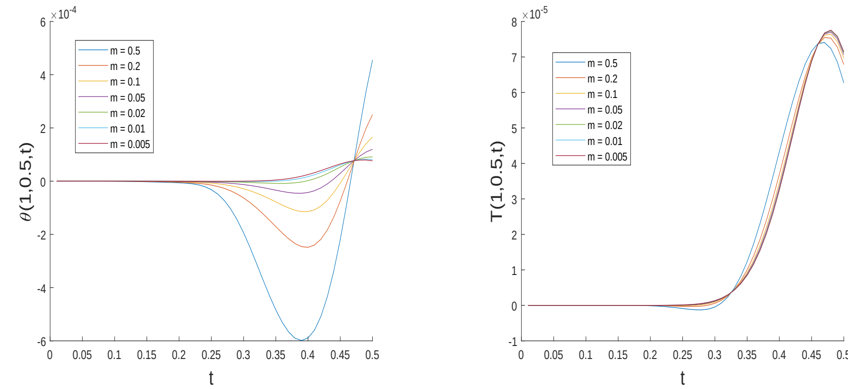

FIGURE 4. Example 2: Evolution in time of the temperature and inductive temperature at point $\boldsymbol{x} = (1, 0.5)$ for different values of parameter m.

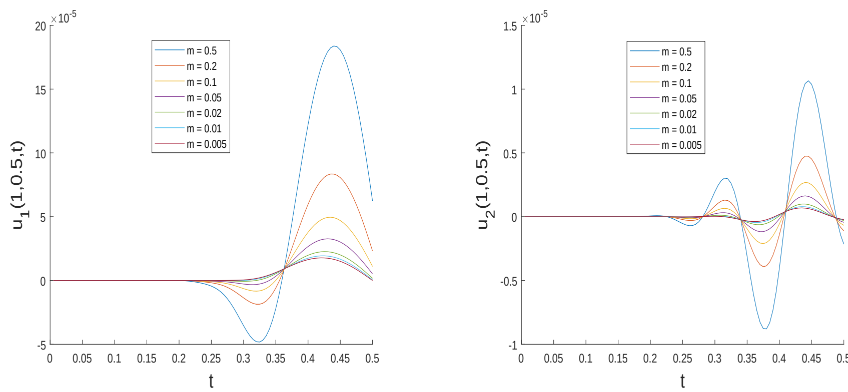

FIGURE 5. Example 2: Evolution in time of the horizontal and vertical displacements at point $\boldsymbol{x} = (1, 0.5)$ for different values of parameter m.

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